# AN EXPLICIT FORMULA FOR THE CUBIC SZEGŐ EQUATION 

PATRICK GÉRARD AND SANDRINE GRELLIER


#### Abstract

We derive an explicit formula for the general solution of the cubic Szegő equation and of the evolution equation of the corresponding hierarchy. As an application, we prove that all the solutions corresponding to finite rank Hankel operators are quasiperiodic.


## 1. Introduction

This paper is a continuation of the study of dynamical properties of an integrable system introduced by the authors in [2, 3. As an evolution equation, the cubic Szegő equation is a simple model of nondispersive dynamics. More precisely, it can be identified as a first order Birkhoff normal form for a certain nonlinear wave equation; see [4]. As a Hamiltonian equation, it was proved in [2] to admit a Lax pair and finite dimensional invariant submanifolds corresponding to some finite rank conditions. In [3], action angle variables were introduced on generic subsets of the phase space, and on open dense subsets of the finite rank submanifolds. However, unlike the KdV equation or the one dimensional cubic nonlinear Schrödinger equation, this integrable system displays some degeneracy, since the collection of its conservation laws does not control the high regularity of the solution, as observed in [2]. An important consequence of this instability phenomenon is that the action angle variables cannot be extended to the whole phase space, even when restricted to one of the finite rank submanifolds. Our purpose in this paper is to prove a formula for the general solution of the initial value problem for this equation. In the case of generic data, this formula reduces to the one given by the action angle variables above. However, the formula enables us to study the nongeneric case too, and allows us in particular to establish the quasiperiodicity of all solutions lying in one of the above finite rank submanifolds, despite the already mentioned lack of a global system of action-angle variables. Finally, this formula is also very useful to revisit the instability phenomenon displayed in [2]. We now introduce the general setting of this equation.
1.1. The setting. Let $\mathbb{T}=\mathbb{R} / 2 \pi \mathbb{Z}$, endowed with the Haar integral

$$
\int_{\mathbb{T}} f:=\frac{1}{2 \pi} \int_{0}^{2 \pi} f(x) d x
$$

[^0]On $L^{2}(\mathbb{T})$, we use the inner product

$$
(f \mid g):=\int_{\mathbb{T}} f \bar{g} .
$$

The family of functions $\left(\mathrm{e}^{i k x}\right)_{k \in \mathbb{Z}}$ is an orthonormal basis of $L^{2}(\mathbb{T})$, on which the components of $f \in L^{2}(\mathbb{T})$ are the Fourier coefficients

$$
\hat{f}(k):=\left(f \mid \mathrm{e}^{i k x}\right) .
$$

We introduce the closed subspace

$$
L_{+}^{2}(\mathbb{T}):=\left\{u \in L^{2}(\mathbb{T}): \forall k<0, \hat{u}(k)=0\right\} .
$$

Notice that elements $u \in L_{+}^{2}(\mathbb{T})$ identify to traces of holomorphic functions $\underline{u}$ on the unit disc $D$ such that

$$
\sup _{r<1} \int_{0}^{2 \pi}\left|\underline{u}\left(r e^{i x}\right)\right|^{2} d x<\infty
$$

via the correspondence

$$
\underline{u}(z):=\sum_{k=0}^{\infty} \hat{u}(k) z^{k}, z \in D, u(x)=\lim _{r \rightarrow 1} \underline{u}\left(r e^{i x}\right),
$$

which establishes a bijective isometry between $L_{+}^{2}(\mathbb{T})$ and the Hardy space of the disc.

We denote by $\Pi$ the orthogonal projector from $L^{2}(\mathbb{T})$ onto $L_{+}^{2}(\mathbb{T})$, known as the Szegő projector:

$$
\Pi\left(\sum_{k=-\infty}^{\infty} \hat{f}(k) \mathrm{e}^{i k x}\right)=\sum_{k=0}^{\infty} \hat{f}(k) \mathrm{e}^{i k x} .
$$

On $L_{+}^{2}(\mathbb{T})$, we introduce the symplectic form

$$
\omega\left(h_{1}, h_{2}\right)=\operatorname{Im}\left(h_{1} \mid h_{2}\right)
$$

The densely defined energy functional

$$
E(u):=\frac{1}{4} \int_{\mathbb{T}}|u|^{4}
$$

formally corresponds to the Hamiltonian evolution equation,

$$
\begin{equation*}
i \dot{u}=\Pi\left(|u|^{2} u\right), \tag{1}
\end{equation*}
$$

which we called the cubic Szegő equation. In [2], we solved the initial value problem for this equation on the intersections of Sobolev spaces with $L_{+}^{2}(\mathbb{T})$. More precisely, define, for $s \geq 0$,

$$
H_{+}^{s}(\mathbb{T}):=H^{s}(\mathbb{T}) \cap L_{+}^{2}(\mathbb{T})=\left\{u \in L_{+}^{2}(\mathbb{T}): \sum_{k=0}^{\infty}|\hat{u}(k)|^{2}\left(1+k^{2}\right)^{s}<\infty\right\}
$$

Then equation (11) defines a smooth flow on $H_{+}^{s}(\mathbb{T})$ for $s>\frac{1}{2}$, and a continuous flow on $H_{+}^{\frac{1}{2}}(\mathbb{T})$. The main result of this paper provides an explicit formula for the solution of this initial value problem.
1.2. Hankel operators and the explicit formula. Let $u \in H_{+}^{\frac{1}{2}}(\mathbb{T})$. We denote by $H_{u}$ the $\mathbb{C}$-antilinear operator defined on $L_{+}^{2}(\mathbb{T})$ as

$$
H_{u}(h)=\Pi(u \bar{h}), h \in L_{+}^{2}(\mathbb{T}) .
$$

In terms of Fourier coefficients, this operator reads

$$
\widehat{H_{u}(h)}(n)=\sum_{p=0}^{\infty} \hat{u}(n+p) \overline{\hat{h}(p)}
$$

In particular, its Hilbert-Schmidt norm $\|\cdot\|_{H S}$ is finite since $u \in H_{+}^{\frac{1}{2}}(\mathbb{T})$ and

$$
\begin{equation*}
\left\|H_{u}\right\|_{\mathcal{L}\left(L_{+}^{2}\right)} \leq\left\|H_{u}\right\|_{H S} \simeq\|u\|_{H^{1 / 2}} \tag{2}
\end{equation*}
$$

We call $H_{u}$ the Hankel operator of symbol $u$. Notice that this definition is different from the standard ones used in references [9, [11, where Hankel operators were rather defined as linear operators from $L_{+}^{2}$ into its orthogonal complement. The link between these two definitions can be easily established by means of the involution

$$
f^{\sharp}(x)=\mathrm{e}^{-i x} \overline{f(x)} .
$$

Notice that, with our definition, $H_{u}$ satisfies the following self-adjointness identity:

$$
\begin{equation*}
\left(H_{u}\left(h_{1}\right) \mid h_{2}\right)=\left(H_{u}\left(h_{2}\right) \mid h_{1}\right), h_{1}, h_{2} \in L_{+}^{2}(\mathbb{T}) \tag{3}
\end{equation*}
$$

In particular, $H_{u}$ is a $\mathbb{R}$-linear symmetric operator for the real inner product

$$
\left\langle h_{1}, h_{2}\right\rangle:=\operatorname{Re}\left(h_{1} \mid h_{2}\right) .
$$

A fundamental property of Hankel operators is their connection with the shift operator $S$, defined on $L_{+}^{2}(\mathbb{T})$ as

$$
S u(x)=\mathrm{e}^{i x} u(x) .
$$

This property reads

$$
S^{*} H_{u}=H_{u} S=H_{S^{*} u},
$$

where $S^{*}$ denotes the adjoint of $S$. We denote by $K_{u}$ this operator, and call it the shifted Hankel operator of symbol $u$. Notice that $K_{u}$ is Hilbert-Schmidt and symmetric as well. As a consequence, operators $H_{u}^{2}$ and $K_{u}^{2}$ are $\mathbb{C}$-linear trace class positive operators on $L_{+}^{2}(\mathbb{T})$. Moreover, they are related by the following important identity:

$$
\begin{equation*}
K_{u}^{2}=H_{u}^{2}-(\cdot \mid u) u \tag{4}
\end{equation*}
$$

Theorem 1. Let $u_{0} \in H_{+}^{\frac{1}{2}}(\mathbb{T})$, and let $u \in C\left(\mathbb{R}, H_{+}^{\frac{1}{2}}(\mathbb{T})\right)$ be the solution of equation (1) such that $u(0)=u_{0}$. Then

$$
\underline{u}(t, z)=\left(\left(I-z \mathrm{e}^{-i t H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*}\right)^{-1} \mathrm{e}^{-i t H_{u_{0}}^{2}} u_{0} \mid 1\right)
$$

The proof of this theorem will be given in section 3 It is a nontrivial consequence of the Lax pair structure recalled in section2, Our second result concerns the special case of data $u_{0}$ such that $H_{u_{0}}$ is of finite rank. In this case, operators $S^{*}, H_{u_{0}}^{2}, K_{u_{0}}^{2}$ act on a finite dimensional space containing $u_{0}$, and the implementation of the above formula reduces to diagonalization of matrices.
1.3. Finite rank manifolds and quasiperiodicity. Let $d$ be a positive integer. We denote by $\mathcal{V}(d)$ the set of $u \in H_{+}^{\frac{1}{2}}(\mathbb{T})$ such that

$$
\operatorname{rk} H_{u}=\left[\frac{d+1}{2}\right], \operatorname{rk} K_{u}=\left[\frac{d}{2}\right],
$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Using Kronecker's theorem [6, 11, [9], one can show that $\mathcal{V}(d)$ is a complex Kähler submanifold of $L_{+}^{2}(\mathbb{T})$ of dimension $d$ (see the appendix of [2]), consisting of rational functions of $\mathrm{e}^{i x}$. More precisely, $\mathcal{V}(d)$ consists of functions of the form

$$
u(x)=\frac{A\left(\mathrm{e}^{i x}\right)}{B\left(\mathrm{e}^{i x}\right)},
$$

where $A, B$ are polynomials with no common factors, $B$ has no zero in the closed unit disc, $B(0)=1$, and

- If $d=2 N$ is even, the degree of $A$ is at most $N-1$ and the degree of $B$ is exactly $N$.
- If $d=2 N+1$ is odd, the degree of $A$ is exactly $N$ and the degree of $B$ is at most $N$.
Using the Lax pair structure recalled in section 2, $\mathcal{V}(d)$ is invariant through the flow of (11). We now state the second result of this paper. In the sequel, $\mathbb{S}^{1}$ denotes the unit circle of complex numbers of modulus 1.

Theorem 2. For every $u_{0} \in \mathcal{V}(d)$, the map

$$
t \in \mathbb{R} \mapsto u(t) \in \mathcal{V}(d)
$$

is quasiperiodic. More precisely, there exist a positive integer n, real numbers $\omega_{1}, \cdots, \omega_{n}$, and a smooth mapping

$$
\Phi:\left(\mathbb{S}^{1}\right)^{n} \rightarrow \mathcal{V}(d)
$$

such that, for every $t \in \mathbb{R}$,

$$
u(t)=\Phi\left(\mathrm{e}^{i \omega_{1} t}, \cdots, \mathrm{e}^{i \omega_{n} t}\right) .
$$

In particular, for every $s>\frac{1}{2}$,

$$
\begin{equation*}
\sup _{t \in \mathbb{R}}\|u(t)\|_{H^{s}}<+\infty . \tag{5}
\end{equation*}
$$

Notice that property (5) was established in Theorem 7.1 of [2] under the additional generic assumption that $u_{0}$ belongs to $\mathcal{V}(d)_{\text {gen }}$, namely that the vectors $H_{u_{0}}^{2 n}(1), n=1, \ldots, N=\left[\frac{d+1}{2}\right]$, are linearly independent. Our general formula allows us to extend property (5) to all data in $\mathcal{V}(d)$. However, it should be emphasized that, while it is clear from the arguments of Lemma 5 in [2] that estimate (5) is uniform if $u_{0}$ varies in a compact subset of $\mathcal{V}(d)_{\text {gen }}$, (5) does not follow from an a priori estimate on the whole of $\mathcal{V}(d)$, in the sense that one can find families of data $\left(u_{0}^{\varepsilon}\right)$ in $\mathcal{V}(d)$, belonging to a compact subset of $\mathcal{V}(d)$, in particular bounded in all $H^{s}$, and such that

$$
\sup _{\varepsilon} \sup _{t \in \mathbb{R}}\left\|u^{\varepsilon}(t)\right\|_{H^{s}}=\infty, s>\frac{1}{2}
$$

see corollary 5 of [2]. We shall revisit this phenomenon in section 4 thanks to the explicit formula of Theorem 1 .

Another natural observation is that property (5) is in sharp contrast with the results obtained by O. Pocovnicu for the cubic Szegő equation on the real line [12. Indeed, in [12], Pocovnicu obtains an explicit formula for all rational solutions of the cubic Szegő equation on the line, and displays a nongeneric rational solution of this equation such that, for every $s>\frac{1}{2}$,

$$
\|u(t)\|_{H^{s}} \underset{t \rightarrow \infty}{\longrightarrow} \infty
$$

This illustrates how the growing phenomenon of high Sobolev norms may differ depending on the domain of the equation.

Finally, let us mention that the generalization of property (5) to nonfinite rank solutions is an open problem.

Note added in proof. Since this paper was accepted, we proved that property (5) fails for generic initial data in $H^{s}$ for every $s>\frac{1}{2}$.
1.4. Organization of the paper. Section 2 is devoted to recalling the crucial Lax pair structure attached to equation (11). As a fundamental consequence, $H_{u(t)}$ and $K_{u(t)}$ remain unitarily equivalent to their respective initial data. In section 3, we take advantage of this structure to derive Theorem [1 In section [4 we apply this theorem to the particular case of data $u_{0}$ belonging to $\mathcal{V}(3)$, which sheds a new light on the instability phenomenon. The next two sections are devoted to the proof of Theorem 2. As a preparation, we first generalize the explicit formula to Hamiltonian flows associated to energies

$$
J^{y}(u):=\left(\left(I+y H_{u}^{2}\right)^{-1}(1) \mid 1\right),
$$

where $y$ is a positive parameter. The quasiperiodicity theorem then follows by observing, through an interpolation argument, that the map $\Phi$ in the statement of Theorem 22 can be defined as the value at time 1 of the Hamiltonian flow corresponding to a suitable linear combination of energies $J^{y}$.

## 2. The Lax pair structure

In this section, we recall the Lax pairs associated to the cubic Szegő equation; see [2], 3]. First we introduce the notion of a Toeplitz operator. Given $b \in L^{\infty}(\mathbb{T})$, we define $T_{b}: L_{+}^{2} \rightarrow L_{+}^{2}$ as

$$
T_{b}(h)=\Pi(b h), h \in L_{+}^{2} .
$$

Notice that $T_{b}$ is bounded and $T_{b}^{*}=T_{\bar{b}}$. The starting point is the following lemma.
Lemma 1. Let $a, b, c \in H_{+}^{s}, s>\frac{1}{2}$. Then

$$
H_{\Pi(a \bar{b} c)}=T_{a \bar{b}} H_{c}+H_{a} T_{b \bar{c}}-H_{a} H_{b} H_{c} .
$$

Proof. Given $h \in L_{+}^{2}$, we have

$$
\begin{aligned}
H_{\Pi(a \bar{b})}(h) & =\Pi(a \bar{b} c \bar{h})=\Pi(a \bar{b} \Pi(c \bar{h}))+\Pi(a \bar{b}(I-\Pi)(c \bar{h})) \\
& =T_{a \bar{b}} H_{c}(h)+H_{a}(g) g:=b \overline{(I-\Pi)(c \bar{h})} .
\end{aligned}
$$

Since $g \in L_{+}^{2}$,

$$
g=\Pi(g)=\Pi(b \bar{c} h)-\Pi(b \overline{\Pi(c \bar{c})})=T_{b \bar{c}}(h)-H_{b} H_{c}(h) .
$$

This completes the proof.

Using Lemma 1 with $a=b=c=u$, we get

$$
\begin{equation*}
H_{\Pi\left(|u|^{2} u\right)}=T_{|u|^{2}} H_{u}+H_{u} T_{|u|^{2}}-H_{u}^{3} \tag{6}
\end{equation*}
$$

Theorem 3. Let $u \in C^{\infty}\left(\mathbb{R}, H_{+}^{s}\right), s>\frac{1}{2}$, be a solution of (1). Then

$$
\begin{aligned}
\frac{d H_{u}}{d t} & =\left[B_{u}, H_{u}\right], B_{u}:=\frac{i}{2} H_{u}^{2}-i T_{|u|^{2}} \\
\frac{d K_{u}}{d t} & =\left[C_{u}, K_{u}\right], C_{u}:=\frac{i}{2} K_{u}^{2}-i T_{|u|^{2}} .
\end{aligned}
$$

Proof. Using equation (11) and identity (61),

$$
\frac{d H_{u}}{d t}=H_{-i \Pi\left(|u|^{2} u\right)}=-i H_{\Pi\left(|u|^{2} u\right)}=-i\left(T_{|u|^{2}} H_{u}+H_{u} T_{|u|^{2}}-H_{u}^{3}\right) .
$$

Using the antilinearity of $H_{u}$, this leads to the first identity. For the second one, we observe that

$$
\begin{equation*}
K_{\Pi\left(|u|^{2} u\right)}=H_{\Pi\left(|u|^{2} u\right)} S=T_{|u|^{2}} H_{u} S+H_{u} T_{|u|^{2}} S-H_{u}^{3} S \tag{7}
\end{equation*}
$$

Moreover, notice that

$$
T_{b}(S h)=S T_{b}(h)+(b S h \mid 1)
$$

In the case $b=|u|^{2}$, this gives

$$
T_{|u|^{2}} S h=S T_{|u|^{2}} h+\left(|u|^{2} S h \mid 1\right) .
$$

Moreover,

$$
\left(|u|^{2} S h \mid 1\right)=(u \mid u \overline{S h})=\left(u \mid K_{u}(h)\right) .
$$

Consequently,

$$
H_{u} T_{|u|^{2}} S h=K_{u} T_{|u|^{2}} h+\left(K_{u}(h) \mid u\right) u .
$$

Coming back to (7), we obtain

$$
K_{\Pi\left(|u|^{2} u\right)}=T_{|u|^{2}} K_{u}+K_{u} T_{|u|^{2}}-\left(H_{u}^{2}-(\cdot \mid u) u\right) K_{u}
$$

Using identity (4), this leads to

$$
\begin{equation*}
K_{\Pi\left(|u|^{2} u\right)}=T_{|u|^{2}} K_{u}+K_{u} T_{|u|^{2}}-K_{u}^{3} . \tag{8}
\end{equation*}
$$

The second identity is therefore a consequence of antilinearity and of

$$
\frac{d K_{u}}{d t}=-i K_{\Pi\left(|u|^{2} u\right)^{2}}
$$

In the sequel, we denote by $\mathcal{L}\left(L_{+}^{2}\right)$ the Banach space of bounded linear operators on $L_{+}^{2}$. Observing that $B_{u}, C_{u}$ are linear and antiselfadjoint, we obtain, following a classical argument due to Lax [7,
Corollary 1. Under the conditions of Theorem 3, define $U=U(t)$ and $V=V(t)$ as the solutions of the following linear ODEs on $\mathcal{L}\left(L_{+}^{2}\right)$ :

$$
\frac{d U}{d t}=B_{u} U, \frac{d V}{d t}=C_{u} V, U(0)=V(0)=I
$$

Then $U(t), V(t)$ are unitary operators and

$$
H_{u(t)}=U(t) H_{u(0)} U(t)^{*}, K_{u(t)}=V(t) K_{u(0)} V(t)^{*}
$$

Remark 1. The notion of a Lax pair is now familiar in the theory of integrable systems. The most famous examples concern the KdV equation 7 and the one dimensional cubic Schrödinger equation [13]. It may seem strange and unusual here that the cubic Szegő equation admits two different Lax pairs. In fact, the eigenvalues of the selfadjoint operators $H_{u}^{2}$ and $K_{u}^{2}$ corresponding to these two Lax pairs are essentially independent, leading to a complete set of actions, as proved in [3]. Indeed, according to Theorems 1.1 and 1.2 of [3], if $u$ is generic, one can fix the eigenvalues $\left(\sigma_{j}^{2}\right)$ of $K_{u}^{2}$ independently of the eigenvalues $\left(\rho_{j}^{2}\right)$ of $H_{u}^{2}$, except for the interlacement constraint

$$
\rho_{1}^{2}>\sigma_{1}^{2}>\rho_{2}^{2}>\sigma_{2}^{2}>\ldots
$$

This suggests that the Lax pair for the cubic Szegő equation should be rather regarded as the pair $\left(L_{u}, D_{u}\right)$, where $L_{u}$ is the operator $\operatorname{diag}\left(H_{u}, K_{u}\right)$ acting on $L_{+}^{2} \times L_{+}^{2}$, and $D_{u}:=\operatorname{diag}\left(B_{u}, C_{u}\right)$.

## 3. Proof of the formula

In this section, we prove Theorem (1) Our starting point is the following identity, valid for every $v \in L_{+}^{2}$ :

$$
\begin{equation*}
\underline{v}(z)=\left(\left(I-z S^{*}\right)^{-1} v \mid 1\right), z \in D . \tag{9}
\end{equation*}
$$

Indeed, the Taylor coefficient of order $n$ of the right hand side is

$$
\left(\left(S^{*}\right)^{n} v \mid 1\right)=\left(v \mid S^{n} 1\right)=\hat{v}(n)
$$

which coincides with the Taylor coefficient of order $n$ of the left hand side. Let $u \in C^{\infty}\left(\mathbb{R}, H_{+}^{s}\right)$ be a solution of (11), $s>\frac{1}{2}$. Applying (9) to $v=u(t)$ and using the unitarity of $U(t)$, we get

$$
\underline{u}(t, z)=\left(\left(I-z S^{*}\right)^{-1} u(t) \mid 1\right)=\left(U(t)^{*}\left(I-z S^{*}\right)^{-1} u(t) \mid U(t)^{*} 1\right),
$$

which yields

$$
\begin{equation*}
\underline{u}(t, z)=\left(\left(I-z U(t)^{*} S^{*} U(t)\right)^{-1} U(t)^{*} u(t) \mid U(t)^{*} 1\right) . \tag{10}
\end{equation*}
$$

We shall identify successively $U(t)^{*} 1, U(t)^{*} u(t)$, and the restriction of $U(t)^{*} S^{*} U(t)$ on the range of $H_{u_{0}}$. We begin with $U(t)^{*} 1$,

$$
\frac{d}{d t} U(t)^{*} 1=-U(t)^{*} B_{u}(1)
$$

and

$$
B_{u}(1)=\frac{i}{2} H_{u}^{2}(1)-i T_{|u|^{2}}(1)=-\frac{i}{2} H_{u}^{2}(1)
$$

Hence

$$
\frac{d}{d t} U(t)^{*} 1=\frac{i}{2} U(t)^{*} H_{u}^{2}(1)=\frac{i}{2} H_{u_{0}}^{2} U(t)^{*} 1
$$

where we have used Corollary 1 This yields

$$
\begin{equation*}
U(t)^{*} 1=\mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}}(1) \tag{11}
\end{equation*}
$$

Consequently,

$$
U(t)^{*} u(t)=U(t)^{*} H_{u(t)}(1)=H_{u_{0}} U(t)^{*}(1)=H_{u_{0}} \mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}}(1)
$$

and therefore

$$
\begin{equation*}
U(t)^{*} u(t)=\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}}\left(u_{0}\right) \tag{12}
\end{equation*}
$$

Finally,

$$
U(t)^{*} S^{*} U(t) H_{u_{0}}=U(t)^{*} S^{*} H_{u(t)} U(t)=U(t)^{*} K_{u(t)} U(t)
$$

and therefore

$$
\begin{equation*}
U(t)^{*} S^{*} U(t) H_{u_{0}}=U(t)^{*} V(t) K_{u_{0}} V(t)^{*} U(t) \tag{13}
\end{equation*}
$$

On the other hand,

$$
\begin{aligned}
\frac{d}{d t} U(t)^{*} V(t) & =-U(t)^{*} B_{u(t)} V(t)+U(t)^{*} C_{u(t)} V(t)=U(t)^{*}\left(C_{u(t)}-B_{u(t)}\right) V(t) \\
& =\frac{i}{2} U(t)^{*}\left(K_{u(t)}^{2}-H_{u(t)}^{2}\right) V(t)=\frac{i}{2}\left(U(t)^{*} V(t) K_{u_{0}}^{2}-H_{u_{0}}^{2} U(t)^{*} V(t)\right)
\end{aligned}
$$

We infer

$$
U(t)^{*} V(t)=\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i \frac{t}{2} K_{u_{0}}^{2}} .
$$

Plugging this identity into (13), we obtain

$$
\begin{aligned}
U(t)^{*} S^{*} U(t) H_{u_{0}} & =\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i \frac{t}{2} K_{u_{0}}^{2}} K_{u_{0}} \mathrm{e}^{-i \frac{t}{2} K_{u_{0}}^{2}} \mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}} \\
& =\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} K_{u_{0}} \mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}} \\
& =\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*} H_{u_{0}} \mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}} \\
& =\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*} \mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} H_{u_{0}} .
\end{aligned}
$$

We conclude that, on the range of $H_{u_{0}}$,

$$
\begin{equation*}
U(t)^{*} S^{*} U(t)=\mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}} S^{*} \mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \tag{14}
\end{equation*}
$$

It remains to plug identities (11), (12), (14) into (10). We finally obtain

$$
\begin{aligned}
\underline{u}(t, z) & =\left(\left.\left(I-z \mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*} \mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}}\right)^{-1} \mathrm{e}^{-i \frac{t}{2} H_{u_{0}}^{2}}\left(u_{0}\right) \right\rvert\, \mathrm{e}^{i \frac{t}{2} H_{u_{0}}^{2}}(1)\right) \\
& =\left(\left(I-z \mathrm{e}^{-i t H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*}\right)^{-1} \mathrm{e}^{-i t H_{u_{0}}^{2}}\left(u_{0}\right) \mid 1\right)
\end{aligned}
$$

which is the claimed formula in the case of data $u_{0} \in H_{+}^{s}, s>\frac{1}{2}$. The case $u_{0} \in H_{+}^{\frac{1}{2}}$ follows by a simple approximation argument. Indeed, we know from [2], Theorem 2.1, that, for every $t \in \mathbb{R}$, the mapping $u_{0} \mapsto u(t)$ is continuous on $H_{+}^{\frac{1}{2}}$. On the other hand, the maps $u_{0} \mapsto H_{u_{0}}, K_{u_{0}}$ are continuous from $H_{+}^{\frac{1}{2}}$ into $\mathcal{L}\left(L_{+}^{2}\right)$ as linear operators satisfying (2). Since $H_{u_{0}}^{2}, K_{u_{0}}^{2}$ are selfadjoint, the operator

$$
\mathrm{e}^{-i t H_{u_{0}}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*}
$$

has norm at most 1. Hence, for $z \in D$, the right hand side of the formula is continuous from $H_{+}^{\frac{1}{2}}$ into $\mathbb{C}$.

## 4. An example

This section is devoted to revisiting sections 6.1, 6.2 of [2] by means of the explicit formula. Given $\varepsilon \in \mathbb{R}$, we define

$$
u_{0}^{\varepsilon}(x)=\mathrm{e}^{i x}+\varepsilon .
$$

It is easy to check that $u_{0}^{\varepsilon} \in \mathcal{V}(3)$. Hence the corresponding solution $u^{\varepsilon}$ of (1) is valued in $\mathcal{V}(3)$, and consequently reads

$$
u^{\varepsilon}(t, x)=\frac{a^{\varepsilon}(t) \mathrm{e}^{i x}+b^{\varepsilon}(t)}{1-p^{\varepsilon}(t) \mathrm{e}^{i x}}
$$

with $a^{\varepsilon}(t) \in \mathbb{C}^{*}, b^{\varepsilon}(t) \in \mathbb{C}, p^{\varepsilon}(t) \in D, a^{\varepsilon}(t)+b^{\varepsilon}(t) p^{\varepsilon}(t) \neq 0$. We are going to calculate these functions explicitly. We start with the special case $\varepsilon=0$. In this case, $\left|u_{0}^{0}\right|=1$; hence

$$
u^{0}(t, x)=\mathrm{e}^{-i t} u_{0}^{0}(x)
$$

so

$$
a^{0}(t)=\mathrm{e}^{-i t}, b^{0}(t)=0, p^{0}(t)=0
$$

We come to $\varepsilon \neq 0$. The operators $H_{u_{0}}^{2}, K_{u_{0}}^{2}, S^{*}$ act on the range of $H_{u_{0}^{\varepsilon}}$, which is the two dimensional vector space spanned by $1, \mathrm{e}^{i x}$. In this basis, the matrices of these three operators are respectively

$$
\mathcal{M}\left(H_{u_{0}}^{2}\right)=\left(\begin{array}{cc}
1+\varepsilon^{2} & \varepsilon \\
\varepsilon & 1
\end{array}\right), \mathcal{M}\left(K_{u_{0}}^{2}\right)=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right), \mathcal{M}\left(S^{*}\right)=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) .
$$

The eigenvalues of $H_{u_{0}}^{2}$ are

$$
\rho_{ \pm}^{2}=1+\frac{\varepsilon^{2}}{2} \pm \varepsilon \sqrt{1+\frac{\varepsilon^{2}}{4}}
$$

hence the matrix of the exponential is given by

$$
\begin{aligned}
\mathcal{M}\left(\mathrm{e}^{-i t H_{u_{0}}^{2}}\right) & =\frac{\mathrm{e}^{-i t \rho_{+}^{2}}-\mathrm{e}^{-i t \rho_{-}^{2}}}{\rho_{+}^{2}-\rho_{-}^{2}} \mathcal{M}\left(H_{u_{0}}^{2}\right)+\frac{\rho_{-}^{2} \mathrm{e}^{-i t \rho_{+}^{2}}-\rho_{+}^{2} \mathrm{e}^{-i t \rho_{-}^{2}}}{\rho_{-}^{2}-\rho_{+}^{2}} I \\
& =\frac{\mathrm{e}^{-i \Omega t}}{2 \omega}\left(-2 i \sin (\omega t) \mathcal{M}\left(H_{u_{0}}^{2}\right)+(2 \omega \cos (\omega t)+2 i \Omega \sin (\omega t)) I\right)
\end{aligned}
$$

where

$$
\omega:=\varepsilon \sqrt{1+\frac{\varepsilon^{2}}{4}}, \Omega:=1+\frac{\varepsilon^{2}}{2} .
$$

We obtain
$\mathrm{e}^{-i t H_{u_{0}}^{2}}\left(u_{0}\right)=\frac{\mathrm{e}^{-i \Omega t}}{2 \omega}\left(-2 i \varepsilon \Omega \sin (\omega t)+2 \varepsilon \omega \cos (\omega t)+\left(2 \omega \cos (\omega t)-i \varepsilon^{2} \sin (\omega t)\right) \mathrm{e}^{i x}\right)$,
$\mathcal{M}\left(\mathrm{e}^{-i t H_{u_{0}}^{2}} \mathrm{e}^{i t K_{u_{0}}^{2}} S^{*}\right)=\frac{\mathrm{e}^{-i t \frac{\varepsilon^{2}}{2}}}{2 \omega}\left(\begin{array}{cc}0 & 2 \omega \cos (\omega t)-i \varepsilon^{2} \sin (\omega t) \\ 0 & -2 i \varepsilon \sin (\omega t)\end{array}\right)$,
and finally

$$
\begin{aligned}
& a^{\varepsilon}(t)=\mathrm{e}^{-i t\left(1+\varepsilon^{2}\right)}, b^{\varepsilon}(t)=\mathrm{e}^{-i t\left(1+\varepsilon^{2} / 2\right)}\left(\varepsilon \cos (\omega t)-i \frac{2+\varepsilon^{2}}{\sqrt{4+\varepsilon^{2}}} \sin (\omega t)\right), \\
& p^{\varepsilon}(t)=-\frac{2 i}{\sqrt{4+\varepsilon^{2}}} \sin (\omega t) \mathrm{e}^{-i t \varepsilon^{2} / 2}, \omega:=\frac{\varepsilon}{2} \sqrt{4+\varepsilon^{2}}
\end{aligned}
$$

The important feature of such dynamics concerns the regime $\varepsilon \rightarrow 0$. Though $p^{0}(t) \equiv 0, p^{\varepsilon}(t)$ may visit small neighborhoods of the unit circle at large times. Specifically, at time $t^{\varepsilon}=\pi /(2 \omega) \sim \pi /(2 \varepsilon)$, we have $\left|p^{\varepsilon}(t)\right| \sim 1-\varepsilon^{2}$. A consequence is that the momentum density,

$$
\begin{aligned}
\mu_{n}\left(t^{\varepsilon}\right) & :=n\left|\hat{u}^{\varepsilon}\left(t^{\varepsilon}, n\right)\right|^{2}=n\left|a^{\varepsilon}\left(t^{\varepsilon}\right)+b^{\varepsilon}\left(t^{\varepsilon}\right) p^{\varepsilon}\left(t^{\varepsilon}\right)\right|^{2}\left|p^{\varepsilon}\left(t^{\varepsilon}\right)\right|^{2(n-1)} \\
& =n \frac{\varepsilon^{4}}{\left(4+\varepsilon^{2}\right)^{2}}\left(1-\frac{\varepsilon^{2}}{4+\varepsilon^{2}}\right)^{n-1},
\end{aligned}
$$

which satisfies

$$
\sum_{n=1}^{\infty} \mu_{n}\left(t^{\varepsilon}\right)=\operatorname{Tr}\left(K_{u^{\varepsilon}\left(t^{\varepsilon}\right)}^{2}\right)=\operatorname{Tr}\left(K_{u_{0}^{\varepsilon}}^{2}\right)=1,
$$



Figure 4.1. The trajectory of $p^{\varepsilon}$ for small $\varepsilon$.
becomes concentrated at high frequencies

$$
n \simeq \frac{1}{\varepsilon^{2}}
$$

This induces the following instability of $H^{s}$ norms:

$$
\left\|u^{\varepsilon}\left(t^{\varepsilon}\right)\right\|_{H^{s}} \simeq \frac{1}{\varepsilon^{2 s-1}}, s>\frac{1}{2}
$$

which is a phenomenon of the same nature as the one displayed by Colliander, Keel, Staffilani, Takaoka and Tao in [1. This proves in particular that conservation laws do not control $H^{s}$ regularity for $s>\frac{1}{2}$. This is in strong contrast with the case of other integrable PDEs such as the KdV equation ([5], [7, [8, [9]) and the one dimensional cubic NLS ([13]). Notice that, as already mentioned at the end of subsection 1.3 of the introduction, the family $\left(u_{0}^{\varepsilon}\right)$ approaches $u_{0}^{0}$, which is a nongeneric element of $\mathcal{V}(3)$, since $H_{u_{0}}^{2}$ admits 1 as a double eigenvalue.

Another example of integrable system is the cubic Szegő equation on the real line, studied by O. Pocovnicu in [12]. In this case, the author displays an example of a nongeneric rational solution $u$ such that

$$
\|u(t)\|_{H^{s}} \simeq \frac{1}{t^{2 s-1}}, t \rightarrow+\infty, s>\frac{1}{2}
$$

Quite similarly to what is discussed in this section, the phenomenon relies on a pole of the solution approaching the real line at distance $t^{-2}$ for large $t$. However, in that case, it can be achieved on an individual trajectory, without appealing to an extra parameter $\varepsilon$.

Coming back to the cubic Szegő equation on the circle, these comments naturally lead to the question of large time behavior of the $H^{s}$ norms of individual solutions for $s>\frac{1}{2}$. We are going to answer this question in the special case of finite rank solutions by proving the quasiperiodicity theorem in the next two sections.

## 5. Generalization to the Szegő hierarchy

The Szegő hierarchy was introduced in [2] and used in [3]. For the convenience of the reader, and because our notation is slightly different, we shall recall the main
facts here. For $y>0$ and $u \in H_{+}^{\frac{1}{2}}$, we set

$$
J^{y}(u)=\left(\left(I+y H_{u}^{2}\right)^{-1}(1) \mid 1\right) .
$$

Notice that the connection with the Szegő equation is made by

$$
E(u)=\frac{1}{4}\left(\partial_{y}^{2} J_{\mid y=0}^{y}-\left(\partial_{y} J_{\mid y=0}^{y}\right)^{2}\right) .
$$

For every $s>\frac{1}{2}, J^{y}$ is a smooth real valued function on $H_{+}^{s}$, and its Hamiltonian vector field is given by

$$
X_{J^{y}}(u)=2 i y w^{y} H_{u} w^{y}, w^{y}:=\left(I+y H_{u}^{2}\right)^{-1}(1),
$$

which is a Lipschitz vector field on bounded subsets of $H_{+}^{s}$. This fact is a consequence of the following lemma, where we collect basic estimates. We recall that the Wiener algebra $W$ is the space of $f \in L_{+}^{2}$ such that

$$
\|f\|_{W}:=\sum_{k=0}^{\infty}|\hat{f}(k)|<\infty .
$$

Lemma 2. Let $f, u, v \in L_{+}^{2}$. Then

$$
\begin{aligned}
\left\|H_{u} f\right\|_{W} & \leq\|u\|_{W}\|f\|_{W} \\
\left\|H_{u} f\right\|_{H^{s-\frac{1}{2}}} & \leq\|u\|_{H^{s}}\|f\|_{L^{2}}, s \geq \frac{1}{2} \\
\left\|H_{u} f\right\|_{H^{s}} & \leq\|u\|_{H^{s}}\|f\|_{W}, s \geq 0 \\
\left\|w^{y}\right\|_{H^{s}} & \leq 1+y\|u\|_{H^{s}}^{2}, s>1 \\
\|f g\|_{H^{s}} & \leq C_{s}\left(\|f\|_{W}\|g\|_{H^{s}}+\|g\|_{W}\|f\|_{H^{s}}\right) \\
\left\|X_{J^{y}}(u)-X_{J^{y}}(v)\right\|_{H^{s}} & \leq C_{s}(R, y)\|u-v\|_{H^{s}}, s>1,\|u\|_{H^{s}}+\|v\|_{H^{s}} \leq R .
\end{aligned}
$$

Proof. The first three estimates are straightforward consequences of the formula

$$
\widehat{H_{u} f}(k)=\sum_{\ell=0}^{\infty} \hat{u}(k+\ell) \overline{\hat{f}(\ell)} .
$$

The fourth estimate comes from these estimates and the fact that

$$
w^{y}=1-y H_{u}^{2} w^{y},\left\|w^{y}\right\|_{L^{2}} \leq 1 .
$$

The fifth estimate is obtained by decomposing

$$
\widehat{f g}(k)=\sum_{\ell=0}^{\infty} \hat{f}(k-\ell) \hat{g}(\ell)=\sum_{|k-\ell| \leq \ell} \hat{f}(k-\ell) \hat{g}(\ell)+\sum_{|k-\ell|>\ell} \hat{f}(k-\ell) \hat{g}(\ell) .
$$

As for the last estimate, we set

$$
w^{y}[u]:=\left(I+y H_{u}^{2}\right)^{-1}(1) .
$$

We write
$\left\|w^{y}[u]-w^{y}[v]\right\|_{L^{2}}=y\left\|\left(I+y H_{u}^{2}\right)^{-1}\left(H_{v}^{2}-H_{u}^{2}\right)\left(I+y H_{v}^{2}\right)^{-1}(1)\right\|_{L^{2}} \leq y R\|u-v\|_{H^{s}}$.
Then, by again using the first two inequalities,

$$
w^{y}[u]-w^{y}[v]=y\left(H_{v}^{2}\left(w^{y}[v]\right)-H_{u}^{2}\left(w^{y}[u]\right)\right)
$$

leads to

$$
\left\|w^{y}[u]-w^{y}[v]\right\|_{H^{s}} \leq C(R, y)\|u-v\|_{H^{s}}
$$

Using moreover the fact that $H^{s}$ is an algebra, this yields the desired estimate.

By the Cauchy-Lipschitz theorem, the evolution equation

$$
\begin{equation*}
\dot{u}=X_{J^{y}}(u) \tag{15}
\end{equation*}
$$

admits local in time solutions for every initial data in $H_{+}^{s}$ for $s>1$, and the lifetime is bounded from below if the data are bounded in $H_{+}^{s}$. We shall see that this evolution equation admits a Lax pair structure similar to the one in section 2,

Theorem 4. For every $u \in H_{+}^{s}$, we have

$$
\begin{aligned}
H_{i X_{J y}(u)} & =H_{u} F_{u}^{y}+F_{u}^{y} H_{u}, \\
K_{i X_{J y}(u)} & =K_{u} G_{u}^{y}+G_{u}^{y} K_{u}, \\
G_{u}^{y}(h) & :=-y w^{y} \Pi\left(\overline{w^{y}} h\right)+y^{2} H_{u} w^{y} \Pi\left(\overline{H_{u} w^{y}} h\right), \\
F_{u}^{y}(h) & :=G_{u}^{y}(h)-y^{2}\left(h \mid H_{u} w^{y}\right) H_{u} w^{y} .
\end{aligned}
$$

If $u \in C^{\infty}\left(\mathcal{I}, H_{+}^{s}\right)$ is a solution of equation (15) on a time interval $\mathcal{I}$, then

$$
\begin{aligned}
\frac{d H_{u}}{d t} & =\left[B_{u}^{y}, H_{u}\right], \frac{d K_{u}}{d t}=\left[C_{u}^{y}, K_{u}\right] \\
B_{u}^{y} & =-i F_{u}^{y}, C_{u}^{y}=-i G_{u}^{y}
\end{aligned}
$$

Proof.
Lemma 3. We have the following identity:

$$
H_{a H_{u}(a)}(h)=H_{u}(a) H_{a}(h)+H_{u}(a \Pi(\bar{a} h)-(h \mid a) a) .
$$

Proof.

$$
H_{a H_{u}(a)}(h)=\Pi\left(a H_{u}(a) \bar{h}\right)=H_{u}(a) H_{a}(h)+\Pi\left(H_{u}(a)(I-\Pi)(a \bar{h})\right)
$$

On the other hand,

$$
(1-\Pi)(a \bar{h})=\overline{\Pi(\bar{a} h)}-(a \mid h) .
$$

The lemma follows by plugging the latter formula into the former one.
Let us complete the proof. Using the identity

$$
w^{y}=1-y H_{u}^{2} w^{y},
$$

and Lemma 3 with $a=H_{u}\left(w^{y}\right)$, we get

$$
\begin{aligned}
& H_{w^{y} H_{u}\left(w^{y}\right)}(h)=H_{H_{u}\left(w^{y}\right)}(h)-y H_{H_{u}\left(w^{y}\right) H_{u}^{2}\left(w^{y}\right)}(h) \\
& ==H_{H_{u}\left(w^{y}\right)}(h)-y H_{u}^{2}\left(w^{y}\right) H_{H_{u}\left(w^{y}\right)}(h) \\
& \quad-y H_{u}\left(H_{u}\left(w^{y}\right) \Pi\left(\overline{H_{u}\left(w^{y}\right)} h\right)-\left(h \mid H_{u}\left(w^{y}\right)\right) H_{u}\left(w^{y}\right)\right) \\
& = \\
& =w^{y} H_{H_{u}\left(w^{y}\right)}(h)-y H_{u}\left(H_{u}\left(w^{y}\right) \Pi\left(\overline{H_{u}\left(w^{y}\right)} h\right)-\left(h \mid H_{u}\left(w^{y}\right)\right) H_{u}\left(w^{y}\right)\right) \\
& = \\
& w^{y} \Pi\left(\overline{w^{y}} H_{u} h\right)-y H_{u}\left(H_{u}\left(w^{y}\right) \Pi\left(\overline{H_{u}\left(w^{y}\right)} h\right)-\left(h \mid H_{u}\left(w^{y}\right)\right) H_{u}\left(w^{y}\right)\right) .
\end{aligned}
$$

We therefore have obtained

$$
H_{w^{y} H_{u}\left(w^{y}\right)}=L_{u}^{y} H_{u}+H_{u} R_{u}^{y},
$$

where $L_{u}^{y}$ and $R_{u}^{y}$ are the following selfadjoint operators:

$$
L_{u}^{y}(h)=w^{y} \Pi\left(\overline{w^{y}} h\right), R_{u}^{y}(h)=-y\left(H_{u}\left(w^{y}\right) \Pi\left(\overline{H_{u}\left(w^{y}\right)} h\right)-\left(h \mid H_{u}\left(w^{y}\right)\right) H_{u}\left(w^{y}\right)\right) .
$$

Consequently, using the symmetry of $H_{w^{y} H_{u}\left(w^{y}\right)}$ for the real inner product,

$$
H_{w^{y} H_{u}\left(w^{y}\right)}=\frac{1}{2}\left(L_{u}^{y}+R_{u}^{y}\right) H_{u}+H_{u} \frac{1}{2}\left(L_{u}^{y}+R_{u}^{y}\right)
$$

Multiplying by $-2 y$, we obtain the desired formula, since

$$
F_{u}^{y}=-y\left(L_{u}^{y}+R_{u}^{y}\right)
$$

We now come to the second identity. From the first one, we get

$$
\begin{equation*}
K_{i X_{J y}(u)}=H_{i X_{J y}(u)} S=H_{u} F_{u}^{y} S+F_{u}^{y} K_{u} . \tag{16}
\end{equation*}
$$

For every $h, v \in L_{+}^{2}$, we use

$$
\Pi(\bar{v} S h)=S \Pi(\bar{v} h)+(S h \mid v)
$$

and infer

$$
\begin{aligned}
F_{u}^{y} S h & =-y w^{y} \Pi\left(\overline{w^{y}} S h\right)+y^{2} H_{u} w^{y} \Pi\left(\overline{H_{u} w^{y}} S h\right)-y^{2}\left(S h \mid H_{u} w^{y}\right) H_{u} w^{y} \\
& =S G_{u}^{y} h-y\left(S h \mid w^{y}\right) w^{y}=S G_{u}^{y} h+y^{2}\left(S h \mid H_{u}^{2} w^{y}\right) w^{y} \\
& =S G_{u}^{y} h+y^{2}\left(H_{u}\left(w^{y}\right) \mid K_{u}(h)\right) w^{y},
\end{aligned}
$$

where we have used $w^{y}=1-y H_{u}^{2} w^{y}$ again. Plugging this identity into (16), we obtain the claim.

The last formulae are straightforward consequences of the antilinearity of $H_{u}$ and $K_{u}$.

Using Theorem 4 in a similar way to section 2, we derive
Corollary 2. Under the conditions of Theorem 4, assuming moreover $0 \in \mathcal{I}$, define $U^{y}=U^{y}(t)$ and $V^{y}=V^{y}(t)$ as the solutions of the following linear ODEs on $\mathcal{L}\left(L_{+}^{2}\right)$ :

$$
\frac{d U^{y}}{d t}=B_{u}^{y} U^{y}, \frac{d V^{y}}{d t}=C_{u}^{y} V^{y}, U^{y}(0)=V^{y}(0)=I
$$

Then $U^{y}(t), V^{y}(t)$ are unitary operators and

$$
H_{u(t)}=U^{y}(t) H_{u(0)} U^{y}(t)^{*}, K_{u(t)}=V^{y}(t) K_{u(0)} V^{y}(t)^{*}
$$

At this stage, we are going to slightly generalize the setting, for the needs of the next section. Let $y_{1}, \ldots, y_{n}$ be positive numbers and $a_{1}, \ldots, a_{n}$ be real numbers. We consider the functional

$$
\hat{J}(u)=\sum_{k=1}^{n} a_{k} J^{y_{k}}(u)=\left(f\left(H_{u}^{2}\right) 1 \mid 1\right), f(s):=\sum_{k=1}^{n} \frac{a_{k}}{1+y_{k} s},
$$

and the evolution equation

$$
\begin{equation*}
\dot{u}=X_{\hat{J}}(u) . \tag{17}
\end{equation*}
$$

By linearity from Theorem [4 it is clear that the solution of (17) satisfies

$$
\begin{equation*}
\frac{d H_{u}}{d t}=\left[\hat{B}_{u}, H_{u}\right], \frac{d K_{u}}{d t}=\left[\hat{C}_{u}, K_{u}\right] \tag{18}
\end{equation*}
$$

with

$$
\begin{equation*}
\hat{B}_{u}=\sum_{k=1}^{n} a_{k} B_{u}^{y_{k}}, \hat{C}_{u}=\sum_{k=1}^{n} a_{k} C_{u}^{y_{k}} \tag{19}
\end{equation*}
$$

Corollary 3. Let u be a solution of equation (17) on some time interval $\mathcal{I}$ containing 0 , and define $\hat{U}=\hat{U}(t)$ and $\hat{V}=\hat{V}(t)$ as the solutions of the following linear ODEs on $\mathcal{L}\left(L_{+}^{2}\right)$ :

$$
\frac{d \hat{U}}{d t}=\hat{B}_{u} \hat{U}, \frac{d \hat{V}}{d t}=\hat{C}_{u} \hat{V}, \hat{U}(0)=\hat{V}(0)=I
$$

Then $\hat{U}(t), \hat{V}(t)$ are unitary operators and

$$
H_{u(t)}=\hat{U}(t) H_{u(0)} \hat{U}(t)^{*} K_{u(t)}=\hat{V}(t) K_{u(0)} \hat{V}(t)^{*}
$$

As a consequence of this corollary, if we start from an initial datum $u(0)$ such that $H_{u(0)}$ is a trace class operator, then $H_{u(t)}$ is trace class for every $t$, with the same trace norm. By Peller's theorem [11, Chap. 6, Theorem 1.1], the trace norm of $H_{u}$ is equivalent to the norm of $u$ in the Besov space $B_{1,1}^{1}$, which is contained in $W$ and contains $H_{+}^{s}$ for every $s>1$. Consequently, if $u(0) \in H_{+}^{s}$ for some $s>1$, then $u(t)$ stays bounded in $W$. We claim that, if $u(0)$ is in $\mathcal{V}(d)$, the evolution can be continued for all time. Moreover, since the ranks of $H_{u(t)}$ and $K_{u(t)}$ are conserved in view of Corollary 3, this evolution takes place in $\mathcal{V}(d)$ if $u(0) \in \mathcal{V}(d)$.

Corollary 4. The equation (17) defines a smooth flow on $\mathcal{V}(d)$ for every $d$.
In view of the Gronwall lemma, the statement is an easy consequence of the following estimate.

Lemma 4. Let $R, y \geq 0, s>1$ be given. There exists $C(d, R, y, s)>0$ such that, for every $u \in \mathcal{V}(d)$ with $\|u\|_{W} \leq R$,

$$
\left\|X_{J^{y}}(u)\right\|_{H^{s}} \leq C(d, R, y, s)\left(1+\|u\|_{H^{s}}\right)
$$

Proof. By using Lemma 2, we are reduced to prove

$$
\left\|w^{y}\right\|_{W} \leq B(d, R, y)
$$

We set $N=\left[\frac{d+1}{2}\right]$. The above estimate is an easy consequence of

$$
\left(I+H_{u}^{2}\right)^{-1}=\sum_{k=0}^{N} \alpha_{k} H_{u}^{2 k}
$$

with $\left|\alpha_{k}\right| \leq 1$ for $k=0, \ldots, N$. In fact, the Cayley-Hamilton theorem yields

$$
\left(H_{u}^{2}\right)^{N+1}=\sum_{k=1}^{N}(-1)^{k-1} S_{k}\left(H_{u}^{2}\right)^{N-k+1}, S_{k}:=\sum_{\ell_{1}<\cdots<\ell_{k}} \rho_{\ell_{1}}^{2} \ldots \rho_{\ell_{k}}^{2},
$$

and one can easily check that

$$
\alpha_{k}=(-1)^{k} \frac{1+\sum_{j=1}^{N-k} S_{j}}{1+\sum_{j=1}^{N} S_{j}}, k=0, \ldots, N,
$$

where $\rho_{1}^{2} \geq \cdots \geq \rho_{N}^{2}$ are the positive eigenvalues of $H_{u}^{2}$, listed with their multiplicities.

Remark 2. For general data $u(0) \in H_{+}^{s}$, one can prove similarly that the solution can be continued for all time if $y\|u(0)\|_{H^{s}}$ is small enough, or just if $y \operatorname{Tr}\left|H_{u(0)}\right|$ is small enough.

Our next step is to derive an explicit formula for the solution of (17) along the same lines as in section 3 The starting points are the formulae

$$
\begin{aligned}
B_{u}^{y}(1) & =i y J^{y}(u) w^{y} \\
C_{u}^{y}-B_{u}^{y} & =-i y^{2}\left(\cdot \mid H_{u} w^{y}\right) H_{u} w^{y} \\
& =i y J^{y}(u)\left(\left(I+y H_{u}^{2}\right)^{-1}-\left(I+y K_{u}^{2}\right)^{-1}\right),
\end{aligned}
$$

where we have used the identity $K_{u}^{2}=H_{u}^{2}-(\cdot \mid u) u$. This leads to

$$
\begin{aligned}
\hat{B}_{u}(1) & =i g\left(H_{u}^{2}\right)(1), g(s):=\sum_{k=1}^{n} \frac{a_{k} y_{k} J^{y_{k}}(u)}{1+y_{k} s}, \\
\hat{C}_{u}-\hat{B}_{u} & =i\left(g\left(H_{u}^{2}\right)-g\left(K_{u}^{2}\right)\right) .
\end{aligned}
$$

Arguing exactly as in section 3 we obtain the following formula.
Theorem 5. The solution $u$ of equation (17) with initial data $u(0)=u_{0} \in \mathcal{V}(d)$ is given by

$$
\begin{equation*}
\underline{u}(t, z)=\left(\left(I-z \mathrm{e}^{2 i t g\left(H_{u_{0}}^{2}\right)} \mathrm{e}^{-2 i t g\left(K_{u_{0}}^{2}\right)} S^{*}\right)^{-1} \mathrm{e}^{2 i t g\left(H_{u_{0}}^{2}\right)} u_{0} \mid 1\right), z \in D, \tag{20}
\end{equation*}
$$

where

$$
g(s):=\sum_{k=1}^{n} \frac{a_{k} y_{k} J^{y_{k}}(u)}{1+y_{k} s} .
$$

## 6. Proof of the quasiperiodicity theorem

In this section, we prove Theorem 2, Let $u_{0} \in \mathcal{V}(d)$ be given. From Theorem [1, it is easy to see that, after diagonalizing $H_{u_{0}}^{2}$ and $K_{u_{0}}^{2}$, the solution indeed has the form

$$
u(t)=\Phi\left(\mathrm{e}^{i \omega_{1} t}, \ldots, \mathrm{e}^{i \omega_{n} t}\right)
$$

and, on the other hand, obviously belongs to $\mathcal{V}(d)$ for all $t$. In order to prove that $t \mapsto u(t)$ is a quasiperiodic function valued into $\mathcal{V}(d)$, we need to establish some stronger property, namely that one can find a smooth function $\Phi$ defined on the torus $\left(\mathbb{S}^{1}\right)^{n}$ and valued in $\mathcal{V}(d)$, such that the above formula holds.

We denote by $\Sigma$ the union of the spectra of $H_{u_{0}}^{2}$ and $K_{u_{0}}^{2}$. We set $n=\sharp \Sigma$, and we identify $\left(\mathbb{S}^{1}\right)^{n}$ to $\left(\mathbb{S}^{1}\right)^{\Sigma}$, or the set of functions from $\Sigma$ to $\mathbb{S}^{1}$. For every function $F: \Sigma \rightarrow \mathbb{S}^{1}$, the operator $F\left(H_{u_{0}}^{2}\right) \bar{F}\left(K_{u_{0}}^{2}\right) S^{*}$ has norm at most 1 ; hence we can define

$$
\Phi(F)(z)=\left(\left(I-z F\left(H_{u_{0}}^{2}\right) \bar{F}\left(K_{u_{0}}^{2}\right) S^{*}\right)^{-1} F\left(H_{u_{0}}^{2}\right) u_{0} \mid 1\right), z \in D
$$

Notice that Theorem 1 exactly claims that $\underline{u}(t, z)=\Phi\left(F_{t}\right)$, where $F_{t}$ is the function $\Sigma \rightarrow \mathbb{S}^{1}$ defined by $F_{t}(s)=\mathrm{e}^{-i t s}$. Hence, in order to prove quasiperiodicity, it is enough to prove that $\Phi(F)$ defines an element of $\mathcal{V}(d)$ which depends smoothly on $F$. First notice that, from the above formula, $\Phi(F)$ is a rational function with coefficients smoothly dependent on $F$. Hence we just have to prove that, for every $F, \Phi(F)$ defines an element of $\mathcal{V}(d)$.

Let $F \in\left(\mathbb{S}^{1}\right)^{\Sigma}$. For each $s \in \Sigma$, we set

$$
F(s)=\mathrm{e}^{i \omega(s)}
$$

where $\omega(s) \in[0,2 \pi)$. Let $y_{1}, \ldots, y_{n}$ be $n$ positive numbers pairwise distinct. Then the matrix

$$
\left(\frac{1}{1+y_{k} s}\right)_{k=1, \ldots, n, s \in \Sigma}
$$

is invertible, and hence the linear system

$$
\omega(s)=2 \sum_{k=1}^{n} \frac{a_{k} y_{k} J^{y_{k}}\left(u_{0}\right)}{1+y_{k} s}, s \in \Sigma
$$

has a unique solution $a_{1}, \ldots, a_{n}$. Using Theorem 5 $\Phi(F)$ is the value at time $t=1$ of the solution $u$ of equation (17) with parameters $a_{1}, \ldots, a_{n}, y_{1}, \ldots, y_{n}$. By Corollary 4 it belongs to $\mathcal{V}(d)$. This proves quasiperiodicity.

Since $\Phi$ is a continuous mapping, $\Phi\left(\left(\mathbb{S}^{1}\right)^{\Sigma}\right)$ is a compact subset of $\mathcal{V}(d)$. On the other hand, for every $s$, the $H^{s}$ norm is continuous on $\mathcal{V}(d)$. It is therefore bounded on this compact subset, which contains the integral curve issued from $u_{0}$. This completes the proof of Theorem 2

Remark 3. It is tempting to adapt the above proof of quasiperiodicity to nonfinite rank solutions. However, even assuming that one can define a flow on $H_{+}^{s}$ for all $y$ with convenient estimates for large $y$, this strategy meets a serious difficulty. Indeed, on the one hand, the construction of a Hamiltonian flow on $H_{+}^{s}$ for

$$
\hat{J}(u)=\left(f\left(H_{u}^{2}\right) 1 \mid 1\right)
$$

requires a minimal regularity for $f$, say $C^{1}$, which, if $f$ is represented as

$$
f(s)=\int_{0}^{\infty} \frac{a(y)}{1+y s} d \mu(y)
$$

for some positive measure $\mu$ and some function $a$ on $\mathbb{R}_{+}$, imposes a decay condition as

$$
\int_{0}^{\infty} y|a(y)| d \mu(y)<\infty
$$

On the other hand, $\Sigma$ is made of a sequence of positive numbers converging to 0 and of its limit, and the interpolation problem

$$
F(s)=\exp \left(2 i \int_{0}^{\infty} \frac{y a(y) J^{y}\left(u_{0}\right)}{1+y s} d \mu(y)\right)
$$

would have a solution only if $\omega: \Sigma \rightarrow \mathbb{S}^{1}$ is continuous on $\Sigma$. Unfortunately, the space $C\left(\Sigma, \mathbb{S}^{1}\right)$ is not compact, neither for the simple convergence nor for the uniform convergence. Therefore the question of large time dynamics of nonfinite rank solutions of the cubic Szegő equation remains widely open.

## Acknowledgements

This paper benefited from discussions with several colleagues, in particular P. Deift, T. Kappeler, H. Koch, S. Kuksin and M. Zworski. The authors wish to thank them deeply. The authors are grateful to the anonymous referee for making helpful suggestions and comments.

## References

[1] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation, Invent. Math. 181 (2010), no. 1, 39-113, DOI 10.1007/s00222-010-0242-2. MR2651381 (2011f:35320)
[2] Patrick Gérard and Sandrine Grellier, The cubic Szegő equation (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) 43 (2010), no. 5, 761-810. MR2721876 (2012b:37188)
[3] Patrick Gérard and Sandrine Grellier, Invariant tori for the cubic Szegö equation, Invent. Math. 187 (2012), no. 3, 707-754. MR2944951
[4] Patrick Gérard and Sandrine Grellier, Effective integrable dynamics for a certain nonlinear wave equation, Anal. PDE 5 (2012), no. 5, 1139-1155, DOI 10.2140/apde.2012.5.1139. MR3022852
[5] Thomas Kappeler and Jürgen Pöschel, KdV $\mathcal{G} K A M$, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 45, SpringerVerlag, Berlin, 2003. MR1997070 (2004g:37099)
[6] L. Kronecker, Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen, Monatsber. Königl. Preuss. Akad. Wiss. (Berlin), 535-600 (1881). Reprinted in Leopold Kronecker's Werke, vol. 2, 113-192, Chelsea, 1968.
[7] Peter D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21 (1968), 467-490. MR0235310 (38 \#3620)
[8] Peter D. Lax, Periodic solutions of the KdV equation, Comm. Pure Appl. Math. 28 (1975), 141-188. MR0369963 (51 \#6192)
[9] Nikolai K. Nikolski, Operators, functions, and systems: an easy reading. Vol. 1, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz; Translated from the French by Andreas Hartmann. MR 1864396 (2003i:47001a)
[10] S. P. Novikov, A periodic problem for the Korteweg-de Vries equation. I (Russian), Funkcional. Anal. i Priložen. 8 (1974), no. 3, 54-66. MR0382878|(52 \#3760)
[11] Vladimir V. Peller, Hankel operators and their applications, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. MR1949210|(2004e:47040)
[12] Oana Pocovnicu, Explicit formula for the solution of the Szegö equation on the real line and applications, Discrete Contin. Dyn. Syst. 31 (2011), no. 3, 607-649, DOI 10.3934/dcds.2011.31.607. MR2825631 (2012h:35330)
[13] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and onedimensional self-modulation of waves in nonlinear media (Russian, with English summary), Ž. Èksper. Teoret. Fiz. 61 (1971), no. 1, 118-134; English transl., Soviet Physics JETP 34 (1972), no. 1, 62-69. MR0406174 (53 \#9966)

Université Paris-Sud XI, Laboratoire de Mathématiques d’Orsay, CNRS, UMR 8628, 91405 Orsay Cedex, France and Institut Universitaire de France

E-mail address: Patrick.Gerard@math.u-psud.fr
Fédération Denis Poisson, MAPMO-UMR 6628, Département de Mathématiques, Université d'Orleans, 45067 Orléans Cedex 2, France

E-mail address: Sandrine.Grellier@univ-orleans.fr


[^0]:    Received by the editors June 19, 2013 and, in revised form, October 6, 2013.
    2010 Mathematics Subject Classification. Primary 37K15; Secondary 47B35.
    Key words and phrases. Cubic Szegő equation, inverse spectral transform, quasiperiodicity, energy transfer to high frequencies, instability.

    Part of this work was completed while the authors were visiting CIRM in Luminy. They are grateful to this institution for its warm hospitality.

