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AN IMMERSED S^2 SELF-SHRINKER

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ABSTRACT. We construct an immersed and non-embedded S^2 self-shrinker.

1. INTRODUCTION

An immersion F from a two-dimensional manifold M into $\mathbb{R}^3$ is a self-shrinker if it satisfies

$$\Delta_g F = -\frac{1}{2}F^{\perp}, \tag{1.1}$$

where g is the metric on M induced by the immersion, Δ_g is the Laplace-Beltrami operator, and $F^{\perp}(p)$ is the projection of $F(p)$ into the normal space N_pM. When $F : M \to \mathbb{R}^3$ satisfies (1.1), the family of submanifolds $M_t = \sqrt{-t}F(M)$ is a solution of mean curvature flow for $t \in (-\infty, 0)$. In the case where M is compact, the rescalings M_t shrink to the origin as t approaches 0 (hence the name self-shrinker). It is a consequence of Huisken's monotonicity formula [10] that a solution of mean curvature flow behaves asymptotically like a self-shrinker at a type I singularity. So, not only do self-shrinkers provide precious examples of solutions of mean curvature flow, but they also describe the behavior of mean curvature flow at certain singular points where the curvature blows up. The simplest examples of self-shrinkers in $\mathbb{R}^3$ are the sphere of radius 2 centered at the origin (the standard sphere), cylinders with an axis through the origin and radius $\sqrt{2}$, and planes through the origin. In this paper, we construct an immersed and non-embedded S^2 self-shrinker in $\mathbb{R}^3$.

Theorem 1.1. *There exists an immersion $F : S^2 \to \mathbb{R}^3$ satisfying $\Delta_g F = -\frac{1}{2}F^{\perp}$, and F is not an embedding.*

In 1989, Angenent [2] constructed an embedded self-shrinker with the topology type of a torus and provided numerical evidence for the existence of an immersed and non-embedded S^2 self-shrinker. (We note that the S^2 self-shrinker in Angenent's numerics is different from the one we construct.) In 1994, Chopp [4] described an algorithm for constructing surfaces that are approximately self-shrinkers and provided numerical evidence for the existence of a number of self-shrinkers, including compact, embedded self-shrinkers of genus 5 and 7. More recently, Kapouleas, Kleene, and Møller [11] and Nguyen [14]–[16] used desingularization constructions to produce examples of complete, non-compact, embedded self-shrinkers with high genus in $\mathbb{R}^3$. Møller [13] also used desingularization techniques to construct compact, embedded, high genus self-shrinkers in $\mathbb{R}^3$. Møller's high genus examples, along with Angenent's torus and the standard sphere, are the only known

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examples of compact self-shrinkers in $\mathbb{R}^3$. In contrast to these examples are several rigidity theorems for compact self-shrinkers. Huisken [10] showed that the only compact, mean-convex self-shrinker in $\mathbb{R}^3$ is the standard sphere. In their study of generic singularities of mean curvature flow, Colding and Minicozzi [5] showed that the only compact, embedded F-stable self-shrinker in $\mathbb{R}^3$ is the standard sphere. As part of their classification of complete, embedded self-shrinkers with rotational symmetry, Kleene and Møller [12] showed that the standard sphere is the only embedded S^2 self-shrinker with rotational symmetry. In an independent work [6], we proved this result by showing that an embedded S^2 self-shrinker with rotational symmetry must be mean-convex. It is unknown whether or not the standard sphere is the only embedded S^2 self-shrinker in $\mathbb{R}^3$.

The basic idea of the proof of Theorem 1.1 is to construct a curve in the (x, z)-plane with self-intersections whose rotation about the z-axis is an S^2 self-shrinker. In this setting, the self-shrinker equation (1.1) reduces to a differential equation. When the curve can be written in the form $(x, \gamma(x))$, the differential equation is

$$\frac{\gamma''}{1+(\gamma')^2} = \left(\frac{1}{2}x - \frac{1}{x}\right)\gamma' - \frac{1}{2}\gamma. \tag{1.2}$$

Using comparison arguments we describe the behavior of solutions of the differential equation for a range of initial conditions. Then, following the approach of Angenent in [2], we use a continuity argument to find an initial condition that corresponds to a solution whose rotation about the z-axis is an immersed and non-embedded S^2 self-shrinker.

The curve we construct in the proof of Theorem 1.1 (see Figure 1) is symmetric with respect to reflections across the x-axis, and it is enough to describe this curve as it travels from the positive z-axis to the point where it intersects the x-axis perpendicularly. We start the construction by studying solutions of (1.2) with $\gamma(0) > 0$ and $\gamma'(0) = 0$. Notice that one of the terms in the differential equation involves $\frac{1}{x}$, and hence this equation has a singularity at $x = 0$. We begin Section 2 by discussing the existence, uniqueness, and continuous dependence on initial height of solutions when x is near 0. As we move away from the origin, we can use existence theorems for differential equations to show that a solution γ will exist until it blows up. Next, we show that γ is decreasing and concave down, and for small initial height, it must cross the x-axis before it blows up. By a theorem of Lu Wang [17], we know that γ blows up at a finite point x_*, and we use a comparison argument to estimate γ' and show there is a finite point z_* so that $\gamma(x_*) = z_*$. We finish Section 2 by showing $x_* \to \infty$ and $z_* \to 0$ as the initial height approaches 0.

In Section 3 we study the behavior of the curve $(x, \gamma(x))$ near the point (x_*, z_*). Writing the curve $(x, \gamma(x))$ in the form $(\alpha(z), z)$, we get a solution of the differential equation

$$\frac{\alpha''}{1+(\alpha')^2} = \left(\frac{1}{\alpha} - \frac{1}{2}\alpha\right) + \frac{1}{2}z\alpha'. \tag{1.3}$$

At (x_*, z_*), we have $\alpha(z_*) = x_*$ and $\alpha'(z_*) = 0$, and by the existence theory for differential equations, we can continue the curve $(x, \gamma(x))$ past the blow-up point (x_*, z_*) along $(\alpha(z), z)$. Following $(\alpha(z), z)$, we show that the curve makes a turn at (x_*, z_*) and heads back towards the z-axis. The curve heading back towards the z-axis can be written as $(x, \beta(x))$, where $\beta(x)$ is a solution of (1.2). Applying existence, uniqueness, and continuity theorems to these differential equations, we

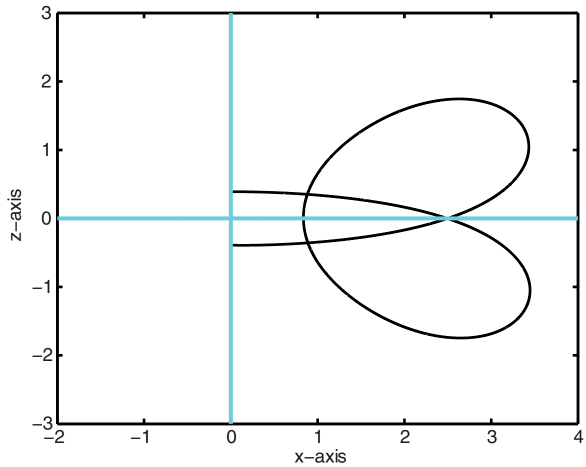

FIGURE 1. A numerical approximation of a curve whose rotation about the z-axis is an immersed and non-embedded S^2 self-shrinker in $\mathbb{R}^3$.

discuss how the curves γ and α and the point (x_*, z_*) depend continuously on the initial height. We also show that $z_* < 0$ when $\gamma(0) \in (0, 2)$. Although this last result is not essential to the construction, it is a nice consequence of the rigidity of compact, mean-convex self-shrinkers due to Huisken [10].

In Section 4 we study the solutions $\beta(x)$ as they travel from (x_*, z_*) towards the z-axis. We know there exists a point $x_{**} \geq 0$ so that β is a solution of (1.2) on (x_{**}, x_*) and either β blows up as x approaches x_{**} or $x_{**} = 0$. We show for small initial height $\gamma(0)$ that β achieves a negative minimum at a point $x_m \in (x_{**}, x_*)$, β is concave up, $x_{**} > 0$, and $0 < \beta(x_{**}) < \infty$. To prove $\beta(x_{**}) > 0$, we give a direct argument, which shows how the singular term in (1.2) forces β to cross the x-axis when x_{**} is small. This direct crossing argument is different from the limiting argument used by Angenent in [2], and the analysis of β in this section leads to a different construction of Angenent's torus self-shrinker. We also note that Møller [13] constructed a torus self-shrinker with explicit estimates on the cross-sections, which he used to construct the high genus compact, embeddded self-shrinkers.

Finally, in Section 5 we finish the proof of Theorem 1.1. We let γ_b be the solution of (1.2) with $\gamma_b(0) = b$, and define β_b, x_*^b, x_{**}^b, and x_m^b as above. Following Angenent's argument in [2], we consider the initial height b_0 given by

$$b_0 = \sup\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists x_m^b \in (x_{**}^b, x_*^b) \text{ so that } \beta_b'(x_m^b) = 0 \text{ and } \beta_b(x_{**}^b) > 0\}.$$

Using continuity arguments, we show that β_{b_0} intersects the x-axis perpendicularly at $x_{**}^{b_0}$. Thus, the curve $\gamma_{b_0} \cup \beta_{b_0} \cup -\beta_{b_0} \cup -\gamma_{b_0}$ is a smooth curve in the right-half plane that intersects the z-axis perpendicularly at precisely two points (see Figure 1), and the rotation of this curve about the z-axis is an immersed and non-embedded S^2 self-shrinker in $\mathbb{R}^3$.

Remark 1.2. The proof works in higher dimensions to give an immersed and non-embedded S^n self-shrinker in $\mathbb{R}^{n+1}$. In this setting, the $\frac{1}{x}$ singular term in (1.2) is replaced with $\frac{n-1}{x}$. In the one-dimensional case, the singular term vanishes and solutions can cross over the line $x = 0$ without the slope restriction that holds in higher dimensions. The compact one-dimensional self-shrinkers were completely classified by both Abresch and Langer [1] and Epstein and Weinstein [8]. In this case, the standard circle is the only embedded S^1 self-shrinker, and there are many immersed and non-embedded S^1 self-shrinkers.

2. The first branch

In this section we study solutions of (1.2) with $\gamma(0) = b > 0$ and $\gamma'(0) = 0$. We begin by discussing the existence, uniqueness, and continuous dependence on initial height of solutions. After this, we use a variety of comparison estimates to describe the basic behavior of γ when $b > 0$. Finally, we finish the section with a detailed description of γ when the initial height $b > 0$ is small.

2.1. **Existence of solutions near $x = 0$.** Notice that one of the terms in (1.2) involves $\frac{1}{x}$, and hence this equation has a singularity at $x = 0$. We have the following proposition addressing the existence, uniqueness, and continuous dependence of solutions on initial height when x is near 0.

Proposition 2.1. *For any $b \in \mathbb{R}$, there exists $A = A(b) > 0$ and a unique analytic function γ defined on $[0, 1/A]$ so that $\gamma(0) = b$, $\gamma'(0) = 0$, and γ is a solution of* (1.2)*. Moreover, γ and γ' depend continuously on b as follows: For each $M > 0$ there exists $A = A(M) > 0$ with the property that for $\varepsilon > 0$ there is a $\delta > 0$ so that if $|b_1 - b_2| < \delta$ and $|b_i| \leq M$, then $|\gamma_1(x) - \gamma_2(x)| < \varepsilon$ and $|\gamma_1'(x) - \gamma_2'(x)| < \varepsilon$ for all $x \in [0, 1/A]$, where γ_i is the unique analytic solution of* (1.2) *with $\gamma_i(0) = b_i$.*

Proof. We mention two proofs of the proposition. First, using a power series argument specific to the equation (1.2) we established the existence, uniqueness, and continuity results as stated in the proposition. This argument is included in the appendix. Afterwards, Robin Graham informed us of the general reference [3], where the Cauchy problem for singular systems of partial differential equations was studied. Applying Theorem 2.2 from [3] also shows that (1.2) has a unique analytic solution in a neighborhood of 0. We would like to thank Robin Graham for this reference. $\square$

2.2. **Basic shape of γ.** Let γ be the solution of (1.2) with $\gamma(0) = b > 0$ and $\gamma'(0) = 0$. Then $\gamma''(0) = -b/4$ so that γ starts out concave down. Taking derivatives of (1.2), we have the following equations:

$$\frac{\gamma'''}{1+(\gamma')^2} = \frac{2\gamma'(\gamma'')^2}{(1+(\gamma')^2)^2} + \left(\frac{1}{2}x - \frac{1}{x}\right)\gamma'' + \frac{1}{x^2}\gamma' \tag{2.1}$$

and

$$\begin{aligned} \frac{\gamma^{(iv)}}{1+(\gamma')^2} \quad = \quad & \frac{6\gamma'\gamma''\gamma''' + 2(\gamma'')^3}{(1+(\gamma')^2)^2} - \frac{8(\gamma')^2(\gamma'')^3}{(1+(\gamma')^2)^3} + \left(\frac{1}{2}x - \frac{1}{x}\right)\gamma''' \\ & + \left(\frac{1}{2} + \frac{2}{x^2}\right)\gamma'' - \frac{2}{x^3}\gamma'. \end{aligned} \tag{2.2}$$

Claim 2.2. $\gamma'' < 0$.

Proof. Since $\gamma''(0) = -\frac{b}{4}$, we know that $\gamma'' < 0$ near 0. Suppose $\gamma''(x) = 0$ for some $x > 0$. Choose $\bar{x}$ so that $\gamma''(\bar{x}) = 0$ and $\gamma''(x) < 0$ for $x \in [0, \bar{x})$. Then $\gamma'''(\bar{x}) \geq 0$. Also, $\gamma'(\bar{x}) < 0$ (since $\gamma'(0) = 0$). Using (2.1), we see that

$$0 \leq \frac{\gamma'''(\bar{x})}{1 + \gamma'(\bar{x})^2} = \frac{1}{\bar{x}^2}\gamma'(\bar{x}) < 0,$$

which is a contradiction. $\square$

In [17] Lu Wang proved that an entire self-shrinker graph must be a plane. It follows that γ cannot be defined on all of $[0, \infty)$, and therefore by the existence theory for differential equations there must be a point $x_* < \infty$ so that γ blows up at x_* (blows up in the sense that either $|\gamma|$ or $|\gamma'|$ goes to ∞ as x goes to x_*). Since $\gamma'' < 0$ and $x_* < \infty$, it follows that $\lim_{x \to x_*} \gamma'(x) = -\infty$.

Claim 2.3. $x_* > \sqrt{2}$.

Proof. Suppose $x_* < \sqrt{2}$. Then using (1.2), we see that $\lim_{x \to x_*} \gamma''(x) = \infty$, which contradicts the fact that $\gamma'' < 0$. On the other hand, the existence of the cylinder self-shrinker prevents x_* from being equal to $\sqrt{2}$. To see this, suppose $x_* = \sqrt{2}$. Since $\gamma'' < 0$, we know that $\gamma(x) > 0$ for $x \in [0, x_*)$, and therefore there exists $z_* \geq 0$ so that $\lim_{x \to x_*} \gamma(x) = z_*$. Near the point $(\sqrt{2}, z_*)$, we write the curve $(x, \gamma(x))$ as $(\alpha(z), z)$, where α satisfies the differential equation (1.3). Now, $\alpha(z_*) = \sqrt{2}$ and $\alpha'(z_*) = 0$, and by the uniqueness of solutions for this differential equation, α must be the constant function $\alpha(z) = \sqrt{2}$ (which corresponds to the cylinder self-shrinker). This contradicts the fact that $(x, \gamma(x))$ agrees with $(\alpha(z), z)$ near $(\sqrt{2}, z_*)$. $\square$

Lemma 2.4. $\lim_{x \to x_*} \gamma(x) > -\infty$.

Proof. Let γ be a solution of (1.2) with $\gamma(0) = b > 0$ and $\gamma'(0) = 0$. Let $x_* > \sqrt{2}$ be the point where γ blows up. Fix $\delta > 0$ so that $x_* - \delta > \sqrt{2}$, and let $m > 0$ be such that $(\frac{1}{2}x - \frac{1}{x}) \geq m$ when $x \in (x_* - \delta, x_*)$. Choose $M > 0$ so that $m \geq \frac{3}{2M^2}$ and $M \geq -\gamma'(x_* - \delta)$.

For $\varepsilon > 0$, define $g_\varepsilon(x)$ on $(x_* - \delta, x_* - \varepsilon)$ by

$$g_\varepsilon(x) = \frac{M}{\sqrt{(x_* - \varepsilon) - x}}.$$

Then

$$g_\varepsilon''(x) = \frac{3}{2M^2} g_\varepsilon(x)^2 g_\varepsilon'(x) \leq \left(\frac{1}{2}x - \frac{1}{x}\right) g_\varepsilon(x)^2 g_\varepsilon'(x),$$

for $x \in (x_* - \delta, x_* - \varepsilon)$. We will use the function g_ε to show that $-\gamma'$ blows up no faster than $M/\sqrt{x_* - x}$. Let $f(x) = -\gamma'(x)$. Then $f \geq 0$, $f' > 0$, and by (2.1),

$$f''(x) \geq \left(\frac{1}{2}x - \frac{1}{x}\right) f(x)^2 f'(x),$$

when $x \geq \sqrt{2}$. We will show $f \leq g_\varepsilon$. By construction,

$$f(x_* - \delta) \leq M < g_\varepsilon(x_* - \delta)$$

and

$$f(x_* - \varepsilon) < \lim_{x \to (x_* - \varepsilon)} g_\varepsilon(x).$$

Therefore, if $f > g_\varepsilon$ at some point, then $f - g_\varepsilon$ achieves a positive maximum at some point $x' \in (x_* - \delta, x_* - \varepsilon)$. This leads to $(f - g_\varepsilon)'(x') = 0$ and $(f - g_\varepsilon)''(x') \leq 0$. Consequently,

$$0 \geq (f - g_\varepsilon)''(x') \geq \left(\frac{1}{2}x' - \frac{1}{x'}\right) f'(x') \left(f(x')^2 - g_\varepsilon(x')^2\right) > 0,$$

which is a contradiction.

It follows that $f \leq g_\varepsilon$ on $(x_* - \delta, x_* - \varepsilon)$. Taking $\varepsilon \to 0$, we conclude that

$$\gamma'(x) \geq \frac{-M}{\sqrt{x_* - x}},$$

for $x \in (x_* - \delta, x_*)$. Therefore, $\lim_{x \to x_*} \gamma(x) > -\infty$. □

Remark 2.5. At this point, we can give a basic description of the γ curves: For $b > 0$, let γ_b denote the solution of (1.2) with $\gamma_b(0) = b$ and $\gamma_b'(0) = 0$. Then γ_b is decreasing and concave down, and there exists a point $x_*^b \in (\sqrt{2}, \infty)$ so that γ_b is defined on $[0, x_*)$ and $\lim_{x \to x_*^b} \gamma_b'(x) = -\infty$. There also exists a point $z_*^b \in (-\infty, b)$ so that $\gamma_b(x_*^b) = z_*^b$.

2.3. **Estimates for small initial height.** In this section we prove estimates for x_*^b and z_*^b when the initial height $b > 0$ is small.

Proposition 2.6. *For $b > 0$, let γ_b denote the solution of* (1.2) *with $\gamma_b(0) = b$ and $\gamma_b'(0) = 0$. Let x_*^b denote the point where γ_b blows up, and let $z_*^b = \gamma_b(x_*^b)$.*

There exists $\bar{b} > 0$ so that if $b \in (0, \bar{b}]$, then

$$x_*^b \geq \sqrt{\log \frac{2}{\pi b^2}},$$

$$\frac{-12}{\sqrt{\log \frac{2}{\pi b^2}}} \leq z_*^b < 0,$$

and there exists a point $x_0^b \in [2, 2\sqrt{2}]$ so that $\gamma_b(x_0^b) = 0$.

Before we prove the proposition, we prove some results about solutions of (1.2) when the initial height is small. Let γ be the solution of (1.2) with $\gamma(0) = b$ and $\gamma'(0) = 0$.

Claim 2.7. *If $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^4}$, then $x_* > 2\sqrt{2}$ and $|\gamma'(x)| \leq \frac{\sqrt{3}}{3}$ for $x \in [0, 2\sqrt{2}]$.*

Proof. Since $\gamma'(0) = 0$, $\gamma'' < 0$, and $\lim_{x \to x_*} \gamma'(x) = -\infty$, we know there exists $x' \in (0, x_*)$ so that $\gamma'(x') = -\frac{\sqrt{3}}{3}$. For $x \in (0, x')$, we have

$$\begin{aligned}
\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\gamma'(x)\right) &= e^{-\frac{x^2}{2}}\gamma''(x) - xe^{-\frac{x^2}{2}}\gamma'(x) \\
&\geq \frac{2}{1 + \gamma'(x)^2} e^{-\frac{x^2}{2}}\gamma''(x) - xe^{-\frac{x^2}{2}}\gamma'(x) \\
&= 2e^{-\frac{x^2}{2}}\left[\left(\frac{1}{2}x - \frac{1}{x}\right)\gamma' - \frac{1}{2}\gamma\right] - xe^{-\frac{x^2}{2}}\gamma'(x) \\
&= -2e^{-\frac{x^2}{2}}\frac{1}{x}\gamma' - e^{-\frac{x^2}{2}}\gamma(x) \\
&\geq -e^{-\frac{x^2}{2}}\gamma(x).
\end{aligned}$$

Integrating from 0 to x',

$$-\frac{\sqrt{3}}{3}e^{-\frac{(x')^2}{2}} \geq -\int_0^{x'} e^{-\frac{x^2}{2}}\gamma(x)dx \geq -b\int_0^{x'} e^{-\frac{x^2}{2}}dx \geq -b\sqrt{\frac{\pi}{2}}.$$

When $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^4}$, we have $e^{-\frac{(x')^2}{2}} < e^{-4}$, and therefore $x' > 2\sqrt{2}$. □

Claim 2.8. *If $|\gamma'(x)| \leq \frac{\sqrt{3}}{3}$ for $x \in [0, \sqrt{2}]$, then $\gamma'''(x) < 0$ for $x \in (0, x_*)$.*

Proof. From the power series expansion for γ at $x = 0$, we know that $\gamma'''(0) = 0$ and $\gamma^{(iv)}(0) < 0$. Therefore, $\gamma'''(x) < 0$ when $x > 0$ is near 0. Also, using (2.1), we see that $\gamma'''(x) < 0$ when $x \geq \sqrt{2}$. Suppose $\gamma'''(x) = 0$ for some $x > 0$. Then there exists $\bar{x} \in (0, \sqrt{2})$ so that $\gamma'''(\bar{x}) = 0$ and $\gamma'''(x) < 0$ for $x \in (0, \bar{x})$. It follows that $\gamma^{(iv)}(\bar{x}) \geq 0$. Notice that $x\gamma''(x) - \gamma'(x)$ is decreasing and hence negative on $(0, \bar{x})$. Then, using (2.2) and the assumption that $|\gamma'(\bar{x})| \leq \frac{\sqrt{3}}{3}$, we see that

$$\frac{\gamma^{(iv)}(\bar{x})}{1+\gamma'(\bar{x})^2} = 2(\gamma''(\bar{x}))^3\frac{1-3(\gamma'(\bar{x}))^2}{(1+\gamma'(\bar{x})^2)^3} + \frac{1}{2}\gamma''(\bar{x}) + 2\frac{\bar{x}\gamma''(\bar{x}) - \gamma'(\bar{x})}{(\bar{x})^3} < 0,$$

which is a contradiction. □

Claim 2.9. *If $|\gamma'(x)| \leq \frac{\sqrt{3}}{3}$ for $x \in [0, 2\sqrt{2}]$ and $b < \frac{1}{2}$, then $\frac{x\gamma'(x)-\gamma(x)}{\sqrt{1+\gamma'(x)^2}}$ is non-increasing on $[0, 2\sqrt{2}]$.*

Proof. Looking at the derivative of $\frac{x\gamma'(x)-\gamma(x)}{\sqrt{1+\gamma'(x)^2}}$:

$$\frac{d}{dx}\left(\frac{x\gamma'(x)-\gamma(x)}{\sqrt{1+\gamma'(x)^2}}\right) = \gamma''(x)\frac{x+\gamma(x)\gamma'(x)}{(1+\gamma'(x)^2)^{3/2}},$$

we see that it is enough to show $x + \gamma(x)\gamma'(x) \geq 0$. Since $x + \gamma(x)\gamma'(x)$ equals 0 when $x = 0$, it is sufficient to show $1 + \gamma(x)\gamma''(x) + \gamma'(x)^2 \geq 0$ on $(0, 2\sqrt{2}]$. For $x \in (0, 2\sqrt{2}]$, assuming $|\gamma'(x)| \leq \frac{\sqrt{3}}{3}$ and $b < \frac{1}{2}$, we have

$$\begin{aligned}\gamma''(x) &= (1+\gamma'(x)^2)\left[\left(\frac{1}{2}x - \frac{1}{x}\right)\gamma'(x) - \frac{1}{2}\gamma(x)\right] \\ &\geq \frac{4}{3}\left[-\sqrt{2}\frac{\sqrt{3}}{3} - \frac{b}{2}\right] \geq -2,\end{aligned}$$

and it follows that $1 + \gamma(x)\gamma''(x) + \gamma'(x)^2 \geq 0$. □

Lemma 2.10. *Suppose $x_* > 2\sqrt{2}$ and there exists a point $x_0 \in [2, 2\sqrt{2}]$ so that $\gamma(x_0) = 0$. Then, for $x \in [x_0, x_*)$,*

$$\gamma(x) > \frac{8}{x}\gamma'(x).$$

Proof. Let $\Phi(x) = \frac{1}{8}x\gamma(x) - \gamma'(x)$. We want to show $\Phi(x) > 0$. We know that $\Phi(x_0) = -\gamma'(x_0) > 0$. We also have

$$\frac{1}{8}x\gamma(x) = \frac{1}{8}x\int_{x_0}^{x}\gamma'(\xi)d\xi > \frac{1}{8}x(x-x_0)\gamma'(x).$$

Since $x_0 \geq 2$, we see that $\Phi(x) > 0$ when $x \leq 4$.

Suppose $\Phi(x) = 0$ for some $x \in [x_0, x_*)$. Then $x > 4$ and there exists a point $\bar{x} \in (4, x_*)$ so that $\Phi(\bar{x}) = 0$ and $\Phi(x) > 0$ for $x \in [x_0, \bar{x})$. This implies that $\Phi'(\bar{x}) \leq 0$ and $\frac{1}{8}\bar{x}\gamma(\bar{x}) = \gamma'(\bar{x})$. Since $\bar{x} > 4$ and $\gamma(\bar{x}) < 0$, we have

$$\begin{aligned}
\Phi'(\bar{x}) &= \frac{1}{8}\gamma(\bar{x}) + \frac{1}{8}\bar{x}\gamma'(\bar{x}) - \gamma''(\bar{x}) \\
&\geq \frac{1}{8}\gamma(\bar{x}) + \frac{1}{8}\bar{x}\gamma'(\bar{x}) - \frac{\gamma''(\bar{x})}{1+\gamma'(\bar{x})^2} \\
&= \frac{1}{8}\gamma(\bar{x}) + \frac{1}{8}\bar{x}\gamma'(\bar{x}) - \left[\left(\frac{1}{2}\bar{x} - \frac{1}{\bar{x}}\right)\gamma'(\bar{x}) - \frac{1}{2}\gamma(\bar{x})\right] \\
&= \gamma(\bar{x})\left(\frac{1}{8} + \frac{1}{64}(\bar{x})^2 - \left[\left(\frac{1}{2}\bar{x} - \frac{1}{\bar{x}}\right)\frac{1}{8}\bar{x} - \frac{1}{2}\right]\right) \\
&= \gamma(\bar{x})\left(\frac{3}{4} - \frac{3}{64}(\bar{x})^2\right) > 0,
\end{aligned}$$

which is a contradiction. □

Proof of Proposition 2.6. Let $b > 0$, and let γ be the solution of (1.2) with $\gamma(0) = b$ and $\gamma'(0) = 0$. We know there exists a point $x_* \in (\sqrt{2}, \infty)$ and a point $z_* \in (-\infty, b)$ so that $\lim_{x\to x_*}\gamma'(x) = -\infty$ and $\gamma(x_*) = z_*$. We assume $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^4}$ (and also $b < \frac{1}{2}$). By Claim 2.7, we know that $x_* > 2\sqrt{2}$ and $|\gamma'(x)| \leq \frac{\sqrt{3}}{3}$ for $x \in [0, 2\sqrt{2}]$. Then by Claim 2.8 we have $\gamma''' < 0$ on $(0, x_*)$. Integrating this inequality from 0 to x repeatedly, we see that

$$\gamma(x) < b(1 - \frac{1}{8}x^2).$$

Since $x_* > 2\sqrt{2}$, it follows that there exists $x_0 \in (0, 2\sqrt{2})$ so that $\gamma(x_0) = 0$.

To estimate x_0 from below, we write equation (1.2) in the form

$$\frac{d}{dx}\left(\frac{x\gamma'(x)}{\sqrt{1+\gamma'(x)^2}}\right) = \frac{1}{2}x \cdot \frac{x\gamma'(x) - \gamma(x)}{\sqrt{1+\gamma'(x)^2}}. \tag{2.3}$$

It follows from Claim 2.9 that $\frac{x\gamma'(x)-\gamma(x)}{\sqrt{1+\gamma'(x)^2}} \geq \frac{x_0\gamma'(x_0)}{\sqrt{1+\gamma'(x_0)^2}}$ for $x \in [0, x_0]$, and thus by integrating (2.3) from 0 to x_0, we get

$$\begin{aligned}
\frac{x_0\gamma'(x_0)}{\sqrt{1+\gamma'(x_0)^2}} &= \int_0^{x_0} \frac{1}{2}x \cdot \frac{x\gamma'(x) - \gamma(x)}{\sqrt{1+\gamma'(x)^2}}dx \\
&\geq \frac{x_0\gamma'(x_0)}{\sqrt{1+\gamma'(x_0)^2}} \int_0^{x_0} \frac{1}{2}x dx.
\end{aligned}$$

Therefore, $1 \leq \frac{(x_0)^2}{4}$ and we see that $x_0 \geq 2$. This proves the last statement in the proposition.

Next, we want to slightly refine the estimate from Claim 2.7 to establish a lower bound for x_* in terms of b. This will simplify the constants that appear in the following calculations. Let $x_1 \in (0, x_*)$ be the point where $\gamma'(x_1) = -1$. Using the same argument we used in the proof of Claim 2.7, we integrate the inequality

$$\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\gamma'(x)\right) \geq -e^{-\frac{x^2}{2}}\gamma(x)$$

from 0 to x_1 to conclude that $-e^{-\frac{(x_1)^2}{2}} \geq -b\sqrt{\frac{\pi}{2}}$ and therefore,

$$x_1 \geq \sqrt{\log \frac{2}{\pi b^2}}.$$

Since $x_0 \in [2, 2\sqrt{2}]$, it follows from Lemma 2.10 that $\gamma(x) > \frac{8}{x}\gamma'(x)$. In particular, at x_1, we have

$$\gamma(x_1) > -\frac{8}{x_1} \geq -\frac{8}{\sqrt{\log \frac{2}{\pi b^2}}}.$$

We will extend this estimate for $\gamma(x_1)$ to an estimate for $\gamma(x_*) = z_*$. We assume $b \leq \sqrt{\frac{2}{\pi e^{25}}}$ so that $x_1 \geq 5$. For $x \geq x_1$, we have

$$\begin{aligned} \gamma''(x) &\leq \gamma'(x)^2 \frac{\gamma''(x)}{1+\gamma'(x)^2} \\ &= \gamma'(x)^2 \left[\left(\frac{1}{2}x - \frac{1}{x}\right)\gamma'(x) - \frac{1}{2}\gamma(x)\right] \\ &< \gamma'(x)^2 \left(\frac{1}{2}x - \frac{5}{x}\right)\gamma'(x) \\ &\leq \frac{1}{4}x\gamma'(x)^3, \end{aligned}$$

where we have used that $x \geq 5$ and $\gamma(x) > \frac{8}{x}\gamma'(x)$.

Integrating the previous inequality from x to x_* implies

$$\gamma'(x)^2 \leq \frac{4}{(x_*)^2 - x^2},$$

for $x \geq x_1$. Since $\gamma'(x) < 0$, we have

$$\gamma'(x) \geq -\frac{2}{\sqrt{(x_*)^2 - x^2}} \geq -\frac{1}{\sqrt{x_* + x_1}} \cdot \frac{2}{\sqrt{x_* - x}}, \tag{2.4}$$

for $x \in [x_1, x_*)$. At x_1, this tells us that

$$-\frac{\sqrt{x_* - x_1}}{\sqrt{x_* + x_1}} \geq -\frac{2}{x_* + x_1}.$$

Finally, integrating (2.4) from x_1 to x_*, we have

$$\gamma(x_*) - \gamma(x_1) \geq -\frac{4}{\sqrt{x_* + x_1}} \cdot \sqrt{x_* - x_1},$$

and therefore

$$\begin{aligned} \gamma(x_*) &\geq \gamma(x_1) - \frac{4}{\sqrt{x_* + x_1}} \cdot \sqrt{x_* - x_1} \\ &\geq -\frac{8}{x_1} - \frac{8}{x_* + x_1} \\ &\geq -\frac{12}{x_1} \geq -\frac{12}{\sqrt{\log \frac{2}{\pi b^2}}}, \end{aligned}$$

which completes the proof of the proposition with $\bar{b} = \sqrt{\frac{2}{\pi e^{25}}}$. $\square$

3. Connecting the first and second branches

Given the basic shape of γ described in the previous section, we know that $\gamma'(x) < 0$ when $x > 0$, and thus, for $x > 0$, the curve $(x, \gamma(x))$ can be written as $(\alpha(z), z)$. Since $\alpha'(z) = 1/\gamma'(\alpha(z))$ and γ is a solution of (1.2), it follows that α is a solution of (1.3) with $\alpha(z_*) = x_*$ and $\alpha'(z_*) = 0$. In particular, $\alpha''(z_*) = \frac{1}{x_*} - \frac{1}{2}x_* < 0$. This shows us that the α curve is concave down at (x_*, z_*) and heads back towards the z-axis as z decreases. More precisely, in a neighborhood of z_*, we have $\alpha'(z) > 0$ when $z < z_*$. It follows that the curve $(\alpha(z), z)$ can be written as a curve $(x, \beta(x))$ where $\beta(x)$ satisfies (1.2). Using the existence theory for differential equations, we know that there exists $x_{**} < x_*$ so that β is a solution of (1.2) on (x_{**}, x_*). Here $x_{**} \geq 0$ is chosen so that β blows up as x approaches x_{**} or $x_{**} = 0$. We note that $\beta(x_*) = z_*$ and $\lim_{x\to x_*} \beta'(x) = \infty$, and these two conditions uniquely determine β as a solution of (1.2).

In the next section we will study the behavior of β. For the remainder of this section, we discuss how the curves γ and α and the point (x_*, z_*) depend continuously on the initial height. From Proposition 2.1 we know that the γ curves depend continuously on the initial height in a neighbrhood of 0. Once we move away from the singularity at 0, if we rewrite (1.2) as a first order system, then a direct application of the existence, uniqueness, and continuity theorems for differential equations extends this continuity:

Proposition 3.1. *For $b > 0$, let γ_b denote the unique solution of* (1.2) *with $\gamma_b(0) = b$ and $\gamma_b'(0) = 0$. Let x_*^b denote the point where γ_b blows up, and let $z_*^b = \gamma_b(x_*^b)$.*

Fix $b_0 > 0$. Then, for $\varepsilon > 0$, there exists $\delta > 0$ so that if $|b - b_0| < \delta$ and $b > 0$, then $\gamma_b(x)$ is defined on $[0, x_*^{b_0} - \varepsilon]$. Moreover,

$$|\gamma_b(x) - \gamma_{b_0}(x)| + |\gamma_b'(x) - \gamma_{b_0}'(x)| < \varepsilon,$$

for $x \in [0, x_*^{b_0} - \varepsilon]$.

We end this section with a proposition which shows how the α curves depend continuously on the initial height. Again, the proof of the proposition is an application of the existence, uniqueness, and continuity theorems for differential equations.

Proposition 3.2. *Fix $b_0 > 0$, and let α_{b_0} be the unique solution of* (1.3) *with $\alpha_{b_0}(z_*^{b_0}) = x_*^{b_0}$ and $\alpha_{b_0}'(z_*^{b_0}) = 0$. Let $\rho > 0$ be chosen so that $[z_*^{b_0} - \rho, z_*^{b_0} + \rho]$ is contained in the maximal interval of existence for α_{b_0}. Then, for $\varepsilon > 0$, there exists $\delta > 0$ so that if $|b - b_0| < \delta$ and $b > 0$, then the unique solution α_b of* (1.3) *with $\alpha_b(z_*^b) = x_*^b$ and $\alpha_b'(z_*^b) = 0$ is defined on $[z_*^{b_0} - \rho, z_*^{b_0} + \rho]$. Moreover, for $z \in [z_*^{b_0} - \rho, z_*^{b_0} + \rho]$,*

$$|\alpha_b(z) - \alpha_{b_0}(z)| + |\alpha_b'(z) - \alpha_{b_0}'(z)| < \varepsilon.$$

As an application of Proposition 2.6 and Proposition 3.2 we use the rigidity of compact, mean-convex self-shrinkers due to Huisken [10] to show that $z_* < 0$ when $\gamma(0) \in (0, 2)$. We note that this result is not essential to the proof of Theorem 1.1.

Corollary 3.3. *Fix $b_0 > 0$. Then, for $\varepsilon > 0$, there exists $\delta > 0$ so that if $|b-b_0| < \delta$ and $b > 0$, then $|x_*^b - x_*^{b_0}| < \varepsilon$ and $|z_*^b - z_*^{b_0}| < \varepsilon$. Furthermore, if $b \in (0, 2)$, then $z_*^b < 0$.*

Proof. The first statement is a consequence of Proposition 3.2. To prove the second statement, let

$$b_0 = \max\{b' \in (0, 2] : z_*^b < 0 \text{ for } b \in (0, b']\}.$$

It follows from Proposition 2.6 that b_0 is well defined. We will show that $b_0 = 2$. By definition of b_0, there exists an increasing sequence b_n converging to b_0 so that $z_*^{b_n} < 0$. Applying the first part of the corollary, we have $z_*^{b_0} \leq 0$. If $z_*^{b_0} = 0$, then the rotation of the curve $\gamma_{b_0} \cup -\gamma_{b_0}$ about the z-axis is a convex, compact self-shrinker. By Huisken's classification of compact, mean-convex self-shrinkers, this must be the sphere of radius 2 centered at the origin, and in this case $b_0 = 2$. If $b_0 < 2$, then $z_*^{b_0} < 0$, and by the first part of the corollary, there exists $\delta > 0$ so that $z_*^b < 0$ for $|b - b_0| < \delta$. This contradicts the choice of b_0 as the maximum of $\{b' \in (0, 2] : z_*^b < 0 \text{ for } b \in (0, b']\}$, and we conclude that $b_0 = 2$. In particular, $z_*^b < 0$ when $b \in (0, 2)$. □

4. The second branch

In this section we study the β curves as they travel from (x_*, z_*) toward the z-axis. For $b > 0$, let γ denote the solution of (1.2) with $\gamma(0) = b$ and $\gamma'(0) = 0$, let x_* denote the point where γ blows up, and let $z_* = \gamma(x_*)$. Also, let β denote the unique solution of (1.2) with $\beta(x_*) = z_*$ and $\lim_{x \to x_*} \beta'(x) = \infty$, and let $x_{**} \in [0, x_*)$ be the point where β blows up, or if no such point exists, set $x_{**} = 0$. We will show for small $b > 0$ that β achieves a negative minimum at a point $x_m \in (x_{**}, x_*)$, β is concave up, $x_{**} > 0$, and $0 < \beta(x_{**}) < \infty$.

4.1. Basic shape of β.

First, we prove some basic properties of the β curves that are consequences of equations (1.2) and (2.1) and the fact that $\lim_{x \to x_*} \beta'(x) = \infty$.

Claim 4.1. *There exists at most one point $x_m \in (x_{**}, x_*)$ for which $\beta'(x_m) = 0$. If such a point exists, then $\beta'' > 0$ on (x_{**}, x_*).*

Proof. Using equation (2.1), we see that β' cannot vanish at more than one point. Now, suppose $\beta'(x_m) = 0$ for some $x_m \in (x_{**}, x_*)$. By uniqueness of solutions, $\beta(x_m) \neq 0$ (otherwise β would be identically 0), and thus $\beta''(x_m) \neq 0$. Since $\lim_{x \to x_*} \beta'(x) = \infty$, we know that $\beta'(x) > 0$ for $x < x_*$ and near x_*, and it follows that $\beta''(x_m) > 0$. Arguing as in the proof of Claim 2.2, we see that $\beta'' > 0$ on (x_m, x_*) and similarly on (x_{**}, x_m). We note that $\beta(x_m) < 0$. □

Claim 4.2. *If there exists $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$, then $\beta(x) < 0$ whenever $x \in (x_{**}, x_m)$ and $x \geq \sqrt{2}$.*

Proof. In this case, we have $\beta'' > 0$ on (x_{**}, x_*). If $\beta(x) = 0$ for some $x \in (x_{**}, x_m)$, then $\beta'(x) < 0$, and using (1.2), we see that $x < \sqrt{2}$. □

Claim 4.3. $x_{**} < \sqrt{2}$.

Proof. We treat the two cases from Claim 4.1. In the first case, there exists a point $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$ and $\beta'' > 0$ on (x_{**}, x_*). By Claim 4.2, we know that $\beta(x) < 0$ when $x \in (x_{**}, x_m)$ and $x \geq \sqrt{2}$. Also, if we let $M = -\beta(x_m)$, then $\beta(x) \geq -M$ when $x \in (x_{**}, x_*)$. Now, for any $\varepsilon > 0$, there exists a constant $m_\varepsilon > 0$ so that $\frac{1}{2}x - \frac{1}{x} > m_\varepsilon$ for $x \geq \sqrt{2} + \varepsilon$, and using (1.2), we have $\beta'(x) > -\frac{M}{2m_\varepsilon}$ when $x \geq \sqrt{2} + \varepsilon$. Therefore, $|\beta(x)|$ and $|\beta'(x)|$ are uniformly bounded for $x \geq \sqrt{2} + \varepsilon$ and away from x_*. By the existence theory for differential equations $x_{**} < \sqrt{2} + \varepsilon$. Taking $\varepsilon \to 0$, we have $x_{**} \leq \sqrt{2}$. To see that $x_{**} < \sqrt{2}$, suppose $x_{**} = \sqrt{2}$. Then there exists $z_{**} \in [-M, 0]$ so that $\lim_{x \to x_{**}} \beta(x) = z_{**}$. It follows that, near the point (x_{**}, z_{**}), the curve $(x, \beta(x))$ can be written as $(\bar{\alpha}(z), z)$ where $\bar{\alpha}$ is a solution of (1.3) with $\bar{\alpha}(z_{**}) = \sqrt{2}$ and $\bar{\alpha}'(z_{**}) = 0$. By the uniqueness of solutions of (1.3) we deduce that $\bar{\alpha}$ is the constant function $\bar{\alpha}(z) = \sqrt{2}$, which is a contradiction.

In the second case, $\beta' > 0$ on (x_{**}, x_*). If $x_{**} = 0$, we are done. Otherwise, $x_{**} > 0$ and consequently $\lim_{x \to x_{**}} \beta'(x) = \infty$. In this case, β'' must be negative at some point, and arguing as in the proof of Claim 2.2, we see that $\beta'' < 0$ near x_{**}. It follows from (1.2) that $x_{**} < \sqrt{2}$ when $\beta < 0$ near x_{**}. On the other hand, if $\beta \geq 0$, then $|\beta|$ is uniformly bounded (since $\beta \leq z_*$) and arguing as we did in the first case, we see that $x_{**} < \sqrt{2}$. □

Lemma 4.4. $\lim_{x \to x_{**}} \beta(x) < \infty$.

Proof. Suppose $\lim_{x \to x_{**}} \beta(x) = \infty$. Since $\beta' > 0$ near x_*, there exists a point $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$. By Claim 4.1, we know that $\beta'' > 0$ and $\beta(x_m) < 0$. It follows that there exists a point $x_\ell \in (x_{**}, x_m)$ so that $\beta(x_\ell) = 0$ with $\beta'(x_\ell) = -m < 0$. By Claim 4.2, we know there exists $\delta > 0$ so that $x_\ell + \delta < \sqrt{2}$, and in particular $\left(\frac{1}{2}x - \frac{1}{x}\right) \leq -\frac{1}{M}$ for some $M > 0$, when $x \leq x_\ell$.

Let $f = -\beta'$. Then $f > 0$ when $x \in (x_{**}, x_\ell)$ and $f' < 0$. Using (2.1), we have

$$f'' \geq -\frac{1}{M} f' \cdot f^2,$$

when $x \in (x_{**}, x_\ell)$. Fix $\varepsilon > 0$, and let

$$g_\varepsilon(x) = \frac{m\sqrt{x_\ell - (x_{**} + \varepsilon)} + \sqrt{3M}}{\sqrt{x - (x_{**} + \varepsilon)}}.$$

Then

$$g_\varepsilon'' = -\frac{3}{2} \frac{1}{(m\sqrt{x_\ell - (x_{**} + \varepsilon)} + \sqrt{3M})^2} g_\varepsilon' \cdot g_\varepsilon^2 \leq -\frac{1}{M} g_\varepsilon' \cdot g_\varepsilon^2,$$

for $x \in (x_{**} + \varepsilon, x_\ell)$.

Now, $g_\varepsilon(x_\ell) > f(x_\ell)$ and $g_\varepsilon(x_{**} + \varepsilon) > f(x_{**} + \varepsilon)$. Suppose $f > g_\varepsilon$ at some point in $(x_{**} + \varepsilon, x_\ell)$. Then $f - g_\varepsilon$ must achieve a positive maximum in $(x_{**} + \varepsilon, x_\ell)$. At such a point

$$0 \geq (f - g_\varepsilon)'' \geq -\frac{1}{M} f'(f^2 - g_\varepsilon^2) > 0,$$

which is a contradiction. Therefore, $g_\varepsilon \geq f$. Taking $\varepsilon \to 0$, we have the estimate

$$f(x) \leq \frac{m\sqrt{x_\ell - x_{**}} + \sqrt{3M}}{\sqrt{x - x_{**}}},$$

for $x \in (x_{**}, x_\ell)$. Integrating from x to x_ℓ,

$$\beta(x) - \beta(x_\ell) \leq 2\left(m\sqrt{x_\ell - x_{**}} + \sqrt{3M}\right)\left(\sqrt{x_\ell - x_{**}} - \sqrt{x - x_{**}}\right).$$

Since $\beta(x_\ell) = 0$, we have

$$\lim_{x \to x_{**}} \beta(x) \leq 2\left(m\sqrt{x_\ell - x_{**}} + \sqrt{3M}\right)\left(\sqrt{x_\ell - x_{**}}\right),$$

which is a contradiction. □

Now that we know β is bounded from above, we can show that $x_{**} > 0$ when there exists $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$.

Claim 4.5. *Suppose there exists $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$. Then $x_{**} > 0$.*

Proof. If there exists $x_m \in (x_{**}, x_*)$ so that $\beta'(x_m) = 0$, then $\beta'' > 0$ and there exists $\varepsilon > 0$ and $x_\varepsilon \in (x_{**}, x_m)$ so that $\frac{\beta'(x_\varepsilon)}{\sqrt{1+\beta'(x_\varepsilon)^2}} = -\varepsilon$. Also, by Lemma 4.4, there exists $M \geq 0$ so that $\beta < M$.

Let $\theta(x) = \arctan \beta'(x)$. Then

$$\frac{d}{dx}(\log \sin \theta(x)) = \frac{1}{2}x - \frac{1}{x} - \frac{\beta(x)}{2\beta'(x)} \leq \frac{1}{2}x - \frac{1}{x} - \frac{M}{2\beta'(x_\varepsilon)},$$

for $x \in (x_{**}, x_\varepsilon)$. Integrating the inequality from x to x_ε:

$$\log\left(\frac{\sin\theta(x_\varepsilon)}{\sin\theta(x)}\right) \leq \frac{1}{4}(x_\varepsilon)^2 + \log\left(\frac{x}{x_\varepsilon}\right) + \frac{M x_\varepsilon}{2(-\beta'(x_\varepsilon))}.$$

Since $\sin\theta(x_\varepsilon) = \frac{\beta'(x_\varepsilon)}{\sqrt{1+\beta'(x_\varepsilon)^2}} = -\varepsilon$ and $\sin\theta(x_{**}) = -1$, we have

$$\varepsilon \leq \left(\frac{x_{**}}{x_\varepsilon}\right) e^{\frac{1}{4}(x_\varepsilon)^2 + \frac{M x_\varepsilon}{2(-\beta'(x_\varepsilon))}},$$

which proves the claim. □

Remark 4.6. At this point, we can give a basic description of the β curves: For $b > 0$, let (x_*^b, z_*^b) be the blow-up point of γ_b, and let β_b be the unique solution of (1.2) with $\beta_b(x_*^b) = z_*^b$ and $\lim_{x \to x_*^b} \beta_b'(x) = \infty$. Then there exists $x_{**}^b \in [0, \sqrt{2})$ so that β_b is defined on (x_{**}^b, x_*^b) and either β_b blows up as $x \to x_{**}^b$ or $x_{**}^b = 0$. Also, β_b is bounded from above, and if there exists x_m^b for which $\beta_b'(x_m^b) = 0$, then $\beta_b'' > 0$ on (x_{**}^b, x_*^b) and $x_{**}^b > 0$.

4.2. **A note on the blow-up of β at x_{**}.** Now that we know the basic shape of β, we discuss the dependence of β and (x_{**}, z_{**}) on the initial height. As in Section 3, the following proposition is a consequence of the existence, uniqueness, and continuity theory for differential equations.

Proposition 4.7. *For $b > 0$, let γ_b denote the solution of* (1.2) *with $\gamma_b(0) = b$ and $\gamma_b'(0) = 0$. Let x_*^b denote the point where γ_b blows up, and let $z_*^b = \gamma_b(x_*^b)$. Let β_b denote the unique solution of* (1.2) *with $\beta_b(x_*^b) = z_*^b$ and $\lim_{x \to x_*^b} \beta'(x) = \infty$, and let $x_{**}^b \in [0, x_*^b)$ be chosen so that β_b is smooth on (x_{**}^b, x_*^b) and either β_b blows up at x_{**}^b or $x_{**}^b = 0$.*

Fix $b_0 > 0$. Then, for $\varepsilon > 0$, there exists $\delta > 0$ so that if $|b - b_0| < \delta$ and $b > 0$, then $\beta_b(x)$ is defined on $[x_{**}^{b_0} + \varepsilon, x_*^{b_0} - \varepsilon]$. Moreover,

$$|\beta_b(x) - \beta_{b_0}(x)| + |\beta_b'(x) - \beta_{b_0}'(x)| < \varepsilon,$$

for $x \in [x_{**}^{b_0} + \varepsilon, x_*^{b_0} - \varepsilon]$.

Applying Proposition 4.7, we have the following continuity result.

Proposition 4.8. *Fix $b_0 > 0$. Suppose there exists $x_m^{b_0} \in (x_{**}^{b_0}, x_*^{b_0})$ so that $\beta_{b_0}'(x_m^{b_0}) = 0$ and hence $x_{**}^{b_0} > 0$. Suppose $\beta_{b_0}(x_{**}^{b_0}) = z_{**}^{b_0}$, where $|z_{**}^{b_0}| < \infty$. Then, for $\varepsilon > 0$, there exists $\delta > 0$, so that if $|b - b_0| < \delta$ and $b > 0$, then the solution β_b blows up at the point $x_{**}^b > 0$ with $\beta_b(x_{**}^b) = z_{**}^b$ and $|z_{**}^b| < \infty$. Furthermore, if $|b - b_0| < \delta$ and $b > 0$, then $|x_{**}^b - x_{**}^{b_0}| < \varepsilon$ and $|z_{**}^b - z_{**}^{b_0}| < \varepsilon$.*

Proof. Since $x_{**}^{b_0} > 0$ and $|z_{**}^{b_0}| < \infty$, we can continue $(x, \beta_{b_0}(x))$ past the blow-up point $(x_{**}^{b_0}, z_{**}^{b_0})$ along a curve $(\bar{\alpha}_{b_0}(z), z)$ just as we did for $(x, \gamma(x))$ at the blow-up point (x_*, z_*). Using Proposition 4.7 we can show that the $\bar{\alpha}$ curves depend continuously on the initial height. In particular, for b in a neighborhood of b_0, the blow-up points (x_{**}^b, z_{**}^b) will exist and depend continuously on b. $\square$

4.3. **Behavior of β for small $b > 0$.** Fix $b \in (0, \bar{b}]$, and let γ be the solution of (1.2) with $\gamma(0) = b$ and $\gamma'(0) = 0$. Let x_* denote the point where γ blows up, and let $z_* = \gamma(x_*)$. Also, let β denote the unique solution of (1.2) with $\beta(x_*) = z_*$ and $\lim_{x \to x_*} \beta'(x) = \infty$. We know there is a point $x_{**} \in [0, \sqrt{2})$ so that β is defined on (x_{**}, x_*) and either blows up as $x \to x_{**}$ or $x_{**} = 0$. We also know that

$$x_*^b \geq \sqrt{\log \frac{2}{\pi b^2}}$$

and

$$\frac{-12}{\sqrt{\log \frac{2}{\pi b^2}}} \leq z_*^b < 0.$$

Claim 4.9. *Suppose $x_* \geq 4$. Then there exists a point $x_m \in [x_* - 2, x_*)$ so that $\beta'(x_m) = 0$.*

Proof. Suppose $\beta'(x) > 0$ for $x \in [x_* - 2, x_*)$. When $x_* \geq 4$, we have $x_* - 2 \geq \sqrt{2}$ and $\beta(x_*) < 0$. Using (1.2), we see that $\beta'' > 0$ in $[x_* - 2, x_*)$ and $\beta''(x_* - 2) \geq -\frac{1}{2}\beta(x_* - 2)$. Then, using (2.1), we have $\beta''' > 0$ in $[x_* - 2, x_*)$. Integrating from $x_* - 2$ to x, we get

$$\beta(x) \geq \beta(x_* - 2)\left[1 - \frac{1}{4}\left(x - (x_* - 2)\right)^2\right],$$

for $x \in [x_* - 2, x_*)$. This tells us that $\beta(x_*) \geq 0$, which is a contradiction. $\square$

Let $\bar{b} > 0$ be given as in the conclusion of Proposition 2.6 so that if $b \in (0, \bar{b}]$, then $\gamma \leq 0$ for $x \in [2\sqrt{2}, x_*)$. We also assume that $\bar{b}$ is chosen so small that $x_m > 2\sqrt{2}$ and $z_* \geq -\frac{1}{8} \geq -\frac{3\sqrt{2}}{4}$ when $b \in (0, \bar{b}]$. (It suffices to choose $\bar{b} < \frac{\sqrt{2}}{\sqrt{\pi} e^{4608}}$.)

Lemma 4.10. *If $b \in (0, \bar{b}]$, then $2z_* \leq \beta(x) < 0$ for $x \in [2\sqrt{2}, x_*]$.*

Proof. Let $\alpha(z)$ denote the curve that connects γ and β. Then α is a solution of (1.3) with $\alpha(z_*) = x_*$ and $\alpha'(z_*) = 0$. Using (1.3) we have $\alpha''(z_*) = \frac{1}{x_*} - \frac{1}{2}x_*$ and $\alpha'''(z_*) = \frac{1}{2}z_*(\frac{1}{x_*} - \frac{1}{2}x_*)$ so that

$$\alpha(z) = x_* + \frac{1}{2}\left(\frac{1}{x_*} - \frac{1}{2}x_*\right)(z - z_*)^2 + \frac{1}{12}z_*\left(\frac{1}{x_*} - \frac{1}{2}x_*\right)(z - z_*)^3 + \mathcal{O}(|z - z_*|^4)$$

as $z \to z_*$. Since $x_* > \sqrt{2}$ and $z_* < 0$, the coefficient of the $(z - z_*)^3$ term is positive. Now, for $x < x_*$ and near x_*, we can find $s, t > 0$ so that

$$\alpha(z_* + t) = x = \alpha(z_* - s),$$

and it follows from the previous formula for $\alpha(z)$ that $t > s$.

To prove the lemma, we consider the function $\delta(x) = \gamma(x) + \beta(x)$. For $x \in [2\sqrt{2}, x_*)$, since $\gamma(x) \leq 0$ and $\beta(x) < 0$, we have

$$\delta(x) < 0.$$

Also, by the previous paragraph,

$$\delta(x) = (z_* + t) + (z_* - s) > 2z_* = \delta(x_*),$$

for $x < x_*$ and near x_*.

Now we are in position to show that $\delta > 2z_*$ on $[2\sqrt{2}, x_*)$. Suppose $\delta(x) = 2z_*$ for some $x \in [2\sqrt{2}, x_*)$. It follows from the previous discussion that δ achieves a negative maximum at some point $\bar{x} \in (2\sqrt{2}, x_*)$. At this point we have $\delta''(\bar{x}) \leq 0$ and

$$\frac{\delta''(\bar{x})}{1+\gamma'(\bar{x})^2} = -\frac{1}{2}\delta(\bar{x}) > 0,$$

which is a contradiction. Therefore, $\delta > 2z_*$ on $[2\sqrt{2}, x_*)$. Since $\gamma(x) \leq 0$ on $[2\sqrt{2}, x_*)$, this completes the proof of the lemma. □

Claim 4.11. *Let $b \in (0, \bar{b}]$. If $\beta < 0$ on (x_{**}, x_*), then*

$$x_{**} \leq \frac{8}{\pi - \sqrt{2}}(-z_*).$$

Proof. By our assumptions on $\bar{b}$, we know that $x_m > 2\sqrt{2}$, $\beta'' > 0$, and $\beta(2\sqrt{2}) \geq 2z_*$. Using equation (1.2), we get the estimate $\beta'(2\sqrt{2}) > \frac{\sqrt{2}}{3}\beta(2\sqrt{2}) \geq \frac{2\sqrt{2}}{3}z_*$ so that $\beta'(2\sqrt{2}) > -1$. For $x \in (x_{**}, \sqrt{2})$, we have

$$\frac{d}{dx}\left(\arctan \beta'(x)\right) = \left(\frac{1}{2}x - \frac{1}{x}\right)\beta'(x) - \frac{1}{2}\beta(x) \leq -\frac{1}{x_{**}}\beta'(x) - \frac{1}{2}\beta(2\sqrt{2}).$$

Integrate from x_{**} to $2\sqrt{2}$,

$$\arctan \beta'(2\sqrt{2}) + \frac{\pi}{2} \leq \left(\frac{1}{x_{**}} + \sqrt{2}\right)(-\beta(2\sqrt{2})).$$

Since $\beta'(2\sqrt{2}) > -1$, and $\beta(2\sqrt{2}) \geq 2z_*$, this becomes

$$\frac{\pi}{4} \leq \left(\frac{1}{x_{**}} + \sqrt{2}\right)2(-z_*).$$

Rearranging this inequality to estimate x_{**} and using $z_* \geq -1/8$, we have

$$x_{**} \leq \frac{8}{\pi - \sqrt{2}}(-z_*).$$

□

Proposition 4.12. *There exists $\tilde{b} > 0$ so that for $b \in (0, \tilde{b}]$ there is a point $x_m^b \in (x_{**}^b, x_*^b)$ so that $\beta_b'(x_m^b) = 0$ and $0 < \beta_b(x_{**}^b) < \infty$.*

Proof. Fix $b \in (0, \bar{b}]$, and let $\beta = \beta_b$. By Claim 4.9 we know there exists $x_m > 2\sqrt{2}$ so that $\beta'(x_m) = 0$. It follows that $\beta'' > 0$ on (x_{**}, x_*) and $\beta' < 0$ on (x_{**}, x_m). From Lemma 4.4, we know that $\beta(x_{**}) < \infty$.

Suppose $\beta(x_{**}) \leq 0$ so that $\beta < 0$ in (x_{**}, x_m). By Claim 4.11 we know that $x_{**} < 1$ for b sufficiently small. Then, using the equation (2.1) for β''', we see that $\beta''' < 0$ in $(x_{**}, \sqrt{2}]$. We know that $\beta''(\sqrt{2}) \geq -\frac{1}{2}\beta(\sqrt{2})$, and thus $\beta''(x) \geq -\frac{1}{2}\beta(\sqrt{2})$, for $x \in (x_{**}, \sqrt{2}]$. Integrating from 1 to $\sqrt{2}$ and using that β is decreasing on $(1, \sqrt{2})$, we have

$$\beta'(1) \leq \frac{\sqrt{2}-1}{2}\beta(1).$$

Let $x \in (x_{**}, 1)$. Under the above assumptions, we may write (1.2) as $\frac{\beta''(x)}{\beta'(x)} \leq \left(\frac{1}{2}x - \frac{1}{x}\right)$, and integrating this inequality from x to 1, we get $\beta'(x) \leq \frac{\beta'(1)}{x}e^{-\frac{1}{4}}$. Integrating again,

$$\beta(x) \geq \beta(1) + \beta'(1)e^{-\frac{1}{4}} \log x.$$

Combining this with $\beta'(1) \leq \frac{\sqrt{2}-1}{2}\beta(1)$, we see that

$$0 > \beta(x) \geq \beta'(1)\left(\frac{2}{\sqrt{2}-1} + e^{-\frac{1}{4}} \log x\right).$$

If b is chosen sufficiently small (so that $x_{**} < e^{-\frac{2e^{1/4}}{\sqrt{2}-1}}$), then we have a contradiction. Therefore, $0 < \beta_b(x^b_{**}) < \infty$ when $b > 0$ is sufficiently small. $\square$

5. An immersed S^2 self-shrinker

In this section we complete the proof of Theorem 1.1. We consider the set

$$\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists x^b_m \in (x^b_{**}, x^b_*) \text{ so that } \beta'_b(x^b_m) = 0 \text{ and } \beta_b(x^b_{**}) > 0\}.$$

By Proposition 4.12, we know that this set is non-empty. Following Angenent's argument in [2], we let b_0 be the supremum of this set:

$$b_0 = \sup\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists x^b_m \in (x^b_{**}, x^b_*) \text{ so that } \beta'_b(x^b_m) = 0 \text{ and } \beta_b(x^b_{**}) > 0\}.$$

Since $\beta_b(x) = -\sqrt{4 - x^2}$ when $b = 2$, we know that $b_0 \leq 2$. We want to show β_{b_0} intersects the x-axis perpendicularly at $x^{b_0}_{**}$.

Claim 5.1. $b_0 < 2$.

Proof. We will prove it is impossible to have $\beta'_{b_0} > 0$ and $\beta''_{b_0} \geq 0$ in $(x^{b_0}_{**}, x^{b_0}_*)$. In particular, this will show $b_0 \neq 2$.

Suppose $\beta'_{b_0} > 0$ and $\beta''_{b_0} \geq 0$ in $(x^{b_0}_{**}, x^{b_0}_*)$. Then $x^{b_0}_{**} = 0$, and there exists $m > 0$ so that $\beta_{b_0}(x) \leq -m$ for $x \in (0, 1]$. Let b_n be an increasing sequence that converges to b_0, and let β_n be the solution of (1.2) corresponding to the initial height b_n. Fix $\varepsilon > 0$. By Proposition 4.7, there exists $N = N(\varepsilon) > 0$ so that for $n > N$, we have $x^n_{**} < \varepsilon$ and $\beta_n(\varepsilon) < -m$. We know that β_n intersects the x-axis at some point $x^n_\ell < \varepsilon$, and we see that the curve $(x, \beta_n(x))$ may be written as the curve $(\bar{\alpha}_n(z), z)$ for $z \in [-m/2, 0]$, where $\bar{\alpha}_n$ is a solution of (1.3). In fact, we have the estimates

$$0 < \bar{\alpha}_n(z) < \varepsilon$$

and

$$-\frac{2\varepsilon}{m} \leq \bar{\alpha}'_n(z) \leq 0.$$

Using (1.3), we get an estimate for $\bar{\alpha}''_n(z)$ when $z \in [-m/2, 0]$:

$$\bar{\alpha}''_n(z) \geq \frac{1}{\varepsilon} - \frac{1}{2}\varepsilon \geq \frac{1}{2\varepsilon},$$

for small $\varepsilon > 0$. Integrating repeatedly from z to 0:

$$\bar{\alpha}_n(z) \geq \bar{\alpha}_n(0) + \bar{\alpha}'_n(0)z + \frac{1}{4\varepsilon}z^2 \geq \frac{1}{4\varepsilon}z^2.$$

Taking $z = -m/2$, we have $\bar{\alpha}_n(-m/2) \geq \frac{m^2}{16\varepsilon}$. This implies that the point $x' \in (x^n_{**}, \varepsilon)$ for which $\beta_n(x') = -m/2$ satisfies $x' \geq \frac{m^2}{16\varepsilon}$. Therefore, $\frac{m^2}{16\varepsilon} \leq x' < \varepsilon$, which is a contradiction when $\varepsilon > 0$ is sufficiently small. $\square$

Claim 5.2. *There exists* $x_m^{b_0} \in (x_{**}^{b_0}, x_*^{b_0})$ *so that* $\beta'_{b_0}(x_m^{b_0}) = 0$.

Proof. Suppose not. Then $\beta'_{b_0} > 0$ in $(x_{**}^{b_0}, x_*^{b_0})$. By the proof of Claim 5.1 we know that it is impossible to have $\beta'_{b_0} > 0$ and $\beta''_{b_0} \geq 0$ in $(x_{**}^{b_0}, x_*^{b_0})$. Therefore, we must have $\beta''_{b_0} < 0$ at some point. Let b_n be an increasing sequence that converges to b_0, and let β_n be the solution of (1.2) corresponding to the initial height b_n. Since each curve β_n crosses the x-axis, we know from Claim 4.1 that $\beta''_n > 0$. Since $\beta''_{b_0} < 0$ at some point x', we can use Proposition 4.7 to find an $N > 0$ so that $\beta''_n(x') < 0$ for $n > N$, which contradicts the fact that $\beta''_n > 0$. ☐

Since there exists $x_m^{b_0} \in (x_{**}^{b_0}, x_*^{b_0})$ with $\beta'_{b_0}(x_m^{b_0}) = 0$, we know that $\beta''_{b_0} > 0$ on $(x_{**}^{b_0}, x_*^{b_0})$ and $x_{**}^{b_0} > 0$. We also know that there is a point $z_{**}^{b_0}$ with $|z_{**}^{b_0}| < \infty$ so that $\beta_{b_0}(x_{**}^{b_0}) = z_{**}^{b_0}$.

Claim 5.3. $z_{**}^{b_0} = 0$.

Proof. We know that $x_{**}^{b_0} > 0$ and $|z_{**}^{b_0}| < \infty$. We also know that there exists $x_m^{b_0} \in (x_{**}^{b_0}, x_*^{b_0})$ so that $\beta'_{b_0}(x_m^{b_0}) = 0$. It follows from Proposition 4.7 (and Claim 4.1) that there is a $\delta' > 0$ so that when $|b - b_0| < \delta'$ there exists $x_m^b \in (x_{**}^b, x_*^b)$ with $\beta'_b(x_m^b) = 0$.

If $z_{**}^{b_0} > 0$, then applying Proposition 4.8, we can find $\delta \in (0, \delta')$ so that the curve β_b has a finite blow-up point (x_{**}^b, z_{**}^b) with $z_{**}^b > 0$ when $|b - b_0| < \delta$. Then $b_0 + \delta/2 \in \{\tilde{b} : \forall b \in (0, \tilde{b}], \exists x_m^b \in (x_{**}^b, x_*^b) \text{ so that } \beta'_b(x_m^b) = 0 \text{ and } \beta_b(x_{**}^b) > 0\}$, which contradicts the definition of b_0. On the other hand, if $z_{**}^{b_0} < 0$, then Proposition 4.8 tells us there exists $\delta \in (0, \delta')$ so that the curve β_b has a finite blow-up point (x_{**}^b, z_{**}^b) with $z_{**}^b < 0$ when $|b - b_0| < \delta$, but this also contradicts the definition of b_0. Therefore, $z_{**}^{b_0} = 0$. ☐

It follows that the curve $(x, \beta_{b_0}(x))$ intersects the x-axis perpendicularly at the point $(x_{**}^{b_0}, 0)$, where $x_{**}^{b_0} \in (0, \sqrt{2})$. Now we can describe the curve $\mathcal{C}$ in the (x, z)-plane whose rotation about the z-axis is an immersed S^2 self-shrinker.

Proof of Theorem 1.1. Let $\mathcal{C}$ be the curve in the (x, z)-plane obtained by following along $(x, \gamma_{b_0}(x))$ as x goes from 0 to $x_*^{b_0}$, following along $(x, \beta_{b_0}(x))$ as x goes from $x_*^{b_0}$ to $x_{**}^{b_0}$, and then following the reflections $(x, -\beta_{b_0}(x))$ and $(x, -\gamma_{b_0}(x))$. That is,

$$\mathcal{C} = \gamma_{b_0} \cup \beta_{b_0} \cup -\beta_{b_0} \cup -\gamma_{b_0}$$

(see Figure 1). The curve $\mathcal{C}$ intersects the z-axis perpendicularly at the two points $(0, b_0)$ and $(0, -b_0)$, and the rotation of $\mathcal{C}$ about the z-axis is smooth in a neighborhood of these points. In addition, at the vertical tangent points where the γ and β curves meet, $\mathcal{C}$ can be represented as an α curve, and we see that $\mathcal{C}$ is a smooth curve, whose rotation M about the z-axis is a smooth manifold. In fact, M is the image of a smooth immersion from S^2 into $\mathbb{R}^3$. By construction, the γ, α, and β curves are solutions of the differential equation that corresponds to self-shrinkers with rotational symmetry. Also, the γ_{b_0} and $-\beta_{b_0}$ curves intersect transversally. Therefore, the surface M is an immersed and non-embedded S^2 self-shrinker in $\mathbb{R}^3$. ☐

6. Appendix: Existence of solutions near $x = 0$

In this section, we use power series to construct solutions to the differential equation

$$\frac{\gamma''}{1+(\gamma')^2} = \left(\frac{1}{2}x - \frac{1}{x}\right)\gamma' - \frac{1}{2}\gamma, \tag{6.1}$$

when $\gamma(0) \in \mathbb{R}$ and $\gamma'(0) = 0$.

We look for solutions of the form:

$$\gamma(x) = \sum_{i=0}^{\infty} a_i x^i.$$

If we assume that we can differentiate the power series term by term so that

$$\gamma'(x) = \sum_{i=0}^{\infty}(i+1)a_{i+1}x^i$$

and

$$\gamma''(x) = \sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i,$$

then the condition that γ satisfies (6.1):

$$\gamma'' = -\frac{1}{2}\gamma + \frac{1}{2}x\gamma' - \frac{1}{x}\gamma' - \frac{1}{2}\gamma(\gamma')^2 + \frac{1}{2}x(\gamma')^3 - \frac{1}{x}(\gamma')^3,$$

is a condition on the coefficients $\{a_i\}$. Namely, $a_0 = \gamma(0)$, $a_1 = 0$, and

$$\begin{aligned}
(m+2)(m+1)a_{m+2} \quad = \quad & -\frac{1}{2}a_m + \frac{1}{2}ma_m - (m+2)a_{m+2} \\
& -\frac{1}{2}\sum_{i+j+k=m}(i+1)(j+1)a_{i+1}a_{j+1}a_k \\
& +\frac{1}{2}\sum_{i+j+k=m-1}(i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1} \\
& -\sum_{i+j+k=m+1}(i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}.
\end{aligned}$$

The previous equation simplifies to:

$$\begin{aligned}
(m+2)^2 a_{m+2} \quad = \quad & \frac{1}{2}(m-1)a_m - \frac{1}{2}a_0\sum_{i+j=m}(i+1)(j+1)a_{i+1}a_{j+1} \\
& +\frac{1}{2}\sum_{i+j+k=m-1}(i+1)(j+1)k\cdot a_{i+1}a_{j+1}a_{k+1} \\
& -\sum_{i+j+k=m+1}(i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}.
\end{aligned} \tag{6.2}$$

Claim 6.1. $a_{2i+1} = 0$

Proof. This follows from the above formula for a_m and induction. We know $a_1 = 0$. Suppose $a_{2i+1} = 0$ for all $i < m$ and consider a_{2m+1}. Every term in the expression for a_{2m+1} contains a term of the form a_{2i+1}, and thus $a_{2m+1} = 0$. □

In order to construct a solution of (6.1), we need $\sum a_i x^i$ to be a convergent power series, and hence we need an estimate on the coefficients a_{2m}.

Claim 6.2. *For each $M > 0$, there exists $A = A(M) > 0$ so that if $|a_0| \leq M$, then*

$$|a_{2m}| \leq \frac{A^{2m-1}}{(2m)^3}.$$

Proof. Fix $M > 0$, and use (6.2) to choose $A > 0$ so that the estimate holds for $m = 1, 2, 3$. Arguing inductively, suppose $a_{2i} \leq \frac{A^{2i-1}}{(2i)^3}$ when $i \leq m$, and consider a_{2m+2}. From (6.2), we have

$$\begin{aligned}(2m+2)^2 a_{2m+2} \;=\; & \frac{1}{2}(2m-1)a_{2m} \\ & -\frac{1}{2}a_0 \sum_{\substack{i+j=2m \\ i,j \text{ odd}}} (i+1)(j+1)a_{i+1}a_{j+1} \\ & +\frac{1}{2} \sum_{\substack{i+j+k=2m-1 \\ i,j,k \text{ odd}}} (i+1)(j+1)k \cdot a_{i+1}a_{j+1}a_{k+1} \\ & - \sum_{\substack{i+j+k=2m+1 \\ i,j,k \text{ odd}}} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}.\end{aligned} \tag{6.3}$$

To estimate the terms with sums, we need the following inequality:

$$\sum_{\substack{i+j=2N \\ i,j \text{ odd}}} \frac{1}{(i+1)^2}\frac{1}{(j+1)^2} \leq \frac{2}{(2N+2)^2}. \tag{6.4}$$

This inequality follows from the identity:

$$\sum_{\substack{i+j=2N \\ i,j \text{ odd}}} \frac{1}{(i+1)^2}\frac{1}{(j+1)^2} = \frac{2}{(2N+2)^2}\sum_{\substack{i=1 \\ i \text{ odd}}}^{2N-1} \frac{1}{(i+1)^2} + \frac{4}{(2N+2)^3}\sum_{\substack{i=1 \\ i \text{ odd}}}^{2N-1} \frac{1}{i+1}.$$

Applying (6.4) twice, we have the following inequality:

$$\sum_{\substack{i+j+k=2N-1 \\ i,j,k \text{ odd}}} \frac{1}{(i+1)^2}\frac{1}{(j+1)^2}\frac{1}{(k+1)^2} \leq \frac{4}{(2N+2)^2}. \tag{6.5}$$

Now, we can use (6.4), (6.5), and the inductive hypothesis to estimate each term on the right-hand side of formula (6.3):

$$\frac{1}{2}(2m-1)|a_{2m}| \leq \frac{1}{2}\frac{2m-1}{(2m)^3}A^{2m-1} \leq \left[\frac{m+1}{(2m)^2}\right]\frac{1}{2m+2}A^{2m+1},$$

$$\frac{1}{2}\left| a_0 \sum_{\substack{i+j=2m \\ i,j \text{ odd}}} (i+1)(j+1)a_{i+1}a_{j+1}\right| \leq \left[\frac{1}{2m+2}\right]\frac{1}{2m+2}A^{2m+1},$$

$$\frac{1}{2}\left| \sum_{\substack{i+j+k=2m-1 \\ i,j,k \text{ odd}}} (i+1)(j+1)k \cdot a_{i+1}a_{j+1}a_{k+1}\right| \leq \left[\frac{1}{m+1}\right]\frac{1}{2m+2}A^{2m+1},$$

$$\left| \sum_{\substack{i+j+k=2m+1 \\ i,j,k \text{ odd}}} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1} \right| \leq \left[\frac{2}{m+2} \right] \frac{1}{2m+2} A^{2m+1}.$$

Applying these estimates to (6.3) when $m \geq 3$, we see that

$$\begin{aligned} (2m+2)^2 |a_{2m+2}| &\leq \left[\frac{m+1}{(2m)^2} + \frac{1}{2m+2} + \frac{1}{m+1} + \frac{2}{m+2} \right] \frac{1}{2m+2} A^{2m+1} \\ &\leq \frac{1}{2m+2} A^{2m+1}. \end{aligned}$$

□

The estimates on the coefficients a_i imply that the power series $\gamma(x) = \sum a_i x^i$ is an analytic function on $[0, 1/A]$. By the previous discussion, $\gamma(x)$ is the unique analytic solution of (6.1) in $(0, 1/A]$. Furthermore, since the coefficients a_i depend continuously on a_0, the solution $\gamma(x)$ depends continuously on the initial height $\gamma(0)$.

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