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INTRINSIC SQUARE FUNCTION CHARACTERIZATIONS OF MUSIELAK-ORLICZ HARDY SPACES

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ABSTRACT. Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt $A_\infty(\mathbb{R}^n)$ weight uniformly in t. In this article, for any $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, the authors establish the s-order intrinsic square function characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of the intrinsic Lusin area function $S_{\alpha,s}$, the intrinsic g-function $g_{\alpha,s}$ and the intrinsic g^*_λ-function $g^*_{\lambda,\alpha,s}$ with the best known range $\lambda \in (2+2(\alpha+s)/n,\infty)$, which are defined via $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ functions supporting in the unit ball. A φ-Carleson measure characterization of the Musielak-Orlicz Campanato space $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ is also established via the intrinsic function. To obtain these characterizations, the authors first show that these s-order intrinsic square functions are pointwise comparable with those similar-looking s-order intrinsic square functions defined via $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ functions without compact supports, which when $s=0$ was obtained by M. Wilson. All these characterizations of $H^\varphi(\mathbb{R}^n)$, even when $s=0$,

$$\varphi(x,t) := w(x)t^p \text{ for all } t \in [0,\infty) \text{ and } x \in \mathbb{R}^n$$

with $p \in (n/(n+\alpha),1]$ and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$, also essentially improve the known results.

1. INTRODUCTION

The intrinsic square functions were first introduced by Wilson in [35] to settle a conjecture proposed by R. Fefferman and E. M. Stein on the boundedness of the Lusin area function S from the weighted Lebesgue space $L^2_{M(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$, where $0 \le v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and M denotes the Hardy-Littlewood maximal function. Moreover, Wilson [36] proved that these intrinsic square functions are bounded on the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ when $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$ (the class of Muckenhoupt weights). More applications of such intrinsic square functions were also given by Wilson [37,38] and Lerner [23,24].

These intrinsic square functions can be thought of as "grand maximal" square functions, in the style of the "grand maximal function" of C. Fefferman and Stein from [12]: they dominate all the square functions of the form $S(f)$ (and the classical ones as well), but are not essentially bigger than any one of them. Like the Fefferman-Stein and Hardy-Littlewood maximal functions, their generic natures make them pointwise equivalent to each other and extremely easy to work with.

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Moreover, the intrinsic Lusin area function has the distinct advantage of being pointwise comparable at different cone openings, which is a property long known not to hold true for the classical Lusin area function; see Wilson [35, 36, 38] and also Lerner [23, 24].

Recently, Huang and Liu [18] obtained the intrinsic square function characterizations of the weighted Hardy space $H^1_w(\mathbb{R}^n)$ under the *additional assumption that* $f \in L^1_w(\mathbb{R}^n)$, which was further generalized to the weighted Hardy space $H^p_w(\mathbb{R}^n)$ with $p \in (n/(n+\alpha), 1)$ and $\alpha \in (0,1]$ by Wang and Liu [34], under the *additional assumption that* $f \in (\mathrm{Lip}(\alpha,1,0))^*$. Moreover, Wang and Liu [33] obtained the weak type estimates of these intrinsic square functions on the weighted Hardy space $H^p_w(\mathbb{R}^n)$ when $p = n/(n+\alpha)$.

On the other hand, Birnbaum-Orlicz [2] and Orlicz [29] introduced the Orlicz space, which is a natural generalization of $L^p(\mathbb{R}^n)$. Recently, Ky [21] introduced a new *Musielak-Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$, which generalizes both the Orlicz-Hardy space (see, for example, [19, 32]) and the weighted Hardy space (see, for example, [14, 15, 30]). Moreover, more real-variable characterizations of $H^\varphi(\mathbb{R}^n)$ were obtained in [17, 25] and the local Musielak-Orlicz Hardy space, $h^\varphi(\mathbb{R}^n)$, was studied in [39]. Musielak-Orlicz functions are the natural generalization of Orlicz functions that may vary in the spatial variables; see, for example, [27]. The motivation to study function spaces of Musielak-Orlicz type comes from their wide applications in physics and mathematics; see, for example, [4–6, 21, 28] and their references. In particular, some special Musielak-Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ (see [5, 6]), and the endpoint estimates for the div-curl lemma and the commutators of singular integral operators (see [3, 5, 22]).

In this article, we establish various s-order intrinsic square function characterizations of $H^\varphi(\mathbb{R}^n)$, including the intrinsic Lusin area function, the intrinsic g-function and the intrinsic g^*_λ-function. To this end, we first show that the s-order intrinsic square functions, defined via $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ functions supporting in the unit ball, are pointwise comparable with those similar-looking s-order intrinsic square functions, defined via $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ functions without compact supports, which when $s = 0$ were obtained by Wilson [35]. Since the square function characterizations of $H^\varphi(\mathbb{R}^n)$ have been obtained in [25] and the intrinsic square function is larger than the square function pointwise, it suffices to show the boundedness of the intrinsic square functions from $H^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$. We point out that our characterizations of $H^\varphi(\mathbb{R}^n)$, even when $s = 0$,

$$\varphi(x,t) := w(x)t^p \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in [0,\infty) \tag{1.1}$$

with $p \in (n/(n+\alpha), 1]$ and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$, also essentially improve the known results in [18] and [34] by *removing* the additional assumptions that $f \in L^1_w(\mathbb{R}^n)$ or $f \in (\mathrm{Lip}(\alpha,1,0))^*$. Moreover, by using some ideas from [13], the range of λ in our intrinsic g^*_λ-function characterization of $H^\varphi(\mathbb{R}^n)$ coincides with the known best range of the g^*_λ-function characterization for $H^p(\mathbb{R}^n)$, which improves the ranges of λ appearing in the corresponding results in [18] and [34].

To state our main results, we begin with some notions and notation. For $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+ := \{0,1,\ldots\}$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the *family of functions* $\phi \in C^s(\mathbb{R}^n)$

such that $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : \ |x| \le 1\}$,

$$\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$$

for all $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$ and $|\gamma| := \gamma_1 + \cdots + \gamma_n \le s$, and

$$|D^\nu \phi(x_1) - D^\nu \phi(x_2)| \le |x_1 - x_2|^\alpha \quad \text{for all} \quad x_1,\, x_2 \in \mathbb{R}^n,\, \nu \in \mathbb{Z}_+^n \text{ and } |\nu| = s.$$

Here and in what follows, for all $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$ and $x := (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \ \text{ and } \ D^\gamma := \left(\frac{\partial}{\partial x_1}\right)^{\gamma_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\gamma_n}.$$

For all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |f * \phi_t(y)|,$$

where, for all $t \in (0,\infty)$, $\phi_t(\cdot) := \frac{1}{t^n}\phi(\frac{\cdot}{t})$. Then, the *intrinsic g-function*, the *intrinsic Lusin area integral* and the *intrinsic g^*_λ-function* of f are, respectively, defined by setting, for all $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) := \left\{\int_0^\infty [A_{\alpha,s}(f)(x,t)]^2 \, \frac{dt}{t}\right\}^{1/2},$$

$$S_{\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\{y \in \mathbb{R}^n : \ |y-x|<t\}} [A_{\alpha,s}(f)(y,t)]^2 \, \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \tag{1.2}$$

and

$$g^*_{\lambda,\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_{\alpha,s}(f)(y,t)]^2 \, \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

We also introduce another kind of similar-looking square functions, defined via convolutions with kernels that have unbounded supports. For $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (0,\infty)$, let $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ be the family of functions $\phi \in C^s(\mathbb{R}^n)$ such that, for all $x \in \mathbb{R}^n$, $\gamma \in \mathbb{Z}_+^n$ and $|\gamma| \le s$,

$$|D^\gamma \phi(x)| \le (1+|x|)^{-n-\epsilon},$$

$$\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$$

and, for all $x_1,\, x_2 \in \mathbb{R}^n$, $\nu \in \mathbb{Z}_+^n$ and $|\nu| = s$,

$$|D^\nu \phi(x_1) - D^\nu \phi(x_2)| \le |x_1 - x_2|^\alpha [(1+|x_1|)^{-n-\epsilon} + (1+|x_2|)^{-n-\epsilon}]. \tag{1.3}$$

Observe that, in what follows, the parameter ϵ usually has to be chosen to be large enough. For all f satisfying

$$|f(\cdot)|(1+|\cdot|)^{-n-\epsilon} \in L^1(\mathbb{R}^n) \tag{1.4}$$

and $(y,t) \in \mathbb{R}^{n+1}_+$, let

$$\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Then, for all $x \in \mathbb{R}^n$, we define

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(x,t) \right]^2 \frac{dt}{t} \right\}^{1/2},$$

$$\widetilde{S}_{(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) \right]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2} \tag{1.5}$$

and

$$\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) \right]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

When $s = 0$, these intrinsic square functions were first introduced by Wilson [35].

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0,\infty)$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Observe that, different from the classical Orlicz functions being convex, the Orlicz functions in this article may not be convex. The function Φ is said to be of *upper type* (resp. *lower type*) p for some $p \in [0,\infty)$ if there exists a positive constant C such that, for all $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0,1]$),

$$\Phi(st) \le Cs^p \Phi(t).$$

For a given function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper type* (resp. *uniformly lower type*) p for some $p \in [0,\infty)$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ (resp. $s \in [0,1]$),

$$\varphi(x, st) \le Cs^p \varphi(x,t).$$

Moreover, φ is said to be of *positive uniformly upper type* (resp. *uniformly lower type*) if it is of uniformly upper type (resp. uniformly lower type) p for some $p \in (0,\infty)$. The *critical uniformly lower type index* of φ is defined by

$$i(\varphi) := \sup\{p \in (0,\infty) :\ \varphi \text{ is of uniformly lower type } p\}. \tag{1.6}$$

Observe that $i(\varphi)$ may not be attainable; namely, φ may not be of uniformly lower type $i(\varphi)$ (see [25]).

Let $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfy that $x \mapsto \varphi(x,t)$ is measurable for all $t \in [0,\infty)$. Following [21], the function $\varphi(\cdot,t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$, it holds true that

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,dx \left\{ \int_B [\varphi(y,t)]^{-q'/q}\,dy \right\}^{q/q'} < \infty,$$

where $1/q + 1/q' = 1$, or, when $q = 1$, it holds true that

$$\sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,dx \left(\operatorname*{ess\,sup}_{y\in B} [\varphi(y,t)]^{-1} \right) < \infty.$$

Here the first supremums are taken over all $t \in [0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

The *critical weight index* of $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ is defined as follows:

$$q(\varphi) := \inf\{q \in [1,\infty) :\ \varphi \in \mathbb{A}_q(\mathbb{R}^n)\}. \tag{1.7}$$

Observe that if $q(\varphi) \in (1,\infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}(\mathbb{R}^n)$ and there exists $\varphi \notin \mathbb{A}_1(\mathbb{R}^n)$ such that $q(\varphi) = 1$ (see, for example, [20]).

Now we recall the notion of growth functions (see [21]).

Definition 1.1. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a *Musielak-Orlicz function*; namely,
  (i)$_1$ the function $\varphi(x,\cdot) :\ [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$,
  (i)$_2$ the function $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) φ is of positive uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Throughout the whole paper, we *always assume that φ is a growth function* as in Definition 1.1 and, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, we let

$$\varphi(E,t) := \int_E \varphi(x,t)\,dx.$$

The *Musielak-Orlicz space* $L^\varphi(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx < \infty$ with the *quasi-norm*

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) :\ \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right)\,dx \le 1\right\}.$$

Observe that $\varphi(x,\cdot)$ for any $x \in \mathbb{R}^n$ may not be convex in the time variable, and hence $\|\cdot\|_{L^\varphi(\mathbb{R}^n)}$ may not be a Luxembourg norm.

In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For $m \in \mathbb{N}$, let

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\psi \in \mathcal{S}(\mathbb{R}^n) :\ \sup_{\beta\in\mathbb{Z}_+^n,\, |\beta|\le m+1}\ \sup_{x\in\mathbb{R}^n} (1+|x|)^{(m+2)(n+1)}|\partial_x^\beta \psi(x)| \le 1\right\}.$$

Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *nontangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi\in\mathcal{S}_m(\mathbb{R}^n)}\ \sup_{|y-x|<t,\, t\in(0,\infty)} |f * \psi_t(y)|,$$

where, for all $t \in (0,\infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor,$$

where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (1.7) and (1.6), we *denote $f_{m(\varphi)}^*$ simply by f^**.

Now we recall the definition of the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ introduced by Ky [21] as follows.

Definition 1.2. Let φ be a growth function. The *Musielak-Orlicz Hardy space* $H^\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$ with the *quasi-norm*

$$\|f\|_{H^\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}.$$

Since $\|\cdot\|_{L^\varphi(\mathbb{R}^n)}$ is only known to be a quasi-norm, it follows that $H^\varphi(\mathbb{R}^n)$ usually is not a Banach space, but locally convex and therefore good enough to have non-trivial linear functionals. Indeed, Ky [21] also introduced the following Musielak-Orlicz BMO-type space $\mathrm{BMO}^\varphi(\mathbb{R}^n)$ and proved that the dual space of $H^\varphi(\mathbb{R}^n)$ is the Musielak-Orlicz BMO space $\mathrm{BMO}^\varphi(\mathbb{R}^n)$ *in the case when* $m(\varphi)=0$.

Definition 1.3. Let φ be a growth function. The *Musielak-Orlicz* BMO-*type* space $\mathrm{BMO}^\varphi(\mathbb{R}^n)$ is defined to be the space of all $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}^\varphi(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x)-f_B|\,dx<\infty,$$

where the supremum is taken over all the balls $B\subset\mathbb{R}^n$ and f_B denotes the average of f on B, namely,

$$f_B := \frac{1}{|B|}\int_B f(y)\,dy. \tag{1.8}$$

To complete the study of Ky [21] on the dual space of $H^\varphi(\mathbb{R}^n)$, namely, to decide the dual space of $H^\varphi(\mathbb{R}^n)$ *in the case when* $m(\varphi)\in\mathbb{N}$, the following Musielak-Orlicz Campanato spaces $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ were introduced in [26] and the space $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ was proved to be the dual space of $H^\varphi(\mathbb{R}^n)$ for all $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$ in a natural way (see also Lemma 3.5 below).

In what follows, for any $s\in\mathbb{Z}_+$, we use $\mathcal{P}_s(\mathbb{R}^n)$ to denote the *set of all polynomials on* $\mathbb{R}^n$ *with order not more than* s.

Definition 1.4. Let φ be as in Definition 1.1, $q\in[1,\infty)$ and $s\in\mathbb{Z}_+$. A locally integrable function f on $\mathbb{R}^n$ is said to belong to the *Musielak-Orlicz Campanato space* $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ if

$$\begin{aligned}\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} &:= \sup_{B\subset\mathbb{R}^n}\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\\ &\quad\times\left\{\int_B\left[\frac{|f(x)-P^s_Bf(x)|}{\varphi(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\right]^q\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)dx\right\}^{1/q}<\infty,\end{aligned}$$

where the supremum is taken over all the balls $B\subset\mathbb{R}^n$ and P^s_Bg denotes the unique $P\in\mathcal{P}_s(\mathbb{R}^n)$ such that, for all $Q\in\mathcal{P}_s(\mathbb{R}^n)$,

$$\int_B[g(x)-P(x)]Q(x)\,dx=0. \tag{1.9}$$

Remark 1.5. (i) When

$$\varphi(x,t):=t^p \text{ for all } x\in\mathbb{R}^n \text{ and } t\in(0,\infty)$$

with $p\in(0,1]$, via some trivial computations, we know that

$$\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}=|B|^{1/p}$$

and

$$\varphi\left(x,\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)=|B|^{-1}$$

for any ball $B\subset\mathbb{R}^n$ and $x\in\mathbb{R}^n$ and hence, in this case,

$$\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n}|B|^{1-1/p}\left\{\frac{1}{|B|}\int_B|f(x)-P^s_Bf(x)|^q\,dx\right\}^{1/q},$$

where the supremum is taken over all the balls $B \subset \mathbb{R}^n$ and $P_B^s g$ is as in (1.9). That is, in this case, $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ just becomes the classical Campanato space $L_{\frac{1}{p}-1,q,s}(\mathbb{R}^n)$, which was introduced by Campanato [7].

(ii) When φ is as in (1.1) with $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$, via some trivial computations, we see that

$$\|\chi_B\|_{L^\varphi(\mathbb{R}^n)} = [w(B)]^{1/p}$$

and

$$\varphi\left(x, \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right) = w(x)[w(B)]^{-1}$$

for any ball $B \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, where $w(B) := \int_B w(x)\,dx$. Thus, in this case,

$$\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} [w(B)]^{1-1/p} \left\{\frac{1}{w(B)} \int_B |f(x) - P_B^s f(x)|^q [w(x)]^{1-q}\,dx\right\}^{1/q},$$

where the supremum is taken over all the balls $B \subset \mathbb{R}^n$ and $P_B^s g$ is as in (1.9). That is, in this case, $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ coincides with the weighted Campanato space introduced by García-Cuerva [14] as the dual space of the corresponding weighted Hardy space.

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$; see, for example, [13, p. 50]. The growth function φ is said to satisfy a *uniformly locally dominated convergence condition* if the following holds: For every compact $K \subset \mathbb{R}^n$ and sequence $\{f_m\}_{m\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$, if $f_m \to f$ almost everywhere, and $|f_m| \le g$ almost everywhere for some nonnegative measurable function g satisfying that

$$\sup_{t\in(0,\infty)} \int_K g(x)\frac{\varphi(x,t)}{\varphi(K,t)}\,dx < \infty,$$

then

$$\sup_{t\in(0,\infty)} \int_K |f_m(x) - f(x)|\frac{\varphi(x,t)}{\varphi(K,t)}\,dx \to 0$$

as $m \to \infty$.

Observe that the growth functions

$$\varphi(x,t) := w(x)\Phi(t),$$

with $w \in A_\infty(\mathbb{R}^n)$ and Φ being an Orlicz function,

$$\varphi(x,t) := t^p$$

and

$$\varphi(x,t) = \frac{t^p}{[\log(e+|x|) + \log(e+t^p)]^p},$$

with $p \in (0,1]$, for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, satisfy the uniformly locally dominated convergence condition. More examples of growth functions satisfying the uniformly locally dominated convergence condition can be found in [17, 25, 39].

Our main results of this paper are as follows.

Theorem 1.6. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition, $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s),1]$ and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$,* *the dual space of* $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$, f *vanishes weakly at infinity and* $g_{\alpha,s}(f) \in L^{\varphi}(\mathbb{R}^n)$*; moreover, when this is obtained, it holds true that*

$$\frac{1}{C}\|g_{\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)} \le \|f\|_{H^{\varphi}(\mathbb{R}^n)} \le C\|g_{\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)}$$

with C *being a positive constant independent of* f*.*

The same is true if $g_{\alpha,s}(f)$ *is replaced by* $\widetilde{g}_{(\alpha,\epsilon),s}(f)$*.*

Observe that, for all $x \in \mathbb{R}^n$, $S_{\alpha,s}(f)(x)$ and $g_{\alpha,s}(f)(x)$ are pointwise comparable (see Proposition 2.4 below), which, together with Theorem 1.6, immediately implies Corollary 1.7. We omit the details.

Corollary 1.7. *Let* φ *be a growth function satisfying the uniformly locally dominated convergence condition,* $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s),1]$ *and* $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$*. Then* $f \in H^{\varphi}(\mathbb{R}^n)$ *if and only if* $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, f *vanishes weakly at infinity and* $S_{\alpha,s}(f) \in L^{\varphi}(\mathbb{R}^n)$*; moreover, when this is obtained, it holds true that*

$$\frac{1}{C}\|S_{\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)} \le \|f\|_{H^{\varphi}(\mathbb{R}^n)} \le C\|S_{\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)}$$

with C *being a positive constant independent of* f*.*

The same is true if $S_{\alpha,s}(f)$ *is replaced by* $\widetilde{S}_{(\alpha,\epsilon),s}(f)$*.*

Theorem 1.8. *Let* φ *be a growth function satisfying the uniformly locally dominated convergence condition,* $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$ *and* $\lambda \in (2+2(\alpha+s)/n,\infty)$*. Then* $f \in H^{\varphi}(\mathbb{R}^n)$ *if and only if* $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, f *vanishes weakly at infinity and* $g^*_{\lambda,\alpha,s}(f) \in L^{\varphi}(\mathbb{R}^n)$*; moreover, when this is obtained, it holds true that*

$$\frac{1}{C}\|g^*_{\lambda,\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)} \le \|f\|_{H^{\varphi}(\mathbb{R}^n)} \le C\|g^*_{\lambda,\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)}$$

with C *being a positive constant independent of* f*.*

The same is true if $g^*_{\lambda,\alpha,s}(f)$ *is replaced by* $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$*.*

Finally, we establish the intrinsic φ-Carleson measure characterization of the space $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$. We first recall the following φ-Carleson measure which was first introduced in [26].

Definition 1.9. Let φ be a growth function. A measure μ on $\mathbb{R}^{n+1}_+$ is called a *φ-Carleson measure* if

$$\|\mu\|_{\varphi} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}} \left\{\int_{\widehat{B}} \frac{t^n}{\varphi(B(x,t),\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)})}\,|d\mu(x,t)|\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and

$$\widehat{B} := \{(x,t) \in \mathbb{R}^{n+1}_+ :\ B(x,t) \subset B\}$$

denotes the *tent* over B.

Remark 1.10. (i) Notice that when

$$\varphi(x,t) := t \quad \text{for all} \quad x \in \mathbb{R}^n \text{ and } t \in (0,\infty),$$

then, for all $B \subset \mathbb{R}^n$, it holds true that

$$\|\chi_{B_0}\|^{-1}_{L^{\varphi}(\mathbb{R}^n)} = |B|^{-1}$$

and hence

$$\|\mu\|_{\varphi} := \sup_{B\subset\mathbb{R}^n}\left[\frac{\mu(\widehat{B})}{|B|}\right]^{1/2},$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\widehat{B}$ denotes the tent over B. Thus, in this case, the φ-Carleson measure just becomes the classical Carleson measure, which was introduced by Carleson [8, 9] and was used by Fefferman and Stein in [12] to characterize the space $\mathrm{BMO}(\mathbb{R}^n)$.

(ii) When φ is as in Remark 1.5(i), in this case, we then have

$$\|\mu\|_{\varphi} := \sup_{B\subset\mathbb{R}^n}\left[\frac{\mu(\widehat{B})}{|B|^{2/p-1}}\right]^{1/2},$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\widehat{B}$ denotes the tent over B. Thus, in this case, the φ-Carleson measure just becomes the fractional Carleson measure, which was introduced by Essén et al. [11] and was used by Dafni and Xiao in [10] to characterize the space $Q_\alpha(\mathbb{R}^n)$.

(iii) When φ is as in Remark 1.5(ii), in this case, we have

$$\|\mu\|_{\varphi} := \sup_{B\subset\mathbb{R}^n}\left\{\int_{\widehat{B}}\frac{t^n}{w(B(x,t))[w(B)]^{2/p-1}}\,|d\mu(x,t)|\right\}^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $\widehat{B}$ denotes the tent over B.

In what follows, for $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (0,\infty)$ and $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ such that b satisfies (1.4) with f replaced by b, the *measure* μ_b *on* $\mathbb{R}^{n+1}_+$ is defined by setting, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$d\mu_b(x,t) := [\widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)]^2\,\frac{dx\,dt}{t}. \tag{1.10}$$

Theorem 1.11. *Let φ be a growth function, $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$. Then $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ if and only if b satisfies (1.4) with f replaced by b, and μ_b as in (1.10) is a φ-Carleson measure on $\mathbb{R}^{n+1}_+$. Moreover, when this is obtained, there exists a positive constant C, independent of b, such that*

$$\frac{1}{C}\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \le \|\mu_b\|_{\varphi} \le C\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}.$$

Remark 1.12. (i) We point out that if ϕ belongs to $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ or $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, then $\phi \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$; see Proposition 2.3 below. Thus, the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$. Observe that if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then there exists a positive constant C such that $C\phi$ satisfies (1.3) and hence $\phi \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ (by Proposition 2.3). Thus, if $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ and f vanishing weakly at infinity makes sense.

(ii) Recall that the *Lipschitz space* $\mathrm{Lip}(\alpha,1,0)$, for $\alpha \in (0,1]$, is defined by

$$\begin{aligned}&\mathrm{Lip}(\alpha,1,0)\\ &:= \left\{b \in L^1_{\mathrm{loc}}(\mathbb{R}^n):\ \|b\|_{\mathrm{Lip}(\alpha,1,0)} := \sup_B \frac{1}{|B|^{1+\alpha/n}}\int_B |b(y)-b_B|\,dy < \infty\right\},\end{aligned}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and b_B denotes the average of b on B; namely, b_B is as in (1.8) with f replaced by b.

When φ is as in (1.1), Wang and Liu [34] proved that if $f \in (\mathrm{Lip}(\alpha,1,0))^*$, then $f \in H^p_w(\mathbb{R}^n)$ if and only if f vanishes weakly at infinity, and $g_\alpha(f) \in L^p_w(\mathbb{R}^n)$ or $\widetilde{g}_{(\alpha,\epsilon)}(f) \in L^p_w(\mathbb{R}^n)$ or $S_\alpha(f) \in L^p_w(\mathbb{R}^n)$ or $\widetilde{S}_{(\alpha,\epsilon)}(f) \in L^p_w(\mathbb{R}^n)$ or $g^*_{\lambda,\alpha}(f) \in L^p_w(\mathbb{R}^n)$ or $\widetilde{g}^*_{\lambda,(\alpha,\epsilon)}(f) \in L^p_w(\mathbb{R}^n)$ with $\lambda > 3 + 2\alpha/n$. In the present article, we remove the additional assumption that $f \in (\mathrm{Lip}(\alpha,1,0))^*$ in [34].

(iii) For $p \in (0,1]$, Folland and Stein [13] established the Littlewood-Paley g^*_λ–characterization, with $\lambda \in (2/p,\infty)$, of $H^p(\mathbb{R}^n)$, which is the best known range of λ. For the Littlewood-Paley g^*_λ–characterization of the weighted Hardy space $H^p_w(\mathbb{R}^n)$, with $p \in (0,1]$, $q \in [1,\infty)$ and $w \in A_q(\mathbb{R}^n)$, the best known range of λ is $\lambda \in (2q/p,\infty)$; see, for example, [25]. If $p \in (n/(n+\alpha),1]$ and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$, Huang and Liu [18] and Wang and Liu [34] established the intrinsic Littlewood-Paley g^*_λ-characterization of $H^p_w(\mathbb{R}^n)$ with $\lambda > 3 + 2\alpha/n$. This corresponds to the case when $s = 0$ of Theorem 1.8, in which we improve the range of λ to $\lambda > 2(1+\alpha/n)$, which coincides with the best known range of λ. Moreover, observe that it was proved in [35, p. 783] that the intrinsic square functions in (1.2) and (1.5), defined via using cones with other apertures, are pointwise comparable, which can give simpler proofs of [18, Theorem 3] and [34, Theorem 3]. We omit the details.

(iv) When

$$\varphi(x,t) := t \quad \text{for all} \quad x \in \mathbb{R}^n \text{ and } t \in (0,\infty),$$

Theorem 1.11 was obtained by Wilson [38]. In the other case, Theorem 1.11 is new.

This article is organized as follows.

In Section 2, we establish some estimates on the intrinsic square functions, which are the key tools for the proofs of our main results. In Proposition 2.3 below, we show that if ϕ belongs to $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ or $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, then $\phi \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$, which further implies that the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$. In Proposition 2.4 and Theorem 2.6 below, we also show that if $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (\alpha+s,\infty)$, then, for all f satisfying (1.4) and $x \in \mathbb{R}^n$,

$$S_{\alpha,s}(f)(x) \sim g_{\alpha,s}(f)(x) \sim \widetilde{g}_{(\alpha,\epsilon),s}(f)(x).$$

We point out that the key point, appearing in the proof of Theorem 2.6, is that a function in $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ can be decomposed into a sequence of functions belonging to $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, in whose proof we borrow some ideas from Taibleson and Weiss [31] on how to use the minimal polynomial to construct the higher order vanishing moments of the functions in $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$.

Section 3 is devoted to the proofs of Theorems 1.6, 1.8 and 1.11. The key tools used to show Theorem 1.6 are Theorem 2.6 in Section 2 of this article, the Littlewood-Paley g-function characterization of $H^\varphi(\mathbb{R}^n)$ from [25, Theorem 4.4], the atomic characterization of $H^\varphi(\mathbb{R}^n)$ established by Ky [21] (see also Lemma 3.4 below) and the fact that the dual space of $H^\varphi(\mathbb{R}^n)$ is $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ proved in [26, Theorem 3.5] (see also Lemma 3.5 below). In the proof of Theorem 1.8, by borrowing some ideas from Folland and Stein [13] and Aguilera and Segovia [1], we first establish key technical Lemma 3.6, which clarifies the relation between the amplitude β and the Orlicz norm of the intrinsic square function $\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)$ in (3.7) below. This, together with Corollary 1.7, further induces the best range $\lambda \in (2(n+\alpha+s)/n,\infty)$ appearing in Theorem 1.8. Applying Theorem 1.6, we then complete the proof of Theorem 1.11.

Finally we make some conventions on notation. Throughout the whole article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and its *characteristic function* by χ_E. We also set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

2. Some estimates of intrinsic square functions

In this section, we prove some key facts on the intrinsic square functions, which are the key tools for the proof of Theorem 1.6 in Section 3. We begin with recalling Lemma 2.1 on the properties of growth functions, which is from [21, Lemmas 4.1, 4.2 and 4.5], respectively.

Lemma 2.1. *Let φ be a growth function. Then the following hold true:*

(i) *there exists a positive constant C such that, for all $(x, t_j) \in \mathbb{R}^n \times [0,\infty)$ with $j \in \mathbb{N}$,*

$$\varphi\left(x, \sum_{j=1}^{\infty} t_j\right) \le C \sum_{j=1}^{\infty} \varphi(x, t_j);$$

(ii) *for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$,*

$$\widetilde{\varphi}(x,t) := \int_0^t \frac{\varphi(x,s)}{s}\, ds$$

is a growth function equivalent to φ; moreover, $\widetilde{\varphi}(x,\cdot)$ is continuous and strictly increasing;

(iii) *for all $f \in L^{\varphi}(\mathbb{R}^n) \setminus \{0\}$,*

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\|f\|_{L^{\varphi}(\mathbb{R}^n)}}\right) dx = 1;$$

(iv) *if $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$, then there exists a positive constant C such that, for all balls B_1, $B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in (0,\infty)$,*

$$\frac{\varphi(B_2,t)}{\varphi(B_1,t)} \le C \left[\frac{|B_2|}{|B_1|}\right]^q;$$

(v) *if $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ with $q \in [1,\infty)$, then there exists a positive constant C such that, for all balls $B(x_0,r) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$, and $t \in [0,\infty)$,*

$$\int_{[B(x_0,r)]^{\complement}} \frac{\varphi(x,t)}{|x-x_0|^{nq}}\, dx \le C \frac{\varphi(B(x_0,r),t)}{r^{nq}}.$$

The following lemma is from [31, p. 83].

Lemma 2.2. *Let $g \in L^1_{\rm loc}(\mathbb{R}^n)$, $s \in \mathbb{Z}_+$ and B be a ball in $\mathbb{R}^n$. Then there exists a positive constant C, independent of g and B, such that*

$$\sup_{x\in B} |P^s_B g(x)| \le \frac{C}{|B|} \int_B |g(x)|\, dx.$$

The following technical proposition implies that the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$.

Proposition 2.3. *Let φ be a growth function, $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s), 1]$ and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$. If $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, then $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$.*

Proof. For any $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, ball $B := B(x_0, r) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$, and $x \in \mathbb{R}^n$, let

$$p_B(x) := \sum_{\gamma\in\mathbb{Z}_+^n,\,|\gamma|\le s} \frac{D^\gamma f(x_0)}{\gamma!}(x-x_0)^\gamma \in \mathcal{P}_s(\mathbb{R}^n).$$

Then, by Lemma 2.2, Taylor's theorem and (1.3), we see that, for any $x \in B$, there exists $\xi(x) \in B$ such that

$$\begin{aligned}
&\int_B |f(x) - P_B^s f(x)|\,dx \\
&\quad\le \int_B [|f(x) - p_B(x)| + |P_B^s(p_B - f)(x)|]\,dx \\
&\quad\lesssim \int_B |f(x) - p_B(x)|\,dx \\
&\quad\lesssim \int_B \left|\sum_{\gamma\in\mathbb{Z}_+^n,\,|\gamma|=s} \frac{D^\gamma f(\xi(x)) - D^\gamma f(x_0)}{\gamma!}(x-x_0)^\gamma\right| dx \\
&\quad\lesssim r^{n+\alpha+s}\left\{[1+|\xi(x)|]^{-n-\epsilon} + (1+|x_0|)^{-n-\epsilon}\right\}.
\end{aligned} \tag{2.1}$$

For all balls B_1, $B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in [0,\infty)$, by (iii) and (iv) of Lemma 2.1, the uniformly lower type p property of φ and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$, we conclude that

$$\begin{aligned}
&\frac{|B_1|^{1+(\alpha+s)/n}}{\|\chi_{B_1}\|_{L^\varphi(\mathbb{R}^n)}} \\
&\quad\sim \frac{|B_1|^{1+(\alpha+s)/n}}{\|\chi_{B_1}\|_{L^\varphi(\mathbb{R}^n)}} \left[\frac{\varphi(B_2, \|\chi_{B_2}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}{\varphi(B_1, \|\chi_{B_1}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\right]^{1/p} \\
&\quad\lesssim \frac{|B_1|^{1+(\alpha+s)/n}}{\|\chi_{B_1}\|_{L^\varphi(\mathbb{R}^n)}} \left[\frac{\varphi(B_2, \|\chi_{B_1}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}{\varphi(B_1, \|\chi_{B_1}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\right]^{1/p} \frac{\|\chi_{B_1}\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_{B_2}\|_{L^\varphi(\mathbb{R}^n)}} \\
&\quad\lesssim \frac{|B_1|^{1+(\alpha+s)/n}}{\|\chi_{B_2}\|_{L^\varphi(\mathbb{R}^n)}} \left[\frac{|B_2|}{|B_1|}\right]^{1+(\alpha+s)/n} \sim \frac{|B_2|^{1+(\alpha+s)/n}}{\|\chi_{B_2}\|_{L^\varphi(\mathbb{R}^n)}}.
\end{aligned} \tag{2.2}$$

Now, if $|x_0| + r \le 1$, namely, $B \subset B(0,1)$, then, by (2.1) and (2.2), we see that

$$\begin{aligned}
\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - P_B^s f(x)|\,dx &\lesssim \frac{|B|^{1+(\alpha+s)/n}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \\
&\lesssim \frac{|B(0,1)|^{1+(\alpha+s)/n}}{\|\chi_{B(0,1)}\|_{L^\varphi(\mathbb{R}^n)}} \sim 1.
\end{aligned} \tag{2.3}$$

If $|x_0| + r > 1$ and $|x_0| \le 2r$, then $r > 1/3$ and $|B| \sim |B(0, |x_0| + r)|$. Since $|f(x)| \le (1 + |x|)^{-n-\epsilon}$ for all $x \in \mathbb{R}^n$, we have

$$\int_B |f(x) - P_B^s f(x)|\,dx \lesssim \int_{\mathbb{R}^n} (1+|x|)^{-n-\epsilon}\,dx \lesssim 1.$$

Thus, from (2.2), it follows that

$$\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - P_B^s f(x)|\,dx \lesssim \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \lesssim 1. \tag{2.4}$$

If $|x_0|+r>1$ and $|x_0|>2r$, then, for all $x\in B$, it holds true that $1\lesssim |x|\sim |x_0|$, which, together with (2.1), (2.2) and $\epsilon\in(\alpha+s,\infty)$, further implies that

$$\begin{aligned}(2.5)\qquad &\frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\int_B |f(x)-P_B^s f(x)|\,dx\\ &\lesssim \frac{|B|^{\frac{n+\alpha+s}{n}}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}|B(0,|x_0|+r)|^{-\frac{n+\epsilon}{n}}\\ &\lesssim \frac{|B(0,|x_0|+r)|^{1+(\alpha+s)/n}}{|B(0,|x_0|+r)|^{\frac{n+\epsilon}{n}}\|\chi_{B(0,|x_0|+r)}\|_{L^\varphi(\mathbb{R}^n)}}\lesssim 1.\end{aligned}$$

Combining (2.3), (2.4) and (2.5), we see that $f\in\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}\lesssim 1$, which completes the proof of Proposition 2.3. $\square$

Let $\alpha\in(0,1]$, $s\in\mathbb{Z}_+$ and $\epsilon\in(0,\infty)$. For all f satisfying (1.4) and $x\in\mathbb{R}^n$, define

$$\sigma_{\alpha,s}(f)(x):=\left\{\sum_{k\in\mathbb{Z}}\left[A_{\alpha,s}(f)(x,2^k)\right]^2\right\}^{1/2}$$

and

$$\widetilde{\sigma}_{(\alpha,\epsilon),s}(f)(x):=\left\{\sum_{k\in\mathbb{Z}}\left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(x,2^k)\right]^2\right\}^{1/2}.$$

Next we show that the intrinsic square functions $S_{\alpha,s}(f)$, $g_{\alpha,s}(f)$, $\sigma_{\alpha,s}(f)$ and the similar-looking intrinsic square functions are pointwise comparable.

Proposition 2.4. *Let $\alpha\in(0,1]$, $s\in\mathbb{Z}_+$ and $\epsilon\in(0,\infty)$. Then, for all f satisfying (1.4) and all $x\in\mathbb{R}^n$, it holds true that*

$$g_{\alpha,s}(f)(x)\sim S_{\alpha,s}(f)(x)\sim\sigma_{\alpha,s}(f)(x)$$

and

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x)\sim\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\sim\widetilde{\sigma}_{(\alpha,\epsilon),s}(f)(x)$$

with the implicit positive constants independent of f.

Proof. Let $\mathcal{C}_{\alpha,s}(y,t)$, with $y\in\mathbb{R}^n$ and $t\in(0,\infty)$, be the *family of functions* $\phi:\mathbb{R}^n\to\mathbb{R}$, supported in $B(y,t)$, such that, for all $\gamma\in\mathbb{Z}_+^n$ and $|\gamma|\le s$,

$$\int_{\mathbb{R}^n}\phi(x)x^\gamma\,dx=0$$

and, for all x_1, $x_2\in\mathbb{R}^n$, $\nu\in\mathbb{Z}_+^n$ and $|\nu|=s$,

$$|D^\nu\phi(x_1)-D^\nu\phi(x_2)|\le t^{-n-\alpha}|x_1-x_2|^\alpha.$$

It is easy to see that

$$A_{\alpha,s}(f)(y,t)=\sup_{\phi\in\mathcal{C}_{\alpha,s}(y,t)}\left|\int_{\mathbb{R}^n}f(x)\phi(x)\,dx\right|.$$

By the definition of $\mathcal{C}_{\alpha,s}(y,t)$, we know that, for all $t\in(0,\infty)$,

$$\int_{\{y\in\mathbb{R}^n:\ |y|<t\}}[A_{\alpha,s}(f)(y,t)]^2\,\frac{dy}{t^n}\lesssim[A_{\alpha,s}(f)(0,2t)]^2$$

and

$$\begin{aligned}[A_{\alpha,s}(f)(0,t)]^2 &\lesssim \int_{\{y\in\mathbb{R}^n:\ |y|<t\}} [A_{\alpha,s}(f)(y,2t)]^2\,\frac{dy}{t^n}\\ &\lesssim \int_{\{y\in\mathbb{R}^n:\ |y|<2t\}} [A_{\alpha,s}(f)(y,2t)]^2\,\frac{dy}{t^n}.\end{aligned}$$

Integrating these inequalities in $\frac{dt}{t}$ from 0 to ∞, we then conclude that

$$g_{\alpha,s}(f)(0) \sim S_{\alpha,s}(f)(0),$$

which, together with the translation transformation, further implies that, for all $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) \sim S_{\alpha,s}(f)(x) \sim \sigma_{\alpha,s}(f)(x).$$

Similarly, we also see that, for all $x \in \mathbb{R}^n$,

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \sim \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) \sim \widetilde{\sigma}_{(\alpha,\epsilon),s}(f)(x),$$

which completes the proof of Proposition 2.4. $\square$

Observe that Proposition 2.4 when $s = 0$ was first obtained by Wilson [35, p. 783], which shows the advantages of intrinsic square functions.

To show that $g_{\alpha,s}(f)$ and $\widetilde{g}_{(\alpha,\epsilon),s}(f)$ are pointwise comparable, we need the following technical lemma.

Lemma 2.5. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (\max\{s,\alpha\},\infty)$. Then, for any $\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, there exist positive constants C and a sequence $\{\phi_k\}_{k\in\mathbb{Z}_+}$ of functions such that $C\phi_k \in \mathcal{C}_{\alpha,s}(0,2^k)$ and*

$$\psi = \sum_{k\in\mathbb{Z}_+} 2^{-k(\epsilon-\max\{s,\alpha\})}\phi_k.$$

Proof. Let $h \in C_c^\infty(\mathbb{R}^n)$ be real, radial and non-negative, support in

$$\{x \in \mathbb{R}^n:\ 1/8 \le |x| \le 1/2\},$$

and be normalized such that, for all $x \neq 0$,

$$\sum_{k=-\infty}^{\infty} h(2^{-k}x) = 1.$$

Let

$$\rho_0(\cdot) := 1 - \sum_{k=1}^{\infty} h(2^{-k}\cdot)$$

and, for $k \in \mathbb{N}$,

$$\rho_k(\cdot) := h(2^{-k}\cdot).$$

Then, $\operatorname{supp} \rho_k \subset \{x \in \mathbb{R}^n:\ 2^{k-3} \le |x| \le 2^{k-1}\}$, $\sum_{k=0}^\infty \rho_k = 1$ and, for all $x \in \mathbb{R}^n$,

$$\psi(x) = \sum_{k=0}^{\infty} \rho_k(x)\psi(x).$$

Let

$$M_k := \left[\int_{\mathbb{R}^n} \rho_k(x)\,dx\right]^{-1}.$$

Then $M_k \sim 2^{-kn}$.

For $k \in \mathbb{Z}_+$, let $\{\phi_l^k : \ l \in \mathbb{Z}_+^n \text{ and } |l| \le s\}$ be the orthogonal polynomials with weight ρ_k obtained via the Gram-Schmidt method from

$$\{x^\beta : \ \beta \in \mathbb{Z}_+^n \text{ and } |\beta| \le s\};$$

namely, for all l, $\nu \in \mathbb{Z}_+^n$ and $|l|$, $|\nu| \le s$, $\phi_l^k \in \mathcal{P}_s(\mathbb{R}^n)$ and

$$\left(\phi_\nu^k, \phi_l^k\right)_k := M_k \int_{\mathbb{R}^n} \phi_\nu^k(x)\phi_l^k(x)\rho_k(x)\,dx = \delta_{\nu l},$$

where $\delta_{\nu l} := 1$ if $\nu = l$ and $\delta_{\nu l} := 0$ if $\nu \ne l$.

Let

$$P_k := \sum_{l\in\mathbb{Z}_+^n,\,|l|\le s} \left(\psi, \phi_l^k\right)_k \phi_l^k.$$

Then, for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$, it holds true that

$$\int_{\mathbb{R}^n} [\psi(x) - P_k(x)]Q(x)\rho_k(x)\,dx = 0. \tag{2.6}$$

For $k \in \mathbb{Z}_+$, let $\{\psi_l^k : \ l \in \mathbb{Z}_+^n \text{ and } |l| \le s\}$ be the dual basis of

$$\{x^\beta : \ \beta \in \mathbb{Z}_+^n \text{ and } |\beta| \le s\}$$

with respect to the weight ρ_k; that is, for all l, $\beta \in \mathbb{Z}_+^n$ and $|l|$, $|\beta| \le s$, $\psi_l^k \in \mathcal{P}_s(\mathbb{R}^n)$ and

$$\left(\psi_l^k, x^\beta\right)_k := M_k \int_{\mathbb{R}^n} \psi_l^k(x)x^\beta\rho_k(x)\,dx = \delta_{l\beta}. \tag{2.7}$$

Then, from the fact that, for all l, $\beta \in \mathbb{Z}_+^n$ and $|l|$, $|\beta| \le s$,

$$\left(\sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(\phi_l^k, \psi_\nu^k\right)_k x^\nu, \psi_\beta^k\right)_k = \left(\phi_l^k, \sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(x^\nu, \psi_\beta^k\right)_k \psi_\nu^k\right)_k = \left(\phi_l^k, \psi_\beta^k\right)_k,$$

it follows that, for all $x \in \mathbb{R}^n$,

$$\phi_l^k(x) = \sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(\phi_l^k, \psi_\nu^k\right)_k x^\nu.$$

Thus, it holds true that

$$\begin{aligned} P_k &= \sum_{l\in\mathbb{Z}_+^n,\,|l|\le s} \left(\psi, \phi_l^k\right)_k \phi_l^k \\ &= \sum_{l\in\mathbb{Z}_+^n,\,|l|\le s} \left(\psi, \sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(\phi_l^k, \psi_\nu^k\right)_k x^\nu\right)_k \phi_l^k \\ &= \sum_{l\in\mathbb{Z}_+^n,\,|l|\le s} \sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(\psi, x^\nu\right)_k \left(\phi_l^k, \psi_\nu^k\right)_k \phi_l^k \\ &= \sum_{\nu\in\mathbb{Z}_+^n,\,|\nu|\le s} \left(\psi, x^\nu\right)_k \psi_\nu^k. \end{aligned} \tag{2.8}$$

For all l, $\beta \in \mathbb{Z}_+^n$ and $|l|$, $|\beta| \le s$, by the equality

$$M_1 \int_{\mathbb{R}^n} \psi_l^1(y) y^\beta \rho_1(y)\,dy = \delta_{l\beta} = M_k \int_{\mathbb{R}^n} \psi_l^k(x) x^\beta \rho_k(x)\,dx$$
$$= M_1 \int_{\mathbb{R}^n} (2^{k-1})^{|\beta|} \psi_l^k(2^{k-1}y) y^\beta \rho_1(y)\,dy,$$

we see that

$$\psi_l^k(\cdot) = (2^{k-1})^{-|l|} \psi_l^1(2^{-k+1}\cdot).$$

Thus, for all l, $\beta \in \mathbb{Z}_+^n$, $|l|$, $|\beta| \le s$ and $k \in \mathbb{N}$, we have

$$\|D^\beta \psi_l^k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-(k-1)(|l|+|\beta|)}. \tag{2.9}$$

For all $l \in \mathbb{Z}_+^n$, $|l| \le s$ and $k \in \mathbb{Z}_+$, let

$$N_l^k := \sum_{j=k}^\infty \left(\psi, x^l\right)_j \int_{\mathbb{R}^n} \rho_j(y)\,dy.$$

Then, for all $l \in \mathbb{Z}_+^n$ and $|l| \le s$, we see that

$$N_l^0 = \sum_{j=0}^\infty \int_{\mathbb{R}^n} \psi(x) x^l \rho_j(x)\,dx = \int_{\mathbb{R}^n} \psi(x) x^l\,dx = 0.$$

From the assumption $\epsilon \in (\alpha + s, \infty)$, it follows that, for all $l \in \mathbb{Z}_+^n$, $|l| \le s$ and $k \in \mathbb{N}$,

$$|N_l^k| \le \sum_{j=k}^\infty \left| \int_{\mathbb{R}^n} \psi(x) x^l \rho_j(x)\,dx \right| \lesssim \sum_{j=k}^\infty 2^{j(-\epsilon+|l|)} \lesssim 2^{k(-\epsilon+|l|)}. \tag{2.10}$$

Using (2.10) and (2.9), we know that

$$M_k \|N_l^k \psi_l^k \rho_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k(n+\epsilon)} \to 0, \text{ as } k \to \infty. \tag{2.11}$$

Thus, from (2.8) and (2.11), we deduce that

$$\sum_{k=0}^\infty P_k \rho_k = \sum_{k=0}^\infty \sum_{l \in \mathbb{Z}_+^n,\,|l| \le s} \left(\psi, x^l\right)_k \psi_l^k \rho_k$$
$$= \sum_{l \in \mathbb{Z}_+^n,\,|l| \le s} \sum_{k=0}^\infty N_l^k \left(M_k \psi_l^k \rho_k - M_{k+1} \psi_l^{k+1} \rho_{k+1}\right).$$

Now, write

$$\begin{aligned}
\psi &= \sum_{k=0}^\infty (\psi - P_k + P_k)\rho_k \\
&= \sum_{k=0}^\infty \left\{ (\psi - P_k)\rho_k + \sum_{l \in \mathbb{Z}_+^n,\,|l| \le s} N_l^k \left(M_k \psi_l^k \rho_k - M_{k+1} \psi_l^{k+1} \rho_{k+1}\right) \right\} \\
&=: \sum_{k=0}^\infty 2^{-k(\epsilon - \max\{s,\alpha\})} \widetilde{\phi}_k.
\end{aligned}$$

We next show that there exists a positive constant C such that, for all $k \in \mathbb{Z}_+$, $C\widetilde{\phi}_k \in \mathcal{C}_{\alpha,s}(0, 2^k)$.

Obviously, supp $\widetilde{\phi}_k \subset \{x \in \mathbb{R}^n : \ 2^{k-3} \le |x| \le 2^k\}$. From (2.6) and (2.7), it follows that, for all $l \in \mathbb{Z}_+^n$ and $|l| \le s$,

$$\int_{\mathbb{R}^n} \widetilde{\phi}_k(x) x^l \, dx = 0.$$

By (1.3), we see that, for all $k \in \mathbb{Z}_+$, $\nu \in \mathbb{Z}_+^n$, $|\nu| = s$ and $x_1, x_2 \in \mathbb{R}^n$,

$$|D^\nu(\psi\rho_k)(x_1) - D^\nu(\psi\rho_k)(x_2)| \lesssim 2^{-k(n+\epsilon)}|x_1 - x_2|^\alpha. \tag{2.12}$$

On the other hand, by (2.9), we know that, for all $k \in \mathbb{Z}_+$, $l, \nu \in \mathbb{Z}_+^n$, $|l| \le s$, $|\nu| = s$ and $x_1, x_2 \in \mathbb{R}^n$,

$$|D^\nu(\psi_l^k\rho_k)(x_1) - D^\nu(\psi_l^k\rho_k)(x_2)| \lesssim 2^{-k}|x_1 - x_2| \lesssim 2^{-k\alpha}|x_1 - x_2|^\alpha. \tag{2.13}$$

From this and (2.10), we deduce that, for all $k \in \mathbb{Z}_+$, $l, \nu \in \mathbb{Z}_+^n$, $|l| \le s$, $|\nu| = s$ and $x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} &\left|D^\nu\left(N_l^k M_k \psi_l^k \rho_k\right)(x_1) - D^\nu\left(N_l^k M_k \psi_l^k \rho_k\right)(x_2)\right| \\ &\quad\lesssim 2^{-k(n+\epsilon-s)}2^{-k\alpha}|x_1 - x_2|^\alpha \sim 2^{-k(n+\epsilon+\alpha-s\})}|x_1 - x_2|^\alpha. \end{aligned} \tag{2.14}$$

By (2.13) and (2.8), we also conclude that, for all $k \in \mathbb{Z}_+$, $l, \nu \in \mathbb{Z}_+^n$, $|l| \le s$, $|\nu| = s$ and $x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} &|D^\nu(P_k\rho_k)(x_1) - D^\nu(P_k\rho_k)(x_2)| \\ &\quad= \sum_{l \in \mathbb{Z}_+^n, \, |l| \le s} \left(\psi, x^l\right)_k |D^\nu(\psi_l^k\rho_k)(x_1) - D^\nu(\psi_l^k\rho_k)(x_2)| \\ &\quad\lesssim 2^{-k(n+\epsilon-s)}2^{-k\alpha}|x_1 - x_2|^\alpha \\ &\quad\lesssim 2^{-k(n+\epsilon+\alpha-s)}|x_1 - x_2|^\alpha. \end{aligned} \tag{2.15}$$

Combining (2.12), (2.14), (2.15) and $\epsilon \in (\max\{s, \alpha\}, \infty)$, we see that, for all $k \in \mathbb{Z}_+$, $\nu \in \mathbb{Z}_+^n$, $|\nu| = s$ and $x_1, x_2 \in \mathbb{R}^n$,

$$|D^\nu(\widetilde{\phi}_k)(x_1) - D^\nu(\widetilde{\phi}_k)(x_2)| \lesssim 2^{-k(n+\alpha)}|x_1 - x_2|^\alpha,$$

which further implies that there exists a positive constant C such that

$$C\widetilde{\phi}_k \in \mathcal{C}_{\alpha,s}(0, 2^k).$$

This finishes the proof of Lemma 2.5. $\square$

Using Lemma 2.5, we now prove Theorem 2.6, which, in the case when $s = 0$, was first obtained by Wilson [35, Theorem 2].

Theorem 2.6. *Let $\alpha \in (0, 1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (\max\{s, \alpha\}, \infty)$. Then there exists a positive constant C such that, for all f satisfying* (1.4) *and $x \in \mathbb{R}^n$,*

$$\frac{1}{C} g_{\alpha,s}(f)(x) \le \widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \le C g_{\alpha,s}(f)(x).$$

Proof. Obviously, for any $\alpha \in (0, 1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) \lesssim \widetilde{g}_{(\alpha,\epsilon),s}(f)(x).$$

To finish the proof of Theorem 2.6, we only need to prove the second inequality.

By Lemma 2.5, $\epsilon \in (\max\{s,\alpha\},\infty)$ and Hölder's inequality, we conclude that, for all $\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$,

$$
\begin{aligned}
(2.16)\qquad |f*\psi(0)| &\lesssim \sum_{k\in\mathbb{Z}_+} 2^{-k(\epsilon-\max\{s,\alpha\})} A_{\alpha,s}(f)(0,2^k)\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}_+} 2^{-k(\epsilon-\max\{s,\alpha\})}\left[A_{\alpha,s}(f)(0,2^k)\right]^2\right\}^{1/2}.
\end{aligned}
$$

From the definition of $\mathcal{C}_{\alpha,s}(0,t)$, we deduce that if t and r are positive, then $\phi \in \mathcal{C}_{\alpha,s}(0,t)$ if and only if $\phi_r \in \mathcal{C}_{\alpha,s}(0,rt)$. Therefore, by this observation and (2.16), we see that, for any $\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, it holds true that

$$
|f*\psi_{2^j}(0)| \lesssim \left\{\sum_{k\in\mathbb{Z}_+} 2^{-k(\epsilon-\max\{s,\alpha\})}\left[A_{\alpha,s}(f)(0,2^{k+j})\right]^2\right\}^{1/2},
$$

which, together with $\epsilon \in (\max\{s,\alpha\},\infty)$, implies that, for any sequence $\{\psi^{(j)}\}_{j\in\mathbb{Z}}$ of functions from $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, it holds true that

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |f*\psi^{(j)}_{2^j}(0)|^2 &\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}_+} 2^{-k(\epsilon-\max\{s,\alpha\})}\left[A_{\alpha,s}(f)(0,2^{k+j})\right]^2\\
&\sim \sum_{l\in\mathbb{Z}}\left[A_{\alpha,s}(f)(0,2^l)\right]^2 \sum_{j=-\infty}^{l} 2^{-(l-j)(\epsilon-\max\{s,\alpha\})}\\
&\sim \sum_{l\in\mathbb{Z}}\left[A_{\alpha,s}(f)(0,2^l)\right]^2 \sim [\sigma_{\alpha,s}(f)(0)]^2 \sim [g_{\alpha,s}(f)(0)]^2.
\end{aligned}
$$

Taking the supremum over all the sequences $\{\psi^{(j)}\}_{j\in\mathbb{Z}} \subset \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, we then conclude that

$$
\widetilde{g}_{(\alpha,\epsilon),s}(f)(0) \sim \widetilde{\sigma}_{(\alpha,\epsilon),s}(f)(0) \lesssim \sigma_{\alpha,s}(f)(0) \sim g_{\alpha,s}(f)(0),
$$

which, together with the translation transformation, further implies that, for all $x \in \mathbb{R}^n$,

$$
\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \lesssim g_{\alpha,s}(f)(x).
$$

This finishes the proof of Theorem 2.6. $\square$

3. Proofs of Theorems 1.6, 1.8 and 1.11

In this section, we prove Theorems 1.6, 1.8 and 1.11.

We first prove Theorem 1.6. To this end, we need the following technical lemma.

Lemma 3.1. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (0,\infty)$. Then, $\mathcal{C}_{\alpha,s}(\mathbb{R}^n) \subset \mathcal{C}_{\alpha,s-1}(\mathbb{R}^n)$.*

Proof. To show this lemma, it suffices to show that, for $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and any function ϕ on $\mathbb{R}^n$ satisfying that $\operatorname{supp}\phi \subset B(0,1)$,

$$
\int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0 \text{ for all } \gamma \in \mathbb{Z}^n_+ \text{ and } |\gamma| \le s
$$

and

$$
|D^\nu\phi(x_1) - D^\nu\phi(x_2)| \le |x_1-x_2|^\alpha \quad \text{for all} \quad x_1,\, x_2 \in \mathbb{R}^n,\ \nu \in \mathbb{Z}^n_+ \text{ and } |\nu| = s,
$$

it also satisfies that, for all $x_1,\, x_2 \in \mathbb{R}^n$, $l \in \mathbb{Z}_+^n$ and $|l| = s-1$,

$$|D^l\phi(x_1) - D^l\phi(x_2)| \le |x_1 - x_2|^\alpha. \tag{3.1}$$

For all $x \in B(0,1)$, let $\widetilde{x} := \frac{x}{|x|} \in \partial B(0,1)$. Then, for all $\nu,\, l \in \mathbb{Z}_+^n$, $|\nu| = s$ and $|l| = s-1$, since $|x - \widetilde{x}| = 1 - |x|$, by $\operatorname{supp}\phi \subset B(0,1)$, we see that

$$|D^\nu\phi(x)| = |D^\nu\phi(x) - D^\nu\phi(\widetilde{x})| \le |x - \widetilde{x}|^\alpha = (1-|x|)^\alpha$$

and, by this and the mean value theorem, we further conclude that there exists $\theta \in (0,1)$ such that $\xi = \theta x + (1-\theta)\widetilde{x} \in B(0,1)$ and

$$\begin{aligned} |D^l\phi(x)| &= |D^l\phi(x) - D^l\phi(\widetilde{x})| = |\nabla(D^l\phi)(\xi)(x - \widetilde{x})| \\ &\le \max_{\gamma\in\mathbb{Z}_+^n,\, |\gamma|=s}\{|D^\gamma\phi(\xi)|\}|x - \widetilde{x}| \\ &\le (1-|\xi|)^\alpha(1-|x|) \le (1-|x|)^{\alpha+1}. \end{aligned} \tag{3.2}$$

Now, we prove (3.1) in the following four cases.

Case i) $x_1,\, x_2 \in [B(0,1)]^\complement$. In this case, the conclusion is trivial.

Case ii) $x_1 \in B(0,1)$ *and* $x_2 \in [B(0,1)]^\complement$. In this case,

$$|x_1 - x_2| \ge |x_1 - \widetilde{x}_1| = 1 - |x_1|,$$

which, combined with (3.2), further implies that

$$|D^l\phi(x_1) - D^l\phi(x_2)| = |D^l\phi(x_1)| \le (1-|x_1|)^{\alpha+1} \le |x_1 - x_2|^\alpha.$$

Case iii) $x_1,\, x_2 \in B(0,1)$ *and* $|x_1 - x_2| \le 1$. In this case, by the mean value theorem, we know that there exists $\theta \in (0,1)$ such that $\xi = \theta x_1 + (1-\theta)x_2 \in B(0,1)$ and

$$\begin{aligned} |D^l\phi(x_1) - D^l\phi(x_2)| &= |\nabla(D^l\phi)(\xi)(x_1 - x_2)| \\ &\le \max_{\gamma\in\mathbb{Z}_+^n,\, |\gamma|=s}\{|D^\gamma\phi(\xi)|\}|x_1 - x_2|, \end{aligned}$$

which, together with (3.2), further implies that

$$|D^l\phi(x_1) - D^l\phi(x_2)| \le (1-|\xi|)^{\alpha+1}|x_1 - x_2| \le |x_1 - x_2| \le |x_1 - x_2|^\alpha.$$

Case iv) $x_1,\, x_2 \in B(0,1)$ *and* $|x_1 - x_2| > 1$. In this case, since

$$|x_1 - x_2| + (1-|x_1|) + (1-|x_2|) \le 2$$

and $|x_1 - x_2| > 1$, we see that $(1-|x_1|) + (1-|x_2|) < 1$, which, combined with (3.2), implies that

$$\begin{aligned} |D^l\phi(x_1) - D^l\phi(x_2)| &\le |D^l\phi(x_1)| + |D^l\phi(x_2)| \\ &\le (1-|x_1|)^{\alpha+1} + (1-|x_2|)^{\alpha+1} \le 1 \le |x_1 - x_2|^\alpha. \end{aligned}$$

This finishes the proof of (3.1) and hence Lemma 3.1. ☐

From Lemma 3.1, we deduce the following conclusion.

Proposition 3.2. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, φ be a growth function, $q \in (1,\infty)$ and $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$. Then, there exists a positive constant C such that, for all $t \in [0,\infty)$ and measurable functions f,*

$$\int_{\mathbb{R}^n} [g_{\alpha,s}(f)(x)]^q \varphi(x,t)\,dx \le C\int_{\mathbb{R}^n} |f(x)|^q\varphi(x,t)\,dx.$$

Proof. From Lemma 3.1, we deduce that, for all $x \in \mathbb{R}^n$ and $s \in \mathbb{Z}_+$,

$$g_{\alpha,s}(f)(x) \le g_\alpha(f)(x)$$

which, together with the fact proved in [36, Theorem 7.2] that, for all $t \in [0, \infty)$,

$$\int_{\mathbb{R}^n} [g_\alpha(f)(x)]^q \varphi(x,t)\,dx \lesssim \int_{\mathbb{R}^n} |f(x)|^q \varphi(x,t)\,dx,$$

then completes the proof of Proposition 3.2. □

To prove Theorem 1.6, we also need the atomic characterization of $H^\varphi(\mathbb{R}^n)$ from [21] and its dual space $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ from [26].

Recall that, for any ball B in $\mathbb{R}^n$, the *space* $L^q_\varphi(B)$ for $q \in [1, \infty]$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$, supported in B, such that

$$\|f\|_{L^q_\varphi(B)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\varphi(B,t)}\displaystyle\int_{\mathbb{R}^n} |f(x)|^q\varphi(x,t)\,dx\right]^{1/q} < \infty, & q \in [1,\infty); \\ \|f\|_{L^\infty(\mathbb{R}^n)} < \infty, & q = \infty. \end{cases}$$

Now, we recall the atomic Musielak-Orlicz Hardy spaces introduced by Ky [21] as follows. A triplet (φ, q, s) is said to be *admissible* if $q \in (q(\varphi), \infty]$ and $s \in \mathbb{Z}_+$ satisfies $s \ge m(\varphi)$. A measurable function a is called a (φ, q, s)*-atom* if it satisfies the following three conditions:

(i) $a \in L^q_\varphi(B)$ for some ball B;

(ii) $\|a\|_{L^q_\varphi(B)} \le \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for any $\alpha \in \mathbb{Z}^n_+$ and $|\alpha| \le s$.

The *atomic Musielak-Orlicz Hardy space* $H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be represented as a sum of multiples of (φ, q, s)-atoms, that is,

$$f = \sum_{j=1}^\infty b_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where, for each j, b_j is a multiple of some (φ, q, s)-atom supported in some ball B_j, with the property

$$\sum_{j=1}^\infty \varphi(B_j, \|b_j\|_{L^q_\varphi(B_j)}) < \infty.$$

For any given sequence of multiples of (φ, q, s)-atoms, $\{b_j\}_{j\in\mathbb{N}}$, let

$$\Lambda_q(\{b_j\}_{j\in\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) :\ \sum_{j=1}^\infty \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\}$$

and then define

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)} := \inf\left\{\Lambda_q(\{b_j\}_{j\in\mathbb{N}}) :\ f = \sum_{j=1}^\infty b_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)\right\},$$

where the infimum is taken over all decompositions of f as above.

Remark 3.3. Observe that when φ is as in (1.1) with $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$, then, for all balls $B \subset \mathbb{R}^n$ and measurable functions f, it holds true that

$$\|f\|_{L^q_\varphi(B)} = \frac{\|f\|_{L^q_w(\mathbb{R}^n)}}{[w(B)]^{1/q}}.$$

Thus, for all $f \in H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)$, $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of (φ,q,s)-atoms, $\{a_j\}_{j\in\mathbb{N}}$, associated with balls $\{B_j\}_{j\in\mathbb{N}}$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

we have

$$\Lambda_q(\{\lambda_j a_j\}_{j\in\mathbb{N}}) = \left\{\sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j|\|a_j\|_{L^q_w(\mathbb{R}^n)}}{[w(B)]^{1/q-1/p}}\right]^p\right\}^{1/p} \le \left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p},$$

which further implies that

$$\begin{aligned}\text{(3.3)} \qquad \|f\|_{H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)} &\le \inf\left\{\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p} :\ f = \sum_{j=1}^\infty \lambda_j a_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)\right\} \\ &=: \|f\|_{\widetilde{H}^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)},\end{aligned}$$

where the infimum is taken over all decompositions of f as above.

On the other hand, for any $j \in \mathbb{N}$, and a_j and λ_j as above, let

$$\widetilde{\lambda}_j := \frac{\|a_j\|_{L^q_w(\mathbb{R}^n)}\lambda_j}{[w(B)]^{1/q-1/p}}$$

and

$$\widetilde{a}_j := \frac{[w(B)]^{1/q-1/p} a_j}{\|a_j\|_{L^q_w(\mathbb{R}^n)}}.$$

Then, $\{\widetilde{a}_j\}_{j\in\mathbb{N}}$ is also a sequence of (φ,q,s)-atoms associated with balls $\{B_j\}_{j\in\mathbb{N}}$,

$$f = \sum_{j\in\mathbb{N}} \widetilde{\lambda}_j \widetilde{a}_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

and

$$\left(\sum_{j\in\mathbb{N}} |\widetilde{\lambda}_j|^p\right)^{1/p} = \left\{\sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j|\|a_j\|_{L^q_w(\mathbb{R}^n)}}{[w(B)]^{1/q-1/p}}\right]^p\right\}^{1/p} = \Lambda_q(\{\lambda_j a_j\}_{j\in\mathbb{N}}),$$

which, together with (3.3), further implies that

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)} = \|f\|_{\widetilde{H}^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)}.$$

That is, in this case, the quasi-norm $\|\cdot\|_{H^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)}$ just becomes the quasi-norm $\|\cdot\|_{\widetilde{H}^{\varphi,q,s}_{\mathrm{at}}(\mathbb{R}^n)}$ of the weighted atomic Hardy space in [14].

We use $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ to denote the *set of all finite combinations of* (φ,q,s)*-atoms.* The norm of f in $H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{\varphi,q,s}_{\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\{\Lambda_q(\{b_j\}_{j=1}^k) :\ k \in \mathbb{N} \text{ and } f = \sum_{j=1}^k b_j \text{ in } \mathcal{S}'(\mathbb{R}^n)\right\},$$

where the infimum is taken over all finite decompositions of f. It is easy to see that $H_{\mathrm{fin}}^{\varphi,q,s}(\mathbb{R}^n)$ is dense in $H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n)$.

The following Lemmas 3.4 and 3.5 are, respectively, just [21, Theorem 3.1] and [26, Theorem 3.5], which play a key role in the proof of Theorem 1.6.

Lemma 3.4. *Let (φ, q, s) be admissible. Then, $H^{\varphi}(\mathbb{R}^n) = H_{\mathrm{at}}^{\varphi,q,s}(\mathbb{R}^n)$ with equivalent norms.*

Lemma 3.5. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition and $s \in [m(\varphi), \infty) \cap \mathbb{Z}_+$. Then the dual space of $H^{\varphi}(\mathbb{R}^n)$, denoted by $(H^{\varphi}(\mathbb{R}^n))^*$, is $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ in the following sense:*

(i) *Suppose that $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$. Then the linear functional*

$$L_b:\ f \to L_b(f) := \int_{\mathbb{R}^n} f(x)b(x)\,dx,$$

initially defined for all $f \in H_{\mathrm{fin}}^{\varphi,q,s}(\mathbb{R}^n)$ with some $q \in (q(\varphi), \infty)$, has a bounded extension to $H^{\varphi}(\mathbb{R}^n)$.

(ii) *Conversely, every continuous linear functional on $H^{\varphi}(\mathbb{R}^n)$ arises as in* (i) *with a unique $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$.*

Moreover,

$$\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \sim \|L_b\|_{(H^{\varphi}(\mathbb{R}^n))^*},$$

where the implicit constants are independent of b.

Having these equipment, we can now give the proof of Theorem 1.6.

Proof of Theorem 1.6. Observe that, by Theorem 2.6, we know that for $\epsilon \in (\alpha + s, \infty)$ and all $x \in \mathbb{R}^n$, $g_{\alpha,s}(f)(x)$ and $\widetilde{g}_{(\alpha,\epsilon),s}(f)(x)$ are pointwise comparable. Thus, to finish the proof of Theorem 1.6, we only need to consider $g_{\alpha,s}(f)$ in our proof.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function, $\operatorname{supp} \phi \subset \{x \in \mathbb{R}^n:\ |x| \le 1\}$,

$$\int_{\mathbb{R}^n} \phi(x)x^{\gamma}\,dx = 0 \ \text{ for all } \ |\gamma| \le s$$

and, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^{\infty} |\hat{\phi}(\xi t)|^2 \frac{dt}{t} = 1.$$

Recall that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *Littlewood-Paley g-function* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$g_s(f)(x) := \left[\int_0^{\infty} |f * \phi_t(y)|^2\ \frac{dt}{t}\right]^{1/2}.$$

If $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$ vanishes weakly at infinity and $g_{\alpha,s}(f) \in L^{\varphi}(\mathbb{R}^n)$, then, by the fact that, for all $x \in \mathbb{R}^n$,

$$g_s(f)(x) \lesssim g_{\alpha,s}(f)(x)$$

and [25, Theorem 4.4], together with Remark 1.12(i), we conclude that $f \in H^{\varphi}(\mathbb{R}^n)$ and

$$\|f\|_{H^{\varphi}(\mathbb{R}^n)} \lesssim \|g_s(f)\|_{L^{\varphi}(\mathbb{R}^n)} \lesssim \|g_{\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)}.$$

This finishes the proof of the sufficiency of Theorem 1.6.

It therefore remains to prove the necessity. Let $q := p[1 + (\alpha + s)/n]$. If $f \in H^{\varphi}(\mathbb{R}^n)$, then, by [17, Lemma 4.12], we know that f vanishes weakly at infinity. Also, since φ satisfies the uniformly locally dominated convergence condition, from

Lemma 3.5, it follows that $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$. For any $0 \neq f \in H^\varphi(\mathbb{R}^n)$, let $f = \sum_{j=1}^\infty b_j$ be an atomic decomposition of f as in Lemma 3.4, with $\mathrm{supp}\, b_j \subset B_j$ for all $j \in \mathbb{N}$. Then, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, it holds true that

$$\sum_{j=1}^\infty b_j * \phi = f * \phi \text{ pointwise,}$$

since $f = \sum_{j=1}^\infty b_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, for all $x \in \mathbb{R}^n$, we have

$$g_{\alpha,s}(f)(x) \le \sum_{j=1}^\infty g_{\alpha,s}(b_j)(x).$$

We now claim: it suffices to prove that, for any (φ, q, s)-atom a associated with a ball $B := B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, it holds true that

$$\int_{\mathbb{R}^n} \varphi(x, g_{\alpha,s}(a)(x))\, dx \lesssim \varphi(B, \|a\|_{L^q_\varphi(B)}). \tag{3.4}$$

Indeed, from (3.4) and Lemma 2.1(i), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\left(x, \frac{g_{\alpha,s}(f)(x)}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) dx &\lesssim \sum_{k=1}^\infty \int_{\mathbb{R}^n} \varphi\left(x, \frac{g_{\alpha,s}(b_k)(x)}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) dx \\ &\lesssim \sum_{k=1}^\infty \varphi\left(B_j, \frac{\|b_k\|_{L^q_\varphi(B_k)}}{\Lambda_q(\{b_j\}_{j=1}^\infty)}\right) \lesssim 1,\end{aligned}$$

which implies that

$$\|g_{\alpha,s}(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \Lambda_q(\{b_j\}_{j=1}^\infty)$$

for all atomic decompositions $f = \sum_{j=1}^\infty b_j$, and hence

$$\|g_{\alpha,s}(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H^\varphi(\mathbb{R}^n)}.$$

This is the desired conclusion.

It therefore remains to prove (3.4). Let $\widetilde{B} := 9B$ and write

$$\int_{\mathbb{R}^n} \varphi(x, g_{\alpha,s}(a)(x))\, dx = \int_{\widetilde{B}} \varphi(x, g_{\alpha,s}(a)(x))\, dx + \int_{\widetilde{B}^\complement} \cdots =: \mathrm{I}_1 + \mathrm{I}_2.$$

Since φ is of uniformly upper type 1, by Hölder's inequality, Proposition 3.2 and Lemma 2.1(iv), we see that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_{\widetilde{B}} \left[\frac{g_{\alpha,s}(a)(x)}{\|a\|_{L^q_\varphi(B)}} + 1\right] \varphi(x, \|a\|_{L^q_\varphi(B)})\, dx \\ &\lesssim \frac{1}{\|a\|_{L^q_\varphi(B)}} \left\{\int_{\widetilde{B}} [g_{\alpha,s}(a)(x)]^q\, \varphi(x, \|a\|_{L^q_\varphi(B)})\, dx\right\}^{1/q} \left[\varphi(\widetilde{B}, \|a\|_{L^q_\varphi(B)})\right]^{(q-1)/q} \\ &\quad + \varphi(\widetilde{B}, \|a\|_{L^q_\varphi(B)}) \\ &\lesssim \varphi(B, \|a\|_{L^q_\varphi(B)}).\end{aligned}$$

By $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, we conclude that, for all $\lambda \in (0, \infty)$,

$$\int_B \varphi(y, \lambda) dy \left\{\int_B [\varphi(y, \lambda)]^{-1/(q-1)} dy\right\}^{q-1} \lesssim |B|^q. \tag{3.5}$$

Therefore, for any $\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, $t \in (0,\infty)$ and $x \in \widetilde{B}^{\complement}$, by the vanishing moment condition of a and (3.5), together with Taylor's theorem, we see that

$$\begin{aligned}
(3.6)\quad & |a*\phi_t(x)| \\
&= \frac{1}{t^n}\left|\int_B a(y)\left[\phi\left(\frac{x-y}{t}\right) - \sum_{|\beta|\le s} \frac{D^\beta \phi\left(\frac{x-x_0}{t}\right)}{\beta!}\left(\frac{x_0-y}{t}\right)^\beta\right] dy\right| \\
&\lesssim \int_B |a(y)| \frac{|y-x_0|^{\alpha+s}}{t^{n+\alpha+s}}\, dy \\
&\lesssim \frac{r^{\alpha+s}}{t^{n+\alpha+s}}\left[\int_B |a(y)|^q \varphi(y,\lambda)\, dy\right]^{1/q}\left\{\int_B [\varphi(y,\lambda)]^{-1/(q-1)}\, dy\right\}^{(q-1)/q} \\
&\lesssim \|a\|_{L^q_\varphi(B)}\left(\frac{r}{t}\right)^{n+\alpha+s}.
\end{aligned}$$

Notice that $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n: |x| \le 1\}$. If $x \in \widetilde{B}^{\complement}$ and $\phi_t * a(x) \neq 0$, then there exists a $y \in B$ such that $\frac{|x-y|}{t} \le 1$, and hence

$$t \ge |x-y| \ge |x-x_0| - |x_0-y| \ge \frac{|x-x_0|}{2}.$$

This, combined with (3.6), implies that

$$\begin{aligned}
|g_{\alpha,s}(a)(x)|^2 &= \int_0^\infty \left[\sup_{\phi\in\mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |a*\phi_t(x)|\right]^2 \frac{dt}{t} \\
&\lesssim \|a\|^2_{L^q_\varphi(B)} r^{2(n+\alpha+s)} \int_{\frac{|x-x_0|}{2}}^\infty t^{-2(n+\alpha+s)-1}\, dt \\
&\sim \|a\|^2_{L^q_\varphi(B)}\left[\frac{r}{|x-x_0|}\right]^{2(n+\alpha+s)},
\end{aligned}$$

which, together with Lemma 2.1(v), further implies that

$$\begin{aligned}
\mathrm{I}_2 &= \int_{\widetilde{B}^{\complement}} \varphi(x, g_{\alpha,s}(a)(x))\, dx \lesssim \int_{\widetilde{B}^{\complement}} \left[\frac{r}{|x-x_0|}\right]^{(n+\alpha+s)p} \varphi(x, \|a\|_{L^q_\varphi(B)})\, dx \\
&\lesssim r^{(n+\alpha+s)p}\frac{\varphi(\widetilde{B}, \|a\|_{L^q_\varphi(B)})}{r^{(n+\alpha+s)p}} \lesssim \varphi(B, \|a\|_{L^q_\varphi(B)}).
\end{aligned}$$

This finishes the proof of Theorem 1.6. □

For all $\beta \in (0,\infty)$, $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$ and $x \in \mathbb{R}^n$, let

$$\begin{aligned}
(3.7)\quad & \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) \\
&:= \left\{\int_0^\infty \int_{\{y\in\mathbb{R}^n:\ |y-x|<\beta t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 (\beta t)^{-n}\, \frac{dy\, dt}{t}\right\}^{1/2}.
\end{aligned}$$

To prove Theorem 1.8, we need the following technical lemma.

Lemma 3.6. *Let $q \in [1,\infty)$, φ be a growth function and $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $\beta \in [1,\infty)$, $t \in [0,\infty)$ and measurable functions f,*

$$\int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\right) dx \le C\beta^{n(q-p/2)} \int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right) dx.$$

Proof. For all $\lambda \in (0,\infty)$, let

$$\mathrm{E}_\lambda := \left\{x \in \mathbb{R}^n:\ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \lambda\beta^{n/2}\right\}$$

and

$$\mathrm{U} := \{x \in \mathbb{R}^n:\ M(\chi_{\mathrm{E}_\lambda})(x) > (4\beta)^{-n}\},$$

where M denotes the *Hardy-Littlewood maximal function*; namely, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\,dy$$

and the supremum is taken over all balls $B \ni x$ of $\mathbb{R}^n$. Since $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, by the weighted weak-type (q,q) boundedness of M (see, for example, [15]), we see that

$$\begin{aligned}\varphi(\mathrm{U},\lambda) &= \varphi\left(\{x \in \mathbb{R}^n:\ M(\chi_{\mathrm{E}_\lambda})(x) > (4\beta)^{-n}\},\lambda\right)\\ &\lesssim (4\beta)^{nq}\|\chi_{\mathrm{E}_\lambda}\|^q_{L^q_{\varphi(\cdot,\lambda)}(\mathbb{R}^n)} \sim \beta^{nq}\varphi(\mathrm{E}_\lambda,\lambda)\end{aligned} \tag{3.8}$$

and we now claim that

$$\begin{aligned}&\beta^{n(1-q)}\int_{\mathrm{U}^\complement}[\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)]^2\varphi(x,\lambda)\,dx\\ &\quad\lesssim \int_{\mathrm{E}_\lambda^\complement}[\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2\varphi(x,\lambda)\,dx.\end{aligned} \tag{3.9}$$

If (3.9) holds true, then, from (3.8) and (3.9), it follows that

$$\begin{aligned}&\varphi\left(\left\{x \in \mathbb{R}^n:\ \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\right\},\lambda\right)\\ &\quad\le \varphi(\mathrm{U},\lambda) + \varphi\left(\mathrm{U}^\complement \cap \{x \in \mathbb{R}^n:\ \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\},\lambda\right)\\ &\quad\lesssim \beta^{nq}\varphi(\mathrm{E}_\lambda,\lambda) + \lambda^{-2}\int_{\mathrm{U}^\complement}[\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)]^2\varphi(x,\lambda)\,dx\\ &\quad\lesssim \beta^{nq}\varphi(\mathrm{E}_\lambda,\lambda) + \beta^{n(q-1)}\lambda^{-2}\int_{\mathrm{E}_\lambda^\complement}[\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2\varphi(x,\lambda)\,dx\\ &\quad\sim \beta^{nq}\varphi(\mathrm{E}_\lambda,\lambda) + \beta^{n(q-1)}\lambda^{-2}\int_0^{\lambda\beta^{n/2}} t\varphi\left(\left\{x \in \mathbb{R}^n:\ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\},\lambda\right)\,dt,\end{aligned}$$

which, together with the assumption that $\beta \in [1, \infty)$, Lemma 2.1(ii), the uniformly lower type p and upper type 1 properties of φ, further implies that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\right) dx \\
&\quad\sim \int_0^\infty \frac{1}{\lambda}\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x) > \lambda\right\}, \lambda\right) d\lambda \\
&\quad\lesssim \beta^{nq} \int_0^\infty \frac{1}{\lambda}\varphi(\mathrm{E}_\lambda, \lambda)\, d\lambda \\
&\qquad + \beta^{n(q-1)} \int_0^\infty \lambda^{-3} \int_0^{\lambda\beta^{n/2}} t\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\}, \lambda\right) dt\, d\lambda \\
&\quad\lesssim \beta^{n(q-p/2)} \int_0^\infty \frac{1}{\lambda}\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > \lambda\right\}, \lambda\right) d\lambda \\
&\qquad + \beta^{n(q-1)} \left\{\int_0^\infty \lambda^{-3} \int_0^{\lambda} \lambda\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\}, t\right) dt\, d\lambda \right.\\
&\qquad \left. + \int_0^\infty \lambda^{-3} \int_\lambda^{\lambda\beta^{n/2}} (\lambda/t)^p t\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\}, t\right) dt\, d\lambda\right\} \\
&\quad\lesssim \beta^{n(q-p/2)} \int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right) dx \\
&\qquad + \beta^{n(q-1)} \left\{\int_0^\infty \frac{1}{t}\varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\}, t\right) dt \right.\\
&\qquad \left. + \int_0^\infty \frac{1}{t}\left[\beta^{(2-p)n/2} - 1\right] \varphi\left(\left\{x \in \mathbb{R}^n : \ \widetilde{S}_{(\alpha,\epsilon),s}(f)(x) > t\right\}, t\right) dt\right\} \\
&\quad\lesssim \beta^{n(q-p/2)} \int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right) dx.
\end{aligned}$$

It therefore remains to prove (3.9). Let

$$\rho(y) := \inf\left\{|y - z| : \ z \in \mathrm{U}^{\complement}\right\}.$$

Then, it holds true that

$$\begin{aligned}
(3.10)\qquad &\int_{\mathrm{U}^{\complement}} \left[\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)\right]^2 \varphi(x, \lambda)\, dx \\
&\quad= \int_{\mathrm{U}^{\complement}} \left[\int_0^\infty \int_{\{y \in \mathbb{R}^n : \ |y-x| < \beta t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 (\beta t)^{-n} \frac{dy\, dt}{t}\right] \\
&\quad\quad \times \varphi(x, \lambda)\, dx \\
&\quad= \int_0^\infty \int_{\{y \in \mathbb{R}^n : \ \rho(y) < \beta t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 (\beta t)^{-n} \\
&\quad\quad \times \varphi(\mathrm{U}^{\complement} \cap B(y, \beta t), \lambda)\, \frac{dy\, dt}{t}.
\end{aligned}$$

If

$$\int_{\mathrm{U}^{\complement}} [\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)]^2 \varphi(x, \lambda)\, dx > 0,$$

then, by (3.10), we know that $\mathrm{U}^{\complement} \cap B(y, \beta t) \neq \emptyset$. Thus, there exists $x_0 \in \mathrm{U}^{\complement} \cap B(y, \beta t)$ and, by the definition of U, we further have

$$\frac{|\mathrm{E}_\lambda \cap B(y,t)|}{|B(y,t)|} \le \frac{\beta^n}{|B(y,\beta t)|} \int_{B(y,\beta t)} \chi_{\mathrm{E}_\lambda}(x)\,dx \le \beta^n M(\chi_{\mathrm{E}_\lambda})(x_0) \le 4^{-n},$$

which, together with $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ and Lemma 2.1(iv), further implies that

$$\begin{aligned}\varphi(\mathrm{U}^{\complement} \cap B(y,\beta t), \lambda) &\le \varphi(B(y,\beta t),\lambda) \lesssim \beta^{nq}\varphi(B(y,t),\lambda) \\ &\lesssim \beta^{nq}\left[\frac{|\mathrm{E}_\lambda^{\complement} \cap B(y,t)|}{|B(y,t)|}\right]^q \varphi\left(\mathrm{E}_\lambda^{\complement} \cap B(y,t),\lambda\right) \\ &\lesssim \beta^{nq}\varphi\left(\mathrm{E}_\lambda^{\complement} \cap B(y,t),\lambda\right).\end{aligned} \tag{3.11}$$

Thus, from (3.10) and (3.11), it follows that

$$\begin{aligned}&\int_{\mathrm{U}^{\complement}} [\widetilde{S}_{\beta,(\alpha,\epsilon),s}(f)(x)]^2 \varphi(x,\lambda)\,dx \\ &\quad\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 (\beta t)^{-n}\beta^{nq}\varphi\left(\mathrm{E}_\lambda^{\complement} \cap B(y,t),\lambda\right)\frac{dy\,dt}{t} \\ &\quad\sim \beta^{n(q-1)} \int_{\mathrm{E}_\lambda^{\complement}} \int_0^\infty \int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\varphi(x,\lambda)\,dx \\ &\quad\sim \beta^{n(q-1)} \int_{\mathrm{E}_\lambda^{\complement}} \left[\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right]^2 \varphi(x,\lambda)\,dx.\end{aligned}$$

This finishes the proof of Lemma 3.6. □

Proof of Theorem 1.8. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be as in the proof of Theorem 1.6. Recall that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *Littlewood-Paley g_λ^*-function*, with $\lambda \in (1,\infty)$, of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$g^*_{\lambda,s}(f)(x) := \left[\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |f * \phi_t(y)|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2}.$$

If $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$ vanishes weakly at infinity and $g^*_{\lambda,\alpha,s}(f) \in L^\varphi(\mathbb{R}^n)$, then, by the fact that, for all $x \in \mathbb{R}^n$,

$$g^*_{\lambda,s}(f)(x) \lesssim g^*_{\lambda,\alpha,s}(f)(x) \lesssim \widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)$$

and [25, Theorem 4.8], together with Remark 1.12(i), we know that $f \in H^\varphi(\mathbb{R}^n)$ and

$$\|f\|_{H^\varphi(\mathbb{R}^n)} \lesssim \left\|g^*_{\lambda,s}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \lesssim \left\|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\right\|_{L^\varphi(\mathbb{R}^n)}.$$

This finishes the proof of the sufficiency of Theorem 1.8.

It therefore remains to prove the necessity. Let $q := p[1+(\alpha+s)/n]$. Then, by $\lambda \in (\frac{2(n+\alpha+s)}{n},\infty)$, we have $\lambda \in (2q/p,\infty)$. If $f \in H^\varphi(\mathbb{R}^n)$, as in the proof of Theorem 1.6, we know that f vanishes weakly at infinity and $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$.

For all $f \in H^{\varphi}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
(3.12)\qquad &\left[\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)\right]^2\\
&= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\\
&\quad + \sum_{k=1}^\infty \int_0^\infty \int_{2^{k-1}t\le|x-y|<2^k t} \cdots\\
&\lesssim \left[\widetilde{S}_{(\alpha,\epsilon),s} f(x)\right]^2 + \sum_{k=1}^\infty 2^{-kn(\lambda-1)} \left[\widetilde{S}_{2^k,(\alpha,\epsilon),s} f(x)\right]^2.
\end{aligned}
$$

Then, from (3.12), Lemma 2.1(i) and 3.6, and $\lambda \in (2q/p,\infty)$, we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \varphi\left(x, \widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)\right) dx\\
&\quad\lesssim \sum_{k=0}^\infty \int_{\mathbb{R}^n} \varphi\left(x, 2^{-kn(\lambda-1)/2} \widetilde{S}_{2^k,(\alpha,\epsilon),s}(f)(x)\right) dx\\
&\quad\lesssim \sum_{k=0}^\infty 2^{-knp(\lambda-1)/2} 2^{kn(q-p/2)} \int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right) dx\\
&\quad\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \widetilde{S}_{(\alpha,\epsilon),s}(f)(x)\right) dx.
\end{aligned}
$$

By Lemma 2.1(iii), we see that

$$
\begin{aligned}
\int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)}{\|f\|_{H^{\varphi}(\mathbb{R}^n)}}\right) dx &\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)}{\|f\|_{H^{\varphi}(\mathbb{R}^n)}}\right) dx\\
&\sim \int_{\mathbb{R}^n} \varphi\left(x, \frac{\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)}{\|\widetilde{S}_{(\alpha,\epsilon),s}(f)\|_{L^{\varphi}(\mathbb{R}^n)}}\right) dx \sim 1,
\end{aligned}
$$

which further implies that

$$
\|g^*_{\lambda,\alpha,s}(f)\|_{L^{\varphi}(\mathbb{R}^n)} \lesssim \|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\|_{L^{\varphi}(\mathbb{R}^n)} \lesssim \|f\|_{H^{\varphi}(\mathbb{R}^n)}.
$$

This finishes the proof of Theorem 1.8. $\square$

Proof of Theorem 1.11. Let b satisfy (1.4) with f replaced by b, and let μ_b, defined by setting, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$
d\mu_b(x,t) := \left[\widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)\right]^2 \frac{dx\,dt}{t},
$$

be a φ-Carleson measure on $\mathbb{R}^{n+1}_+$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ as in the proof of Theorem 1.6, let

$$
d\mu_{b,0}(x,t) := |\phi_t * b(x)|^2 \frac{dx\,dt}{t}.
$$

Then, by [26, Theorem 4.2], we know that

$$
\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \lesssim \|\mu_{b,0}\|_{\varphi} \lesssim \|\mu_b\|_{\varphi},
$$

which completes the proof of the sufficiency of Theorem 1.11.

It remains to prove the necessity. Let $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ and $B_0 := B(x_0, r) \subset \mathbb{R}^n$, with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$. If $b \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$, it is well known that b satisfies (1.4) with f replaced by b for any $\epsilon \in (0, \infty)$; see, for example, [16]. Then,

$$b = P^s_{B_0} b + (b - P^s_{B_0} b)\chi_{2B_0} + (b - P^s_{B_0} b)\chi_{\mathbb{R}^n \setminus 2B_0} =: b_1 + b_2 + b_3. \tag{3.13}$$

For b_1, since $\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$ for any $\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, $\gamma \in \mathbb{Z}^n_+$ and $|\gamma| \le s$, we see that $\widetilde{A}_{(\alpha,\epsilon),s}(b_1) \equiv 0$ and hence

$$\int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_1)(x,t)]^2 \frac{t^n}{\varphi(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})} \frac{dx\,dt}{t} = 0. \tag{3.14}$$

For b_2, by Hölder's inequality, for all balls $B \subset \mathbb{R}^n$ and $\theta \in (0, \infty)$, we know that

$$|B| = \int_B [\varphi(x,\theta)]^{1/2} [\varphi(x,\theta)]^{-1/2} \, dx \le [\varphi(B,\theta)]^{1/2} [\varphi^{-1}(B,\theta)]^{1/2}, \tag{3.15}$$

where above and in what follows, for any measurable set $E \subset \mathbb{R}^n$ and $\theta \in (0, \infty)$, we let

$$\varphi^{-1}(E,\theta) := \int_E [\varphi(x,\theta)]^{-1} \, dx.$$

From (3.15), it follows that

$$\begin{aligned} &\int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)]^2 \frac{t^n}{\varphi(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})} \frac{dx\,dt}{t} \\ &\quad \lesssim \int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)]^2 \int_{B(x,t)} [\varphi(y, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})]^{-1} \, dy \frac{dx\,dt}{t^{n+1}} \\ &\quad \lesssim \int_B [\varphi(y, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})]^{-1} \\ &\qquad \times \int_{\{(x,t) \in \mathbb{R}^{n+1}_+ :\ |x-y|<t\}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)]^2 \frac{dx\,dt}{t^{n+1}} \, dy. \end{aligned} \tag{3.16}$$

Since $\varphi \in \mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_2(\mathbb{R}^n)$, it follows that

$$\left[\varphi\left(\cdot, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{-1} \in A_2(\mathbb{R}^n)$$

(the class of Muckenhoupt weights). By this, (3.16), Proposition 3.2 and [26, Theorem 2.7], we have

$$\begin{aligned} &\int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)]^2 \frac{t^n}{\varphi(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})} \frac{dx\,dt}{t} \\ &\quad \lesssim \int_{\mathbb{R}^n} |b_2(y)|^2 \left[\varphi\left(y, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{-1} dy \\ &\quad \sim \int_{2B_0} |b(y) - P^s_{B_0} b(y)|^2 \left[\varphi\left(y, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{-1} dy \\ &\quad \lesssim \int_{2B_0} \left[|b(y) - P^s_{2B_0} b(y)|^2 + |P^s_{2B_0} b(y) - P^s_{B_0} b(y)|^2\right] \\ &\qquad \times \left[\varphi\left(y, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\right]^{-1} dy \\ &\quad \lesssim \|\chi_{B_0}\|^2_{L^\varphi(\mathbb{R}^n)} \|b\|^2_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}, \end{aligned} \tag{3.17}$$

where the last inequality is deduced from $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$, $\varphi(2B_0, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}) \sim 1$ and, for $y \in 2B_0$,

$$\begin{aligned}|P^s_{2B_0}b(y) - P^s_{B_0}b(y)| &= |P^s_{B_0}(b - P^s_{2B_0}b)(y)| \\ &\lesssim \frac{1}{|B_0|}\int_{2B_0} |b(x) - P^s_{2B_0}b(x)|\,dx \\ &\lesssim \frac{\|\chi_{2B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}.\end{aligned}$$

Now, for b_3, by (1.3) and [26, Theorem 2.7], we conclude that, for all $(x,t) \in \widehat{B_0}$,

$$\widetilde{A}_{(\alpha,\epsilon),s}(b_3)(x,t) \lesssim \int_{(2B_0)^\complement} \frac{t^\epsilon|b(y) - P^s_{B_0}b(y)|}{|y-x|^{n+\epsilon}}\,dy \lesssim \frac{t^\epsilon}{r^\epsilon}\frac{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}{|B_0|}\|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)},$$

which, together with (3.15), $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$ and $\varphi(B_0, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}) = 1$, implies that

$$\begin{aligned}&\int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_3)(x,t)]^2 \frac{t^n}{\varphi(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\frac{dx\,dt}{t} \\ &\quad\lesssim \int_{\widehat{B_0}} \frac{t^{2\epsilon}}{r^{2\epsilon}}\varphi^{-1}\left(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)}\right)\frac{dx\,dt}{t^{n+1}}\frac{\|\chi_{B_0}\|^2_{L^\varphi(\mathbb{R}^n)}}{|B_0|^2}\|b\|^2_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \\ &\quad\lesssim \int_0^r \frac{t^{2\epsilon}}{r^{2\epsilon}}\frac{dt}{t^{n+1}}\frac{\varphi^{-1}(B_0, \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}{|B_0|}\|\chi_{B_0}\|^2_{L^\varphi(\mathbb{R}^n)}\|b\|^2_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \\ &\quad\lesssim \|\chi_{B_0}\|^2_{L^\varphi(\mathbb{R}^n)}\|b\|^2_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}.\end{aligned}$$

From this, (3.13), (3.14) and (3.17), we deduce that

$$\begin{aligned}&\frac{1}{\|\chi_{B_0}\|_{L^\varphi(\mathbb{R}^n)}}\left\{\int_{\widehat{B_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)]^2 \frac{t^n}{\varphi(B(x,t), \|\chi_{B_0}\|^{-1}_{L^\varphi(\mathbb{R}^n)})}\frac{dx\,dt}{t}\right\}^{1/2} \\ &\quad\lesssim \|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)},\end{aligned}$$

which, combined with the arbitrariness of $B_0 \subset \mathbb{R}^n$, implies that μ_b, defined by setting, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$d\mu_b(x,t) := [\widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)]^2\,\frac{dx\,dt}{t},$$

is a φ-Carleson measure on $\mathbb{R}^{n+1}_+$ and

$$\|\mu_b\|_\varphi \lesssim \|b\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.11. □

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