TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 5, May 2015, Pages 3447–3480
S 0002-9947(2014)06195-3
Article electronically published on November 20, 2014

A NEW CONSTRUCTION OF MOUFANG QUADRANGLES OF TYPE E_6, E_7 AND E_8

LIEN BOELAERT AND TOM DE MEDTS

ABSTRACT. In the classification of Moufang polygons by J. Tits and R. Weiss, the most intricate case is by far the case of the exceptional Moufang quadrangles of type E_6, E_7 and E_8, and in fact, the construction that they present is ad-hoc and lacking a deeper explanation. We will show how tensor products of two composition algebras can be used to construct these Moufang quadrangles in characteristic different from 2.

As a byproduct, we will obtain a method to construct *any* Moufang quadrangle in characteristic different from 2 from a module for a Jordan algebra.

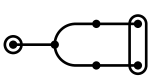
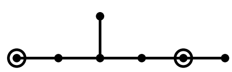
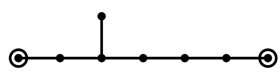
1. INTRODUCTION

In the late sixties, Jacques Tits introduced an (at that time) innovative tool to study semisimple linear algebraic groups of positive relative rank, namely the theory of spherical buildings. Especially in the case of the exceptional groups, these buildings are quite often a main effective tool, and the algebraic description that goes with them is invaluable in order to perform explicit calculations involving exceptional groups.

The cases where the relative rank is at least two are relatively well understood, mainly because of the work of Jacques Tits and Richard Weiss in the theory of Moufang polygons [TW]. However, for the Moufang quadrangles of exceptional type, mainly those of type E_6, E_7 and E_8, the construction is rather ad-hoc, and a deeper explanation is still missing.

The goal of our paper is to give an explicit but at the same time completely intrinsic method to construct a family of rank two groups corresponding to the Moufang quadrangles of type E_6, E_7 and E_8 in characteristic different from 2.

We will be dealing with the forms given by the following Tits indices:

${}^2E_6$:  E_7 :  E_8 : 

Observe that all three forms have the property that their anisotropic kernel is of type D_n (with an additional factor A_1 for the form of type E_7); this implies that these forms will be determined by an anisotropic quadratic form of dimension $2n$ having certain additional properties. (The A_1 factor in the E_7 case gives rise to a quaternion division algebra, which turns up in the description of the Hasse

Received by the editors January 25, 2013 and, in revised form, May 14, 2013.

2010 *Mathematics Subject Classification.* Primary 17A75, 17A40, 17C40, 20G15, 20G41; Secondary 17C27, 51E12.

Key words and phrases. Moufang polygons, Moufang quadrangles, composition algebras, octonion algebras, quadrangular algebras, Jordan algebras, structurable algebras, J-ternary algebras, linear algebraic groups, exceptional groups, E_6, E_7, E_8.

©2014 American Mathematical Society
Reverts to public domain 28 years from publication

invariant of the quadratic form corresponding to the D_n factor.) We will refer to such quadratic forms as forms of type E_6, E_7 and E_8, respectively.

In each case, we will see that the quadratic form can be characterized as the anisotropic part of the *Albert form* of a certain tensor product of composition algebras, and in fact, these algebras themselves will play a crucial role in the understanding of the corresponding algebraic groups; they completely determine the algebraic group up to isogeny.

Our approach will turn out to be applicable in a more general situation, and in fact, we will obtain *every possible* Moufang quadrangle defined over a field of characteristic different from two starting from certain modules over a Jordan algebra. Our construction relies in an essential way on the theory of J-ternary algebras and their Peirce decomposition; see [ABG, Sections 3.12 and 6.61].

We can summarize our main result, namely the explicit construction of the quadrangular algebras of type E_6, E_7 and E_8, as follows.

Construction. Let $\mathrm{char}(k) \neq 2$. We start with a quadratic space (k, V, q) of type E_6, E_7 or E_8 with base point (see also Definition 2.15 below).

By Theorem 3.11, there exist an octonion division algebra C_1 and a composition division algebra C_2 of dimension 2, 4 or 8, respectively, such that C_1 and C_2 contain an isomorphic quadratic field extension, but no isomorphic quaternion algebra, and such that q is similar to the anisotropic part of the Albert form, q_A, of $C_1 \otimes_k C_2$. It follows that there exist $\mathbf{i}_1 \in C_1$ and $\mathbf{i}_2 \in C_2$ such that $\mathbf{i}_1^2 = \mathbf{i}_2^2 = a \in k \setminus \{k^2\}$.

We define a subspace V of the space $\mathcal{S}$ of skew-elements of $C_1 \otimes_k C_2$ of dimension 6, 8 or 12, respectively, as[1]

$$V := \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^{\perp}.$$

We choose an arbitrary $u \in V \setminus \{0\}$ and define the quadratic form

$$Q := \frac{1}{q_A(u)} q_A|_V;$$

this form has base point u and is similar to the quadratic form of type E_6, E_7 or E_8 we started with.

We then define the subspace X_0 of $C_1 \otimes_k C_2$ of dimension 8, 16 or 32 as

$$X_0 := \Big\langle \big(x \otimes y + \tfrac{1}{a}\, \mathbf{i}_1 x \otimes \mathbf{i}_2 y\big) \mid x \in C_1, y \in C_2 \Big\rangle.$$

Next, we define a suitable element $r \in \mathcal{S}$ as in Definition 3.16(iii) below, and we define the bilinear map $X_0 \times L_0 \to X_0$ as

$$x \cdot v = v(r(u(rx))),$$

and the bilinear map $h\colon X_0 \times X_0 \to V$ as

$$h(x, y) = (u(rx))\overline{y} - y((\overline{x}r)u).$$

In Theorem 3.20 we prove that the 7-tuple $(k, V, Q, u, X_0, \cdot, h)$ is a quadrangular algebra of type E_6, E_7 or E_8, respectively. It follows that this is the structure described in [TW, Chapter 13] giving rise to the Moufang quadrangles of type E_6, E_7 and E_8, and hence to the corresponding rank two forms of exceptional linear algebraic groups of type E_6, E_7 and E_8.

[1]The orthogonal complement is taken w.r.t. the bilinear form associated to the Albert form; this quadratic form is defined on the skew elements of $C_1 \otimes_k C_2$.

Organization of the paper. In Section 2 we give some preliminary material on composition algebras, tensor products of composition algebras, quadrangular algebras and Peirce decomposition in Jordan algebras.

In Section 3 we work towards the main theorem of our paper, which is Theorem 3.20 and which gives a construction of quadrangular algebras of type E_6, E_7 and E_8.

In Section 3.1 we show that we can construct a quadrangular algebra in characteristic not 2 starting from a specific kind of module for a Jordan algebra of reduced spin type (see Definition 2.20).

In Section 3.2 we construct, in a similar way, a quadrangular algebra starting from a J-ternary algebra over a Jordan algebra of reduced spin type. This section deals only with fields of characteristic not 2 nor 3, since J-ternary algebras are only defined over such fields.

In Section 3.3 we show that we can apply this procedure to obtain each quadrangular algebra of pseudo-quadratic form type in characteristic not 2. In Section 3.4 we construct (see the construction above) a module for a Jordan algebra of reduced spin type out of the tensor product of two composition algebras and show that this gives rise to quadrangular algebras of type E_6, E_7 and E_8 in characteristic not 2.

In Section 4, inspired by Theorem 3.5, we give a uniform description of all Moufang quadrangles in characteristic different from 2.

2. Preliminaries

We assume throughout the paper that k is a commutative field of characteristic different from 2.

2.1. **Composition algebras.** A *composition algebra* is a, not necessarily commutative nor associative, unital k-algebra C equipped with a quadratic form $q\colon C \to k$ that is multiplicative, i.e. $q(xy) = q(x)q(y)$ for all $x, y \in C$. This quadratic form q is called the *norm form*; its associated bilinear form will be denoted by f. With the norm form we associate an involution on C by defining

$$\sigma\colon C \to C\colon x \mapsto \overline{x} := f(x,1)1 - x.$$

By a classical result (see for example [SV, Theorem 1.6.2]) each composition algebra has dimension 1, 2, 4 or 8:

(i) If $\dim_k C = 1$, then $C = k$, $q(x) = x^2$ and the involution is trivial.

(ii) If $\dim_k C = 2$, then C/k is a *quadratic étale extension* of k. There exists $a \in k$ such that $C = k[\mathbf{i}]/(\mathbf{i}^2 - a)$; the norm form is $\langle 1, -a\rangle$. Either C/k is a separable quadratic field extension and σ is the non-trivial element of $\mathrm{Gal}(C/k)$ or $C \cong k \oplus k$ and σ interchanges the two components.

(iii) If $\dim_k C = 4$, then C/k is a *quaternion algebra* over k. There exist $a, b \in k$ such that $C = k \oplus k\mathbf{i} \oplus k\mathbf{j} \oplus k(\mathbf{ij})$ with multiplication defined by

$$\mathbf{i}^2 = a, \mathbf{j}^2 = b, \mathbf{ij} = -\mathbf{ji}.$$

This quaternion algebra is denoted by $(a,b)_k$. The norm form is equal to $\langle 1, -a\rangle\langle 1, -b\rangle$; the involution fixes k and maps $\mathbf{i} \mapsto -\mathbf{i}$, $\mathbf{j} \mapsto -\mathbf{j}$.

(iv) If $\dim_k C = 8$, then C/k is an *octonion algebra* over k. There exist $a, b, c \in k$ such that $C = Q \oplus Q\mathbf{k}$ where $Q = (a,b)_k$ and multiplication is given by

$$(x_1 + x_2\mathbf{k})(y_1 + y_2\mathbf{k}) = (x_1y_1 + c\overline{y_2}x_2) + (y_2x_1 + x_2\overline{y_1})\mathbf{k} \quad \text{for all } x_i, y_i \in Q.$$

The norm form is $\langle 1,-a\rangle\langle 1,-b\rangle\langle 1,-c\rangle$ and the involution is given by $\overline{x_1+x_2\mathbf{k}} = \overline{x_1} - x_2\mathbf{k}$ for all $x_1, x_2 \in Q$.

In each case, the norm form is a Pfister form; these are forms of dimension 2^n denoted by $\langle\langle a_1,\dots,a_n\rangle\rangle := \bigotimes_{i=1}^n \langle 1, a_i\rangle$ for $a_1,\dots,a_n \in k$. The norm form is anisotropic when C is a division algebra, and it is hyperbolic otherwise (i.e. when C is a split algebra).

The norm form is completely determined by the algebra structure of the composition algebra. It is a well known but somewhat deeper fact (see e.g. [SV]) that the converse also holds; i.e. the composition algebra is determined up to isomorphism by the (similarity class of) the norm.

Quaternion algebras are not commutative, but associative. Octonion algebras are neither commutative nor associative. In the lemma below we summarize some useful identities that hold in each composition algebra.

Lemma 2.1 ([SV, Lemmas 1.3.2, 1.3.3 and 1.4.1])**.** *Let C be an arbitrary composition algebra with norm q, with associated bilinear form f, and involution denoted by $x \mapsto \overline{x}$. Then for all $x, y, z \in C$ we have*

(i) $x^2 - f(x,e)x + q(x)e = 0$,
(ii) $f(xy,z) = f(y,\overline{x}z)$, $\quad f(xy,z) = f(z, y\overline{x})$, $\quad f(xy,z) = f(y\overline{z}, \overline{x})$,
(iii) *each subalgebra generated by two elements is associative,*
(iv) $x(\overline{x}y) = q(x)y$, $\quad (x\overline{y})y = q(y)x$,
(v) $(zx)(yz) = z((xy)z)$, $\quad z(x(zy)) = (z(xz))y$, $\quad x(z(yz)) = ((xz)y)z$.

Property (iii) is called the alternativity; the identities in (v) are called the *Moufang identities*.

2.2. **Tensor products of composition algebras.** We now assume that C_1 and C_2 are two composition algebras over k (possibly of different dimension), with norm forms q_1 and q_2 and involutions σ_1 and σ_2, respectively. Consider

$$C_1 \otimes_k C_2,$$

equipped with the involution

$$\sigma := \sigma_1 \otimes \sigma_2.$$

If $\mathrm{char}(k) \neq 2, 3$ the algebra $(C_1 \otimes_k C_2, \sigma_1 \otimes \sigma_2)$ is a *structurable algebra*. This is a class of algebras that generalizes Jordan algebras and associative algebras with involution. We will not need the exact definition and refer the interested reader to [A1].

Let $\mathcal{S}_i$ be the set of skew elements in C_i, i.e.

$$\mathcal{S}_i = \{x \in C_i \mid \overline{x} := x^{\sigma_i} = -x\},$$

and similarly, let $\mathcal{S}$ be the set of skew elements of $C_1 \otimes_k C_2$, i.e.

$$\mathcal{S} = \{x \in C_1 \otimes_k C_2 \mid \overline{x} := x^{\sigma} = -x\} = (\mathcal{S}_1 \otimes 1) \oplus (1 \otimes \mathcal{S}_2);$$

observe that $\dim_k \mathcal{S} = \dim_k C_1 + \dim_k C_2 - 2$.

Definition 2.2. We will associate a quadratic form q_A to $C_1 \otimes_k C_2$, called the *Albert form*, by setting

$$q_A \colon \mathcal{S} \to k \colon (x \otimes 1) + (1 \otimes y) \mapsto q_1(x) - q_2(y)$$

for all $x \in \mathcal{S}_1$ and $y \in \mathcal{S}_2$. When we denote $q_i' = q_i|_{\mathcal{S}_i}$ for the pure part of the Pfister form q_i, we have $q_A = q_1' \perp \langle -1\rangle q_2'$.

This form is named after A.A. Albert, who studied the case where C_1 and C_2 are both quaternion algebras; i.e. $C_1 \otimes_k C_2$ is a biquaternion algebra.

Definition 2.3. Let $s = s_1 \otimes 1 + 1 \otimes s_2 \in \mathcal{S}$. We define the map $\natural \colon \mathcal{S} \to \mathcal{S}$ by

$$(s_1 \otimes 1 + 1 \otimes s_2)^\natural = s_1 \otimes 1 - 1 \otimes s_2.$$

If $q_A(s) \neq 0$, the *inverse* of s is defined by

$$s^{-1} := -\frac{1}{q_A(s)} s^\natural.$$

Tensor products of two composition algebras are far from associative or alternative, but the skew elements behave nicer than arbitrary elements:

Lemma 2.4. *For all $x \in C_1 \otimes_k C_2$, $s_1, s_2, s \in \mathcal{S}$ we have that*

(i) $s_1(s_2 s_1) = (s_1 s_2) s_1$.
(ii) $(s_1 s_2 s_1)x = s_1(s_2(s_1 x))$.
(iii) *If s is invertible, then $s(s^{-1}x) = x$.*

Proof. These identities can be easily checked using Lemma 2.1. □

Remark 2.5. In the case that both C_1 and C_2 are quaternion algebras, $C_1 \otimes_k C_2$ is associative, and A.A. Albert proved that $C_1 \otimes_k C_2$ is a division algebra if and only if its Albert form is anisotropic (see [L, Theorem III.4.8].)

It is not obvious to generalize this result to arbitrary composition algebras. (Notice that in the theory of structurable algebras, the concept of conjugate invertibility is used.) In [A2, Theorem 5.1] it is proven in the case that $\operatorname{char}(k) = 0$ that the tensor product of two octonion algebras is a conjugate division algebra if and only if the corresponding Albert form is anisotropic. To the best of our knowledge, it is an open problem whether this equivalence also holds for fields of characteristic > 3.

The case where q_A has Witt index one will be needed to study the rank 2 forms of linear algebraic groups of type E_6, E_7, E_8 discussed in the introduction.

Definition 2.6 ([L, Definition 5.11])**.** Two n-fold Pfister forms q_1, q_2 are r-linked if there is an r-fold Pfister form h such that $q_1 \simeq h \otimes q_3$ and $q_2 \simeq h \otimes q_4$ for some Pfister forms q_3, q_4.

The *linkage number* of q_1 and q_2 is the number $r \in \mathbb{N}$ such that q_1 and q_2 are r-linked but not $(r+1)$-linked.

Lemma 2.7. *Let C_1 be an octonion division algebra with norm q_1 and let C_2 be a separable quadratic field extension, quaternion division algebra or an octonion division algebra with norm q_2. The following are equivalent:*

(i) *C_1 and C_2 contain isomorphic separable quadratic field extensions, but C_1 and C_2 do not contain isomorphic quaternion algebras.*
(ii) *The linkage number of q_1 and q_2 is* 1*; i.e. q_1 and q_2 are* 1*-linked but not* 2*-linked.*
(iii) *The Witt index of the Albert form q_A of $C_1 \otimes_k C_2$ is equal to* 1*.*

Proof. Since the Witt index of q_A is one less than the Witt index of $q_1 \perp -q_2$, the equivalence of (ii) and (iii) is given by a result of Elman–Lam (see for example [L, Theorem X.5.13]).

The following observations follow from [SV, Prop. 1.5.1]. Let C be a composition algebra over k with norm q.

Let $\dim(C) = 4$ or 8. Then C contains a separable extension field isomorphic to $k(\mathbf{i})/(\mathbf{i}^2 - a)$ with $a \in k$ if and only if there exists a Pfister form φ, of dimension 2 or 4 respectively, such that $q \simeq \langle\langle -a \rangle\rangle \otimes \varphi$.

Let $\dim(C) = 8$. Then C contains a quaternion algebra isomorphic to $(a, b)_k$ with $a, b \in k$ if and only if $q \simeq \langle\langle -a, -b, -c \rangle\rangle$ for some $c \in k$.

From this it follows immediately that (i) and (ii) are equivalent. □

Remark 2.8. Suppose that C_1 and C_2 contain isomorphic separable quadratic field extensions. Even if they do not contain isomorphic quaternion algebras, it is still possible that C_1 and C_2 contain more than one isomorphic separable quadratic field extension up to isomorphism.

Definition 2.9. We define the *linkage number* of C_1 and C_2 as the linkage number of their norm forms q_1 and q_2.

Lemma 2.7 indicates that we will be particularly interested in pairs of composition algebras C_1, C_2 with linkage number 1.

2.3. **Quadrangular algebras.** A quadrangular algebra is an algebraic structure that was constructed to describe the exceptional Moufang quadrangles. In Section 4 we explain how one constructs Moufang quadrangles out of quadrangular algebras. For more information on quadrangular algebras, including characteristic 2, we refer to [W]. We emphasize that the structure of a quadrangular algebra simplifies significantly in characteristic different from 2; see Remark 2.11. Since this is the only case we will be dealing with, we restrict our definition to $\mathrm{char}(k) \neq 2$.

Definition 2.10. A *quadrangular algebra*, in characteristic different from 2, is a 7-tuple $(k, L, q, 1, X, \cdot, h)$, where

(i) k is a commutative field with $\mathrm{char}(k) \neq 2$,
(ii) L is a k-vector space,
(iii) q is an anisotropic quadratic form from L to k,
(iv) $1 \in L$ is a *base point* for q, i.e. an element such that $q(1) = 1$,
(v) X is a non-trivial k-vector space,
(vi) $(x, v) \mapsto x \cdot v$ is a map from $X \times L$ to X (usually denoted simply by juxtaposition),
(vii) h is a map from $X \times X$ to L, satisfying the following axioms, where

$$f\colon L \times L \to k\colon (x, y) \mapsto f(x, y) := q(x + y) - q(x) - q(y)\,;$$
$$\sigma\colon L \to L\colon v \mapsto f(1, v)1 - v\,;$$
$$v^{-1} := v^\sigma / q(v)\,.$$

(A1) The map $\cdot$ is k-bilinear.
(A2) $x \cdot 1 = x$ for all $x \in X$.
(A3) $(xv)v^{-1} = x$ for all $x \in X$ and all $v \in L^*$.
(B1) h is k-bilinear.
(B2) $h(x, yv) = h(y, xv) + f(h(x, y), 1)v$ for all $x, y \in X$ and all $v \in L$.
(B3) $f(h(xv, y), 1) = f(h(x, y), v)$ for all $x, y \in X$ and all $v \in L$.
(C) $\theta(x, v) := \frac{1}{2} h(x, xv)$.
(D1) Let $\pi(x) = \theta(x, 1)$ for all $x \in X$. Then $x\theta(x, v) = (x\pi(x))v$ for all $x \in X$ and all $v \in L$.
(D2) For all $x \in X \setminus \{0\}$ we have $\pi(x) \neq 0$.

Moreover, we define a map $g\colon X \times X \to k$ by

$$g(x, y) := \tfrac{1}{2} f(h(x, y), 1)$$

for all $x, y \in X$.

Remark 2.11. When one compares our definition of quadrangular algebras with the general definition in [W, Definition 1.17] there are two differences which are due to the fact that the definition simplifies when the characteristic is different from 2.

(i) The axiom (C) in [W] consists of four more involved axioms (see [W, Remark 4.8]). By assuming $\theta(x, v) = \frac{1}{2} h(x, xv)$ we actually assume that the quadrangular algebra is standard. Every quadrangular algebra is equivalent to a standard quadrangular algebra (see [W, Proposition 4.2].)

(ii) In [W], axiom (D2) says $\pi(x) \equiv 0 \pmod{k}$ if and only if $x = 0$ (where k has been identified with its image under the map $t \mapsto t \cdot 1$ from k to L). We show that this is equivalent to our axiom (D2).

Assume the above (D2) holds. Applying (B2) with $x = y$, $v = 1$ we get $f(h(x, x), 1) = 0$. If we suppose $\pi(x) = \frac{1}{2} h(x, x) \in k1$, we have $f(h(x, x), 1) = 2h(x, x) = 0$ and it follows that $\pi(x) = 0$, so $x = 0$.

Theorem 2.12. *A quadrangular algebra in characteristic not* 2 *is either obtained from an anisotropic pseudo-quadratic space over a quadratic pair (see Section* 2.3.1*) or is of type E_6, E_7 or E_8 (see Section* 2.3.2*).*

Proof. Since the characteristic of k is not 2, it follows from [W, 2.3 and 2.4] that the quadrangular algebra is regular; i.e. f is non-degenerate (from [W, 3.14] it follows that it is also proper, i.e. $\sigma \neq 1$). Now it follows from [W, 3.2] that if the quadrangular algebra is not special (i.e. not arising from a pseudo-quadratic space) it is of type E_6, E_7 or E_8. □

2.3.1. *Pseudo-quadratic spaces.*

Definition 2.13 ([W, Definition 1.16])**.** A *pseudo-quadratic space* over a field of characteristic not 2 is a quintuple (L, σ, X, h, π) where

(i) L is a skew field of characteristic different from 2;

(ii) σ is an involution of L, and we let

$$L_\sigma := \{\ell \in L \mid \ell^\sigma = \ell\} = \{\ell + \ell^\sigma \mid \ell \in L\};$$

(iii) X is a right vector space over L;

(iv) $h\colon X \times X \to L$ is a *skew-hermitian form*, i.e.
- h is bi-additive and $h(x, yu) = h(x, y)u$, and
- $h(x, y)^\sigma = -h(y, x)$,

for all $x, y \in X$ and all $u \in L$;

(v) π is a *pseudo-quadratic form* from X to L, i.e.
- $\pi(x + y) \equiv \pi(x) + \pi(y) + h(x, y) \mod L_\sigma$ and
- $\pi(xu) \equiv u^\sigma \pi(x) u \mod L_\sigma$,

for all $x, y \in X$ and all $u \in L$. Since we work in characteristic not 2 we can always assume that the pseudo-quadratic space is standard, i.e. $\pi(x) = \frac{1}{2} h(x, x)$.

A pseudo-quadratic space (L, σ, X, h, π) is called *anisotropic* if

$$\pi(x) \equiv 0 \mod L_\sigma \text{ only if } x = 0.$$

Not every pseudo-quadratic space is a quadrangular algebra; to be a quadrangular algebra the skew-field has to satisfy some additional properties.

Definition 2.14 ([W, Definition 1.12])**.** Let L be a skew-field with involution σ. We call (L, σ) a *quadratic pair*[2] if $k := L_\sigma$ is a field and if either

(i) L/k is a separable quadratic field extension and σ is the generator of the Galois group or
(ii) L is a quaternion algebra over k and σ is the standard involution.

Define $q(u) = uu^\sigma$; then $(k, L, q, 1)$ is a pointed anisotropic non-degenerate quadratic space.

A result of Dieudonné (see for example [W, Theorem 1.15]) says that if σ is not trivial, then either L is generated by L_σ as a ring or (L, σ) is a quadratic pair (in this case L_σ is a field). From this point of view quadratic pairs are an exceptional class of skew-fields with involution.

In [W, Proposition 1.18] it is shown that a non-zero standard anisotropic pseudo-quadratic space over a quadratic pair gives rise to the quadrangular algebra

$$(k, L, q, 1, X, \text{ scalar multiplication}, h).$$

2.3.2. *Quadrangular algebras of type E_6, E_7 and E_8.* For an explicit description of quadrangular algebras of type E_6, E_7 and E_8, we refer to [TW, Chapters 12 and 13]; for a concise description we refer to the first part of [W, Chapter 10]. Some care is needed, since the map g in [TW] is equal to $-g$ in [W]. Here we give only a concise overview of the structure of a quadrangular algebra of type E_6, E_7 or E_8.

Definition 2.15. A *quadratic space* (k, L, q) with base point is of *type E_6, E_7 or E_8* if it is anisotropic and there exists a separable quadratic field extension E/k, with norm denoted by N, such that:

E_6 : there exist $s_2, s_3 \in k^*$ such that
$$(k, L, q) \cong (k, E^3, N \otimes \langle 1, s_2, s_3\rangle);$$
E_7 : there exist $s_2, s_3, s_4 \in k^*$ such that $s_2s_3s_4 \notin N(E)$ and
$$(k, L, q) \cong (k, E^4, N \otimes \langle 1, s_2, s_3, s_4\rangle);$$
E_8 : there exist $s_2, s_3, s_4, s_5, s_6 \in k^*$ such that $-s_2s_3s_4s_5s_6 \in N(E)$ and
$$(k, L, q) \cong (k, E^6, N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6\rangle).$$
We always assume that $s_2s_3s_4s_5s_6 = -1$, which can be achieved by rescaling the quadratic form if necessary.

As we are working in characteristic not 2, we can choose $\gamma \in E$ such that $E = k(\gamma)$ and $\gamma^2 \in k$.

It is shown in [TW, (12.37)] that if

$$(k, E^6, N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6\rangle)$$

is a quadratic space of type E_8, then $(k, E^4, N \otimes \langle 1, s_2, s_3, s_4\rangle)$ is a quadratic space of type E_7 and $(k, E^3, N \otimes \langle 1, s_2, s_3\rangle)$ is a quadratic space of type E_6.

If (k, L, q) is a quadratic space of type E_6, E_7 or E_8 with base point, there exists a scalar multiplication $E \times L \to L$ that extends the scalar multiplication $k \times L \to L$.

[2]This notion, taken from [W, Definition 1.12], is quite different from the notion of a quadratic pair as defined in *The Book of Involutions* [KMRT] and has nothing to do with the notion of a quadratic pair in (finite) group theory either.

Let $(k, L, q, 1, X, \cdot, h)$ be a quadrangular algebra of type E_6, E_7 or E_8. Then (k, L, q) is a quadratic space of type E_6, E_7 or E_8, respectively, with base point denoted by 1. This quadratic space determines the quadrangular algebra entirely (see [W, Theorem 6.24]).

The vector space X has k-dimension 8, 16 or 32, respectively; it is a $C(q, 1)$-module (see Definition 2.16 below). Some of the properties of the maps $\cdot, h, \theta$ and π are given in Definition 2.10. The existence of the vector space X and of the maps $\cdot$, h and θ is shown in [TW, Chapter 13] by giving an explicit ad-hoc construction using the coordinatization of L.

The goal of this article is to provide an alternative description of X, L and the maps $\cdot, h$ starting from the tensor product of composition algebras.

In order to prove the anisotropy of our new construction of the map π (see Theorem 3.20), we need the concept of an irreducible $C(q, 1)$-module.

Definition 2.16. (i) Let (k, V, q) be a quadratic space with base point 1. Then the *Clifford algebra of q with base point* 1 is defined as

$$C(q, 1) := T(V)/\langle u \otimes u^\sigma - q(u) \cdot 1\rangle,$$

where $T(V)$ is the tensor algebra of V, and where σ is defined as in Definition 2.10. It is shown in [TW, 12.51] that $C(q, 1) \cong C_0(q)$, the even Clifford algebra of q. The notion of a Clifford algebra with base point was introduced by Jacobson and McCrimmon; see [TW, Chapter 12] for more details.

(ii) Since q is anisotropic, the axioms (A1)–(A3) of an arbitrary quadrangular algebra say precisely that X is a $C(q, 1)$-module such that the action of $C(q, 1)$ on X is an extension of the action of L on X (see [W, Proposition 2.22]).

(iii) A $C(q, 1)$-module X is *irreducible* if we have $x \cdot C(q, 1) = X$ for all $x \in X \setminus \{0\}$.

Theorem 2.17 ([W, 2.26])**.** *Let (k, V, q) be a quadratic space of type E_6, E_7 or E_8 with base point* 1 *and let X be a right $C(q, 1)$-module. Then X is an irreducible right $C(q, 1)$-module if and only if* $\dim_k(X) = 8$*,* 16 *or* 32*, respectively.*

2.4. **Peirce decomposition in Jordan algebras.** A good reference to study the theory of Jordan algebras is [M]. Our construction of exceptional quadrangular algebras uses the Peirce decomposition of a Jordan algebra. We summarize the main properties and multiplication rules of Peirce subspaces.

Definition 2.18. A *Jordan k-algebra J* is a unital commutative k-algebra such that for all $x, y \in J$ we have $(x^2y)x = x^2(yx)$.

We define the *U-operator* and its linearization for $x, y, z \in J$:

$$U_x y := 2x(xy) - x^2 y, \quad U_{x,z} y := (U_{x+z} - U_x - U_z)y.$$

An element $x \in J$ is *invertible* if and only if there exists a $y \in J$ such that $xy = 1$ and $x^2y = x$; this condition is equivalent to $U_x y = x, U_x y^2 = 1$. The element y is the *inverse* of x.

Definition 2.19 ([M, II.8.1 and II.8.2 on p. 235])**.** Let J be a Jordan k-algebra.

(i) An element $e \in J$ is an *idempotent* if $e^2 = e$. An idempotent is *proper* if it is different from 0 and 1. If e is an idempotent, then $1 - e$ is also an idempotent. Two idempotents e, e' are *supplementary* if $e + e' = 1$. Observe that two supplementary idempotents are always orthogonal, i.e. $ee' = 0$.

(ii) Let e be a proper idempotent in J. The *Peirce decomposition* with respect to e is defined as follows. For each $i \in \mathbb{Z}\big[\frac{1}{2}\big]$, let

$$J_i = \{x \in J \mid ex = ix\};$$

then we have

$$J = J_0 \oplus J_{1/2} \oplus J_1;$$

in particular, $J_\ell = 0$ if $\ell \notin \{0, \frac{1}{2}, 1\}$. For a non-degenerate Jordan algebra we have $J_0 \neq 0$ (see [M, II.10.1.2]). Let $i \in \{0,1\}$ and $j = 1 - i$; then

$$J_i^2 \subseteq J_i, \quad J_i J_{1/2} \subseteq J_{1/2}, \quad J_{1/2}^2 \subseteq J_0 + J_1, \quad J_i J_j = 0.$$

For all $\ell, m \in \{0, \frac{1}{2}, 1\}$, we have

$$U_{J_m} J_\ell \subseteq J_{2m-\ell}. \tag{2.1}$$

This implies that $U_{J_m} J_\ell = 0$ if $2m - \ell \notin \{0, \frac{1}{2}, 1\}$.

To construct quadrangular algebras we will use two types of Jordan algebras that contain supplementary idempotents. These are the Jordan algebras of reduced spin type and the Jordan algebras $\mathcal{H}(M_2(L), \sigma t)$ for a skew field L with involution σ, where t is the transpose map.

Definition 2.20 ([M, II.3.4 on p. 180])**.** Consider a quadratic form $q \colon V \to k$ over k. Starting from the vector space V we construct a Jordan algebra by adjoining two supplementary idempotents to V.

As a vector space, we define J by adjoining two copies of k to V: $J := ke_0 \oplus V \oplus ke_1$. We define the following multiplication:

$$(t_1 e_i)(t_2 e_j) = \delta_{ij} t_1 t_2 e_i, \tag{2.2}$$

$$(te_i)v = \tfrac{1}{2} tv, \tag{2.3}$$

$$vw = \tfrac{1}{2} f(v,w)(e_0 + e_1), \tag{2.4}$$

for all $i, j \in \{0,1\}, v, w \in V, t, t_1, t_2 \in k$. This defines a Jordan algebra[3] on $ke_0 \oplus V \oplus ke_1$, called the *reduced spin factor of the quadratic form* q. We say that J is of *reduced spin type*.

The unit of this Jordan algebra is $e_0 + e_1$, and for all $v, w \in V$ we have

$$U_v e_0 = q(v) e_1, \quad U_v e_1 = q(v) e_0, \quad U_v w = f(v,w)v - q(v)w.$$

It is clear that e_0 and e_1 are supplementary idempotents and that we have the following Peirce subspaces with respect to e_1:

$$J_0 = ke_0, \quad J_{1/2} = V, \quad J_1 = ke_1.$$

Definition 2.21 ([M, Example II.3.2.4])**.** Let L be a skew-field with involution σ. Define $L_\sigma := \mathrm{Fix}_\sigma(L)$ and $k := Z(L)$. The matrix algebra $M_2(L)$ is associative with involution σt. Now let J be the Jordan k-algebra[4]

$$\mathcal{H}(M_2(L), \sigma t) = \left\{ \begin{bmatrix} \alpha_1 & \ell^\sigma \\ \ell & \alpha_2 \end{bmatrix} \mid \alpha_1, \alpha_2 \in L_\sigma, \ell \in L \right\}.$$

[3]Notice that this is the same Jordan algebra as the Jordan algebra of the quadratic form $Q : ke_0 \oplus V \oplus ke_1 \to k : t_0 e_0 + v + t_1 e_1 \mapsto t_0 t_1 - q(v)$.

[4]This Jordan algebra consists of the fixed points in $M_2(L)$ of the involution σt; multiplication is given by $m.n = \frac{1}{2}(mn + nm)$ (where on the right hand side the usual matrix multiplication is used).

We define the supplementary idempotents $e_0 = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], e_1 = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right] \in J$. With respect to e_1, we have

$$J_0 = L_\sigma e_0, \quad J_1 = L_\sigma e_1 \quad \text{and} \quad J_{1/2} = \left\{ \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \mid \ell \in L \right\}.$$

We have

$$\begin{aligned}(\alpha_1 e_i)(\alpha_2 e_j) &= \delta_{ij} \tfrac{1}{2}(\alpha_1\alpha_2 + \alpha_2\alpha_1)e_i,\\ (\alpha e_0)v &= \tfrac{1}{2}\begin{bmatrix} 0 & \alpha\ell^\sigma \\ \ell\alpha & 0 \end{bmatrix},\\ (\alpha e_1)v &= \tfrac{1}{2}\begin{bmatrix} 0 & \ell^\sigma\alpha \\ \alpha\ell & 0 \end{bmatrix},\\ v_1v_2 &= \tfrac{1}{2}(\ell_1^\sigma\ell_2 + \ell_2^\sigma\ell_1)e_0 + \tfrac{1}{2}(\ell_1\ell_2^\sigma + \ell_2\ell_1^\sigma)e_1,\end{aligned}$$

for all $i, j \in \{0,1\}, v = \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 & \ell_1^\sigma \\ \ell_1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & \ell_2^\sigma \\ \ell_2 & 0 \end{bmatrix} \in J_{1/2}$, $\alpha, \alpha_1, \alpha_2 \in L_\sigma$. For the U-operators we find

$$U_v(\alpha e_0) = (\ell\alpha\ell^\sigma)e_1, \quad U_v(\alpha e_1) = (\ell^\sigma\alpha\ell)e_0, \quad U_{v_1}v_2 = \begin{bmatrix} 0 & \ell_1^\sigma\ell_2\ell_1^\sigma \\ \ell_1\ell_2^\sigma\ell_1 & 0 \end{bmatrix}.$$

Remark 2.22. If we consider the above definition in the case that (L,σ) is a quadratic pair (see Definition 2.14), then $k = L_\sigma$. Now there exists a non-degenerate anisotropic quadratic form $q : L \to k : \ell \mapsto \ell\ell^\sigma = \ell^\sigma\ell$, and the Peirce subspaces J_0, J_1 of $\mathcal{H}(M_2(L), \sigma t)$ are one-dimensional.

Define a quadratic form Q on $J_{1/2} \subseteq \mathcal{H}(M_2(L), \sigma t)$ given by $Q(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = q(\ell)$. By comparing the multiplication in Definition 2.21 and the one in Definition 2.20 we conclude that $\mathcal{H}(M_2(L), \sigma t)$ is the reduced spin factor of the quadratic space $(J_{1/2}, k, Q)$.

In the following proposition we use Osborn's capacity two theorem to show that the two families of Jordan algebras we discussed above can be characterized in a unified way. The proof of this proposition uses some results and concepts of Jordan theory that we will not use in the remaining part of this article.

Proposition 2.23. *Let J be a non-degenerate Jordan k-algebra with supplementary proper idempotents e_0 and e_1. Let $J_0, J_{1/2}, J_1$ be the Peirce subspaces of J with respect to e_1. We assume that each element in $J_{1/2} \setminus \{0\}$ is invertible and that there exists $u \in J_{1/2}$ such that $u^2 = 1$.*

- *If $\dim(J_0) = 1$, then J is the reduced spin factor of some non-degenerate anisotropic quadratic space with base point u.*[5]
- *If $\dim(J_0) > 1$, then J is isomorphic to $\mathcal{H}(M_2(L), \sigma t)$ for some skew-field L with involution σ, such that (L, σ) is not a quadratic pair.*

Proof. We will show that the assumptions imply that J is a simple non-degenerate Jordan algebra of capacity 2. An algebra has capacity 2 if the unit is the sum of two supplementary idempotents e_0, e_1 such that the Peirce subspaces J_0, J_1 are division algebras.

[5]Notice that $\mathcal{H}(M_2(L), \sigma t)$ for (L, σ) a quadratic pair is included in this case.

Since $u^2 = 1$ it follows from [M, II.6.1.10] that U_u is a Jordan isomorphism[6] of J such that $(U_u)^2$ is the identity map. Since $U_u(J_1) \subseteq J_0$, U_u is an isomorphism between J_0 and J_1. Therefore it is enough to show that J_0 is a division algebra. It follows from [M, II.6.1.2] that it is sufficient to show that for each element $t \in J_0 \setminus \{0\}$ the operator U_t is surjective on J_0.

Let $t, s \in J_0 \setminus \{0\}$; using [M, II.8.4.1] one can verify that $U_t s = U_u U_{2ut} s$. We have $2ut \in J_{1/2} \setminus \{0\}$: if $2ut = 0$ it would follow that $U_u(t) = -u^2 t = -t$, which implies that $t \in J_0 \cap J_1 = \{0\}$. It follows that U_{2ut} is invertible. Now let $r \in J_0$. Since $U_{2ut}^{-1} U_u r \in J_0$ we have

$$r = U_t(U_{2ut}^{-1} U_u r),$$

and hence U_t is surjective on J_0.

This proves that J has capacity 2.

We now show that J is simple. From [M, II.20.2.4] a non-degenerate algebra with capacity is simple iff its capacity is connected (i.e. if e_0, e_1 are connected [M, II.10.1.3]). In fact e_0, e_1 are even strongly connected since $u \in J_{1/2}$ is an involution, i.e. $u^2 = 1$.

We proved all the conditions of Osborn's capacity two theorem; see [M, II.22.2.1 on p. 351]. This theorem states that a simple non-degenerate Jordan algebra of capacity 2 belongs to exactly one of the following three disjoint classes from which we can exclude the first:

(i) Full type $M_2(L)^+$, for a non-commutative skew-field L. The only idempotents are $e_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $e_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. It is clear that the element $\begin{bmatrix} 0 & 0 \\ \ell & 0 \end{bmatrix} \in J_{1/2}$ is not invertible.

(ii) Hermitian type $\mathcal{H}(M_2(L), \sigma t)$ with L a skew-field with involution σ such that (L, σ) is not a quadratic pair. In this case $\dim(J_0) > 1$.

(iii) Reduced spin factor of a non-degenerate quadratic space (k, V, q). Since the unit of J is $e_0 + e_1$ and $1 = u^2 = q(u)(e_0 + e_1) = q(u)1$, u is a base point of q.

Suppose there exists a $v \in J_{1/2} \setminus \{0\}$ such that $q(v) = 0$. Then it would follow that $v^2 = 0$, which implies that v is not invertible. Therefore q is anisotropic. In this case $\dim(J_0) = 1$. □

3. A coordinate-free construction of quadrangular algebras

In this section we give a coordinate-free construction of quadrangular algebras. Our construction was inspired by several properties of J-ternary algebras; see Definition 3.6 and [ABG, 3.12].

The entire article [ABG] deals with fields of characteristic zero only. However the concept of a J-ternary algebra and its basic properties, such as Peirce decomposition (see Lemma 3.4 and [ABG, 6.61]) can be generalized without any adjustments to fields of characteristic different from 2 and 3.

It is not clear at all how to generalize the theory of J-ternary algebras to fields of characteristic 2 and 3; one reason is that the definition of a J-ternary algebra uses bilinear and trilinear forms.

However in the theory of quadrangular algebras there is no difference between fields of characteristic different from 2 and 3 and fields of characteristic 3. Therefore

[6] This means that $U_u(xy) = U_u(x)U_u(y)$ for all $x, y \in J$.

we want our construction of quadrangular algebras to work in characteristic 3 in the same way as in characteristic not 2 and 3.

Actually, J-ternary algebras contain some axioms that are superfluous for our construction. In Theorem 3.5 we show that we can prove the axioms for quadrangular algebras using only a few axioms concerning a module for a Jordan algebra that has a skew-symmetric form. We replaced all the identities in the definition of a J-ternary system involving the trilinear form by an identity that only uses the bilinear form. Therefore we do not need the assumption that the characteristic is different from 3. In Section 3.2 we show that a J-ternary algebra still satisfies the identity we demand in Theorem 3.5.

In the following subsection we do not start by giving a definition of J-ternary algebras. Instead we start by considering the concepts that we will need to formulate Theorem 3.5.

To include characteristic 2 would be another cup of tea for several reasons: already the definition of quadrangular algebras is much more complicated, and the definition of a Jordan algebra is more delicate.

3.1. **Quadrangular algebras from special Jordan modules.** Let k be a field of characteristic different from 2. In the next lemma we introduce a module for Jordan algebras.

Lemma 3.1. *Let J be a Jordan k-algebra and let X be a k-vector space. Suppose that J acts on X by $\bullet : J \times X \to X$ such that*

(i) $(tj) \bullet x = j \bullet (tx)$,
(ii) $(j + j') \bullet x = j \bullet x + j' \bullet x$,
(iii) $j \bullet (x + y) = j \bullet x + j \bullet y$,
(iv) $1 \bullet x = x$,

for all $j, j' \in J, x, y \in X, t \in k$. Then the following are equivalent:

(v) $U_j j' \bullet x = j \bullet (j' \bullet (j \bullet x))$,
(v') $(jj') \bullet x = \frac{1}{2}(j \bullet (j' \bullet x) + j' \bullet (j \bullet x))$.

Proof. Assume that (v) holds. Since $U_j 1 = j^2$, we have $j^2 \bullet x = j \bullet (j \bullet x)$. Linearizing this expression gives us (v').

Assuming that (v') holds, we have

$$\begin{aligned} U_j j' \bullet x &= (2j(jj') - j^2 j') \bullet x \\ &= \tfrac{1}{2}(2j \bullet ((jj') \bullet x) + 2(jj') \bullet (j \bullet x) \\ &\qquad - j^2 \bullet (j' \bullet x) - j' \bullet (j^2 \bullet x)) \\ &= \tfrac{1}{2}(j^2 \bullet (j' \bullet x) + j' \bullet (j^2 \bullet x) + 2j \bullet (j' \bullet (j \bullet x)) \\ &\qquad - j^2 \bullet (j' \bullet x) - j' \bullet (j^2 \bullet x)) \\ &= j \bullet (j' \bullet (j \bullet x)). \end{aligned}$$

□

Definition 3.2 ([ABG, 3.12])**.** Let J be a Jordan k-algebra and let X be a k-vector space with action $\bullet : J \times X \to X$. Then X is a *special J-module* if the conditions (i)-(v) of the previous lemma are satisfied.

Lemma 3.3. *Let X be a special J-module and let $(\cdot, \cdot) : X \times X \to J$ be a bilinear skew-symmetric form. Then*

$$U_j(x, y) = (j \bullet x, j \bullet y) \iff j(x, y) = \tfrac{1}{2}((j \bullet x, y) + (x, j \bullet y)). \tag{3.1}$$

Proof. $\Rightarrow$: This follows from $U_{j+1}j' - U_j j' - U_1 j' = 2jj'$.

$\Leftarrow$: We have

$$\begin{aligned} U_j(x,y) &= 2j(j(x,y)) - j^2(x,y) \\ &= \tfrac{1}{2}\big((j \bullet (j \bullet x), y) + (x, j \bullet (j \bullet y)) + 2(j \bullet x, j \bullet y)\big) \\ &\qquad - \tfrac{1}{2}\big((j^2 \bullet x, y) + (x, j^2 \bullet y)\big). \end{aligned}$$

The result follows since $j \bullet (j \bullet x) = j^2 \bullet x$. □

In the following lemma we consider the Peirce decomposition of special J-modules; see also [ABG, 6.61].

Lemma 3.4. *Let J be a Jordan k-algebra with supplementary proper idempotents e_0 and e_1. Let $J_0, J_{1/2}, J_1$ be the Peirce subspaces of J with respect to e_1. Let X be a special J-module and define*

$$X_0 := \{x \in X \mid e_0 \bullet x = x\} = \{x \in X \mid e_1 \bullet x = 0\},$$
$$X_1 := \{x \in X \mid e_0 \bullet x = 0\} = \{x \in X \mid e_1 \bullet x = x\}.$$

Then

(i) *We have $e_0 \bullet X = X_0$, $e_1 \bullet X = X_1$ and $X = X_0 \oplus X_1$.*

(ii) *For $i \in \{0,1\}$ and $j = 1 - i$ we have*

$$J_i \bullet X_i \subseteq X_i, \quad J_i \bullet X_j = 0, \quad J_{1/2} \bullet X_i \subseteq X_j. \tag{3.2}$$

(iii) *Let $(\cdot,\cdot) : X \times X \to J$ be a bilinear skew-symmetric form satisfying* (3.1). *Then*

$$(X_i, X_i) \subseteq J_i, \quad (X_i, X_j) \subseteq J_{1/2} \tag{3.3}$$

for $i \in \{0,1\}$ and $j = 1 - i$.

(iv) *Assume there exists an element $u \in J_{1/2}$ such that $u^2 = 1$. The map*

$$X_0 \to X_1 : x \mapsto u \bullet x$$

is a vector space isomorphism, called the connecting homomorphism.

Proof. (i) Let $i \in \{0,1\}$. We have $e_i \bullet (e_i \bullet x) = (e_i e_i) \bullet x = e_i \bullet x$ for all $x \in X$; therefore $e_i \bullet X = X_i$. Since $(e_0 + e_1) \bullet x = x$, we have $X = X_0 \oplus X_1$.

(ii) This follows by evaluating

$$e_1 \bullet (j \bullet x) = 2(e_1 j) \bullet x - j \bullet (e_1 \bullet x)$$

for all combinations of $j \in J_0$, J_1 or $J_{1/2}$ and $x \in X_0$ or X_1, using the fact that $X = X_0 \oplus X_1$.

(iii) This follows from evaluating $e_1(x,y) = \frac{1}{2}((e_1 \bullet x, y) + (x, e_1 \bullet y))$ for $x, y \in X_0$ or X_1.

(iv) It follows from (ii) that $u \bullet x \in X_1$ iff $x \in X_0$. Since $u \bullet (u \bullet x) = (uu) \bullet x = x$ the connecting homomorphism is an isomorphism. □

Theorem 3.5. *Let $\mathrm{char}(k) \neq 2$. Let J be the reduced spin factor of the non-degenerate anisotropic quadratic space (k, V, q) with base point u: $J = ke_0 \oplus V \oplus ke_1$. Let X be a non-trivial special J-module equipped with a bilinear skew-symmetric form $(\cdot,\cdot) : X \times X \to J$ satisfying* (3.1).

Assume that the following hold:

$$\forall x \in X_0, v \in V: \qquad (v \bullet x, x) \bullet x = v \bullet (u \bullet ((u \bullet x, x) \bullet x)), \tag{3.4}$$

$$\forall x \in X_0 \setminus \{0\}: \qquad (u \bullet x, x) \neq 0. \tag{3.5}$$

We define

$$\cdot : X_0 \times V \to X_0 : x \cdot v = v \bullet (u \bullet x),$$
$$h : X_0 \times X_0 \to V : (x, y) \mapsto (u \bullet x, y).$$

Then $(k, V, q, u, X_0, \cdot, h)$ is a quadrangular algebra.

Proof. Notice that $e_0, e_1 \in J$ are supplementary proper idempotents and that $u \in J_{1/2}$ such that $u^2 = q(u)1 = 1$. Thus we can apply Lemma 3.4 with $J_0 = ke_0, J_{1/2} = V, J_1 = ke_1$. It follows from (3.2) and (3.3) that the maps $\cdot$ and h are well defined. To start we show that $U_u(v) = v^\sigma$ for all $v \in J_{1/2}$ with σ as in Definition 2.10:

$$\begin{aligned} U_u(v) &= 2u(uv) - v \\ &= u(f(u, v)1) - v \\ &= f(u, v)u - v = v^\sigma. \end{aligned}$$

We verify that all the axioms of a quadrangular algebra given in Definition 2.10 hold.

(A1) This follows from linearity of $\bullet$.
(A2) Let $x \in X_0$; then $x \cdot u = u \bullet (u \bullet x) = u^2 \bullet x = 1 \bullet x = x$.
(A3) Let $x \in X_0$ and $v \in J_{1/2}$; then

$$\begin{aligned} (x \cdot v) \cdot v^\sigma &= U_u(v) \bullet (u \bullet (v \bullet (u \bullet x))) \\ &= u \bullet (v \bullet (v \bullet (u \bullet x))) \\ &= \tfrac{1}{2} f(v, v)\ u \bullet (1 \bullet (u \bullet x)) \\ &= q(v)x. \end{aligned}$$

(B1) This follows from bilinearity of $(.,.)$.
(B2) Let $x, y \in X_0$ and $v \in J_{1/2}$. Then by applying consecutively (2.4); (3.1) and (A2); Lemma 3.1(v) we find

$$\begin{aligned} & h(x, y \cdot v) = h(y, x \cdot v) + f(h(x, y), u)v \\ \iff \quad & (u \bullet x, v \bullet (u \bullet y)) = (u \bullet y, v \bullet (u \bullet x)) + f((u \bullet x, y), u)v \\ \iff \quad & (u \bullet x, v \bullet (u \bullet y)) \\ & \quad + (v \bullet (u \bullet x), u \bullet y) = 2((u \bullet x, y)u)v \\ \iff \quad & 2v(u \bullet x, u \bullet y) = ((u \bullet x, u \bullet y) + (x, y))v \\ \iff \quad & vU_u(x, y) = (x, y)v. \end{aligned}$$

From (3.3) we know that $(x, y) = te_0$ for some $t \in k$. It follows from Definition 2.20 that

$$vU_u(te_0) = v(te_1) = \tfrac{1}{2}tv = (te_0)v.$$

(B3) Let $x, y \in X_0$ and $v \in J_{1/2}$. Then by applying consecutively (2.4); (3.1) we find

$$\begin{aligned} & f(h(x\cdot v, y), u) = f(h(x,y), v) \\ \iff \quad & f((u \bullet (v \bullet (u \bullet x)), y), u) = f((u\bullet x, y), v) \\ \iff \quad & (u \bullet (v \bullet (u \bullet x)), y)u = (u \bullet x, y)v \\ \iff \quad & (u \bullet (v \bullet (u \bullet x)), u \bullet y) \\ & \quad + (v \bullet (u \bullet x), y) = (u \bullet x, v \bullet y) + (v \bullet (u \bullet x), y) \\ \iff \quad & (u \bullet (v \bullet (u \bullet x)), u \bullet y) = (u \bullet x, v \bullet y). \end{aligned}$$

Since $U_u = \sigma$ on $J_{1/2}$ is an involution, by (3.1) this is equivalent to

$$\begin{aligned} \iff \quad & (v \bullet (u \bullet x), y) = (x, u \bullet (v \bullet y)) \\ \iff \quad & (v \bullet (u \bullet x), v \bullet (v \bullet y)) = q(v)(u \bullet (u \bullet x), u \bullet (v \bullet y)); \end{aligned}$$

the last equivalence follows from equality (2.4). From (3.3) we know that $(u \bullet x, v \bullet y) = te_1$ for some $t \in k$. The last equation reduces to

$$U_v(te_1) = q(v)U_u(te_1).$$

This holds since $U_v(e_1) = q(v)e_0$ and $U_u(e_1) = e_0$ by Definition 2.20.

(C) $\theta(x, v) := \frac{1}{2}(u \bullet x, v \bullet (u \bullet x))$.

(D1) Let $x \in X_0$ and $v \in J_{1/2}$. Since f is non-degenerate, we have $V = ku \oplus u^\perp$. Now

$$\begin{aligned} & x \cdot h(x, x \cdot v) = (x \cdot h(x,x)) \cdot v \\ \iff \quad & (u \bullet x, v \bullet (u \bullet x)) \bullet (u \bullet x) = v \bullet (u \bullet ((u \bullet x, x) \bullet (u \bullet x))). \end{aligned}$$

Since this expression is linear in v and trivial for $v \in ku$, we can assume $v \in u^\perp$ and thus $f(u, v) = uv = 0$ and hence $u \bullet (v \bullet x) = -v \bullet (u \bullet x)$. In this case we continue as follows:

$$\begin{aligned} \iff \quad & -(u \bullet x, u \bullet (v \bullet x)) \bullet (u \bullet x) = v \bullet (U_u(u \bullet x, x) \bullet x) \\ \iff \quad & U_u(x, v \bullet x) \bullet (u \bullet x) = -v \bullet ((x, u \bullet x) \bullet x) \\ \iff \quad & u \bullet ((x, v \bullet x) \bullet x) = -v \bullet ((x, u \bullet x) \bullet x) \\ \iff \quad & (x, v \bullet x) \bullet x = -u \bullet (v \bullet ((x, u \bullet x) \bullet x)) \\ \iff \quad & (x, v \bullet x) \bullet x = v \bullet (u \bullet ((x, u \bullet x) \bullet x)). \end{aligned}$$

This is exactly (3.4).

(D2) This is assumption (3.5). □

3.2. **Quadrangular algebras from J-ternary algebras.** In this subsection we assume $\mathrm{char}(k) \neq 2, 3$ and we prove that an arbitrary 'anisotropic' non-trivial J-ternary algebra, where J is as in Theorem 3.5, satisfies the assumptions of Theorem 3.5. It follows that we can construct quadrangular algebras from J-ternary algebras.

We remind the reader that the entire article [ABG] is written only for fields of characteristic zero, but that the concept of a J-ternary algebra and its basic properties can be generalized without any adjustments to hold in fields of characteristic different from 2 and 3.

Definition 3.6. Let $\mathrm{char}(k) \neq 2, 3$, let J be a Jordan k-algebra, and let X be a special J-module with action $\bullet$.

Assume $\boldsymbol{(}\ ,\ \boldsymbol{)} : X \times X \to J$ is a skew-symmetric bilinear map, and $\boldsymbol{(}\ ,\ ,\ \boldsymbol{)} : X \times X \times X \to X$ is a trilinear product. Then X is called a J-ternary algebra if the following axioms hold for all $j \in J$, $x, y, z, v, w \in X$:

(JT1) $j\boldsymbol{(}x, y\boldsymbol{)} = \frac{1}{2}\boldsymbol{(}j \bullet x, y\boldsymbol{)} + \frac{1}{2}\boldsymbol{(}x, j \bullet y\boldsymbol{)}$ (This is the right-hand side of (3.1).)

(JT2) $j \bullet \boldsymbol{(}x, y, z\boldsymbol{)} = \boldsymbol{(}j \bullet x, y, z\boldsymbol{)} - \boldsymbol{(}x, j \bullet y, z\boldsymbol{)} + \boldsymbol{(}x, y, j \bullet z\boldsymbol{)}$

(JT3) $\boldsymbol{(}x, y, z\boldsymbol{)} = \boldsymbol{(}z, y, x\boldsymbol{)} - \boldsymbol{(}x, z\boldsymbol{)} \bullet y$

(JT4) $\boldsymbol{(}x, y, z\boldsymbol{)} = \boldsymbol{(}y, x, z\boldsymbol{)} + \boldsymbol{(}x, y\boldsymbol{)} \bullet z$

(JT5) $\boldsymbol{(}\boldsymbol{(}x, y, z\boldsymbol{)}, w\boldsymbol{)} + \boldsymbol{(}z, \boldsymbol{(}x, y, w\boldsymbol{)}\boldsymbol{)} = \boldsymbol{(}x, \boldsymbol{(}z, w\boldsymbol{)} \bullet y\boldsymbol{)}$

(JT6) $\boldsymbol{(}x, y, \boldsymbol{(}z, w, v\boldsymbol{)}\boldsymbol{)} = \boldsymbol{(}\boldsymbol{(}x, y, z\boldsymbol{)}, w, v\boldsymbol{)} + \boldsymbol{(}z, \boldsymbol{(}y, x, w\boldsymbol{)}, v\boldsymbol{)} + \boldsymbol{(}z, w, \boldsymbol{(}x, y, v\boldsymbol{)}\boldsymbol{)}$.

Theorem 3.7. *Let* $\mathrm{char}(k) \neq 2, 3$. *Let J be the reduced spin factor of the non-degenerate anisotropic quadratic space (k, V, q) with base point u. Let X be a non-trivial J-ternary algebra such that $\boldsymbol{(}u \bullet x, x\boldsymbol{)} \neq 0$ for all $x \in X_0 \setminus \{0\}$.*

Then X satisfies (3.4). *Therefore $(k, V, q, u, X_0, \cdot, h)$ is a quadrangular algebra, with $\cdot$ and h as in Theorem* 3.5.

Proof. Let $i \in \{0, 1\}$. We will first show that for all $x \in X_i$ and $v \in J_{1/2}$,

$$v \bullet \boldsymbol{(}x, x, x\boldsymbol{)} = 3\boldsymbol{(}v \bullet x, x\boldsymbol{)} \bullet x = 3\boldsymbol{(}v \bullet x, x, x\boldsymbol{)} = \frac{3}{2}\boldsymbol{(}x, x, v \bullet x\boldsymbol{)}. \tag{3.6}$$

From (JT2) we find that $e_1 \bullet \boldsymbol{(}x, v \bullet x, x\boldsymbol{)} = t\boldsymbol{(}x, v \bullet x, x\boldsymbol{)}$ with $t = -1$ if $i = 0$ and $t = 2$ if $i = -1$; therefore

$$\boldsymbol{(}x, v \bullet x, x\boldsymbol{)} = 0.$$

Using (JT3) and (JT4) respectively we get

$$\boldsymbol{(}x, v \bullet x\boldsymbol{)} \bullet x = \boldsymbol{(}v \bullet x, x, x\boldsymbol{)} - \boldsymbol{(}x, x, v \bullet x\boldsymbol{)} \text{ and } \boldsymbol{(}x, v \bullet x\boldsymbol{)} \bullet x = -\boldsymbol{(}v \bullet x, x, x\boldsymbol{)}.$$

Combining these equations, we obtain

$$\boldsymbol{(}x, x, v \bullet x\boldsymbol{)} = 2\boldsymbol{(}v \bullet x, x, x\boldsymbol{)} = 2\boldsymbol{(}v \bullet x, x\boldsymbol{)} \bullet x.$$

From (JT2) we have

$$v \bullet \boldsymbol{(}x, x, x\boldsymbol{)} = \boldsymbol{(}v \bullet x, x, x\boldsymbol{)} + \boldsymbol{(}x, x, v \bullet x\boldsymbol{)}.$$

Combining the last two formulas proves (3.6). Since $\mathrm{char}(k) \neq 3$, it follows from (3.6) that (3.4) is equivalent to

$$v \bullet \boldsymbol{(}x, x, x\boldsymbol{)} = v \bullet (u \bullet (u \bullet \boldsymbol{(}x, x, x\boldsymbol{)})).$$

Since this last equation holds we have proved that (3.4) holds. □

Remark 3.8. (i) We show that for all $x \in X_0$, $\boldsymbol{(}u \bullet x, x\boldsymbol{)} = 0 \iff \boldsymbol{(}x, x, x\boldsymbol{)} = 0$.

First remark that if $v \bullet x = 0$ we have $v = 0$ or $x = 0$: it follows from $v \bullet x = 0$ that $v \bullet (v \bullet x) = q(v)x = 0$. Since q is anisotropic we have $v = 0$ or $x = 0$. Now we have

$$\begin{aligned} & \boldsymbol{(}u \bullet x, x\boldsymbol{)} = 0 \\ \iff & \boldsymbol{(}u \bullet x, x\boldsymbol{)} \bullet x = 0 \\ \iff & u \bullet \boldsymbol{(}x, x, x\boldsymbol{)} = 0 \quad \text{by (3.6)} \\ \iff & \boldsymbol{(}x, x, x\boldsymbol{)} = 0. \end{aligned}$$

(ii) We have to demand that $(x, x, x) \neq 0$ for all $x \in X_0 \setminus \{0\}$, because there exist J-ternary algebras which fulfill all the requirements but where $(x, x, x) = 0$ for some $x \neq 0 \in X_0$. For example, consider [ABG, Example 6.81] with the zero skew hermitian form. Examples like this we clearly want to avoid.

3.3. **Construction of quadrangular algebras of pseudo-quadratic form type.** Let k be a field of characteristic not 2.

We rely on the example [ABG, 6.81] to obtain a quadrangular algebra of pseudo-quadratic form type using Theorem 3.5. In combination with Section 3.4 this will show that all quadrangular algebras of characteristic not 2 can be obtained using the construction in Theorem 3.5.

In Section 4 we will show that each Moufang quadrangle in characteristic not 2 can be obtained from a construction that generalizes the construction in Theorem 3.5.

Let $(L/k, \sigma)$ be a quadratic pair with quadratic form $q(\ell) = \ell\ell^\sigma$ and let (L, σ, X, h, π) be a standard anisotropic pseudo-quadratic space (see Section 2.3.1), so $(k, L, q, u, X, \text{ scalar multiplication}, h)$ is a quadrangular algebra.

Definition 3.9. (i) Define $J = \mathcal{H}(M_2(L), \sigma)$ (see Definition 2.21 and Remark 2.22). This Jordan algebra is a reduced spin factor of the quadratic form

$$Q : J_{1/2} \to k : \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \mapsto q(\ell).$$

As before we define $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in J : u$ is a base point of Q.

(ii) Define $\widetilde{X} = X^2$, the 1×2 row vectors over X.

(iii) We define the action of J on $\widetilde{X}$ as[7] $j \bullet x := xj \in \widetilde{X}$ for $j \in J, x \in \widetilde{X}$.

(iv) Define $\psi : \widetilde{X} \times \widetilde{X} \to M_2(L) : \psi([x_1, x_2], [y_1, y_2]) := \big[h(x_i, y_j)\big]$. Now we define the skew product $\widetilde{X} \times \widetilde{X} \to J$:

$$\begin{aligned}([x_1, x_2], [y_1, y_2]) &:= \psi([x_1, x_2], [y_1, y_2]) - \psi([y_1, y_2], [x_1, x_2]) \\ &= \begin{bmatrix} h(x_1, y_1) - h(y_1, x_1) & h(x_1, y_2) - h(y_1, x_2) \\ -h(y_2, x_1) + h(x_2, y_1) & h(x_2, y_2) - h(y_2, x_2) \end{bmatrix}.\end{aligned}$$

With respect to e_1 we have

$$\widetilde{X}_0 = \{[x, 0] \mid x \in X\}, \widetilde{X}_1 = \{[0, x] \mid x \in X\}.$$

Lemma 3.10. *The space $\widetilde{X}$ is a non-trivial special J-module with skew-symmetric bilinear form $(\cdot, \cdot)$ that satisfies* (3.1)*;* (3.4) *and* (3.5) *hold as well.*

Under the identifications

$$J_{1/2} \cong L : \ell \leftrightarrow \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \text{ and } \widetilde{X}_0 \cong X : [x, 0] \leftrightarrow x,$$

the quadrangular algebra defined in Theorem 3.5 *is exactly the pseudo-quadratic space we started with:*

$$(k, L, q, u, X, \text{ scalar multiplication}, h).$$

[7] On the right hand side the usual matrix multiplication is used.

Proof. Verifying that $\widetilde{X}$ is a special J-module and (3.1) requires some straightforward calculations. We will verify (3.4) and (3.5).

Define $v = \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \in J_{1/2}$, $\widetilde{x} = [x, 0] \in \widetilde{X}_0$. Notice that $u \bullet \widetilde{x} = [0, x] \in X_1$, and

$$(u \bullet \widetilde{x}, \widetilde{x}) = \begin{bmatrix} 0 & -h(x,x) \\ h(x,x) & 0 \end{bmatrix}.$$

Hence $(u \bullet \widetilde{x}, \widetilde{x})$ is equal to 0 if and only if $h(x,x) = 0$. As we are working in a standard anisotropic pseudo-quadratic space, $\pi(x) = \frac{1}{2}h(x,x)$ is anisotropic (see Remark 2.11.2), so (3.5) holds.

Condition (3.4) holds since

$$(v \bullet \widetilde{x}, \widetilde{x}) \bullet \widetilde{x} = [0, -xh(x,x)l^\sigma] = v \bullet [-xh(x,x), 0] = v \bullet (u \bullet ((u \bullet \widetilde{x}, \widetilde{x}) \bullet \widetilde{x})).$$

From Theorem 3.5 we conclude that $(k, J_{1/2}, Q, u, \widetilde{X}_0, \cdot, \widetilde{h})$ is a quadrangular algebra with

$$[x, 0] \cdot \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} = [0, x\ell],$$

$$\widetilde{h}([x,0],[y,0]) = (u \bullet [x,0],[y,0]) = \begin{bmatrix} 0 & -h(y,x) \\ h(x,y) & 0 \end{bmatrix}. \qquad \square$$

3.4. **Construction of quadrangular algebras of type E_6, E_7, E_8.** In this subsection we give a new construction of the vector spaces X, L and the maps $\cdot$ and h that we discussed in Section 2.3.2.

3.4.1. *A characterization of quadratic forms of type E_6, E_7, E_8.* We start by giving a new way to describe quadratic forms of type E_6, E_7 and E_8 (see Definition 2.15). The following illuminating observation was made by Skip Garibaldi.

Theorem 3.11. *Let q be an anisotropic form over k of dimension $6, 8$ or 12. Then q is of type E_6, E_7 or E_8 respectively if and only if there exist an octonion division algebra C_1 and a division composition algebra C_2, of dimension $2, 4$ or 8 respectively, that have linkage number one such that q is similar to the anisotropic part of the Albert form of $C_1 \otimes_k C_2$.*

The 'only if'-direction of this theorem is proved in the following, more technical, lemma.

Lemma 3.12. *We consider a quadratic form q of type E_6, E_7 or E_8. Let N denote the norm of a separable quadratic field extension $E = k(x)/(x^2 - a)$ for $a \notin k^2$.*

(i) *If $q = N \otimes \langle 1, s_2, s_3 \rangle$ of type E_6, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = E.$$

(ii) *If $q = N \otimes \langle 1, s_2, s_3, s_4 \rangle$ of type E_7, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = (a, s_2 s_3 s_4)_k.$$

(iii) *If $q = N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6 \rangle$ of type E_8, define*

$$C_1 = (a, -s_2, -s_3)_k \text{ and } C_2 = (a, -s_4 s_6, -s_5 s_6)_k.$$

Then q is similar to the anisotropic part of the Albert form of $C_1 \otimes_k C_2$, and C_1 and C_2 are division algebras that have linkage number 1.

Proof. Denote the norm form of C_1 by q_1, the norm form of C_2 by q_2 and the Albert form of $C_1 \otimes_k C_2$ by q_A. In the case that q is of type E_8 we will verify that

$$q \perp 2\mathbb{H} \sim q_A \perp \mathbb{H} \sim q_1 \perp -q_2; \tag{3.7}$$

the other two cases are similar. We have $q_1 = N \otimes \langle\langle s_2, s_3\rangle\rangle$ and $q_2 = N \otimes \langle\langle s_4s_6, s_5s_6\rangle\rangle$. Since $t\mathbb{H} \simeq \mathbb{H}$ for $t \in k$ it follows that

$$\begin{aligned} s_2s_3(\langle 1, s_2, s_3, s_4, s_5, s_6\rangle \perp \mathbb{H}) &\simeq s_2s_3\langle s_3, s_2, 1, s_5, s_4, s_6\rangle \perp \mathbb{H} \\ &\simeq \langle s_2, s_3, s_2s_3, s_2s_3s_5, s_2s_3s_4, s_2s_3s_6\rangle \perp \mathbb{H} \\ &\simeq \langle s_2, s_3, s_2s_3, -s_4s_6, -s_5s_6, -s_4s_5\rangle \perp \langle 1, -1\rangle \\ &\simeq \langle\langle s_2, s_3\rangle\rangle \perp -\langle\langle s_4s_6, s_5s_6\rangle\rangle. \end{aligned} \tag{3.8}$$

By multiplying the above identity by N we obtain (3.7).

Note that q is anisotropic. Therefore $q_1 \perp -q_2$ has Witt index 2; it follows that q_1 and q_2 are anisotropic and both C_1 and C_2 are division algebras. It follows from (3.7) that q_A has Witt index 1, and now Lemma 2.7 implies that C_1 and C_2 have linkage number 1. □

Proof of Theorem 3.11. The 'only if'-direction is proven in the lemma above. The 'if'-direction follows in a similar way. We elaborate the case where C_1 and C_2 are octonion division algebras.

Since C_1 and C_2 contain an isomorphic field extension, by [SV, Prop. 1.5.1] we can assume that $C_1 = (a, b_1, c_1)_k$ and $C_2 = (a, b_2, c_2)_k$ for some $a, b_1, b_2, c_1, c_2 \in k$. We denote the Albert form of $C_1 \otimes_k C_2$ by q_A.

Define $N := \langle\langle -a\rangle\rangle$; this is anisotropic since C_1 is division. By going through (3.8) from bottom to top with

$$s_2 := -b_1, s_3 := -c_1, s_4 := \frac{1}{b_1c_1c_2}, s_5 := \frac{1}{b_1c_1b_2}, s_6 := -b_1b_2c_1c_2$$

we find that q_A is similar to $N\otimes\langle 1, s_2, s_3, s_4, s_5, s_6\rangle \perp \mathbb{H}$. Since the Witt index of q_A is one, $N \otimes \langle 1, s_2, s_3, s_4, s_5, s_6\rangle$ is the anisotropic part of q_A; since $s_2s_3s_4s_5s_6 = -1$ it is of type E_8. □

3.4.2. *The construction.* In order to construct quadrangular algebras of type E_6, E_7 and E_8 we follow Example 6.82 in [ABG] closely. In *loc. cit.* a J-ternary algebra is constructed out of $C_1 \otimes_k C_2$ in characteristic zero, but this restriction is not necessary.

First we give a motivation for the approach we will be following in our construction.

Remark 3.13. Let C_1 be an octonion division algebra and C_2 a separable quadratic field extension, quaternion division algebra or octonion division algebra and assume that C_1 and C_2 have linkage number 1.

The dimension of $C_1 \otimes_k C_2$ is 16, 32 or 64, respectively. The space of skew elements is $\mathcal{S} = S_1 \otimes 1 + 1 \otimes S_2$ and has dimension 8, 10 or 14, respectively. Let $(k, L, q, 1, \widetilde{X}, \cdot, h)$ be a quadrangular algebra of type E_6, E_7 or E_8, respectively. We summarize some dimensions:

	E_6	E_7	E_8
$\dim_k \mathcal{S}$	8	10	14
$\dim_k L$	6	8	12
$\dim_k(C_1 \otimes_k C_2)$	16	32	64
$\dim_k \widetilde{X}$	8	16	32

We see that in all three cases $\dim_k \mathcal{S} = \dim_k L + 2$ and $\dim_k(C_1 \otimes C_2) = 2 \dim_k \widetilde{X}$.

In Theorem 3.5 we considered some objects the dimensions of which behave similarly: Let J be a Jordan algebra of reduced spin type with base point and let X be a special J-module. Then $\dim_k(J) = \dim J_{1/2} + 2$ and $\dim_k X = 2 \dim_k X_0$.

From Lemma 3.12, the Albert form from $\mathcal{S}$ to k can be written as the sum of a hyperbolic plane and a quadratic form of type E_6, E_7 or E_8, respectively. Note that a hyperbolic plane is two-dimensional.

In the following pages, we will give $\mathcal{S}$ the structure of a reduced spin factor of a quadratic form of type E_6, E_7 or E_8, respectively, and identify $J_{1/2}$ with L. Then we will give $C_1 \otimes C_2$ the structure of a special J-module equipped with a bilinear skew-hermitian form, and identify $(C_1 \otimes C_2)_0$ with $\widetilde{X}$.

We start by fixing some notation.

Notation 3.14. (i) We fix a basis for the composition algebras C_1 and C_2 that have linkage number 1. We let C_1 be the octonion division algebra

$$C_1 = \langle 1, \mathbf{i}_1, \mathbf{j}_1, \mathbf{i}_1\mathbf{j}_1, \mathbf{k}_1, \mathbf{i}_1\mathbf{k}_1, \mathbf{j}_1\mathbf{k}_1, (\mathbf{i}_1\mathbf{j}_1)\mathbf{k}_1 \rangle.$$

If C_2 is a separable quadratic field extension, we define $C_2 = \langle 1, \mathbf{i}_2 \rangle$. In the case C_2 is a quaternion division algebra, we define $C_2 = \langle 1, \mathbf{i}_2, \mathbf{j}_2, \mathbf{i}_2\mathbf{j}_2 \rangle$. In the case C_2 is an octonion division algebra, we define

$$C_2 = \langle 1, \mathbf{i}_2, \mathbf{j}_2, \mathbf{i}_2\mathbf{j}_2, \mathbf{k}_2, \mathbf{i}_2\mathbf{k}_2, \mathbf{j}_2\mathbf{k}_2, (\mathbf{i}_2\mathbf{j}_2)\mathbf{k}_2 \rangle.$$

Since C_1 and C_2 have linkage number 1, we can choose these bases in such a way that

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 =: a \in K.$$

(ii) From now on we denote by $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}$ the set of skew elements of C_1, C_2 and $C_1 \otimes_k C_2$, respectively.

(iii) We denote the Albert form of $C_1 \otimes_k C_2$ by $q_A : \mathcal{S} \to k$ and its associated bilinear form by f_A.

(iv) Let $V := \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^\perp$ denote the orthogonal complement of the subspace $\langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle$ of $\mathcal{S}$ with respect to the non-degenerate bilinear form f_A.

We want to make $\mathcal{S}$ into a Jordan algebra of reduced spin type. In particular it should contain supplementary proper idempotents e_0 and e_1 and an element $u \in J_{1/2}$ such that u^2 is the identity. It will become clear that the elements constructed in the following lemma will be the ones we need.

Lemma 3.15. *Let $u \in V \setminus \{0\}$ be arbitrary. Then up to scalars, there exists a unique pair (e_0, e_1) of elements in $\mathcal{S}$ such that*

$$q_A(e_0) = q_A(e_1) = 0,$$
$$f_A(e_0, V) = f_A(e_1, V) = 0, \qquad f_A(e_0, e_1) = -q_A(u) \neq 0.$$

Explicitly, there exists an element $\lambda \in k$ *such that* $(\lambda e_0, \lambda^{-1} e_1)$ *is equal to*

$$\left(\mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2, \frac{q_A(u)}{4a}(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2) \right).$$

Proof. Since q_A has Witt index one, q_A is anisotropic on $V = \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle^\perp$. Hence $q_A(u) \neq 0$.

We demand that e_0, e_1 be isotropic elements in $V^\perp = \langle \mathbf{i}_1 \otimes 1, 1 \otimes \mathbf{i}_2 \rangle$. This implies that they are of the form $\lambda(\mathbf{i}_1 \otimes 1 \pm 1 \otimes \mathbf{i}_2)$. Since $f_A(e_0, e_1)$ should be different from 0 we can take without loss of generality $e_0 = \lambda_0(\mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2)$ and $e_1 = \lambda_1(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2)$ for some $\lambda_0, \lambda_1 \in k \setminus \{0\}$. Now we determine the scalars λ_0, λ_1, such that $f_A(e_0, e_1) = -q_A(u)$. We have

$$f_A(\lambda_0(\mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2), \lambda_1(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2)) = \lambda_0 \lambda_1 (-4a).$$

So we find that $\lambda_1 = \frac{q_A(u)}{4a\lambda_0}$. □

Since $\dim \mathcal{S} = \dim V + 2$, we want to make V into a quadratic space. If we want $u.u = 1$ in the Jordan algebra of reduced spin type that we will define, the element u should be the base point of the quadratic form that determines the reduced spin factor. In the following definition we define a Jordan algebra on $\mathcal{S}$. In Lemma 3.17 we will show that this Jordan algebra has a natural interpretation in the endomorphism ring of $C_1 \otimes_k C_2$.

Definition 3.16. Let $u \in V \setminus \{0\}$ and

$$e_0 = \mathbf{i}_1 \otimes 1 + 1 \otimes \mathbf{i}_2, \quad e_1 = \frac{q_A(u)}{4a}(\mathbf{i}_1 \otimes 1 - 1 \otimes \mathbf{i}_2).$$

(i) We define a quadratic form on the vector space V,

$$Q := \frac{1}{q_A(u)} q_A|_V.$$

We denote the corresponding bilinear form by F.

It follows from Theorem 3.11 that (k, V, Q) is a quadratic space of type E_6, E_7 or E_8, respectively, with base point u.

(ii) We have $\mathcal{S} = ke_0 \oplus V \oplus ke_1$. We define the Jordan multiplication as in Definition 2.20:

$$\begin{aligned}(t_1 e_i).(t_2 e_j) &= \delta_{ij} t_1 t_2 e_i,\\ (te_i).v &= \tfrac{1}{2} tv,\\ v.w &= \tfrac{1}{2} F(v,w)(e_0 + e_1),\end{aligned}$$

for all $i, j \in \{0, 1\}, v, w \in V, t, t_1, t_2 \in k$. We denote this Jordan algebra by J; this is the reduced spin factor of (k, V, Q).

(iii) We define

$$r := (e_0 + e_1)^{-1} = -\frac{1}{q_A(e_0 + e_1)}(e_0 + e_1)^\natural \in \mathcal{S},$$

where the inverse and $\natural$ are as in Definition 2.3. Notice that $e_0 + e_1$ is the identity in the Jordan algebra J on $\mathcal{S}$, so the definition of r has nothing to do with the inverse in J.

(iv) Let $s \in \mathcal{S}$ and define $L_s \in \mathrm{End}_k(C_1 \otimes_k C_2)$ as $L_s x := sx$ for all $x \in C_1 \otimes_k C_2$.

We will consider the Jordan algebra of the associative algebra $\mathrm{End}_k(C_1 \otimes_k C_2)$, denoted by $\mathrm{End}_k(C_1 \otimes_k C_2)^+$. We show that the algebra of reduced spin type we defined above is isomorphic to a Jordan subalgebra of $\mathrm{End}_k(C_1 \otimes_k C_2)^+$.

Lemma 3.17 ([ABG, Example 6.82]). *Let $s_1, s_2 \in \mathcal{S}$. We have*

$$\frac{1}{2}(L_{s_1}L_rL_{s_2}L_r + L_{s_2}L_rL_{s_1}L_r) = L_{s_1.s_2}L_r,$$

where $s_1.s_2$ denotes the multiplication in the algebra J defined in Definition 3.16(ii)*.*

Therefore $L_{\mathcal{S}}L_r$ is a Jordan subalgebra of $\mathrm{End}_k(C_1 \otimes_k C_2)^+$ *isomorphic to J.*

Proof. We will make use of [A3, Proposition 3.3 (3.8)]. In [A3] only characteristic 0 is considered; however this proposition can be generalized to characteristic different from 2 without any adjustments. The proof of this proposition uses basic identities of octonions (see Lemma 2.1) and the identity $s_1(s_2(s_1x)) = (s_1s_2s_1)x$ for $x \in C_1 \otimes_k C_2$ (see Lemma 2.4).

Linearizing [A3, Prop 3.3 (3.8)] gives

$$\begin{aligned} &L_{s_1}L_{(e_0+e_1)^\natural}L_{s_2} + L_{s_2}L_{(e_0+e_1)^\natural}L_{s_1} \\ &\qquad = -f_A(s_1, e_0+e_1)L_{s_2} - f_A(s_2, e_0+e_1)L_{s_1} + f_A(s_1, s_2)L_{e_0+e_1}. \end{aligned}$$

Since $r = (e_0+e_1)^{-1} = -\frac{1}{q_A(e_0+e_1)}(e_0+e_1)^\natural = \frac{1}{q_A(u)}(e_0+e_1)^\natural$, we find that

$$\begin{aligned} &\frac{1}{2}(L_{s_1}L_rL_{s_2}L_r + L_{s_2}L_rL_{s_1}L_r) \\ &= \frac{1}{2q_A(u)}(-f_A(s_1, e_0+e_1)L_{s_2}L_r - f_A(s_2, e_0+e_1)L_{s_1}L_r + f_A(s_1, s_2)L_{e_0+e_1}L_r). \end{aligned}$$

It follows from $u \in V = \langle e_0, e_1\rangle^\perp$, $q_A(e_0) = q_A(e_1) = 0, f_A(e_0, e_1) = -q_A(u)$ that for $i, j \in \{0, 1\}$ and for all $v, w \in V$,

$$\begin{aligned} \tfrac{1}{2}(L_{e_i}L_rL_{e_j}L_r + L_{e_j}L_rL_{e_i}L_r) &= \delta_{ij}L_{e_i}L_r, \\ \tfrac{1}{2}(L_{e_i}L_rL_vL_r + L_vL_rL_{e_i}L_r) &= \tfrac{1}{2}L_vL_r, \\ \tfrac{1}{2}(L_vL_rL_wL_r + L_wL_rL_vL_r) &= \frac{f_A(v,w)}{2q_A(u)}L_{e_0+e_1}L_r. \end{aligned}$$

This is exactly the multiplication of J. □

In order to define an action of J on $C_1 \otimes_k C_2$, we use the isomorphism of the previous lemma.

Definition 3.18. We define the bilinear action

$$\bullet : \mathcal{S} \times (C_1 \otimes_k C_2) \to C_1 \otimes_k C_2 : (s, x) \mapsto L_sL_rx = s(rx).$$

We define the *skew-symmetric bilinear map*

$$\mathbf{(.,.)} : (C_1 \otimes_k C_2) \times (C_1 \otimes_k C_2) \to \mathcal{S} : (x, y) \mapsto x\overline{y} - y\overline{x}.$$

Remark 3.19. (i) After some computation we find

$$\begin{aligned} e_0 \bullet (x_1 \otimes x_2) &= \tfrac{1}{2}\left(x_1 \otimes x_2 + \frac{1}{a}\mathbf{i}_1x_1 \otimes \mathbf{i}_2x_2\right), \\ e_1 \bullet (x_1 \otimes x_2) &= \tfrac{1}{2}\left(x_1 \otimes x_2 - \frac{1}{a}\mathbf{i}_1x_1 \otimes \mathbf{i}_2x_2\right), \end{aligned}$$

for all $x_1 \in C_1, x_2 \in C_2$. Note that this is independent of the choice of the base point u.

(ii) Let C be a composition algebra and define the bilinear skew-symmetric map

$$\psi : C \times C \to \mathcal{S} : (x, y) \mapsto x\overline{y} - y\overline{x}.$$

Then for all $x_1, y_1 \in C_1, x_2, y_2 \in C_2$ we have,

$$\boldsymbol{(}x_1 \otimes x_2, y_1 \otimes y_2\boldsymbol{)} = f_2(x_2, y_2)\psi(x_1, y_1) \otimes 1 + 1 \otimes f_1(x_1, y_1)\psi(x_2, y_2).$$

In the following theorem we show that the construction given in the introduction does indeed give rise to a quadrangular algebra of type E_6, E_7 or E_8. In the proof we make a distinction between the cases $\operatorname{char}(k) \neq 2$ and $\operatorname{char}(k) \neq 2, 3$. When $\operatorname{char}(k) \neq 2, 3$, $C_1 \otimes_k C_2$ is a structurable algebra, and we can use the theory of structurable algebras.

If $\operatorname{char}(k) = 3$ we cannot make use of the theory of structurable algebras; therefore we prove this in a direct way only making use of identities in octonions. Regrettably, this gives rise to lengthy computations, and for one particular identity we had to rely on computer algebra software [Sage]. This proof does not use the fact that the characteristic is equal to 3, but only that it is different from 2.

Theorem 3.20. *Let* $\operatorname{char}(k) \neq 2$. *Let* $e_0, e_1, u \in \mathcal{S}$ *be as in Lemma* 3.15*, let the quadratic form* Q *be of type* E_6, E_7, E_8 *and the reduced spin factor* J *be as in Definition* 3.16*. Let* $X := C_1 \otimes_k C_2$*, let* $\bullet$ *and* $\boldsymbol{(}.,.\boldsymbol{)}$ *be defined as above.*

Then X *is a special* J*-module and* $\boldsymbol{(}.,.\boldsymbol{)}$ *satisfies each side of* (3.1)*. Conditions* (3.4) *and* (3.5) *of Theorem* 3.5 *are satisfied. As in Theorem* 3.5 *we define*

$$\cdot : X_0 \times V \to X_0 : x \cdot v = v \bullet (u \bullet x),$$
$$h : X_0 \times X_0 \to V : (x, y) \mapsto \boldsymbol{(}u \bullet x, y\boldsymbol{)}.$$

Then $(k, V, Q, u, X_0, \cdot, h)$ *is a quadrangular algebra of type* E_6, E_7, E_8.

Proof. We have from Lemma 2.4 that $(e_0+e_1)\bullet x = x$ for all $x \in X$. The fact that X is a special J-module now follows from Lemma 3.17. It follows from Theorem 2.12 that if $(k, V, Q, u, X_0, \cdot, h)$ is a quadrangular algebra, it has to be of type E_6, E_7, E_8 due to the dimension of V.

char(k) $\neq$ 2, 3. In $\operatorname{char}(k) \neq 2, 3$ we can use the theory of structurable algebras to prove (3.1) and (3.4).

In [ABG, Remark 6.7] it is pointed out that each structurable algebra $\mathcal{A}$ (in our case $C_1 \otimes_k C_2$) with an invertible skew element is a J-ternary algebra with $J = L_{\mathcal{S}}L_r \subset \operatorname{End}_k(\mathcal{A})^+$. The action of the Jordan algebra on X and the skew bilinear map are defined as in Definition 3.18 above; the trilinear product is defined as

$$X \times X \times X \to X : (x, y, z) \mapsto -V_{x,ry}z := (x(\overline{y}r))z + (z(\overline{y}r))x + (z\overline{x})(ry).$$

[ABG] only considers fields of characteristic 0. We checked that every structurable algebra, with an invertible skew element, is a J-ternary algebra in characteristic different from 2 and 3. This proof is omitted in [ABG, Remark 6.7] and uses deep identities in structurable algebras. We thank Bruce Allison for giving us a detailed explanation of how to prove this fact.

For the proof of (3.5) we refer to the general characteristic case below. It now follows from Theorem 3.7 that $(k, V, Q, u, X_0, \cdot, h)$ is a quadrangular algebra.

char(k) $\neq$ 2. We first verify that the identity in the right hand side of (3.1) holds; this takes a rather lengthy but straightforward computation:

Since the condition is linear in x and y, one can choose $x = x_1 \otimes x_2$ and $y = y_1 \otimes y_2$ for $x_1, y_1 \in C_1, x_2, y_2 \in C_2$. Let $s = s_1 \otimes 1 + 1 \otimes s_2 \in \mathcal{S}$ and denote $r = r_1 \otimes 1 + 1 \otimes r_2$ instead of using its definition with coordinates. Using Remark 3.19(ii) it is not hard to show that the following identities hold for $i \in \{1, 2\}$:

- $f_i(s_i, \psi(x_i, y_i)) = -2f_i(s_i x_i, y_i)$,
- $\psi(s_i x_i, y_i) + \psi(s_i y_i, x_i) = 2s_i f_i(x_i, y_i)$,
- $f_i(s_i(r_i x_i), y_i) + f_i(s_i(r_i y_i), x_i) = -f_i(s_i, r_i) f_i(x_i, y_i)$.

Using these identities, (3.1) can be simplified to

$$\begin{aligned}\psi(s_i(r_i x_i), y_i) &- \psi(s_i(r_i y_i), x_i)\\ &= -f_i(r_i, \psi(x_i, y_i))s_i + f_i(s_i, \psi(x_i, y_i))r_i - f_i(s_i, r_i)\psi(x_i, y_i),\end{aligned}$$

and this identity can be checked using Lemma 2.1, especially the Moufang identities (v).

We were not able to verify (3.4) by hand. The problem is that (3.4) has degree 3 in x, so we cannot assume that x is of the form $e_0 \bullet (x_1 \otimes x_2)$. We did a computation based on a coordinatization of X; we used the software [Sage] to do the symbolic computations.

Now x is an arbitrary element in $X_0 = e_0 \bullet X$; therefore x is a sum of elements of the form $x_1 \otimes x_2 + \frac{1}{a}\mathbf{i}_1 x_1 \otimes \mathbf{i}_2 x_2$ (see Remark 3.19). We implemented octonions and the tensor product of two octonions in Sage in a symbolic way, and we verified that (3.4) holds.

The only fact that remains to be verified is (3.5). In fact, this is exactly axiom (D2), and in the proof of Theorem 3.5 the condition (3.5) is not used to prove any of the other axioms. Since we already know that the axioms A-B-C-D1 are true, we will use these to prove $\boldsymbol{(} u \bullet x, x \boldsymbol{)} \neq 0$ for all $x \in X_0 \setminus \{0\}$.

First we show that

$$\text{there exists an } x \in X_0 \text{ such that } \boldsymbol{(} u \bullet x, x \boldsymbol{)} \neq 0. \tag{3.9}$$

Let $x = e_0 \bullet (x_1 \otimes x_2) \in X_0, u = s_1 \otimes 1 + 1 \otimes s_2 \in V$. With some calculation using Lemma 2.1, Remark 3.19 and the coordinate expression for r we find that

$$\boldsymbol{(} u \bullet x, x \boldsymbol{)} = \frac{1}{4a}(q_2(x_2)\psi(s_1 x_1, \mathbf{i}_1 x_1) \otimes 1 + 1 \otimes q_1(x_1)\psi(s_2 x_2, \mathbf{i}_2 x_2)).$$

Since C_1 and C_2 are division algebras, it is enough to show that for all $y \in C_1 \setminus \{0\}$ we have $\psi(s_1 y, \mathbf{i}_1 y) \neq 0$. We assume that $y \neq 0$ and $\psi(s_1 y, \mathbf{i}_1 y) = 0$ and deduce a contradiction:

$$\begin{aligned}
& & \psi(s_1 y, \mathbf{i}_1 y) &= 0 & \\
&\Rightarrow & (s_1 y)(\overline{y}\mathbf{i}_1) - (\mathbf{i}_1 y)(\overline{y} s_1) &= 0 & \\
&\Rightarrow & 2(s_1 y)(\overline{y}\mathbf{i}_1) &= f_1(\mathbf{i}_1 y, s_1 y)1 & &\text{since } y\overline{z} + z\overline{y} = f_1(y,z)1\\
&\Rightarrow & s_1 y &= \tfrac{1}{2} f_1(\mathbf{i}_1 y, s_1 y)(\overline{y}\mathbf{i}_1)^{-1} & &\text{since } \overline{y}\mathbf{i}_1 \neq 0\\
&\Rightarrow & s_1 y &= -\frac{f_1(\mathbf{i}_1 y, s_1 y)}{2q_1(\overline{y}\mathbf{i}_1)}\mathbf{i}_1 y & &\text{since } y^{-1} = \frac{1}{q_1(y)}\overline{y}\\
&\Rightarrow & s_1 &= -\frac{f_1(\mathbf{i}_1 y, s_1 y)}{2q_1(\overline{y}\mathbf{i}_1)}\mathbf{i}_1. &
\end{aligned}$$

This is a contradiction since $s_1 \perp \mathbf{i}_1$.

The rest of the proof is inspired by the proof given in [TW, Theorem 13.47].

We fix an arbitrary $x \neq 0 \in X_0$. Notice that we no longer assume that x has the form $e_0 \bullet (x_1 \otimes x_2)$. We suppose that $\big(u \bullet x, x\big) = 0$ and aim to get a contradiction. It follows[8] from (3.4) that $\big(v \bullet x, x\big) = 0$ for all $v \in V$.

We first show that there exists an element $y \in X_0$ such that $\big(u \bullet x, y\big) \neq 0$. Suppose that $\big(u \bullet x, X_0\big) = 0$. It follows from (B2) that for all $y \in X_0, v \in V$,

$$\begin{aligned}\big(u \bullet x, v \bullet (u \bullet y)\big) &= \big(u \bullet y, v \bullet (u \bullet x)\big) \\ &= -\big(v \bullet (u \bullet x), u \bullet y\big) \\ &= -U_u\big(u \bullet (v \bullet (u \bullet x)), y\big).\end{aligned}$$

Therefore $\big(u \bullet (x \cdot v), X_0\big) = 0$, and by repeating this procedure we obtain

$$\big(u \bullet (x \cdot C(Q, u)), X_0\big) = 0.$$

From Definition 2.16(ii) and Theorem 2.17 it follows that X_0 is an irreducible $C(Q, u)$-module; therefore we obtain $\big(u \bullet X_0, X_0\big) = 0$. This contradicts (3.9).

From now on we assume that $y \in X_0$ is such that $\big(u \bullet x, y\big) \neq 0$. Next we show that

$$(x \cdot \big(u \bullet x, y\big)) \cdot v = x \cdot \big(u \bullet x, y \cdot v\big). \tag{3.10}$$

Since this identity is trivial for $u = v$, we assume $v \perp u$. Then (3.10) is equivalent to

$$\begin{aligned}\iff \quad & v \bullet u \bullet \big(u \bullet x, y\big) \bullet u \bullet x = \big(u \bullet x, v \bullet u \bullet y\big) \bullet u \bullet x \\ \iff \quad & v \bullet U_u\big(u \bullet x, y\big) \bullet x = -\big(u \bullet x, u \bullet v \bullet y\big) \bullet u \bullet x \\ \iff \quad & v \bullet \big(x, u \bullet y\big) \bullet x = -U_u\big(x, v \bullet y\big) \bullet u \bullet x \\ \iff \quad & v \bullet \big(x, u \bullet y\big) \bullet x = -u \bullet \big(x, v \bullet y\big) \bullet x \\ \iff \quad & v \bullet u \bullet \big(u \bullet y, x\big) \bullet x = \big(v \bullet y, x\big) \bullet x. \end{aligned} \tag{3.11}$$

We consider (3.4) for $y + tx$ for a parameter $t \in k$, we compare the terms that have degree one in t using the assumption that $\big(v \bullet x, x\big) = 0$ for all $v \in V$, and we get

$$\begin{aligned} & \big(v \bullet x, y\big) \bullet x + \big(v \bullet y, x\big) \bullet x = v \bullet u \bullet (\big(u \bullet x, y\big) + \big(u \bullet y, x\big)) \bullet x \\ \iff \quad & 2\big(v \bullet y, x\big) \bullet x + 2(v\big(x, y\big)) \bullet x = 2v \bullet u \bullet (\big(u \bullet y, x\big) + (u\big(x, y\big))) \bullet x \\ \iff \quad & \big(v \bullet y, x\big) \bullet x = v \bullet u \bullet \big(u \bullet y, x\big) \bullet x,\end{aligned}$$

since $v\big(x, y\big) \in J_1$ and $x \in X_0$. This proves (3.11).

From (3.10) we have $(x \cdot \big(u \bullet x, y\big)) \cdot V \subseteq x \cdot V$. Since $\big(u \bullet x, y\big) \neq 0$, the dimension of those two vector spaces is equal, and we find that

$$(x \cdot \big(u \bullet x, y\big)) \cdot V = x \cdot \big(u \bullet x, y \cdot V\big) = x \cdot V. \tag{3.12}$$

For arbitrary $w \in V$ it follows from (3.10) that

$$(x \cdot \big(u \bullet x, y \cdot w\big)) \cdot v = x \cdot \big(u \bullet x, (y \cdot w) \cdot v\big) \in x \cdot V.$$

From (3.12) we find that $(x \cdot V) \cdot V = x \cdot V$ and hence

$$x \cdot C(q, u) = x \cdot V \neq X_0,$$

contradicting the irreducibility of X_0. This finishes the proof of Theorem 3.20. □

[8] Since q_A is anisotropic on $V: v \bullet x = 0 \iff v \bullet v \bullet x = q_A(v)x = 0 \iff v = 0$ or $x = 0$.

Remark 3.21. The map $g : X_0 \times X_0 \to k : (x, y) \mapsto \frac{1}{2} f(h(y, x), 1)$ takes an elegant expression. Indeed,

$$\begin{aligned} g(x,y)e_0 &= \tfrac{1}{2} f(\boldsymbol{(}u \bullet y, x\boldsymbol{)}, u)e_0 \\ &= (\boldsymbol{(}u \bullet y, x\boldsymbol{)} u)e_0 \\ &= \tfrac{1}{2}(\boldsymbol{(}y, x\boldsymbol{)} + \boldsymbol{(}u \bullet y, u \bullet x\boldsymbol{)})e_0. \end{aligned}$$

Since $\boldsymbol{(}y, x\boldsymbol{)} \in ke_0$ and $\boldsymbol{(}u \bullet y, u \bullet x\boldsymbol{)} \in ke_1$, we conclude that $g(x, y)e_0 = \frac{1}{2}\boldsymbol{(}y, x\boldsymbol{)}$. When we identify k and ke_0, we have

$$g(x, y) = \tfrac{1}{2}\boldsymbol{(}y, x\boldsymbol{)}.$$

It follows from the previous theorem that we have, in characteristic not 2, a new coordinate-free definition of the various maps introduced in [TW, Chapter 13].

Remark 3.22. The reader might wonder what will happen if we apply our construction in the case that both C_1 and C_2 are composition algebras of dimension 2 or 4 with mutual linkage number 1. In the three different cases that arise in this way, we get the following dimensions for the different relevant vector spaces:

	$E \otimes E$	$E \otimes Q$	$Q_1 \otimes Q_2$
$\dim_k \mathcal{S}$	2	4	6
$\dim_k L$	0	2	4
$\dim_k(C_1 \otimes_k C_2)$	4	8	16
$\dim_k \widetilde{X}$	2	4	8

In the first case, the vector space L is trivial, so our construction no longer applies (we cannot find an element $u \in V \setminus \{0\}$ needed in Definition 3.16).

In the second case, the space $\tilde{X}$ gets the structure of a two-dimensional vector space over E, and the corresponding quadrangular algebra is isomorphic to a quadrangular algebra of pseudo-quadratic form type with underlying vector space $\tilde{X}$. Notice that $E \otimes Q \cong M_2(E)$ since E and Q are 1-linked.

Similarly, in the third case, the space $\tilde{X}$ gets the structure of a two-dimensional vector space over a quaternion division algebra Q_3, and the corresponding quadrangular algebra is isomorphic to a quadrangular algebra of pseudo-quadratic form type with underlying vector space $\tilde{X}$. The algebra Q_3 is the quaternion algebra with norm form similar to q_A, and in this case $Q_1 \otimes Q_2 \cong M_2(Q_3)$.

3.4.3. *A final remark.* In an earlier paper [BD], we found another related but quite different class of structurable algebras that seems to play an important role in the understanding of the exceptional Moufang quadrangles of type E_6, E_7 and E_8. That structurable algebra is, in each case, obtained by *doubling* another algebra (instead of *halving* an algebra as we did in the current paper).

More precisely, to each Moufang quadrangle Ω of type E_6, E_7 or E_8, we can associate a structurable algebra Y, the isotopy class of which is a complete invariant of the Moufang quadrangle Ω, and which is obtained by applying the so-called Cayley–Dickson doubling process on the Jordan algebra A^+, where

(i) A is a quaternion algebra Q if Ω is of type E_6;
(ii) A is a tensor product $Q \otimes E$ with Q a quaternion algebra and E a quadratic extension if Ω is of type E_7;
(iii) A is a biquaternion algebra $Q_1 \otimes Q_2$ if Ω is of type E_8.

However, we are not yet aware of a direct way of relating the structurable algebra Y with the structurable algebra $X = C_1 \otimes C_2$ which we have investigated in the current paper.

4. A unified construction for Moufang quadrangles in characteristic not 2

4.1. **Preliminaries on Moufang quadrangles.** A *Moufang polygon* is a notion from incidence geometry introduced by Jacques Tits. We give only a brief summary of the theory of Moufang quadrangles, and we refer to [TW] for more details. The importance will immediately become clear in Theorem 4.1 below.

A *generalized quadrangle* Γ is a connected bipartite graph with diameter 4 and girth 8. We call a generalized polygon *thick* if every vertex has at least three neighbors. A *root* in Γ is a (non-stammering) path of length 4 in Γ.

Let Γ be a thick generalized quadrangle, and let $\alpha = (x_0, \dots, x_4)$ be a root of Γ. Then the group U_α of all automorphisms of Γ fixing all neighbors of x_1, x_2, x_3 (called a *root group*) acts freely on the set of vertices incident with x_0 but different from x_1. If U_α acts transitively on this set (and hence regularly), then we say that α is a *Moufang root*.

A *Moufang quadrangle* is a generalized quadrangle for which every root is Moufang. We then also say that Γ satisfies the *Moufang condition*.

Moufang quadrangles have been classified by J. Tits and R. Weiss [TW]. Loosely speaking, the result is the following.

Theorem 4.1 ([TW]). *Every Moufang quadrangle arises from an absolutely simple linear algebraic group of relative rank two or from a corresponding classical group or group of mixed type.*

In particular, every Moufang quadrangle is of "algebraic origin", and in fact, the Moufang quadrangles provide a useful tool to help in the understanding of the corresponding groups; this is particularly true for the Moufang quadrangles arising from linear algebraic groups of exceptional type. For instance, the Kneser–Tits problem for groups of type $E_{8,2}^{66}$ has recently been solved using the theory of Moufang polygons [PTW].

In order to describe a Moufang quadrangle in terms of algebraic data, we will use so-called *root group sequences.* A root group sequence for a Moufang quadrangle is a sequence of four root groups, labeled $U_1, \dots, U_4$, together with *commutator relations* describing how elements of two different root groups U_i and U_j commute. In each case, the commutator of an element of U_i and U_j (with $i < j$) belongs to the group $\langle U_{i+1}, \dots, U_{j-1} \rangle$. The following result is crucial.

Theorem 4.2. *Let Γ be a Moufang quadrangle. Then Γ is completely determined by the root groups $U_1, \dots, U_4$ together with their commutator relations.*

Proof. See [TW, Chapter 7]. □

For more details about this procedure and how the Moufang polygons can be reconstructed from the root group sequences, we refer to [TW] or to the survey article [DV].

For each type of Moufang quadrangle, we will describe an *algebraic structure* which will allow us to parametrize the root groups and describe the commutator relations.

In principle, it is possible to define a single algebraic structure to describe all possible Moufang quadrangles; this gives rise to the so-called *quadrangular systems* which have been introduced by the second author [D1]. These structures, however, have some disadvantages from an algebraic point of view; most notably, the definition does not mention an underlying field of definition (although it is possible to construct such a field from the data), and the axiom system looks very wild and complicated, with no less than 20 defining identities.

Below is the original classification as given by Tits and Weiss in [TW], distinguishing six different (non-disjoint) classes:

(1) Moufang quadrangles of indifferent type;
(2) Moufang quadrangles of quadratic form type;
(3) Moufang quadrangles of involutory type;
(4) Moufang quadrangles of pseudo-quadratic form type;
(5) Moufang quadrangles of type E_6, E_7 and E_8;
(6) Moufang quadrangles of type F_4.

The Moufang quadrangles of types (2)–(4) are often called *classical*, those of type (5) and (6) are called *exceptional* and those of type (1) are of *mixed type.* Since the Moufang quadrangles of type (1) and (6) exist only over fields of characteristic 2, and moreover are not directly related to rank two forms of algebraic groups, we exclude those two classes from our further discussion.

In the following section we will give a uniform description of the remaining four classes of Moufang quadrangles, over fields of characteristic different from 2, starting from a special Jordan module.

4.2. **Construction of Moufang quadrangles from special Jordan modules.** We will show that each type of Moufang quadrangle in characteristic not 2 can be described in a unified way from a special J-module. We generalize the procedure that we used in Theorem 3.5 to obtain quadrangular algebras. In order to obtain all Moufang quadrangles we allow that $\dim(J_0) > 1$ and we allow the special J-module to be the trivial module. It follows from Theorem 4.2 that it is sufficient to describe the four root groups and the commutator relations of the root groups to describe the Moufang quadrangle completely.

Construction 4.3. Let J be a non-degenerate Jordan algebra that contains supplementary proper idempotents e_0 and e_1. Let $J_0, J_{1/2}, J_1$ be the Peirce subspaces of J with respect to e_1. We assume that each element in $J_{1/2} \setminus \{0\}$ is invertible and that there exists $u \in J_{1/2}$ such that $u^2 = 1$.

Let X be a special J-module equipped with a skew-symmetric bilinear form $\boldsymbol{(}\cdot,\cdot\boldsymbol{)} : X \times X \to J$.

- Define the abelian group $V := (J_{1/2}, +)$.
- Define the (not necessary abelian) group $W := X_0 \times J_0$ with addition

$$[a_1, t_1] \boxplus [a_2, t_2] = [a_1 + a_2, t_1 + t_2 + \tfrac{1}{2}\boldsymbol{(}a_2, a_1\boldsymbol{)}].$$

Notice that the inverse is $\boxminus[a, t] = [-a, -t]$.

Let U_1 and U_3 be two groups isomorphic to W, and let U_2 and U_4 be two groups isomorphic to V. Denote the corresponding isomorphisms by

$$\begin{array}{l} x_1 : W \to U_1 : [a,t] \mapsto x_1(a,t) ; \\ x_2 : V \to U_2 : v \mapsto x_2(v) ; \\ x_3 : W \to U_3 : [a,t] \mapsto x_3(a,t) ; \\ x_4 : V \to U_4 : v \mapsto x_4(v) ; \end{array}$$

we say that U_1 and U_3 are *parametrized* by W and that U_2 and U_4 are *parametrized* by V.

Now, we implicitly define the group $U_+ = \langle U_1, U_2, U_3, U_4 \rangle$ by the following commutator relations:

$$\begin{aligned} [x_1(a_1,t_1), x_3(a_2,t_2)^{-1}] &= x_2\big(\boldsymbol{(}u \bullet a_1, a_2\boldsymbol{)}\big) , \\ [x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, 2(v_1 v_2)e_0\big) , \\ [x_1(a,t), x_4(v)^{-1}] &= x_2\big(\tfrac{1}{2}\boldsymbol{(}u \bullet a, v \bullet (u \bullet a)\boldsymbol{)} + 2(U_u t)v\big) x_3\big(v \bullet (u \bullet a), U_v U_u t\big), \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1,2,3\} , \end{aligned}$$

for all $[a,t], [a_1,t_1], [a_2,t_2] \in W$ and all $v, v_1, v_2 \in V$.

It follows from Lemma 2.23 that J should be either of reduced spin type or of type $\mathcal{H}(M_2(L), \sigma t)$. For each of these two cases, we will distinguish between the zero J-module and a non-zero special J-module. Case by case, we will show that in this way the root groups U_1, U_2, U_3, U_4 and commutation relations given above coincide with the description given in Chapter 16 of [TW] of the Moufang quadrangles in characteristic not 2.

Remark 4.4. In [D2] quadrangular systems are introduced. These are structures that are defined by 24 axioms, which describe in a unified way all Moufang quadrangles (including characteristic 2.) We believe it should be possible to start with Construction 4.3, impose a few more axioms that look like the ones in Theorem 3.5 and prove all the axioms defining a quadrangular system. However the verifications of the axioms that use the map κ (this is a kind of "multiplicative inverse" in the group W) get very complicated.

Moufang quadrangles of quadratic form type. Let J be a reduced spin factor of an anisotropic, non-degenerate quadratic space (k, V, q) with base point u. Let X be the zero module over J.

Remember that $J_0 = ke_0, J_{1/2} = V, J_1 = ke_1$.

- Define the abelian group $V = (J_{1/2}, +)$.
- Define the group $W = X_0 \times J_0 = \{[0, te_0] | t \in k\} \cong k$ with addition $[0, t_1 e_0] \boxplus [0, t_2 e_0] = [0, (t_1 + t_2)e_0]$. Therefore W is isomorphic to the additive group of k with corresponding isomorphism $W \cong k : [0, te_0] \leftrightarrow t$. So we will write $x_1(t) := x_1(0, te_0)$ and $x_3(t) := x_3(0, te_0)$.

Let U_1 and U_3 be parametrized by W and U_2 and U_4 be parametrized by V. Let $t, t_1, t_2 \in k$, $v, v_1, v_2 \in V$. Using the formulas for the multiplication and the U-operator in a Jordan algebra of reduced spin type (see Definition 2.20) we find for the commutator relations:

$$\begin{aligned}
[x_1(t_1), x_3(t_2)^{-1}] &= [x_1(0, t_1e_0), x_3(0, t_2e_0)^{-1}] = x_2\big(0\big) = 1 \ , \\
[x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, f(v_1, v_2)e_0\big) = x_3\big(f(v_1, v_2)\big) \ , \\
[x_1(t), x_4(v)^{-1}] &= [x_1(0, te_0), x_4(v)^{-1}] = x_2\big(2(U_u te_0)v\big)x_3\big(0, U_vU_u te_0\big) \\
&= x_2\big(2(te_1)v\big)x_3\big(0, U_v te_1\big) = x_2\big(tv\big)x_3\big(0, q(v)te_0\big) \\
&= x_2\big(tv\big)x_3\big(q(v)t\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ .
\end{aligned}$$

We obtain exactly the same description as in [TW, Example 16.3].

If $d = \dim_K V$ is finite, then these Moufang quadrangles arise from linear algebraic groups; they are of absolute type $\mathsf{B}_{\ell+2}$ if $d = 2\ell + 1$ is odd, and of type $\mathsf{D}_{\ell+2}$ if $d = 2\ell$ is even.

Moufang quadrangles of type E_6, E_7, E_8. This case was actually already handled in Theorem 3.20, since from quadrangular algebras one can define the root groups and commutation relations of the corresponding Moufang quadrangles; see [W, Chapter 11]. Now we quickly verify that we indeed get the right commutator relations using Construction 4.3.

Let J be a reduced spin factor of an anisotropic, non-degenerate quadratic space (k, V, q) with base point u, let $X = C_1 \otimes_k C_2$ and let the skew-symmetric form $\boldsymbol{(}.,.\boldsymbol{)}$ be as in Section 3.4. Since quadrangular algebras of type E_6, E_7 and E_8 are determined by the similarity class of their quadratic space, the quadrangular algebras we constructed in Theorem 3.20 are identical to the ones in [TW, Chapter 13]. Therefore the following maps coincide with the maps defined in [TW, Chapter 13]:

$$a \cdot v = v \bullet (u \bullet a), \quad h(a, b) = \boldsymbol{(}u \bullet a, b\boldsymbol{)}, \quad g(a, b)e_0 = \tfrac{1}{2}\boldsymbol{(}b, a\boldsymbol{)}.$$

Now define

- the abelian group $V = (J_{1/2}, +)$;
- the group $W = X_0 \times J_0 \cong X_0 \times k$ with addition $[a_1, t_1e_0] \boxplus [a_2, t_2e_0] = [a_1 + a_2, t_1e_0 + t_2e_0 + \frac{1}{2}\boldsymbol{(}a_2, a_1\boldsymbol{)}]$. When we identify $J_0 \cong k : te_0 \leftrightarrow t$, we get

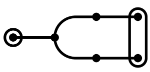
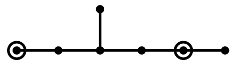
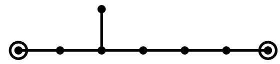
$$[a_1, t_1] \boxplus [a_2, t_2] = [a_1 + a_2, t_1 + t_2 + g(a_1, a_2)].$$

We will write $x_1(a, t) := x_1(a, te_0)$ and $x_3(a, t) := x_3(a, te_0)$.

Let U_1 and U_3 be parametrized by W and U_2 and U_4 be parametrized by V. Let $t, t_1, t_2 \in k$, $v, v_1, v_2 \in V$; we find the following commutator relations:

$$\begin{aligned}
[x_1(a_1, t_1), x_3(a_2, t_2)^{-1}] &= x_2\big(\boldsymbol{(}u \bullet a_1, a_2\boldsymbol{)}\big) = x_2\big(h(a_1, a_2)\big) \ , \\
[x_2(v_1), x_4(v_2)^{-1}] &= x_3\big(0, f(v_1, v_2)e_0\big) = x_3\big(0, f(v_1, v_2)\big) \ , \\
[x_1(a, t), x_4(v)^{-1}] &= [x_1(a, te_0), x_4(v)^{-1}] \\
&= x_2\big(\tfrac{1}{2}\boldsymbol{(}u \bullet a, a \cdot v\boldsymbol{)} + 2(te_1)v\big)x_3\big(a \cdot v, U_v te_1\big) \\
&= x_2\big(\theta(a, v) + tv\big)x_3\big(a \cdot v, q(v)t\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ .
\end{aligned}$$

We obtain exactly the same description as in [TW, Example 16.6]. The Tits indices of the corresponding linear algebraic groups are as follows:

${}^2E_6$:  E_7 :  E_8 : 

Moufang quadrangles of involutory type. As in [TW, Chapter 11], we define an involutory set in characteristic different from 2, as a triple (L, L_σ, σ), where L is a field or a skew-field, σ is an involution of L, $L_\sigma = \{\ell \in L \mid \ell^\sigma = \ell\} = \{\ell + \ell^\sigma \mid \ell \in L\}$. (Notice that the definition in characteristic 2 is much more involved.)

Let $J = \mathcal{H}(M_2(L), \sigma t)$ (see Definition 2.21) and let X be the zero module.

- Define the abelian group $V = (J_{1/2}, +) \cong (L, +)$ with isomorphism
$$\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \leftrightarrow \ell.$$
We will write $x_2(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_2(\ell)$ and $x_4(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_4(\ell)$.
- Define the group $W = X_0 \times J_0 = \{[0, \alpha e_0] | \alpha \in L_\sigma\} \cong L_\sigma$ with addition $[0, \alpha_1 e_0] \boxplus [0, \alpha_2 e_0] = [0, (\alpha_1 + \alpha_2) e_0]$. Therefore W is isomorphic to the additive group of L_σ; we use the isomorphism $W \cong L_\sigma : [0, \alpha e_0] \leftrightarrow \alpha$. We will write $x_1(0, \alpha e_0) = x_1(\alpha)$ and $x_3(0, \alpha e_0) = x_3(\alpha)$.

Let U_1 and U_3 be parametrized by W and U_2 and U_4 be parametrized by V. Let $\alpha, \alpha_1, \alpha_2 \in L_\sigma$ and consider the following elements of V:
$$v = \begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 & \ell_1^\sigma \\ \ell_1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & \ell_2^\sigma \\ \ell_2 & 0 \end{bmatrix} \in J_{1/2}.$$
Using the formulas for the multiplication and the U-operator in $\mathcal{H}(M_2(L), \sigma t)$, we find for the commutator relations:
$$\begin{aligned}
[x_1(\alpha_1), x_3(\alpha_2)^{-1}] &= [x_1(0, \alpha_1 e_0), x_3(0, \alpha_2 e_0)^{-1}] = x_2(0) = 1 \ , \\
[x_2(\ell_1), x_4(\ell_2)^{-1}] &= [x_2(v_1), x_4(v_2)^{-1}] = x_3\big(0, ((\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1) e_0 + (\ell_1 \ell_2^\sigma + \ell_2 \ell_1^\sigma) e_1) e_0\big) \\
&= x_3\big(0, (\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1) e_0\big) = x_3\big(\ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1\big) \ , \\
[x_1(\alpha), x_4(\ell)^{-1}] &= [x_1(0, \ell e_0), x_4(v)^{-1}] = x_2\big(2(U_u \alpha e_0) v\big) x_3\big(0, U_v U_u \alpha e_0\big) \\
&= x_2\big(2(\alpha e_1) v\big) x_3\big(0, U_v \alpha e_1\big) = x_2\left(\begin{bmatrix} 0 & \ell^\sigma \alpha \\ \alpha \ell & 0 \end{bmatrix}\right) x_3\big(0, \ell^\sigma \alpha \ell e_0\big) \\
&= x_2\big(\alpha \ell\big) x_3\big(\ell^\sigma \alpha \ell\big) \ , \\
[U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ .
\end{aligned}$$
This is exactly [TW, Example 16.2]. If L is finite-dimensional over its center, of degree d, then these Moufang quadrangles arise from algebraic groups; they are outer forms of A_{4d-1} if the involution is of the second kind, and they are (inner or outer forms) of absolute type D_{2d} if the involution is of the first kind.

Moufang quadrangles of pseudo-quadratic type. These are obtained in similar fashion as the quadrangular algebras in Section 3.3, but here we start from an arbitrary skew-field with involution instead of starting from a quadratic pair. We repeat part of the setup from Section 3.3.

Let L be a skew-field with involution σ. Let (L, σ, X, h, π) be a standard pseudo-quadratic space (see Section 2.3.1), so $\pi(a) = \frac{1}{2} h(a, a)$ for all $a \in X$.

Let $J = \mathcal{H}(M_2(L), \sigma t)$ (see Definition 2.21) and let $\widetilde{X} = X^2$, the 1×2 row vectors over X. For the action of J on $\widetilde{X}$, for $j \in J, a \in \widetilde{X}$ we have $j \bullet a = aj \in \widetilde{X}$.

As before we define $e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in J$. We have $\widetilde{X}_0 = \{[a,0] \mid a \in X\}, \widetilde{X}_1 = \{[0,a] \mid a \in X\}$.

We define the skew product $\widetilde{X} \times \widetilde{X} \to J$ as

$$\big([a_1,a_2],[b_1,b_2]\big) = \begin{bmatrix} h(a_1,b_1) - h(b_1,a_1) & h(a_1,b_2) - h(b_1,a_2) \\ -h(b_2,a_1) + h(a_2,b_1) & h(a_2,b_2) - h(b_2,a_2) \end{bmatrix}.$$

- Define the abelian group $V = (J_{1/2}, +) \cong (L, +)$ with isomorphism

$$\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix} \leftrightarrow \ell.$$

  We will write $x_2(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_2(\ell)$ and $x_4(\begin{bmatrix} 0 & \ell^\sigma \\ \ell & 0 \end{bmatrix}) = x_4(\ell)$.
- Define the group $W = \widetilde{X}_0 \times J_0 \cong X \times L_\sigma$. When we identify $J_0 \cong L_\sigma$: $\alpha e_0 \leftrightarrow \alpha$ and $\widetilde{X}_0 \cong X : [a,0] \leftrightarrow a$, the addition on W is given by

$$[a_1,\alpha_1] \boxplus [a_2,\alpha_2] = [a_1 + a_2, \alpha_1 + \alpha_2 + \tfrac{1}{2}(h(a_2,a_1) - h(a_1,a_2))].$$

  We will write $x_1(a,\alpha) := x_1([a,0], \alpha e_0)$ and $x_3(a,\alpha) := x_3([a,0], \alpha e_0)$.

For the commutator relations we obtain

$$\begin{aligned} [x_1(a_1,\alpha_1), x_3(a_2,\alpha_2)^{-1}] &= x_2\big(h(a_1,a_2)\big) , \\ [x_2(\ell_1), x_4(\ell_2)^{-1}] &= x_3\big(0, \ell_1^\sigma \ell_2 + \ell_2^\sigma \ell_1\big) , \\ [x_1(a,\alpha), x_4(\ell)^{-1}] &= x_2\big(\theta(a,\ell) + \alpha\ell\big) x_3\big(a\ell, \ell^\sigma \alpha \ell\big) , \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1,2,3\} , \end{aligned}$$

for all $[a,t], [a_1,t_1], [a_2,t_2] \in W$ and all $\ell, \ell_1, \ell_2 \in L$.

In [TW, Example 16.5] U_1 and U_3 are parametrized by the subset $T = \{[a,t] \mid \exists \alpha \in L_\sigma : \pi(a) = t + \alpha\} \subset X \times L$. We consider the bijection

$$\phi : T \to X \times L_\sigma : [a,t] \mapsto [a, -\pi(a) + t].$$

When we translate the group law and commutator relations in [TW, Example 16.5] from T to $X \times L_\sigma$ using ϕ, we indeed obtain the expressions written above.

If L is finite-dimensional over its center, of degree d, and X is finite-dimensional over L, then these Moufang quadrangles arise from algebraic groups. If the involution is of the second kind, they are outer forms of absolute type A_ℓ. If the involution is of the first kind, they are of absolute type C_ℓ or D_ℓ.

Acknowledgments

Some of the authors' ideas were inspired by fruitful discussions with Skip Garbaldi, in particular during a longer visit of the first author at Emory University, whose hospitality is gratefully acknowledged. Skip Garibaldi's observation mentioned in Theorem 3.11 was a crucial first step in the whole project. The authors also thank Bruce Allison for fruitful discussions about J-ternary algebras. Last but not least, the authors are greatly indebted to the referee for a spectacularly detailed and careful reading of their paper.

References

[A1] B. N. Allison, *A class of nonassociative algebras with involution containing the class of Jordan algebras*, Math. Ann. **237** (1978), no. 2, 133–156, DOI 10.1007/BF01351677. MR507909 (81h:17003)

[A2] B. N. Allison, *Structurable division algebras and relative rank one simple Lie algebras*, Lie algebras and related topics (Windsor, Ont., 1984), CMS Conf. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 1986, pp. 139–156. MR832197 (87j:17001)

[A3] B. N. Allison, *Tensor products of composition algebras, Albert forms and some exceptional simple Lie algebras*, Trans. Amer. Math. Soc. **306** (1988), no. 2, 667–695, DOI 10.2307/2000817. MR933312 (89e:17004)

[ABG] B. N. Allison, G. Benkart, and Y. Gao, *Lie algebras graded by the root systems* BC_r, $r \geq 2$, Mem. Amer. Math. Soc. **158** (2002), no. 751, x+158, DOI 10.1090/memo/0751. MR1902499 (2003h:17038)

[BD] L. Boelaert and T. De Medts, *Exceptional Moufang quadrangles and structurable algebras*, Proc. London Math. Soc. (3) **107** (2013), no. 3, 590–626. MR3100778

[D1] T. De Medts, *A characterization of quadratic forms of type* E_6, E_7, *and* E_8, J. Algebra **252** (2002), no. 2, 394–410, DOI 10.1016/S0021-8693(02)00064-9. MR1925144 (2003f:11046)

[D2] T. De Medts, *An algebraic structure for Moufang quadrangles*, Mem. Amer. Math. Soc. **173** (2005), no. 818, vi+99, DOI 10.1090/memo/0818. MR2109785 (2005k:51003)

[DV] T. De Medts and H. Van Maldeghem, *Moufang generalized polygons*, Topics in diagram geometry, Quad. Mat., vol. 12, Dept. Math., Seconda Univ. Napoli, Caserta, 2003, pp. 59–126. MR2066523 (2005d:51007)

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. MR1632779 (2000a:16031)

[L] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR2104929 (2005h:11075)

[M] K. McCrimmon, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004. MR2014924 (2004i:17001)

[PTW] R. Parimala, J.-P. Tignol, and R. M. Weiss, *The Kneser-Tits conjecture for groups with Tits-index* $\mathsf{E}_{8,2}^{66}$ *over an arbitrary field*, Transform. Groups **17** (2012), no. 1, 209–231, DOI 10.1007/s00031-011-9165-2. MR2891217

[Sage] W. A. Stein et al., *Sage Mathematics Software* (Version 4.6.1), The Sage Development Team, 2011, `http://www.sagemath.org`.

[SV] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. MR1763974 (2001f:17006)

[TW] J. Tits and R. M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR1938841 (2003m:51008)

[W] R. M. Weiss, *Quadrangular algebras*, Mathematical Notes, vol. 46, Princeton University Press, Princeton, NJ, 2006. MR2177056 (2007a:17010)

Department of Mathematics, Ghent University, Krijgslaan 281, S22, B-9000 Gent, Belgium

E-mail address: `lboelaer@cage.UGent.be`

Department of Mathematics, Ghent University, Krijgslaan 281, S22, B-9000 Gent, Belgium

E-mail address: `tdemedts@cage.UGent.be`