TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 5, May 2015, Pages 3119–3138
S 0002-9947(2014)06014-5
Article electronically published on December 3, 2014

SOME NEW RESULTS ON DIFFERENTIAL INCLUSIONS FOR DIFFERENTIAL FORMS

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Abstract. In this article we study some necessary and sufficient conditions for the existence of solutions in $W_0^{1,\infty}(\Omega;\Lambda^k)$ of the differential inclusion

$$d\omega \in E \quad \text{a.e. in } \Omega$$

where $E \subset \Lambda^{k+1}$ is a prescribed set.

In this article we discuss the existence of a $k-$form ω, $0 \leq k \leq n-1$, verifying

$$\begin{cases} d\omega \in E & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $E \subset \Lambda^{k+1}$ is a given set of $(k+1)-$forms. For the precise notation we refer to Section 1.

This problem has been mostly studied in the case $k = 0$ ($d\omega$ can then be identified with $\operatorname{grad}\omega$); see [3], [4], [5], [9], [10], [11] and, for an extensive bibliography on the subject, see [7].

The case $k = 1$ ($d\omega$ is then identified with $\operatorname{curl}\omega$) has also received some attention; see [1], [2], [8], [12].

The general case, $0 \leq k \leq n-1$, was first considered in [1].

We improve here on [1] in two directions. The first result concerns the existence part (cf. Theorem 3.7).

Theorem 0.1. *Let $0 \leq k \leq n-1$ be two integers, $\Omega \subset \mathbb{R}^n$ a bounded open set, $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.*

(i) There exists $\omega \in W_0^{1,\infty}\left(\Omega;\Lambda^k\right)$ of the form $\omega(x) = u(x)\, b$ where $u \in W_0^{1,\infty}(\Omega)$ such that

$$d\omega = (\operatorname{grad} u) \wedge b \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

(ii) The following holds:

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}[E \cap (\mathbb{R}^n \wedge b)].$$

This result was already obtained in [1] but only for b of the form

$$b = b^1 \wedge \cdots \wedge b^k$$

where $b^1, \cdots, b^k \in \Lambda^1$. Our present theorem allows us (cf. Corollary 3.9) to get a complete picture when

$$\dim \operatorname{span} E = n - k.$$

Received by the editors July 24, 2012 and, in revised form, November 5, 2012.
2010 *Mathematics Subject Classification.* Primary 35F60.
Part of the present work was done while the first and third authors were visiting EPFL, whose hospitality is gratefully acknowledged.

Our second contribution concerns necessary conditions (cf. Theorem 2.5).

Theorem 0.2. *Let $0 \leq k \leq n-1$ be two integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ be such that*

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Then

$$\dim \operatorname{span} E \geq n-k$$

and more precisely

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) \subset \operatorname{span} E.$$

Moreover, if

$$\dim \operatorname{span} E = n-k,$$

then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) = \operatorname{span} E \quad \text{and} \quad \int_\Omega \omega = b^1 \wedge \cdots \wedge b^k$$

for some $b^1, \cdots, b^k \in \Lambda^1$.

1. Notation

We gather here the notation which we will use throughout this article. For more details on exterior algebra and differential forms see [6] and for convex analysis see [7] or [13].

(1) Let k, n be two integers.
- We write $\Lambda^k(\mathbb{R}^n)$ (or simply Λ^k) to denote the vector space of all alternating $k-$linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k-\text{times}} \to \mathbb{R}$. For $k=0$, we set $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Note that $\Lambda^k(\mathbb{R}^n) = \{0\}$ for $k > n$ and, for $k \leq n$, $\dim\left(\Lambda^k(\mathbb{R}^n)\right) = \binom{n}{k}$.
- $\wedge$, $\lrcorner$, $\langle ; \rangle$ and, respectively, $*$ denote the exterior product, the interior product, the scalar product and, respectively, the Hodge star operator.
- For $b \in \Lambda^k$, $\operatorname{rank}[b]$ denotes the rank of the exterior $k-$form b.
- If $\left\{e^1, \cdots, e^n\right\}$ is a basis of $\mathbb{R}^n$, then, identifying Λ^1 with $\mathbb{R}^n$,
$$\left\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\right\}$$
is a basis of Λ^k.
- For $E \subset \Lambda^k$, $\operatorname{span} E$ denotes the subspace spanned by E.
- Let W be a subspace of Λ^k. We write $\dim W$ to denote the dimension of W and $W^\perp$ to denote the orthogonal complement of W.
- For $b \in \Lambda^k$, we write, identifying again Λ^1 with $\mathbb{R}^n$,
$$\mathbb{R}^n \wedge b = \Lambda^1 \wedge b = \left\{x \wedge b : x \in \Lambda^1\right\} \subset \Lambda^{k+1}.$$

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
- The spaces $C^1\left(\Omega; \Lambda^k\right)$, $W^{1,p}\left(\Omega; \Lambda^k\right)$ and $W_0^{1,p}\left(\Omega; \Lambda^k\right)$, $1 \leq p \leq \infty$, are defined in the usual way.

- For $\omega \in W^{1,p}\left(\Omega;\Lambda^k\right)$, $\int_\Omega \omega$ denotes the exterior $k-$form obtained by integrating componentwise the differential form ω. Explicitly, for $1 \leq i_1 < \cdots < i_k \leq n$,
$$\left(\int_\Omega \omega\right)_{i_1\cdots i_k} = \int_\Omega \omega_{i_1\cdots i_k}\,.$$
- For $\omega \in W^{1,p}\left(\Omega;\Lambda^k\right)$, the exterior derivative $d\omega$ belongs to $L^p(\Omega;\Lambda^{k+1})$ and is defined by
$$(d\omega)_{i_1\cdots i_{k+1}} = \sum_{j=1}^{k+1}(-1)^{j+1}\frac{\partial \omega_{i_1\cdots i_{j-1}i_{j+1}\cdots i_{k+1}}}{\partial x_{i_j}}\,,$$
for $1 \leq i_1 < \cdots < i_{k+1} \leq n$. If $k=0$, then $d\omega \simeq \operatorname{grad}\omega$. If $k=1$, then for $1 \leq i < j \leq n$,
$$(d\omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\,,$$
i.e. $d\omega \simeq \operatorname{curl}\omega$.

(3) For subsets $C, V \subset \Lambda^k$,
- $\operatorname{co} C$ denotes the convex hull of C;
- $\operatorname{int}_V C$ denotes the interior of C with respect to the topology relative to V.

(4) For a convex set $C \subset \Lambda^k$,
- $\operatorname{aff} C$ denotes the affine hull of C which is the intersection of all affine subsets of Λ^k containing C;
- $\operatorname{ri} C$ denotes the relative interior of C which is the interior of C with respect to the topology relative to the affine hull of C. Equivalently $\operatorname{ri} C = \operatorname{int}_{\operatorname{aff} C} C$;
- $\operatorname{rbd} C$ denotes the relative boundary of C which is $\overline{C} \setminus \operatorname{ri} C$.

2. Necessary conditions

2.1. **Preliminaries.**

Lemma 2.1. *Let* $0 \leq k \leq n-1$ *and let* $b \in \Lambda^k \setminus \{0\}$. *Then*
$$\dim\left(\mathbb{R}^n \wedge b\right) \geq n-k.$$
Furthermore,
$$\dim\left(\mathbb{R}^n \wedge b\right) = n-k \quad \Leftrightarrow \quad b = b^1 \wedge \cdots \wedge b^k \text{ for some } b^i \in \Lambda^1.$$

Proof.

Step 1. We prove the first part. By definition of the interior product and the Hodge star operator (cf. Definition 2.11 in [6]), we have that
$$\mathbb{R}^n \lrcorner\, (*b) = (-1)^{k^2} * \left(\mathbb{R}^n \wedge b\right).$$
Hence, since (cf. Proposition 2.32 (i) in [6])
$$\dim\left(\mathbb{R}^n \lrcorner\, (*b)\right) = \operatorname{rank}\left[*b\right],$$
and since (cf. Proposition 2.37 (ii) of [6])
$$\operatorname{rank}\left[*b\right] \geq n-k,$$
we have proved the first part of the lemma.

Step 2. We prove the second part. First note that if $b = b^1 \wedge \cdots \wedge b^k$ where $b^i \in \Lambda^1$, it is elementary to see that

$$\dim (\mathbb{R}^n \wedge b) = n - k.$$

We prove the converse. In this case

$$n - k = \dim (\mathbb{R}^n \wedge b) = \dim (\mathbb{R}^n \lrcorner (*b)),$$

and so, as in Step 1, rank $[*b] = n - k$ and hence (cf. Proposition 2.43 (ii) in [6]) there exist $c^1, \cdots, c^{n-k} \in \Lambda^1$ such that

$$*b = c^1 \wedge \cdots \wedge c^{n-k}.$$

Using Proposition 2.19 in [6], it is not difficult to see that

$$b = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$. The lemma is therefore proved. $\square$

Proposition 2.2. *Let $0 \le k \le n-1$ be two integers and let $f : \mathbb{R}^n \to \Lambda^k$ be continuous at* 0 *and such that $f(0) \neq 0$. Then*

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\}$$

and therefore

$$\dim \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} \ge n - k.$$

Moreover, if

$$\dim \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} = n - k,$$

then

$$\mathbb{R}^n \wedge f(0) = \operatorname{span} \{x \wedge f(x) : x \in \mathbb{R}^n\} \quad \text{and} \quad f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

Proof.

Step 1. From Lemma 2.1, since $f(0) \in \Lambda^k$ and $f(0) \neq 0$, we have

$$\dim \operatorname{span}\{x \wedge f(0) : x \in \mathbb{R}^n\} \ge n - k.$$

Moreover, if $\dim \operatorname{span} \{x \wedge f(0) : x \in \mathbb{R}^n\} = n - k$, then

$$f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

Step 2. We prove the first assertion. Let $x \in \mathbb{R}^n \setminus \{0\}$ be fixed. Note that, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$x \wedge f(\lambda x) = \frac{1}{\lambda} [\lambda x \wedge f(\lambda x)] \in \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

Since $\operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}$ is closed and f is continuous at 0, it follows, letting $\lambda \to 0$, that

$$x \wedge f(0) \in \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

Therefore

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\}$$

which directly shows that (cf. Step 1)

$$\dim \operatorname{span} \{y \wedge f(y) : y \in \mathbb{R}^n\} \ge \dim (\mathbb{R}^n \wedge f(0)) \ge n - k.$$

Step 3. We finally prove the extra assertion. We already know (cf. Step 2) that

$$\mathbb{R}^n \wedge f(0) \subset \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}.$$

Since, by hypothesis,

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} = n - k,$$

we directly deduce from Step 1 that

$$\mathbb{R}^n \wedge f(0) = \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\}$$

and that

$$f(0) = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$. The proof is therefore complete. □

Remark 2.3. (i) With a very similar proof one can show that if $f : \mathbb{R}^n \to \Lambda^k$ is differentiable at some point $x_0 \in \mathbb{R}^n$, then, for every $x \in \mathbb{R}^n$,

$$x_0 \wedge Df(x_0; x) + x \wedge f(x_0) \subset \operatorname{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}$$

where $Df(x_0; x)$ denotes the directional derivative of f at x_0 in the direction of x. In particular if $Df(x_0) = 0$, then

$$\mathbb{R}^n \wedge f(x_0) \subset \operatorname{span}\{y \wedge f(y) : y \in \mathbb{R}^n\}.$$

(ii) It can also be proved that if $f : \mathbb{R}^n \to \Lambda^k$ is continuous and $x_0 \in \mathbb{R}^n$ is such that

$$x_0 \wedge f(x_0) \neq 0,$$

then, necessarily (even if $f(0) = 0$),

$$\dim \operatorname{span}\{x \wedge f(x) : x \in \mathbb{R}^n\} \geq n - k.$$

The following lemma will be used in the proofs of Theorems 2.6 and 3.7 and Corollary 3.9.

Lemma 2.4. *Let $0 \leq k \leq n-1$ be two integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ be such that*

$$d\omega \in E \quad a.e. \ in \ \Omega.$$

Then

$$0 \in \overline{\operatorname{co} E}.$$

Moreover, if $0 \notin \operatorname{ri} \operatorname{co}(E)$, then there exists $D \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus D) = 0$, *$d\omega(D) \subset E$ and*

$$\dim \operatorname{span}(d\omega(D)) < \dim \operatorname{span} E.$$

Proof. Since $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ and hence

$$\int_\Omega d\omega = 0,$$

we deduce, using Proposition 2.36 in [7] and Jensen's inequality, that

$$0 \in \overline{\operatorname{co} E}.$$

Suppose that $0 \notin \operatorname{ri} \operatorname{co}(E)$ and hence (using the previous observation)

$$0 \in \operatorname{rbd} \operatorname{co} E.$$

Using the Separation Theorem (cf. Theorem 2.10 in [7]), we easily deduce from the previous equation that there exists $b \in (\operatorname{span} E) \setminus \{0\}$ such that

$$\langle h; b \rangle \geq 0 \quad \text{for every } h \in \operatorname{co} E.$$

In particular $\langle d\omega; b \rangle \geq 0$ a.e. in Ω. But, as $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ and

$$\int_\Omega \langle d\omega; b \rangle = 0,$$

we therefore obtain that

$$\langle d\omega; b \rangle = 0 \quad \text{a.e. in } \Omega.$$

Let $D \subset \Omega$ be the set where the previous equation holds. Taking D smaller if necessary we can assume without loss of generality that $d\omega(D) \subset E$ (and of course $\operatorname{meas}(\Omega \setminus D) = 0$) and hence

$$\operatorname{span} d\omega(D) \subset \operatorname{span} E.$$

As $\langle d\omega\,(x)\,; b \rangle = 0$ for every $x \in D$ and $b \in (\operatorname{span} E) \setminus \{0\}$, we deduce that

$$\operatorname{span} d\omega(D) \underset{\neq}{\subset} \operatorname{span} E$$

and thus

$$\dim \operatorname{span} d\omega(D) < \dim \operatorname{span} E$$

as wished. □

2.2. **The main results.** We first state the main results of this section.

Theorem 2.5. *Let $0 \leq k \leq n-1$ be two integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $E \subset \Lambda^{k+1}$ and $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ be such that*

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) \subset \operatorname{span} E \quad \text{and} \quad \dim \operatorname{span} E \geq n-k.$$

Moreover, if

$$\dim \operatorname{span} E = n-k,$$

then

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega\right) = \operatorname{span} E \quad \text{and} \quad \int_\Omega \omega = b^1 \wedge \cdots \wedge b^k$$

for some $b^i \in \Lambda^1$.

The following result improves Theorem 3.6 of [1], since E is not assumed to be finite and F is given explicitly.

Theorem 2.6. *Let $0 \leq k \leq n-1$ be integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $E \subset \Lambda^{k+1} \setminus \{0\}$. Let $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ be such that*

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

Then there exists $F \subset E$ such that

$$0 \in \operatorname{ri} \operatorname{co} F.$$

More precisely F can be taken as $d\omega(D)$ for some $D \subset \Omega$ with $\operatorname{meas}(\Omega \setminus D) = 0$.

We start with the proof of Theorem 2.5.

Proof. Let $\mathbb{P} : \Lambda^{k+1} \to \Lambda^{k+1}$ denote the projection onto the orthogonal complement of $\operatorname{span} E$. Since $\omega \in W_0^{1,\infty}\left(\Omega;\Lambda^k\right)$, extending ω by 0 to $\mathbb{R}^n$, it follows that

$$\mathbb{P}(d\omega) = 0 \quad \text{a.e. in } \mathbb{R}^n.$$

Applying the Fourier transform we obtain (recalling that the Fourier transform of ω is continuous)

$$\mathbb{P}\left(x \wedge \left[\int_{\mathbb{R}^n} \omega(y)\cos\left(2\pi\langle x;y\rangle\right)dy\right]\right) = 0 \quad \text{for every } x \in \mathbb{R}^n$$

which is equivalent to

$$x \wedge \left[\int_{\mathbb{R}^n} \omega(y)\cos\left(2\pi\langle x;y\rangle\right)dy\right] \in \operatorname{span} E \quad \text{for every } x \in \mathbb{R}^n.$$

Letting

$$f\left(x\right) = \int_{\mathbb{R}^n} \omega(y)\cos\left(2\pi\langle x;y\rangle\right)dy$$

we get

$$f\left(0\right) = \int_{\Omega} \omega \neq 0.$$

Then, applying Proposition 2.2 to the above f, we have indeed established the theorem. ☐

We next prove Theorem 2.6.

Proof. Let $D \subset \Omega$ (not necessarily unique) be such that

$$\operatorname{meas}(\Omega \setminus D) = 0, \quad d\omega\left(D\right) \subset E$$

and, for every $D_1 \subset D$ with $\operatorname{meas}(\Omega \setminus D_1) = 0$, then

$$\dim \operatorname{span} d\omega\left(D_1\right) = \dim \operatorname{span} d\omega\left(D\right). \tag{2.1}$$

If such a D did not exist, we would find, after a finite induction on the dimension, that

$$d\omega = 0 \quad \text{a.e. in } \Omega$$

which contradicts the fact that $0 \notin E$. Letting $F = d\omega(D) \subset E$, it remains to show that

$$0 \in \operatorname{ri} \operatorname{co} F.$$

For the sake of contradiction suppose that this is not the case. Hence using Lemma 2.4 (with E replaced by F) there exists a set $D_1 \subset D$ (in the conclusion of Lemma 2.4 the set D_1 is only contained in Ω but taking it smaller, if necessary, we can assume that $D_1 \subset D$) such that $\operatorname{meas}(\Omega \setminus D_1) = 0$ and

$$\dim \operatorname{span} d\omega(D_1) < \dim \operatorname{span} F.$$

This is the desired contradiction since, using (2.1),

$$\dim \operatorname{span} d\omega(D_1) = \dim \operatorname{span} F.$$

The proof is therefore complete. ☐

2.3. **Further remarks.** In this section we want to discuss the hypothesis

$$\int_\Omega \omega \neq 0$$

that was made in Theorem 2.5. We will concentrate on the case $k = 1$. We start with an elementary lemma.

Lemma 2.7. *Let $E \subset \Lambda^2 (\mathbb{R}^n)$ be such that there exists a non-degenerate $g \in \Lambda^2$ (i.e. $h \lrcorner g \neq 0$ for every $h \in \Lambda^1 \setminus \{0\}$) and*

$$g \in (\operatorname{span} E)^\perp .$$

Then the two following statements hold true.

(i) There exists no $b \in \Lambda^1 (\mathbb{R}^n) \setminus \{0\}$ such that

$$\mathbb{R}^n \wedge b \subset \operatorname{span} E.$$

(ii) There exists no $\omega \in W_0^{1,\infty}(\Omega; \Lambda^1)$ such that

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Proof. (i) We proceed by contradiction and assume that $\mathbb{R}^n \wedge b \subset \operatorname{span} E$ with $b \neq 0$. Since $g \in (\operatorname{span} E)^\perp$, we deduce that

$$g \in (\mathbb{R}^n \wedge b)^\perp$$

which is equivalent to

$$\langle g; a \wedge b \rangle = 0 \quad \text{for every } a \in \Lambda^1.$$

This in turn is equivalent to (cf. Proposition 2.16 in [6])

$$\langle b \lrcorner g; a \rangle = 0 \quad \text{for every } a \in \Lambda^1.$$

Since the previous equation is the same as

$$b \lrcorner g = 0,$$

we deduce, appealing to the fact that g is non-degenerate, that $b = 0$. This is the desired contradiction.

(ii) Indeed, if such a solution exists, then (cf. Theorem 2.5)

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega \right) \subset \operatorname{span} E.$$

Define $b = \int_\Omega \omega$ and apply the previous point (i) to get the result. □

The next proposition shows that the hypothesis $\int_\Omega \omega \neq 0$ cannot be removed, in general, in Theorem 2.5.

Proposition 2.8. *Let $B \subset \mathbb{R}^4$ be the unit ball centered at 0. Then there exists $E \subset \Lambda^2 \left(\mathbb{R}^4\right) \setminus \{0\}$ with the following properties.*

(i) There exists $\omega_0 \in W_0^{1,\infty} \left(B; \Lambda^1\right)$ such that

$$d\omega_0 \in E \quad \text{a.e. in } B.$$

(ii) There exists no $b \in \Lambda^1 \left(\mathbb{R}^4\right) \setminus \{0\}$ such that

$$\mathbb{R}^4 \wedge b \subset \operatorname{span} E.$$

(iii) For every $\omega \in W_0^{1,\infty}\left(B;\Lambda^1\right)$ such that $d\omega \in \operatorname{span} E$ a.e. in B then, necessarily,

$$\int_B \omega = 0.$$

Remark 2.9. (i) Using the Vitali covering theorem (see [10]), the same set E also works for any open set Ω (instead of B).

(ii) This phenomenon only occurs where $n \geq 4$. Indeed if $n \leq 3$ we can always find (cf. Theorem 3.11) a solution with a non-zero average (as far as there exists a non-trivial solution).

(iii) The previous proposition has an interesting implication for differential inclusions where we seek solutions $u \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^N\right)$, and $\Omega \subset \mathbb{R}^n$, verifying

$$\nabla u \in F \quad \text{a.e. in } \Omega.$$

In the scalar case $N = 1$, we can always ensure (cf. Theorem 3.5) that if there is a non-trivial solution u of the differential inclusion, then there are some solutions with non-zero average. This is no longer the case in the vectorial context when $n, N > 1$. Indeed let $n = N = 4$ and define $F \subset \mathbb{R}^{4\times 4} \approx \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ by

$$F = \left\{(u_1,u_2,u_3,u_4) \in \mathbb{R}^{4\times 4} : u_1 \wedge dx^1 + u_2 \wedge dx^2 + u_3 \wedge dx^3 + u_4 \wedge dx^4 \in E\right\}$$

where E is as in the proposition. Noticing that, for $\omega = u_1 dx^1 + \cdots + u_4 dx^4$,

$$d\omega \in E \quad \Leftrightarrow \quad \nabla u = (\nabla u_1, \nabla u_2, \nabla u_3, \nabla u_4) \in F$$

we have the result.

(iv) If $\dim \operatorname{span} E = n-1$, then, as far as there exists a non-trivial solution (cf. Theorem 4.13 [1]), we always have (without assuming the existence of a solution with non-zero average) $\operatorname{span} E = \mathbb{R}^n \wedge b$ for some $b \in \Lambda^1 \setminus \{0\}$.

Proof. (i) Let

$$\omega_0(x) = \left(|x|^2 - 1\right)\left(x_1\, dx^1 + x_2\, dx^2 + 2x_3\, dx^3 + 2x_4\, dx^4\right).$$

Obviously $\omega_0 \in W_0^{1,\infty}(B;\Lambda^1)$ and

$$d\omega_0(x) = 2x_1x_3\, dx^1 \wedge dx^3 + 2x_1x_4\, dx^1 \wedge dx^4 + 2x_2x_3\, dx^2 \wedge dx^3 + 2x_2x_4\, dx^2 \wedge dx^4.$$

Let

$$\Sigma = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_4 = 0\}.$$

Note in particular that

$$d\omega_0(x) \neq 0 \quad \text{for every } x \in \mathbb{R}^4 \setminus \Sigma.$$

Observe that ω_0 and

$$E = d\omega_0\left(B \setminus \Sigma\right)$$

satisfy all the requirements of the first statement of the proposition, since, trivially, $0 \notin E$, $\omega_0 \in W_0^{1,\infty}\left(B;\Lambda^1\right)$ and $d\omega_0 \in E$ a.e. in B.

(ii) We now prove the second statement. First observe that for every $h \in E$ we have $h_{12} = h_{34} = 0$. We thus deduce that for every $h \in \operatorname{span} E$ we also have $h_{12} = h_{34} = 0$. In other words $dx^1 \wedge dx^2, dx^3 \wedge dx^4 \in (\operatorname{span} E)^\perp$ and hence, in particular,

$$g = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \in (\operatorname{span} E)^\perp.$$

Note that g is non-degenerate. We can therefore invoke Lemma 2.7 (i) to get the claim.

(iii) The last statement of the proposition is a direct consequence of Theorem 2.5 and of the previous point (ii). □

3. Necessary and sufficient conditions

3.1. **Preliminaries.**

We start with an elementary lemma whose proof is straightforward.

Lemma 3.1. *Let X be a normed linear space, $Y \subset X$ a subspace and $A \subset X$. Then $0 \in \operatorname{int}_Y \operatorname{co}(A \cap Y)$ if and only if*

$$\operatorname{span}(A \cap Y) = Y \quad \textit{and} \quad 0 \in \operatorname{ri}\operatorname{co}(A \cap Y).$$

We next recall Caratheodory's theorem (see Corollary 2.16 in [7]).

Lemma 3.2 (Caratheodory's theorem)**.** *Let X be a normed linear space and $E \subset X$. Then $0 \in \operatorname{ri}\operatorname{co} E$ if and only if there exist $m \in \mathbb{N}$, $m \geq \dim \operatorname{span} E + 1$, $z^i \in E$, $t^i > 0$ for every $1 \leq i \leq m$, such that*

$$\sum_{i=1}^{m} t^i z^i = 0, \quad \sum_{i=1}^{m} t^i = 1 \quad \textit{and} \quad \operatorname{span}\left\{z^1, \cdots, z^m\right\} = \operatorname{span} E.$$

In what follows we will constantly identify Λ^1 with $\mathbb{R}^n$.

Lemma 3.3. *Let $0 \leq k \leq n-1$ be two integers, and let $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$ be such that*

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right].$$

Then there exists $F \subset \mathbb{R}^n$ such that

$$E \cap (\mathbb{R}^n \wedge b) = F \wedge b \quad \textit{and} \quad 0 \in \operatorname{int}\operatorname{co} F.$$

Proof.

Step 1. Let us define

$$F = \{x \in \mathbb{R}^n : x \wedge b \in E\}.$$

Evidently, $E \cap (\mathbb{R}^n \wedge b) = F \wedge b$. Since $0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right]$, it follows from Lemma 3.1 that

$$\operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] = \mathbb{R}^n \wedge b \quad \text{and} \quad 0 \in \operatorname{ri}\operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right]. \tag{3.1}$$

Using Lemma 3.2 we find $m \in \mathbb{N}$, $m \geq \dim \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] + 1$, $z^i \in E \cap (\mathbb{R}^n \wedge b)$, $t^i > 0$ for every $1 \leq i \leq m$, such that

$$\sum_{i=1}^{m} t^i z^i = 0, \quad \sum_{i=1}^{m} t^i = 1 \tag{3.2}$$

and

$$\operatorname{span}\left\{z^1, \cdots, z^m\right\} = \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] = \mathbb{R}^n \wedge b. \tag{3.3}$$

As $z^i \in E \cap (\mathbb{R}^n \wedge b)$, for every $1 \leq i \leq m$, there exists $a^i \in F$ such that

$$z^i = a^i \wedge b \quad \text{for every } 1 \leq i \leq m. \tag{3.4}$$

It therefore follows from (3.2) that

$$\left(\sum_{i=1}^{m} t^i a^i\right) \wedge b = 0. \tag{3.5}$$

We next define

$$\mathcal{W} = \{x \in \mathbb{R}^n : x \wedge b = 0\}.$$

Note that it follows from Lemma 2.1 that $0 \leq \dim \mathcal{W} \leq k$. We now consider two cases (cf. Steps 2 and 3).

Step 2. We prove the lemma when $\dim \mathcal{W} = 0$. Note that in this case $\dim (\mathbb{R}^n \wedge b) = n$ and hence $m \geq n+1$. Since $\dim \mathcal{W} = 0$, we deduce from (3.5) that

$$\sum_{i=1}^{m} t^i a^i = 0.$$

Observe that $\operatorname{span}\{a^1, \cdots, a^m\} = \mathbb{R}^n$. Indeed if $\operatorname{span}\{a^1, \cdots, a^m\}$ was a proper subspace of $\mathbb{R}^n$, so would $\operatorname{span}\{z^1, \cdots, z^m\}$ be a proper subspace of $\mathbb{R}^n \wedge b$ which contradicts (3.3). Therefore, using Lemma 3.2, we have that $0 \in \operatorname{int} \operatorname{co} F$. This proves the lemma when $\dim \mathcal{W} = 0$ (or equivalently when b does not have any $1-$form as a factor).

Step 3. We prove the lemma when $\dim \mathcal{W} = r$, where $1 \leq r \leq k$.

Step 3.1. In this case, we have that

$$\dim (\mathbb{R}^n \wedge b) = n - r$$

and thus $m \geq n - r + 1$. Let $\{b^1, \cdots, b^r\}$ be an orthonormal basis of $\mathcal{W}$. Without loss of generality, we can assume that

$$b^j = e^j \quad \text{for every } 1 \leq j \leq r$$

where $\{e^1, \cdots, e^n\}$ is the standard basis of $\mathbb{R}^n$. Since $e^i \wedge b = 0$ for every $1 \leq i \leq r$, we have, invoking Cartan's Lemma (cf. Theorem 2.42 in [6]), that b can be written as

$$b = e^1 \wedge \cdots \wedge e^r \wedge \overline{b}$$

where $\overline{b} \in \Lambda^{k-r}$ does not have any $1-$form as a factor; i.e. we have that

$$x \wedge \overline{b} \neq 0 \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}\,.$$

Note that

$$z^i = a^i \wedge b = a^i \wedge e^1 \wedge \cdots \wedge e^r \wedge \overline{b} = \left(\sum_{j=r+1}^{n} a^i_j e^j \right) \wedge e^1 \wedge \cdots \wedge e^r \wedge \overline{b}.$$

Hence, without loss of generality, we assume that $\{a^1, \cdots, a^m\}$ in (3.4) also satisfy

$$a^i_j = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq r. \tag{3.6}$$

In addition, since $x \wedge \overline{b} \neq 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$ and using (3.6), we deduce that

$$\dim \big(\operatorname{span}\{a^1, \cdots, a^m\}\big) = \dim \big(\operatorname{span}\{z^1, \cdots, z^m\}\big) = n - r. \tag{3.7}$$

Step 3.2. To show that $0 \in \operatorname{int} \operatorname{co} F$, let us define $v^{i,j} \in \mathbb{R}^n$, for $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, as

$$v^{i,1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}, \quad v^{i,2} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}, \cdots, \; v^{i,2^r} = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ -1 \\ a^i_{r+1} \\ \vdots \\ a^i_n \end{pmatrix}. \tag{3.8}$$

Clearly $v^{i,j} \in F$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$, since

$$v^{i,j} \wedge b = a^i \wedge b \in E \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r.$$

Furthermore, defining

$$t^{i,j} = \frac{t^i}{2^r} \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r$$

it is easy check that $m2^r \geq n+1$ for every $1 \leq r \leq n-1$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} v^{i,j} = 0$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{2^r} t^{i,j} = 1.$$

If we show (cf. Step 3.3) that

$$\operatorname{span}\{v^{i,j} : 1 \leq i \leq m;\ 1 \leq j \leq 2^r\} = \mathbb{R}^n, \tag{3.9}$$

then, using Lemma 3.2, we will have proved that $0 \in \operatorname{int} \operatorname{co} F$ and the proof will be finished.

Step 3.3. We show (3.9). Let $y \in \mathbb{R}^n$ be such that $\langle y; v^{i,j} \rangle = 0$, for every $1 \leq i \leq m$ and $1 \leq j \leq 2^r$. It is enough to prove that $y = 0$ to have the claim. Let $1 \leq i \leq m$ be fixed. For every $1 \leq p \leq r$, we choose $1 \leq j, l \leq 2^r$ such that

$$v^{i,j} - v^{i,l} = 2e^p.$$

Then

$$0 = \langle y; v^{i,j} - v^{i,l} \rangle = 2\langle y; e^p \rangle \quad \text{for every } p \in \{1, \cdots, r\}$$

which shows that $y_p = 0$, for every $1 \leq p \leq r$. The system

$$\langle y; v^{i,j} \rangle = 0 \quad \text{for every } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2^r$$

is therefore equivalent to the system

$$\sum_{j=r+1}^{n} a^i_j y_j = 0 \quad \text{for every } 1 \leq i \leq m. \tag{3.10}$$

We now define a matrix $A \in \mathbb{R}^{m \times (n-r)}$ as

$$A_j^i = a_{j+r}^i \quad \text{for every } 1 \le i \le m \text{ and } 1 \le j \le n-r.$$

Then (3.10) can be written as

$$A\bar{y} = 0$$

where $\bar{y} = (y_{r+1}, \cdots, y_n)$.

To show that $\bar{y} = 0$, it is enough to prove that $\operatorname{rank} A = n - r$ (recall that $m \ge n - r + 1$) which directly follows from (3.6) and (3.7). Therefore, $\bar{y} = 0$ and hence, $y = 0$. This proves the claim and concludes the proof. □

Definition 3.4. A map $\omega : \Omega \to \Lambda^k(\mathbb{R}^n)$ is said to be (locally finite) piecewise affine if there exist $A_i \in \mathbb{R}^{\binom{n}{k} \times n}$, $\alpha_i \in \mathbb{R}^{\binom{n}{k}}$ and $\Omega_i \subset \Omega$ open sets where i runs through an at most countable set I such that

(1) $\omega(x) = A_i x + \alpha_i$ in Ω_i for every $i \in I$,
(2) $\operatorname{meas}\left(\Omega \setminus \bigcup_{i \in I} \Omega_i\right) = 0$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i, j \in I$ with $i \ne j$,
(3) for every compact $K \subset \Omega$

$$\{i \in I : K \cap \Omega_i \ne \emptyset\} \quad \text{is finite.}$$

We recall a result concerning the scalar gradient case (see Lemma 2.11 in [10] and its proof).

Theorem 3.5 (Gradient case). *Let $\Omega \subset \mathbb{R}^n$ be open and $F \subset \mathbb{R}^n$. Then there exists $u \in W_0^{1,\infty}(\Omega)$ such that*

$$\operatorname{grad} u \in F \quad \textit{a.e. in } \Omega$$

if and only if

$$0 \in F \cup \operatorname{int} \operatorname{co} F.$$

If moreover $0 \in \operatorname{int} \operatorname{co} F$, then u can be taken piecewise affine, $u \ge 0$ and

$$\int_\Omega u > 0.$$

Finally we give the following elementary lemma.

Lemma 3.6. *Let $0 \le k \le n$ be two integers and let $\omega \in C^\infty\left(\mathbb{R}^n; \Lambda^k\right)$ be such that*

$$\omega(x) = Ax + a \quad \textit{for every } x \in \mathbb{R}^n$$

where $a \in \mathbb{R}^{\binom{n}{k}}$ and $A \in \mathbb{R}^{\binom{n}{k} \times n}$ is such that

$$\operatorname{rank} A \le 1.$$

Then there exists $b \in \Lambda^1 \setminus \{0\}$ such that

$$b \wedge d\omega = 0.$$

Proof. Since we will use the lemma only when $k = 1$, we prove it in this context; the general case is similar. The result being trivial if $\operatorname{rank} A = 0$, we assume that $\operatorname{rank} A = 1$. Hence there exist $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}^n \setminus \{0\}$ such that

$$A_j^i = \alpha_i b_j \quad \text{for every } 1 \le i, j \le n.$$

We claim that

$$b \wedge d\omega = 0$$

which will prove the lemma. Since, for $1 \leq j_1 < j_2 \leq n$,

$$(d\omega)_{j_1 j_2} = \alpha_{j_1} b_{j_2} - \alpha_{j_2} b_{j_1} ,$$

we deduce that, for every $1 \leq r_1 < r_2 < r_3 \leq n$,

$$\begin{aligned}(b \wedge d\omega)_{r_1 r_2 r_3} &= (\alpha_{r_1} b_{r_2} - \alpha_{r_2} b_{r_1})\, b_{r_3} - (\alpha_{r_1} b_{r_3} - \alpha_{r_3} b_{r_1})\, b_{r_2} \\ &\quad + (\alpha_{r_2} b_{r_3} - \alpha_{r_3} b_{r_2})\, b_{r_1} \\ &= 0.\end{aligned}$$

This proves the claim and concludes the lemma. □

3.2. **Statement of the main results.**

Theorem 3.7 (Necessary and sufficient condition)**.** *Let $0 \leq k \leq n-1$ be two integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $b \in \Lambda^k \setminus \{0\}$ and $E \subset \Lambda^{k+1}$. Then the following statements are equivalent.*

(i) There exists $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega u \neq 0.$$

(ii) The following holds:

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co}[E \cap (\mathbb{R}^n \wedge b)].$$

Remark 3.8. (i) The u will be constructed piecewise affine with $u \geq 0$.

(ii) Letting $\omega(x) = u(x)\, b$, we get that $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$,

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

Corollary 3.9. *Let $0 \leq k \leq n-1$ be two integers, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $E \subset \Lambda^{k+1}$ be such that*

$$\dim \operatorname{span} E = n - k.$$

Then the following statements are equivalent.

(i) There exists $\omega \in W_0^{1,\infty}(\Omega; \Lambda^k)$ such that

$$d\omega \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega \neq 0.$$

(ii) There exist $b = b^1 \wedge \cdots \wedge b^k \neq 0$ where $b^i \in \Lambda^1$ such that

$$0 \in \operatorname{ri} \operatorname{co} E \quad \text{and} \quad \operatorname{span} E = \mathbb{R}^n \wedge b. \tag{3.11}$$

Remark 3.10. Using Lemma 3.1, note that (3.11) implies (since in this case $E = E \cap (\mathbb{R}^n \wedge b)$)

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co} E.$$

In the case $n = 3$ the result, obtained in Theorem 4.15 of [1], takes the following optimal form. Note that in this case we can identify $d\omega$ with $\operatorname{curl} \omega$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$.

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set and $E \subset \mathbb{R}^3$. Then the three following assertions are equivalent.*

(i) There exists $\omega \in W_0^{1,\infty}\left(\Omega; \mathbb{R}^3\right)$ such that

$$\operatorname{curl} \omega \in E \quad a.e.\ \Omega.$$

(ii) There exists a piecewise affine $\omega \in W_0^{1,\infty}\left(\Omega; \mathbb{R}^3\right)$ such that

$$\operatorname{curl} \omega \in E \ a.e.\ \Omega \quad and \quad \int_\Omega \omega \neq 0.$$

(iii) There exists $F \subset E$ such that

$$\dim \operatorname{span} F \geq 2 \quad and \quad 0 \in \operatorname{ri} \operatorname{co} F.$$

It is interesting to compare Theorems 3.7 and 3.11. Indeed one should not infer from the first theorem that any solution of $d\omega \in E$ is of the form $\omega = u\, b$, as the following proposition shows.

Proposition 3.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Then there exists a set $E \subset \Lambda^2 \setminus \{0\}$ such that*

$$0 \in \operatorname{int} \operatorname{co} E$$

with the following properties.

(i) There exists no $b \in \Lambda^1 \setminus \{0\}$ such that

$$0 \in \operatorname{int}_{\mathbb{R}^3 \wedge b} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge b\right)\right]$$

and therefore there is no $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \quad a.e.\ in\ \Omega.$$

(ii) There exists $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^2\right)$ such that

$$d\omega \in E \quad a.e.\ in\ \Omega.$$

The general situation when $n \geq 4$ is considerably harder in view of the following considerations. The situation in Theorem 3.11 is very specific to the dimension $n = 3$. In the next proposition we will exhibit a set $E \subset \Lambda^2\left(\mathbb{R}^4\right)$ with

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2\right)\right]$$

for which there cannot exist a piecewise affine solution ω of $d\omega \in E$. However when $n = 3$, since $\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2 = \Lambda^2\left(\mathbb{R}^3\right)$, we always have

$$\operatorname{int}_{\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2\right)\right] = \operatorname{int} \operatorname{co} E.$$

Therefore, appealing to Theorem 3.11, if

$$0 \in \operatorname{int}_{\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge e^1 + \mathbb{R}^3 \wedge e^2\right)\right]$$

we can find a piecewise affine solution ω of $d\omega \in E$.

Proposition 3.13. *There exists a set $E \subset \Lambda^2\left(\mathbb{R}^4\right)$ with*

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co} \left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2\right)\right]$$

with the following property: for every bounded open set $\Omega \subset \mathbb{R}^4$ there exists no piecewise affine $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^1\right)$ such that

$$d\omega \in E \quad a.e.\ in\ \Omega.$$

3.3. **Proof of the main results.** We start by showing Theorem 3.7.

Proof.

Part 1. $(i) \Rightarrow (ii)$. Using Lemma 3.1 we have to show that

$$\mathbb{R}^n \wedge b = \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] \tag{3.12}$$

and

$$0 \in \operatorname{ri} \operatorname{co}\left[E \cap (\mathbb{R}^n \wedge b)\right]. \tag{3.13}$$

Let $\omega = u\, b \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$. Obviously

$$d\omega \in E \cap (\mathbb{R}^n \wedge b) \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega \omega = \left(\int_\Omega u\right) b \neq 0.$$

Hence using Theorem 2.5 (with E replaced by $E \cap (\mathbb{R}^n \wedge b)$), we obtain that

$$\mathbb{R}^n \wedge b \subset \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right]$$

which obviously implies (3.12). We now show (3.13). For the sake of contradiction suppose that (3.13) does not hold. Then using Lemma 2.4 (with E replaced by $E \cap (\mathbb{R}^n \wedge b)$) there exists a set $D \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus D) = 0$, $d\omega(D) \subset E \cap (\mathbb{R}^n \wedge b)$ and

$$\dim \operatorname{span} d\omega(D) < \dim \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right]. \tag{3.14}$$

But, since $d\omega \in d\omega(D)$ a.e. in Ω, we deduce, using again Theorem 2.5, that

$$\mathbb{R}^n \wedge b \subset \operatorname{span}\left(d\omega(D)\right).$$

This is the desired contradiction since, using (3.12) and (3.14), we have

$$\dim\left(\mathbb{R}^n \wedge b\right) \leq \dim \operatorname{span}\left(d\omega(D)\right) < \dim \operatorname{span}\left[E \cap (\mathbb{R}^n \wedge b)\right] = \dim\left(\mathbb{R}^n \wedge b\right).$$

Part 2. $(ii) \Rightarrow (i)$. Using Lemma 3.3, we find $F \subset \mathbb{R}^n$ such that

$$E \cap (\mathbb{R}^n \wedge b) = F \wedge b \quad \text{and} \quad 0 \in \operatorname{int} \operatorname{co} F. \tag{3.15}$$

Appealing to Theorem 3.5, we find $u \in W_0^{1,\infty}(\Omega)$ such that

$$\operatorname{grad} u \in F \quad \text{a.e. in } \Omega. \tag{3.16}$$

Moreover u can be chosen piecewise affine and such that $\int_\Omega u \neq 0$ (and also such that $u \geq 0$). Let us now define $\omega \in W_0^{1,\infty}\left(\Omega; \Lambda^k\right)$ by

$$\omega(x) = u(x)\, b \quad \text{for every } x \in \Omega.$$

It is easy to check that

$$d\omega = (\operatorname{grad} u) \wedge b \text{ a.e. in } \Omega.$$

It therefore follows from (3.15) and (3.16) that

$$d\omega \in E \cap (\mathbb{R}^n \wedge b) \subset E \quad \text{a.e. in } \Omega.$$

This finishes Part (ii) and the proof. $\square$

We now prove Corollary 3.9.

Proof.

Part 1. $(i) \Rightarrow (ii)$. Since $\dim \operatorname{span} E = n - k$, using Theorem 2.5, there exist $b^1, \cdots, b^k \in \Lambda^1$ such that

$$\mathbb{R}^n \wedge \left(\int_\Omega \omega \right) = \operatorname{span} E \quad \text{and} \quad \int_\Omega \omega = b^1 \wedge \cdots \wedge b^k$$

which shows the second part of (3.11). Proceeding exactly as in Part 1 of the proof of Theorem 3.7, we can prove that

$$0 \in \operatorname{ri} \operatorname{co} E.$$

This concludes Part 1.

Part 2. $(ii) \Rightarrow (i)$. Using Lemma 3.1, we have that (3.11) implies (since $E = E \cap (\mathbb{R}^n \wedge b)$)

$$0 \in \operatorname{int}_{\mathbb{R}^n \wedge b} \operatorname{co} \left[E \cap (\mathbb{R}^n \wedge b) \right].$$

Hence, by Theorem 3.7, there exists $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega u \neq 0.$$

Thus $\omega = u\, b$ has the desired properties. This concludes Part 2 and the proof. $\square$

We next turn to the proof of Proposition 3.12.

Proof. Let

$$E = \left\{ e^1 \wedge e^2, -e^1 \wedge e^2 + e^1 \wedge e^3, -e^1 \wedge e^2 - e^1 \wedge e^3 + e^2 \wedge e^3, \right. \\ \left. -e^1 \wedge e^2 - e^1 \wedge e^3 - e^2 \wedge e^3 \right\}.$$

It is easy to see that

$$0 \notin \operatorname{int}_{\mathbb{R}^3 \wedge b} \operatorname{co} \left[E \cap \left(\mathbb{R}^3 \wedge b \right) \right] \quad \text{for every } b \in \Lambda^1$$

but that

$$0 \in \operatorname{int} \operatorname{co} E.$$

(i) The first statement combined with Theorem 3.7 shows that there is no $u \in W_0^{1,\infty}(\Omega)$ such that

$$(\operatorname{grad} u) \wedge b \in E \quad \text{a.e. in } \Omega.$$

(ii) The second statement implies (cf. Theorem 3.11) the existence of $\omega \in W_0^{1,\infty}(\Omega; \Lambda^1)$ such that

$$d\omega \in E \quad \text{a.e. in } \Omega.$$

The proof of the proposition is therefore complete. $\square$

Finally we establish Proposition 3.13.

Proof.

Step 1. Let

$$a^1 = e^1 \wedge e^2, \quad a^2 = \left(e^1 + e^2\right) \wedge e^3, \quad a^3 = \left(e^1 + 2e^2\right) \wedge e^3,$$
$$a^4 = \left(e^1 + 3e^2\right) \wedge e^4, \quad a^5 = \left(e^1 + 4e^2\right) \wedge e^4,$$
$$a^6 = -\sum_{j=1}^{5} a^i = -\left(e^1 \wedge e^2 + 2e^1 \wedge e^3 + 2e^1 \wedge e^4 + 3e^2 \wedge e^3 + 7e^2 \wedge e^4\right)$$

and finally let

$$E = \left\{a^1, \cdots, a^6\right\}.$$

We will show that E has all the desired properties (cf. Steps 2 and 3).

Step 2. Note that

$$\operatorname{span} E = \mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2.$$

Since obviously

$$\sum_{j=1}^{6} \frac{1}{6}\, a^j = 0,$$

it follows directly from Lemmas 3.1 and 3.2 that

$$0 \in \operatorname{int}_{\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2} \operatorname{co}\left[E \cap \left(\mathbb{R}^4 \wedge e^1 + \mathbb{R}^4 \wedge e^2\right)\right].$$

A simple calculation shows that, for every $1 \leq j \leq 5$,

$$\left(a^j - a^6\right) \wedge \left(a^j - a^6\right) \neq 0.$$

This says that

$$\operatorname{rank}\left[a^j - a^6\right] = 4 \quad \text{for every } 1 \leq j \leq 5$$

which is equivalent, in view of Cartan's Lemma (cf. Theorem 2.42 in [6]), to saying that, for every $1 \leq j \leq 5$,

$$b \wedge \left(a^j - a^6\right) \neq 0 \quad \text{for every } b \in \Lambda^1 \setminus \{0\}. \tag{3.17}$$

Step 3. Let $\Omega \subset \mathbb{R}^4$ be a bounded open set. We claim that there does not exist $\omega \in W_0^{1,\infty}(\Omega; \Lambda^1)$ piecewise affine such that

$$d\omega \in E = \{a^1, \cdots, a^6\} \quad \text{a.e. in } \Omega.$$

By contradiction assume that such a map exists. Therefore (cf. Definition 3.4) there exist $A_i \in \mathbb{R}^{4\times 4}$, $\alpha_i \in \mathbb{R}^4$ and $\Omega_i \subset \Omega$ open sets where i runs through an at most countable set I such that

(1) $\omega(x) = A_i x + \alpha_i$ in Ω_i for every $i \in I$,
(2) $\operatorname{meas}\left(\Omega \setminus \bigcup_{i \in I} \Omega_i\right) = 0$ and $\Omega_i \cap \Omega_j = \emptyset$ for every $i, j \in I$ with $i \neq j$,
(3) for every compact $K \subset \Omega$

$$\{i \in I : K \cap \Omega_i \neq \emptyset\} \quad \text{is finite.}$$

It is well known and easy to see that if $\partial\Omega_i \cap \partial\Omega_j$ contains a 3 dimensional subset of a hyperplane, then necessarily

$$\operatorname{rank}\left[A_i - A_j\right] \leq 1. \tag{3.18}$$

From now on we assume that Ω is connected, otherwise we reason separately on every connected component of Ω. Since the partition of Ω is locally finite (cf. Point 3 above) it is easy to see that for every $i, j \in I$ with $i \neq j$ there exist $l_1 = i$, $l_2, \cdots, l_N = j$ such that, for every $2 \leq m \leq N$, either $A_{l_{m-1}} = A_{l_m}$ or

$$\partial\Omega_{l_{m-1}} \cap \partial\Omega_{l_m} \quad \text{contains a 3 dimensional subset of a hyperplane.} \tag{3.19}$$

Since (cf. Lemma 2.4) $0 \in \overline{\operatorname{co} E} = \operatorname{co} E$ and since

$$\{a^1, \cdots, a^5\} \quad \text{are linearly independent,}$$

we deduce that there exists $\overline{i}$ such that

$$d\omega = a^6 \quad \text{in } \Omega_{\overline{i}}.$$

Let $I_1 \subset I$ be defined by

$$\{i \in I : d\omega \in \{a^6\} \text{ in } \Omega_i\}.$$

We claim that $I_1 = I$ which implies that

$$d\omega \in \{a^6\} \quad \text{a.e. in } \Omega$$

and this is the desired contradiction. Choose $i \in I$ with $i \neq \bar{i}$. Combining (3.18) and (3.19), there exist $l_1 = i$, $l_2, \cdots, l_N = \bar{i}$ such that, for every $2 \leq j \leq N$,

$$\operatorname{rank}\left[A_{l_{j-1}} - A_{l_j}\right] \leq 1.$$

For every $1 \leq j \leq N$, let $r_j \in \{1, \cdots, 6\}$ be such that

$$d\omega = a^{r_j} \quad \text{in } \Omega_{l_j}.$$

Note that, in particular, $r_N = 6$. Using Lemma 3.6 we find that, for every $2 \leq j \leq N$, there exists $b^j \in \Lambda^1 \setminus \{0\}$ such that

$$b^j \wedge (a^{r_{j-1}} - a^{r_j}) = 0.$$

Combining the previous equation with (3.17) we immediately deduce that

$$r_1 = \cdots = r_N = 6$$

and hence $i \in I_1$. This shows that $I_1 = I$ and proves the proposition. $\square$

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