TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 5, May 2015, Pages 3423–3446
S 0002-9947(2014)06498-2
Article electronically published on November 24, 2014

MULTI-SCALING LIMITS FOR RELATIVISTIC DIFFUSION EQUATIONS WITH RANDOM INITIAL DATA

GI-REN LIU AND NARN-RUEIH SHIEH

Abstract. Let $u(t,\mathbf{x}),\ t>0,\ \mathbf{x}\in\mathbb{R}^n$, be the spatial-temporal random field arising from the solution of a relativistic diffusion equation with the spatial-fractional parameter $\alpha\in(0,2)$ and the mass parameter $\mathfrak{m}>0$, subject to a random initial condition $u(0,\mathbf{x})$ which is characterized as a subordinated Gaussian field. In this article, we study the large-scale and the small-scale limits for the suitable space-time re-scalings of the solution field $u(t,\mathbf{x})$. Both the Gaussian and the non-Gaussian limit theorems are discussed. The small-scale scaling involves not only scaling on $u(t,\mathbf{x})$ but also re-scaling the initial data; this is a new type result for the literature. Moreover, in the two scalings the parameter $\alpha\in(0,2)$ and the parameter $\mathfrak{m}>0$ play distinct roles for the scaling and the limiting procedures.

1. Introduction

The relativistic operator

$$\mathfrak{m}-(\mathfrak{m}^{\frac{2}{\alpha}}-\Delta)^{\frac{\alpha}{2}},$$

with $\alpha=1$ and $\mathfrak{m}>0$, appeared in the 1970's (to our knowledge) in an article of Lieb [23] studying the stability theory of matters. Its connection with Lévy processes is investigated by Carmona *et al.* in [8]. In Ryznar [29], the version for general $0<\alpha<2$ is studied; see also Baeumer *et al.* [4], Kumara *et al.* [18], and Chen *et al.* [9] for more recent studies. The operator should be understood as a pseudo-differential operator, as that in [8] and [34]. The Fourier transform of the heat kernel associated with the operator shows an interesting multi-scaling property, as that observed in [30, 31].

The purpose of this article is to present this multi-scaling property from the associated PDE. Namely, we consider the following Cauchy problem for the relativistic diffusion equation (RDE for brevity), *subject to some random initial data*, and aim to discuss the multi-scaling limits for the spatial-temporal random field arising from the solution of this random initial value problem:

$$\frac{\partial}{\partial t}u(t,\mathbf{x})=(\mathfrak{m}-(\mathfrak{m}^{\frac{2}{\alpha}}-\Delta)^{\frac{\alpha}{2}})u(t,\mathbf{x}),\quad u(0,\mathbf{x})=u_0(\mathbf{x}),\ t\geq 0,\ \mathbf{x}\in\mathbb{R}^n. \tag{1.1}$$

Received by the editors July 7, 2012 and, in revised form, October 1, 2012 and May 7, 2013.
2010 *Mathematics Subject Classification.* Primary 60G60, 60H05, 62M15, 35K15.
Key words and phrases. Large-scale limits, small-scale limits, relativistic diffusion equations, random initial data, multiple Itô-Wiener integrals, subordinated Gaussian fields, Hermite ranks.
The first author was partially supported by a Taiwan NSC grant for graduate students.

RDEs (1.1) also appear to be an essential role in the theory of computer vision; see a special volume edited by Kimmel *et al.* [16], in which PDE and scale-space methods are the focus and RDEs are particularly employed.

We describe the structure and the goal of this article as follows. We consider the random initial data u_0 to be subordinated Gaussian random fields and study the large-scale and the small-scale limits for the properly re-scaled solution field. We prove that the two parameters α and $\mathfrak{m} > 0$ play distinct roles in the two scaling behaviors. For the large-scale limit (Theorem 1 and Theorem 3), it is the mass $\mathfrak{m} > 0$ that dominates the space-time scaling and also the limiting field, which brings the $\mathfrak{m} > 0$ into its structure. While for the small-scale limit (Theorem 2 and Theorem 4), it is the spatial index α that dominates both the scaling factor and the limiting field, and it appears to be irrelevant for $\mathfrak{m}$ if it is positive or zero.

In our discussions, the large-scale Theorem 1 and Theorem 3 are respectively comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in [7] and to a certain non-Gaussian Central Limit Theorem in [11, 33]. For the small-scale Theorem 2 and Theorem 4, they involve not only the space-time scaling $u(t, \mathbf{x}),\ t > 0,\ \mathbf{x} \in \mathbb{R}^n$, but also need to re-scale the initial data; to our knowledge, these are new type results for the literature; see [24] for the authors' very recent study.

As for the methodology for proofs, for the Gaussian limits we employ the moments and the Feymann-type diagrams used notably in [7], and for the non-Gaussian limits we employ the truncation of Hermite expansions used notably in [1, 2].

We remark that in the non-relativistic case, i.e. $\mathfrak{m} = 0$, the large-scale limits for the random initial value problem with multiple Itô-Wiener integrals as input have been discussed in Anh and Leonenko [1, 2]; subsequent works, together with Burgers' equation, in this direction by the authors and collaborators can be seen in [3, 5, 14, 20–22, 28] and the references therein. However, the multi-scaling limits due to the different roles of the mass and the fractional-index, the target of this article, are *not all* in the cited papers. Moreover, in this article we are able to drop-off the usually imposed isotropic assumption of the initial datum.

We should also mention that in an article discussing tempered stable Lévy processes by Rosiński [27], the author proves rigorously, among others, the statement that such a process in a short time looks like a stable process, while in a large time scale it looks like a Brownian motion. This article has shown nicely how the multi-scaling limits appear in the context of stochastic processes. (We are indebted to the referee for indicating to us the article [27] and the relevant concepts.)

In Section 2, we present some preliminaries; we state our main results in Section 3, and all the proofs of our results are given in Section 4.

Finally, we mention that the study of the PDEs with random initial conditions can be traced back to [15] and [26]. Besides the above-mentioned literature, there also has been very significant progress on Burgers' equation with different types of random input; see the monograph of Woyczyński [36] and Chapter 6 of Bertoin [6].

2. Preliminaries

2.1. **Green function for RDEs.** As understood, we regard the spatial operator in the RDE (1.1) as a pseudo-differential operator; see for example the book and the paper by Wong [34, 35]. The Green function, denoted by $G_{\alpha,\mathfrak{m}}(t, \mathbf{x}),\ t > 0, \mathbf{x} \in \mathbb{R}^n$, for the Cauchy problem (1.1) is thus determined by the (spatial) Fourier transform

$\widehat{G}_{\alpha,\mathfrak{m}}(t,\lambda),\ \alpha\in(0,2),\ \mathfrak{m}>0$, which is given by

$$\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}\rangle}G_{\alpha,\mathfrak{m}}(t,\mathbf{x})d\mathbf{x}=e^{-t\{(\mathfrak{m}^{\frac{2}{\alpha}}+|\lambda|^2)^{\frac{\alpha}{2}}-\mathfrak{m}\}},\ \lambda\in\mathbb{R}^n. \tag{2.1}$$

See Carmona *et al.* [8] for $\alpha=1$ and Ryznar [29] for general $\alpha\in(0,2)$ ([29] also considers the boundary problem). These papers also study $G_{\alpha,\mathfrak{m}}(t,\mathbf{x}),\ \mathfrak{m}>0$, as the transition probability density of a Lèvy process $X_{\alpha,\mathfrak{m}}(t)$ which is the subordination of the Brownian motion by a certain subordinator. The explicit expression for the Green function is known only in the case $\alpha=1$; see for example the recent works [4, 18], which give explicit calculations to show that the subordinator is normal inverse Gaussian.

The solution of (1.1) is given in the form

$$u(t,\mathbf{x};u_0(\cdot))=\int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(t,\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y}. \tag{2.2}$$

In this work, our initial data is a second-order homogeneous random field on $\mathbb{R}^n$, and thus the solution of (1.1) should be understood as a mean-square solution, resulting in a spatial-temporal random solution field $u(t,\mathbf{x})$. See [28, Proposition 1] for some discussions on the mean-square solutions of parabolic PDEs with mean-square random initial data.

2.2. **Subordinated Gaussian fields as initial data.** Let $(\Omega,\mathcal{F},\mathcal{P})$ be an underlying probability space such that all random elements appearing in this article are measurable with respect to it.

We specify the initial data $u_0(\mathbf{x})$ be a subordinated Gaussian field, introduced by Dobrushin [10], as follows; see also [1, 2] for more recent discussions.

Condition A. The initial data of (1.1) is assumed to be a random field on $\mathbb{R}^n$ given by

$$u_0(\mathbf{x})=h(\zeta(\mathbf{x})),\ \mathbf{x}\in\mathbb{R}^n, \tag{2.3}$$

where $\zeta(\mathbf{x})$ is a mean-square continuous and homogeneous Gaussian random field with mean zero and variance 1, and its spectral measure $F(d\lambda)$ has the (spectral) density $f(\lambda),\ \lambda\in\mathbb{R}^n$; moreover, $h:\mathbb{R}\to\mathbb{R}$ is a (non-random) function such that

$$\mathbb{E}h^2(\zeta(\mathbf{0}))=\int_{\mathbb{R}} h^2(r)p(r)dr<\infty;\quad p(r)=\frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}},\ r\in\mathbb{R}. \tag{2.4}$$

Under Condition A, by the Bochner-Khintchine theorem, we have the following spectral representation for the covariance function of the Gaussian field $\zeta(\mathbf{x})$:

$$R(\mathbf{x})=\mathrm{Cov}(\zeta(\mathbf{0}),\zeta(\mathbf{x}))=\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}\rangle}f(\lambda)d\lambda. \tag{2.5}$$

Moreover, by the Karhunen Theorem, $\zeta(\mathbf{x})$ has the representation

$$\zeta(\mathbf{x})=\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}\rangle}\sqrt{f(\lambda)}W(d\lambda),\ \mathbf{x}\in\mathbb{R}^n, \tag{2.6}$$

where $W(d\lambda)$ is the standard complex-valued Gaussian white noise on the Fourier domain $\mathbb{R}^n$, that is, a centered orthogonal-scattered Gaussian random measure on $\mathbb{R}^n$ such that $W(\Delta_1)=\overline{W(-\Delta_1)}$ and $\mathbb{E}W(\Delta_1)\overline{W(\Delta_2)}=\mathrm{Leb}(\Delta_1\cap\Delta_2)$ for any $\Delta_1,\Delta_2\in\mathcal{B}(\mathbb{R}^n)$. See, for example, the book of Leonenko [19, Theorem 1.1.3] for the above facts. We need the following expansion of $h(r)$ in the Hilbert space $L^2(\mathbb{R}, p(r)dr)$:

$$h(r) = C_0 + \sum_{l=1}^{\infty} C_l \frac{H_l(r)}{\sqrt{l!}}, \tag{2.7}$$

where

$$C_l = \int_{\mathbb{R}} h(r) \frac{H_l(r)}{\sqrt{l!}} p(r)dr \tag{2.8}$$

and $\{H_l(r),\ l = 0, 1, 2, \dots\}$ are the Hermite polynomials, that is,

$$H_l(r) = (-1)^l e^{\frac{r^2}{2}} \frac{d^l}{dr^l} e^{-\frac{r^2}{2}}, \quad \text{for} \quad l \in \{0, 1, 2, \dots\}.$$

Accordingly, the *Hermite rank* of the function $h(\cdot)$ is defined by

$$m := \inf\{l \geq 1 : \ C_l \neq 0\}.$$

It is well-known that (see, for example, Major [25, Corollary 5.5 and p. 30])

$$\mathbb{E}[H_{l_1}(\zeta(\mathbf{y}))H_{l_2}(\zeta(\mathbf{z}))] = \delta_{l_2}^{l_1} l_1! R^{l_1}(\mathbf{y} - \mathbf{z}), \quad \mathbf{y},\ \mathbf{z} \in \mathbb{R}^n \tag{2.9}$$

($\delta_{\sigma_2}^{\sigma_1}$ is the Kronecker symbol), and

$$H_l(\zeta(\mathbf{x})) = \int'_{\mathbb{R}^{n\times l}} e^{i\langle \mathbf{x}, \lambda_1 + \cdots + \lambda_l\rangle} \prod_{k=1}^{l} \sqrt{f(\lambda_k)} W(d\lambda_k). \tag{2.10}$$

In the above, (2.10) means the *multiple Itô-Wiener integral* representation, and the integration $\int'$ means that it excludes the diagonal hyperplanes $\mathbf{z}_i = \mp \mathbf{z}_j,\ i, j = 1, \dots, l, i \neq j$.

We impose two different conditions on the singularity of the spectral density $f(\lambda)$ at $\mathbf{0}$ which yield, respectively, the Gaussian and the non-Gaussian scaling limits.

Condition B. The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(\mathbf{x})$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}} \text{ for some } \kappa > \frac{n}{m}, \tag{2.11}$$

where m is the Hermite rank of the function h, and the $B(\cdot) \in \mathrm{C}(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$.

Condition C. The spectral density function $f(\lambda)$ of the Gaussian random field $\zeta(\mathbf{x})$ in Condition A can be written as

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}}, \quad 0 < \kappa < \frac{n}{m}, \tag{2.12}$$

where m is the Hermite rank of the function h, and the $B(\cdot) \in \mathrm{C}(\mathbb{R}^n)$ is of suitable decay at infinity to ensure $f \in L^1(\mathbb{R}^n)$, and moreover $B(\mathbf{0}) > 0$.

Note that in the two conditions, we do not assume that the $B(\cdot)$ is radial in $\cdot$, so that the field $u_0(\mathbf{x})$ is not necessary to be isotropic. We also mention that Condition B means that the density f either is regular at $\mathbf{0}$ or has a singularity for which the order is less than $n(1 - 1/m)$, while Condition C means that f has a singularity at $\mathbf{0}$ for which the order is higher than $n(1 - 1/m)$.

By (2.5) and the convolutions, we have, for each $l \geq 1$,

$$R^l(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}\rangle} f^{*l}(\lambda)d\lambda,\ l \in \mathbb{N}, \tag{2.13}$$

where $f^{*l}(\lambda)$ is the l-fold convolution of f defined recursively as $f^{*1} = f$ and

$$f^{*l}(\lambda) = \int_{\mathbb{R}^n} f(\lambda-\eta)f^{*(l-1)}(\eta)d\eta,\ l \geq 2.$$

The following analytic lemma asserts the behavior of $f^{*l},\ l \in \mathbb{N}$; for completeness, we give its proof in Appendix A.

Lemma 1. *Suppose that the spectral density function f has the form*

$$f(\lambda) = \frac{B(\lambda)}{|\lambda|^{n-\kappa}},\ \kappa > 0,$$

for some non-negative bounded and continuous function $B(\lambda)$ so that $f \in L^1(\mathbb{R}^n)$. Then for any $k \geq 2$ there exists a bounded function $B_k \in \mathrm{C}(\mathbb{R}^n \backslash \{\mathbf{0}\})$ such that the k-fold convolution f^{*k} of f can be written as

$$f^{*k}(\lambda) = \begin{cases} B_k(\lambda)|\lambda|^{k\kappa-n}, & \text{for } k\kappa < n, \\ B_k(\lambda)\ln(2+\frac{1}{|\lambda|}), & \text{for } k\kappa = n, \\ B_k(\lambda) \in \mathrm{C}(\mathbb{R}^n), & \text{for } k\kappa > n. \end{cases} \tag{2.14}$$

Moreover, for any $k_1 > k_2 > n/\kappa$ the inequality $\sup_{\lambda\in\mathbb{R}^n} B_{k_1}(\lambda) \leq \sup_{\lambda\in\mathbb{R}^n} B_{k_2}(\lambda)$ holds.

To understand the difference of Conditions B and C, in view of Lemma 1, Condition B implies that the k-fold convolution f^{*k}, $k \geq m$, has no singularity at the origin $\lambda = \mathbf{0}$, which in turn asserts that the spectral density of the random initial data u_0 has no singularity at $\lambda = \mathbf{0}$, while Condition C asserts that the initial data u_0 has a spectral density which is singularity at $\lambda = \mathbf{0}$. The situation can be described as, respectively, the long-range and the short-range dependence of the initial field u_0, a central notion in vast applications; one may refer to the special volume by Doukhan, Oppenheim, and Taqqu [12].

3. Main results

The significant difference between Condition B and Condition C, as remarked at the end of the last section, is employed to obtain the Gaussian and respectively the non-Gaussian scaling limits. We will present them in two subsections.

In the context henceforth, the notation $\Rightarrow$ denotes the convergence of random variables (respectively, random families) in the sense of distribution (respectively, finite-dimensional distributions).

3.1. Gaussian limits with initial data in (A, B).

As mentioned in Section 1, we will present the large-scale and the small-scale limit theorems. We remark that our Theorems 1 and 2 in this subsection are comparable to the Central Limit Theorem for local functionals of random fields with weak dependence in Breuer and Major [7]. The novel feature is that the mass $\mathfrak{m} > 0$ and the fractional-index α play different roles in the two-scales.

Theorem 1. *Let $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, be the mean-square solution of* (1.1) *with $\mathfrak{m}>0$ and let the initial data $u_0(\mathbf{x})=h(\zeta(\mathbf{x}))$ satisfy Conditions A and B with the Hermite rank $m\geq 1$. Then when $T\to\infty$,*

$$T^{\frac{n}{4}}\left\{u(Tt,\sqrt{T}\mathbf{x};u_0(\cdot))-C_0\right\}\Rightarrow U(t,\mathbf{x}),$$

where $U(t,\mathbf{x})$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, is a Gaussian field with the following spectral representation:

$$U(t,\mathbf{x})=\int_{\mathbb{R}^n}e^{i\langle\lambda,\mathbf{x}\rangle}\sigma_m e^{-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2}W(d\lambda),\ \sigma_m=\Big(\sum_{r=m}^{\infty}f^{*r}(\mathbf{0})C_r^2\Big)^{\frac{1}{2}}, \tag{3.1}$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on $\mathbb{R}^n$ (*cf.* (2.6)).

For the small-scale limit, we need to re-scale the initial data too; thus the notation $u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot)$ imposed on u_0 means that the variable of u_0 is under the indicated dilation factor $\varepsilon^{-\frac{1}{\alpha}-\chi}$.

Theorem 2. *Let $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, be the mean-square solution of* (1.1) *with $\mathfrak{m}>0$ and let the initial data $u_0(\mathbf{x})=h(\zeta(\mathbf{x}))$ satisfy Conditions A and B with the Hermite rank $m\geq 1$. For any $\chi>0$, when $\varepsilon\to 0$,*

$$\varepsilon^{-\frac{n\chi}{2}}\left\{u(\varepsilon t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x};u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot))-C_0\right\}\Rightarrow V(t,\mathbf{x}), \tag{3.2}$$

where $V(t,\mathbf{x})$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, is a Gaussian field with the following spectral representation:

$$V(t,\mathbf{x})=\int_{\mathbb{R}^n}e^{i\langle\lambda,\mathbf{x}\rangle}\sigma_m e^{-t|\lambda|^\alpha}W(d\lambda),\ \sigma_m=\Big(\sum_{r=m}^{\infty}f^{*r}(\mathbf{0})C_r^2\Big)^{\frac{1}{2}}, \tag{3.3}$$

where $W(d\lambda)$ is a complex-valued standard Gaussian noise measure on $\mathbb{R}^n$.

Remark. The typical case for Theorem 2 is $\alpha=1,\chi=1/2$. In this critical case, the scaling order for Theorems 1 and 2 is the same, namely $n/4$. However, the spatial scaling is square-root in Theorem 1, while it is linear in Theorem 2. Moreover, the integral kernel for the limiting field in the two theorems is Gauss vs. Poisson. The latter situation can be comparable with an analytic discussion in Wong [34].

3.2. **Non-Gaussian limits with initial data in (A,C).** As in the above subsection, we have the large-scale and the small-scale limits; however the high singularity order in Condition C asserts that our limiting fields are now non-Gaussian. The non-Gaussian limits are of the convolution type, which can be seen in the pioneering papers of Taqqu [33] and Dobrushin and Major [11], and more recently in Anh and Leonenko [1,2].

Theorem 3. *Let $u(t,\mathbf{x};u_0(\cdot))$, $t>0$, $\mathbf{x}\in\mathbb{R}^n$, be the mean-square solution of* (1.1) *whose initial data $\{u_0(\mathbf{x})=h(\zeta(\mathbf{x})),\ \mathbf{x}\in\mathbb{R}^n\}$ satisfy Conditions A and C with $\kappa\in(0,\frac{n}{m})$ and $1<m$, where m is the Hermite rank of the non-random function h on $\mathbb{R}$, which has the Hermite coefficients $C_j, j=0,1,\ldots.$ Then when $T\to\infty$,*

$$T^{\frac{m\kappa}{4}}\left\{u(Tt,\sqrt{T}\mathbf{x};h(\zeta(\cdot)))-C_0\right\}\Rightarrow U_m(t,\mathbf{x}), \tag{3.4}$$

where $U_m(t,\mathbf{x})$ is represented by the following multiple Wiener integrals:

(3.5)

$$U_m(t,\mathbf{x})=B^{\frac{m}{2}}(\mathbf{0})\frac{C_m}{\sqrt{m!}}\int'_{\mathbb{R}^{n\times m}}e^{i\langle\mathbf{x},\lambda_1+\cdots+\lambda_m\rangle}\frac{\exp(-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda_1+\cdots+\lambda_m|^2)}{(|\lambda_1|\cdots|\lambda_m|)^{\frac{n-\kappa}{2}}}\prod_{l=1}^{m}W(d\lambda_l),$$

where $\int'_{\mathbb{R}^{n\times m}}\cdots$ denotes an m-fold Wiener integral with respect to the complex Gaussian white noise $W(\cdot)$ on $\mathbb{R}^n$.

Theorem 4. *Let $u(t,\mathbf{x};u_0(\cdot))$ be the mean-square solution to* (1.1) *whose initial data $\{u_0(\mathbf{x})=h(\zeta(\mathbf{x})),\ \mathbf{x}\in\mathbb{R}^n\}$ satisfy Conditions A and C with $\kappa\in(0,\frac{n}{m})$ and $1<m$, where m is the Hermite rank of the function h. Then, for any fixed parameter $\chi>0$, when $\varepsilon\to 0$,*

$$\varepsilon^{-\frac{m\kappa\chi}{2}}\Big\{u(\varepsilon t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x};h(\zeta((\varepsilon^{-\frac{1}{\alpha}-\chi})\cdot)))-C_0\Big\}\Rightarrow V_m(t,\mathbf{x}),\tag{3.6}$$

where $V_m(t,\mathbf{x})$ is represented by the multiple Wiener integrals

$$V_m(t,\mathbf{x})=B^{\frac{m}{2}}(\mathbf{0})\frac{C_m}{\sqrt{m!}}\int'_{\mathbb{R}^{n\times m}}e^{i\langle\mathbf{x},\lambda_1+\cdots+\lambda_m\rangle}\frac{\exp(-t|\lambda_1+\cdots+\lambda_m|^{\alpha})}{(|\lambda_1|\cdots|\lambda_m|)^{\frac{n-\kappa}{2}}}\prod_{l=1}^{m}W(d\lambda_l).\tag{3.7}$$

Remark. In [2] the authors considered a hybrid differential operator in the spatial variable (the Riesz-Bessel operator) as follows:

$$-(-\Delta)^{\alpha/2}(I-\Delta)^{\gamma/2},\ \alpha\in(0,2),\ \gamma\geq 0.$$

However, in their main Theorem 2.3, a large-scale limit in our context, only the Riesz parameter α plays the role and the Bessel parameter γ is invisible. This intriguing situation is now justified by the RFD (1.1), in which we could say that it is "physically correct" to consider the relativistic operator $(\mathfrak{m}-(\mathfrak{m}^{\frac{2}{\alpha}}-\Delta)^{\frac{\alpha}{2}})$ rather than the Bessel operator in the form presented in [2].

4. Proofs of theorems

The following two-scale property of the relativistic Green function $G_{\alpha,\mathfrak{m}}$ is the key to our results; when one deals with the Laplacian or the fractional Laplacian operator, it is instead only the mono-scaling. We describe this two-scale property in terms of Fourier transforms:

$$\widehat{G}_{\alpha,\mathfrak{m}}(Tt,T^{-\frac{1}{2}}\lambda)=\exp\Big\{Tt(\mathfrak{m}-(\mathfrak{m}^{\frac{2}{\alpha}}+T^{-1}|\lambda|^2)^{\frac{\alpha}{2}})\Big\}\to\exp\Big\{-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2\Big\},\tag{4.1}$$

as $T\to\infty$; (4.1) is a consequence of Taylor's expansion,

$$\begin{aligned}\mathfrak{m}-(\mathfrak{m}^{\frac{2}{\alpha}}+T^{-1}|\lambda|^2)^{\frac{\alpha}{2}}&=\mathfrak{m}-\Big(\mathfrak{m}+\frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-1}|\lambda|^2+\frac{\alpha}{4}(\frac{\alpha}{2}-1)c_T^{\frac{\alpha}{2}-2}T^{-2}|\lambda|^4\Big)\\&=-\frac{\alpha}{2}(\mathfrak{m}^{\frac{2}{\alpha}})^{\frac{\alpha}{2}-1}T^{-1}|\lambda|^2+\frac{\alpha}{4}(1-\frac{\alpha}{2})c_T^{\frac{\alpha}{2}-2}T^{-2}|\lambda|^4\end{aligned}$$

for some $c_T\in(\mathfrak{m}^{\frac{2}{\alpha}},\mathfrak{m}^{\frac{2}{\alpha}}+T^{-1}|\lambda|^2)$. In contrast to the large-scale (4.1), we have the following small-scale, as $\varepsilon\to 0$:

$$\widehat{G}_{\alpha,\mathfrak{m}}(\varepsilon t,\varepsilon^{-\frac{1}{\alpha}}\lambda)=e^{\varepsilon t\mathfrak{m}}e^{-\varepsilon t(\mathfrak{m}^{\frac{2}{\alpha}}+\varepsilon^{-\frac{2}{\alpha}}|\lambda|^2)^{\frac{\alpha}{2}}}\to e^{-t|\lambda|^{\alpha}}.\tag{4.2}$$

We observe that (4.2) indeed holds no matter whether $\mathfrak{m}$ is >0 or $=0$.

Proofs of Theorems 1 *and* 2. We apply the Hermite expansion (2.7) to $u(t,x)$. For the large-scale, we set

$$\begin{aligned}X_T(t,\mathbf{x}) &= T^{\frac{n}{4}}u(Tt,\sqrt{T}\mathbf{x};u_0(\cdot))-C_0\\ &= T^{\frac{n}{4}}\sum_{k=m}^{\infty}\frac{C_k}{\sqrt{k!}}\int_{\mathbb{R}^n}G_{\alpha,\mathfrak{m}}(Tt,\sqrt{T}\mathbf{x}-\mathbf{y})H_k(\zeta(\mathbf{y}))d\mathbf{y},\end{aligned}$$

and for the small-scale, we set

$$\begin{aligned}Y_\varepsilon(t,\mathbf{x}) :=&\varepsilon^{-\frac{n\chi}{2}}u(\varepsilon t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x};u_0(\varepsilon^{-\frac{1}{\alpha}-\chi}\cdot))-C_0\\ =&\varepsilon^{-\frac{n\chi}{2}}\sum_{l=m}^{\infty}\frac{C_l}{\sqrt{l!}}\int_{\mathbb{R}^n}G_{\alpha,\mathfrak{m}}(\varepsilon t,\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y})H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}.\end{aligned}$$

Below, we proceed only with the proof of Theorem 2, the small-scale limit, and see how the re-scaling of the initial data is needed to obtain the desired limit. The proof of Theorem 1 is parallel and does not require the re-scaling of the initial data. Since the proof in the following does not require the $\mathfrak{m}$ to be strictly positive, our Theorem 2 also provides a small-scale version of the large-scale, i.e. the usual, limit result in [2]. The methodology of the proof can be traced back to [7].

For any $M\in\mathbb{N}$ and any set of real numbers $\{a_1,a_2,\cdots,a_M\}$, denote

$$\xi_\varepsilon := \sum_{j=1}^{M}a_jY_\varepsilon(t_j,\mathbf{x}_j), \tag{4.3}$$

where $\{t_1,\cdots,t_M\}\subset\mathbb{R}_+$ and $\{\mathbf{x}_1,\cdots,\mathbf{x}_M\}\subset\mathbb{R}^n$ are arbitrary. In order to apply the Method of Moments to prove the statement of Theorem 2, we need to verify the following:

$$\lim_{\varepsilon\to0}\mathbb{E}\xi_\varepsilon^p=\begin{cases}0, & p=2\nu+1,\\ (p-1)!!\Big\{\mathbb{E}\Big[\big(\sum_{j=1}^{M}a_jV(t_j,\mathbf{x}_j)\big)^2\Big]\Big\}^\nu, & p=2\nu,\end{cases} \tag{4.4}$$

where $V(t,\mathbf{x})$ is defined in (3.3). We remark that the high (i.e. $p>2$) moments are needed, since ξ_ε is not Gaussian, though the desired limit is Gaussian. Firstly, we split ξ_ε into two parts:

$$\xi_\varepsilon=\xi_{\varepsilon,\le N}+\xi_{\varepsilon,>N}, \tag{4.5}$$

where (henceforth, we will suppress the indices α and $\mathfrak{m}$ for $G_{\alpha,\mathfrak{m}}$ and $\widehat{G}_{\alpha,\mathfrak{m}}$)

$$\xi_{\varepsilon,>N}=\sum_{j=1}^{M}a_j\varepsilon^{-\frac{n\chi}{2}}\sum_{l=N+1}^{\infty}\frac{C_l}{\sqrt{l!}}\int_{\mathbb{R}^n}G(\varepsilon t_j,\varepsilon^{\frac{1}{\alpha}}\mathbf{x}_j-\mathbf{y})H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}, \tag{4.6}$$

and we prove that $E[\xi^2_{\varepsilon,>N}] \to 0$, whenever N is chosen large enough. Observe that for any $N \geq m-1$, by (2.9),

(4.7)

$$\mathbb{E}(\xi_{\varepsilon,>N})^2 = \mathbb{E}\Big[\big(\sum_{j=1}^{M} a_j \varepsilon^{-\frac{n\chi}{2}} \sum_{l=N+1}^{\infty} \frac{C_l}{\sqrt{l!}} \int_{\mathbb{R}^n} G(\varepsilon t_j, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_j - \mathbf{y}) H_l(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}))d\mathbf{y}\big)^2\Big]$$

$$= \sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2}\varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^{2n}} G(\varepsilon t_{j_1}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_1} - \mathbf{y}_1) G(\varepsilon t_{j_2}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_2} - \mathbf{y}_2)$$
$$\times\ R^l(\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y}_1 - \mathbf{y}_2)))$$
$$= \sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2}\varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(\varepsilon(t_{j_1}+t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z}) R^l(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{z})d\mathbf{z},$$

where the last equality is followed by a change of variables, the symmetry property $G(t,\mathbf{z}) = G(t,-\mathbf{z})$ of the transition probability density function G, and its semigroup property

$$\int_{\mathbb{R}^n} G(\varepsilon t_{j_1}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_1} - (\mathbf{z}-\mathbf{z}'))G(\varepsilon t_{j_2}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_2} - \mathbf{z}')d\mathbf{z}'$$
$$= G(\varepsilon(t_{j_1}+t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z}).$$

Continuing to (4.7), by the spectral representation (2.13) for the k-th power of the covariance function $R(\cdot)$, we see that (4.7) is equal to

(4.8)

$$\sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2}\varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} G(\varepsilon(t_{j_1}+t_{j_2}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) - \mathbf{z})$$
$$\times \int_{\mathbb{R}^n} e^{i\langle \varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{z},\lambda\rangle} f^{*l}(\lambda)d\lambda d\mathbf{z}$$
$$= \sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2}\varepsilon^{-n\chi} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i\varepsilon^{-\chi}\langle\lambda,\mathbf{x}_{j_1}-\mathbf{x}_{j_2}\rangle} \widehat{G}(\varepsilon(t_{j_1}+t_{j_2}), \varepsilon^{-\frac{1}{\alpha}-\chi}\lambda) f^{*l}(\lambda)d\lambda$$
$$= \sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2} \sum_{l=N+1}^{\infty} C_l^2 \int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_{j_1}-\mathbf{x}_{j_2}\rangle}$$
$$\times\ \exp\{\varepsilon(t_{j_1}+t_{j_2})[\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + |\varepsilon^{-\frac{1}{\alpha}}\lambda|^2)^{\frac{\alpha}{2}}]\} f^{*l}(\varepsilon^{\chi}\lambda)d\lambda$$
$$\to \sum_{j_1,j_2=1}^{M} a_{j_1}a_{j_2} \sum_{l=N+1}^{\infty} C_l^2 f^{*l}(\mathbf{0}) \int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_{j_1}-\mathbf{x}_{j_2}\rangle}\exp\{-(t_{j_1}+t_{j_2})|\lambda|^{\alpha}\}d\lambda < \infty$$

when $\varepsilon \to 0$, where $f^{*l}(\cdot)$, $l \geq m$, are continuous and uniformly bounded on $\mathbb{R}^n$ since Condition B and Lemma 1 imply:

(4.9)

$$f^{*l}(\lambda) = \int_{\mathbb{R}^n} f^{*m}(\lambda-\eta) f^{*(l-m)}(\eta)d\eta \leq \|B_m\|_\infty \int_{\mathbb{R}^n} f^{*(l-m)}(\eta)d\eta = \|B_m\|_\infty \quad \forall l > m.$$

From (4.8), for any $\delta > 0$ there exists $N_0 \in \mathbb{N}$, $\varepsilon_0 > 0$ such that

$$\mathbb{E}(\xi_{\varepsilon,>N})^2 < \delta, \text{ for any } N \geq N_0, \ \varepsilon < \varepsilon_0, \tag{4.10}$$

which implies that we need to prove a truncated version of (4.4) as follows:

$$\lim_{\varepsilon\to 0} \mathbb{E}\xi^p_{\varepsilon,\leq N_0} = \begin{cases} 0, & p = 2\nu+1, \\ (p-1)!!\Big\{\mathbb{E}\Big[\big(\sum\limits_{j=1}^{M} a_j V_{m,N_0}(t_j,\mathbf{x}_j)\big)^2\Big]\Big\}^\nu, & p = 2\nu, \end{cases} \tag{4.11}$$

where

(4.12)

$$V_{m,N_0}(t,\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}\rangle}\sigma_{m,N_0}e^{-t|\lambda|^\alpha}W(d\lambda) \ \text{ with } \sigma_{m,N_0} = \Big(\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2\Big)^{\frac{1}{2}}.$$

By (4.5) for the definition of $\xi_{\varepsilon,\leq N_0} (= \xi_\varepsilon - \xi_{\varepsilon,>N_0})$ and our rescaling of the initial data, we have

$$\begin{aligned}
\mathbb{E}(\xi_{\varepsilon,\leq N_0})^p =& \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1,\cdots,j_p=1}^{M} \sum_{l_1,\cdots,l_p=m}^{N_0} \Big[\prod_{i=1}^{p} a_{j_i}\frac{C_{l_i}}{\sqrt{l_i!}}\Big] \\
&\times \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_i} - \mathbf{y}_i)\Big\}\Big[\mathbb{E}\prod_{i=1}^{p} H_{l_i}(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{y}_i))\Big] d\mathbf{y}_1\cdots d\mathbf{y}_p \\
=& \varepsilon^{-\frac{pn\chi}{2}} \sum_{j_1,\cdots,j_p=1}^{M} \sum_{l_1,\cdots,l_p=m}^{N_0} \Big[\prod_{i=1}^{p} a_{j_i}\frac{C_{l_i}}{\sqrt{l_i!}}\Big] \qquad (4.13)\\
&\times \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}}\mathbf{x}_{j_i} - \varepsilon^{\frac{1}{\alpha}}\mathbf{y}_i)\Big\}\Big[\mathbb{E}\prod_{i=1}^{p} H_{l_i}(\zeta(\varepsilon^{-\chi}\mathbf{y}_i))\Big] d\mathbf{y}_1\cdots d\mathbf{y}_p.
\end{aligned}$$

To analyze $\mathbb{E}(\xi_{\varepsilon,\leq N_0})^p$, $p = 2\nu$ (the odd $p = 2\nu+1$ is unnecessary, since all the involved random variables are centered), we employ the diagram method (see [7] or [13, p. 72]). A graph Γ with $l_1 + \cdots + l_p$ vertices is called a (complete) diagram of order $(l_1, \cdots, l_p)$ if:

(a) the set of vertices V of the graph Γ is of the form $V = \bigcup_{j=1}^{p} W_j$, where $W_j = \{(j,l) : 1 \leq l \leq l_j\}$ is the j-th level of the graph Γ, $1 \leq j \leq p$;

(b) each vertex is of degree 1; that is, each vertex is just an endpoint of an edge;

(c) if $((j_1,l_1), (j_2,l_2)) \in \Gamma$, then $j_1 \neq j_2$; that is, the edges of the graph Γ may connect only different levels.

Let $\mathrm{T} = \mathrm{T}(l_1,\cdots,l_p)$ be a set of (complete) diagrams of Γ of order $(l_1,\cdots,l_p)$. Denote by $E(\Gamma)$ the set of edges of the graph $\Gamma \in \mathrm{T}$. For the edge $e = ((j_1,l'_1), (j_2,l'_2)) \in E(\Gamma)$, $j_1 < j_2$, $1 \leq l'_1 \leq l_1$, $1 \leq l'_2 \leq l_2$, we set $d_1(e) = j_1$, $d_2(e) = j_2$, to denote the location of the edge e in Γ. We call a diagram Γ *regular* if its levels can be split into pairs in such a manner that no edge connects the levels belonging to different pairs. Denote by $\mathrm{T}^* = \mathrm{T}^*(l_1,\cdots,l_p)$ the set of all regular diagrams in T. Therefore, if $\Gamma \in \mathrm{T}^*$ is a regular diagram, then it can be divided into $p/2$ sub-diagrams (denoted by $\Gamma_1,\cdots,\Gamma_{p/2}$), which cannot be separated again; in this case, we naturally define $d_1(\Gamma_i) \equiv d_1(e)$ and $d_2(\Gamma_i) \equiv d_2(e)$ for any $e \in E(\Gamma_i)$, $i = 1,\ldots,\nu = p/2$. We denote by $\sharp E(\Gamma)$ (resp. $\sharp E(\Gamma_j)$) the number of edges belonging to the specific diagram Γ (resp. the sub-diagram Γ_j).

Based on the notation above let

$$D_p = \{(J,L) : J = (j_1, \cdots, j_p), 1 \le j_i \le M, \\ L = (l_1, \cdots, l_p), m \le l_i \le N_0, i = 1, \cdots, p\},$$

we see that (4.13) can be rewritten as

$$\mathbb{E}(\xi_{\varepsilon,\le N_0})^p = \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in \mathrm{T}^*} F_\Gamma(J,L,\varepsilon) + \sum_{(J,L)\in D_p} K(J,L) \sum_{\Gamma\in \mathrm{T}\setminus\mathrm{T}^*} F_\Gamma(J,L,\varepsilon), \tag{4.14}$$

where

$$K(J,L) = \prod_{i=1}^{p} a_{j_i} \frac{C_{l_i}}{\sqrt{l_i!}}, \tag{4.15}$$

$$F_\Gamma(J,L,\varepsilon) = \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big\{ \prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{j_i}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{j_i} - \mathbf{y}_i)) \Big\} \\ \times \Big[\prod_{e\in E(\Gamma)} R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)} - \mathbf{y}_{d_2(e)})) \Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p.$$

Next, we want to verify two things:

$$\begin{cases} (1)\ \lim\limits_{\varepsilon\to 0} \sum\limits_{(J,L)\in D_p} K(J,L) \sum\limits_{\Gamma\in\mathrm{T}^*} F_\Gamma(J,L,\varepsilon) = (p-1)!! \Big\{ \mathbb{E}\Big[\big(\sum\limits_{j=1}^{M} a_j V_{m,N_0}(t_j,\mathbf{x}_j) \big)^2 \Big] \Big\}^{p/2}, \\ (2)\ \lim\limits_{\varepsilon\to 0} \sum\limits_{(J,L)\in D_p} K(J,L) \sum\limits_{\Gamma\in\mathrm{T}\setminus\mathrm{T}^*} F_\Gamma(J,L,\varepsilon) = 0. \end{cases}$$

Proof of (1). As argued above, for each $\Gamma \in \mathrm{T}^*$, the case $p = 2\nu$, $\nu \in \mathbb{N}$, has a unique decomposition into sub-diagrams $\Gamma = (\Gamma_1, \cdots, \Gamma_\nu)$, for which each one cannot be further decomposed. Accordingly, we can rewrite $F_\Gamma(J,L,\varepsilon)$ as the following $\nu = p/2$ products:

$$\begin{aligned} &F_\Gamma(J,L,\varepsilon) \\ &= \varepsilon^{-\frac{pn\chi}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^{2n}} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{d_1(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{y})) \varepsilon^{\frac{n}{\alpha}} G(\varepsilon t_{d_2(\Gamma_i)}, \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_2(\Gamma_i)} - \mathbf{y}')) \\ &\qquad \times R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}(\mathbf{y} - \mathbf{y}')) \\ &= \varepsilon^{-\frac{pn\chi}{2}} \prod_{i=1}^{\nu} \int_{\mathbb{R}^n} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} - \mathbf{z})) R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}\mathbf{z}) d\mathbf{z}. \end{aligned} \tag{4.16}$$

We note that

$$R^{\sharp E(\Gamma_i)}(\varepsilon^{-\chi}\mathbf{z}) = \varepsilon^{n\chi} \int_{\mathbb{R}^n} e^{i\langle \mathbf{z},\lambda\rangle} f^{*\sharp E(\Gamma_i)}(\varepsilon^{\chi}\lambda) d\lambda,\ i = 1, \cdots, \nu, \tag{4.17}$$

and $\sharp E(\Gamma_i) > n/\kappa$ (since $\kappa > n/m$ in Condition B). By the Fourier transform of G,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{i\langle \mathbf{z},\lambda\rangle} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)}), \varepsilon^{\frac{1}{\alpha}}(\mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)} - \mathbf{z})) d\mathbf{z} \\ &= e^{i\langle \lambda, \mathbf{x}_{d_1(\Gamma_i)} - \mathbf{x}_{d_2(\Gamma_i)}\rangle} \exp\{\varepsilon(t_{d_1(\Gamma_i)} + t_{d_2(\Gamma_i)})[\mathfrak{m} - (\mathfrak{m}^{\frac{2}{\alpha}} + |\varepsilon^{-\frac{1}{\alpha}}\lambda|^2)^{\frac{\alpha}{2}}]\}, \end{aligned} \tag{4.18}$$

and applying the small-scale of G illustrated in (4.2), we have

(4.19)
$$\lim_{\varepsilon\to 0} F_\Gamma(J,L,\varepsilon)=\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\int e^{i\langle\lambda,\mathbf{x}_{d_1(\Gamma_i)}-\mathbf{x}_{d_2(\Gamma_i)}\rangle}\exp\{-(t_{d_1(\Gamma_i)}+t_{d_2(\Gamma_i)})|\lambda|^\alpha\}d\lambda.$$

Meanwhile, if the $L=\{l_1,\cdots,l_{2\nu}\}$ in the defining equation (4.15) of $K(J,L)$ corresponds to a regular diagram Γ in $\mathrm{T}(l_1,\cdots,l_{2\nu})$, then

$$K(J,L)=\prod_{i=1}^{\nu} a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)}\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}. \tag{4.20}$$

Therefore, by (4.19) and (4.20),

$$\begin{aligned}(4.21)\qquad &\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in\mathrm{T}^*}F_\Gamma(J,L,\varepsilon)\\ &=\sum_{(J,L)\in D_{2\nu}}\sum_{\Gamma\in\mathrm{T}^*}\Big[\prod_{i=1}^{\nu}a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)}\int e^{i\langle\lambda,\mathbf{x}_{d_1(\Gamma_i)}-\mathbf{x}_{d_2(\Gamma_i)}\rangle}\\ &\qquad\times\exp\{-(t_{d_1(\Gamma_i)}+t_{d_2(\Gamma_i)})|\lambda|^\alpha\}d\lambda\Big]\Big[\prod_{i=1}^{\nu}f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big].\end{aligned}$$

We note that all components in the first bracket in (4.21) are independent of the index set L and the summation $\sum_{\Gamma\in\mathrm{T}^*}$ depends only on $\sum_L$; therefore

(4.22)
$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in\mathrm{T}^*}F_\Gamma(J,L,\varepsilon)\\ &=\sum_L\sum_{\Gamma\in\mathrm{T}^*}\sum_J\Big[\prod_{i=1}^{\nu}a_{d_1(\Gamma_i)}a_{d_2(\Gamma_i)}\int e^{i\langle\lambda,\mathbf{x}_{d_1(\Gamma_i)}-\mathbf{x}_{d_2(\Gamma_i)}\rangle}\exp\{-(t_{d_1(\Gamma_i)}+t_{d_2(\Gamma_i)})|\lambda|^\alpha\}d\lambda\Big]\\ &\quad\times\Big[\prod_{i=1}^{\nu}f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big]\\ &=\Big[\sum_{j,j'=1}^{M}a_ja_{j'}\int e^{i\langle\lambda,\mathbf{x}_j-\mathbf{x}_{j'}\rangle}\exp\{-(t_j+t_{j'})|\lambda|^\alpha\}d\lambda\Big]^\nu\\ &\quad\times\sum_L\sum_{\Gamma\in\mathrm{T}^*}\Big[\prod_{i=1}^{\nu}f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big].\end{aligned}$$

To handle the summation in the above, we note that $\prod_{i=1}^{\nu}f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}$ only depends on $\{\sharp E(\Gamma_i), i=1,\cdots,\nu\}$, not on the structures of sub-diagrams Γ_i, $i=1,\cdots,\nu$; thus we may rewrite the above summation based on the following observation. Let s be the number of different integers $r_1,\ldots,r_s$ in $\{l_1,\cdots,l_{2\nu}\}$ with $m\le r_1<\ldots<r_s\le N_0$. A regular diagram requires $1\le s\le\nu$, which also implies that the set $\{l_1,\cdots,l_{2\nu}\}$ can be split into s subsets $Q_1,\ldots,Q_s$ and all elements within Q_i have the common value r_i, $i=1,\ldots,s$. For the number of *pairs*

within each subset Q_i, we denote it by q_i, which satisfies $q_i \geq 1$, $i = 1, \ldots, s$, and $q_1 + \cdots + q_s = \nu$. Thus, the above summation is

$$\sum_{1\leq s\leq \nu}(s!)\sum_{m\leq r_1<\cdots<r_s=N_0}\ \sum_{q_1+\cdots+q_s=\nu}\frac{(2\nu)!}{(2q_1)!\cdots(2q_s)!}[\cdots].$$

However, for any $(s; r_1, \ldots, r_s; q_1, \ldots, q_s)$ in the above sum, there correspond $\frac{(2q_1)!\cdots(2q_s)!}{2^\nu q_1!\cdots q_s!}(r_1!)^{q_1}\cdots(r_s!)^{q_s}$ different regular diagrams. Therefore,

(4.23)

$$\begin{aligned}
&\sum_L\sum_{\Gamma\in \mathrm{T}^*}\Big[\prod_{i=1}^{\nu} f^{*\sharp E(\Gamma_i)}(\mathbf{0})\frac{C^2_{\sharp E(\Gamma_i)}}{\sharp E(\Gamma_i)!}\Big]\\
&=\sum_{1\leq s\leq\nu}(s!)\sum_{m\leq r_1<\cdots<r_s=N_0}\ \sum_{q_1+\cdots+q_s=\nu}\frac{(2\nu)!}{2^\nu q_1!\cdots q_s!}(r_1!)^{q_1}\cdots(r_s!)^{q_s}\Big[\prod_{i=1}^{s}\big(f^{*r_i}(\mathbf{0})\frac{C^2_{r_i}}{r_i!}\big)^{q_i}\Big]\\
&=(2\nu-1)!!\sum_{1\leq s\leq\nu}(s!)\sum_{m\leq r_1<\cdots<r_s=N_0}\ \sum_{q_1+\cdots+q_s=\nu}\frac{\nu!}{q_1!\cdots q_s!}\Big[\prod_{i=1}^{s}\big(f^{*r_i}(\mathbf{0})C^2_{r_i}\big)^{q_i}\Big]\\
&=(2\nu-1)!!\Big[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2\Big]^{\nu}.
\end{aligned}$$

Substituting (4.23) into (4.22) and recalling $\sigma_{m,N_0} = (\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2)^{\frac12}$, we get

(4.24)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_{2\nu}} K(J,L)\sum_{\Gamma\in\mathrm{T}^*}F_\Gamma(J,L,\varepsilon)\\
&=(2\nu-1)!!\Big[\sum_{j,j'=1}^{M} a_j a_{j'}\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_j-\mathbf{x}_{j'}\rangle}\exp\big\{-(t_j+t_{j'})|\lambda|^\alpha\big\}d\lambda\Big]^\nu\Big[\sum_{r=m}^{N_0} f^{*r}(\mathbf{0})C_r^2\Big]^\nu\\
&=(2\nu-1)!!\Big[\mathbb{E}\Big(\sum_{j=1}^{M}a_j\int_{\mathbb{R}^n} e^{i\langle\lambda,\mathbf{x}_j\rangle}\sigma_{m,N_0}e^{-t_j|\lambda|^\alpha}W(d\lambda)\Big)^2\Big]^\nu.
\end{aligned}$$

□

Proof of (2). $\lim_{\varepsilon\to 0}\sum_{(J,L)\in D_p} K(J,L)\sum_{\Gamma\in\mathrm{T}\setminus\mathrm{T}^*}F_\Gamma(J,L,\varepsilon) = 0$.

By (4.11), the number of elements in the summation of $\sum_{(J,L)\in D_p}$ is finite; thus it suffices to show that $\lim_{\varepsilon\to 0}F_\Gamma(J,L,\varepsilon) = 0$, i.e.

(4.25)

$$\varepsilon^{-\frac{pn\chi}{2}}\int_{\mathbb{R}^{np}}\Big\{\prod_{i=1}^{p}\varepsilon^{\frac{n}{\alpha}}G(\varepsilon t_{j_i},\varepsilon^{\frac1\alpha}(\mathbf{x}_{j_i}-\mathbf{y}_i))\Big\}\Big[\prod_{e\in E(\Gamma)}R(\varepsilon^{-\chi}(\mathbf{y}_{d_1(e)}-\mathbf{y}_{d_2(e)}))\Big]d\mathbf{y}_1\cdots d\mathbf{y}_p\to 0$$

for each $\Gamma\in\mathrm{T}(l_1,\cdots,l_p)\setminus\mathrm{T}^*$. Without loss of generality, we may just prove (4.25) for $t_{j_i}=1$ and $\mathbf{x}_{j_i}=\mathbf{0}$, $i=1,\ldots,p$, and also just consider the case $l_1\leq l_2\leq\cdots\leq l_p$.

Let

$$A_{j,j'} := \{e \in E(\Gamma) \mid d_1(e) = j,\ d_2(e) = j'\},\ B(i) := \bigcup_{j'>i} A_{i,j'},\ 1 \le i, j < j' \le p.$$

We observe that the number $\sharp B(i)$ of $B(i)$ must be $\le l_i$, and a non-regular diagram Γ must contain a non-empty $B(i)$ with $\sharp B(i) < l_i$; moreover, it has ([7, (2.20)])

$$\sum_{i=1}^{p} \frac{\sharp B(i)}{l_i} \ge \frac{p}{2}. \tag{4.26}$$

(4.27)

$$\begin{aligned}
&F_\Gamma(J, L, \varepsilon)\\
&= \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i)\Big\}\Big[\prod_{i;B(i)\neq\emptyset} \prod_{e\in B(i)} R(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_{d_2(e)}))\Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p\\
&\le \varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i)\Big\}\\
&\quad \times \Big[\prod_{i;B(i)\neq\emptyset} \sum_{e\in B(i)} \frac{1}{\sharp B(i)} R^{\sharp B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_{d_2(e)}))\Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p\\
&\le c\varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i)\Big\}\\
&\quad \times \Big[\prod_{i;B(i)\neq\emptyset} \sum_{j;A_{i,j}\neq\emptyset} \frac{1}{\sharp B(i)} R^{\sharp B(i)}(\varepsilon^{-\chi}(\mathbf{y}_i - \mathbf{y}_j))\Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p,
\end{aligned}$$

where $c = \prod_{i;B(i)\neq\emptyset} \sum_{j;A_{i,j}\neq\emptyset} \sharp A_{i,j}/\sharp B(i)$.

For any $i \in \{1, \ldots, p-1\}$ with $B(i) \neq \emptyset$, let $j(i)$ be any term in $\{j^{'}; A_{i,j'} \neq \emptyset\}$. To prove (4.27) $\to 0$, by the spectral representation, it suffices to show that

$$\begin{aligned}
&\varepsilon^{-\frac{pn\chi}{2}} \int_{\mathbb{R}^{np}} \Big\{\prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i)\Big\}\\
&\qquad \times \Big[\prod_{i;B(i)\neq\emptyset} \int e^{i\langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)}\rangle} f^{*\sharp B(i)}(\varepsilon^{\chi}\lambda_{i,j(i)}) \varepsilon^{n\chi} d\lambda_{i,j(i)}\Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p
\end{aligned} \tag{4.28}$$

converges to zero when $\varepsilon \to 0$.

Applying Lemma 1 to $k = \sharp B(i)$, the number of $B(i)$, we see that

$$f^{*\sharp B(i)}(\lambda) \le \begin{cases} o(1) & \text{if } \sharp B(i) = l_i, \\ o(|\lambda|^{n(\frac{\sharp B(i)}{l_i} - 1)}) & \text{if } 1 < \sharp B(i) < l_i, \end{cases} \quad \text{when } |\lambda| \to 0. \tag{4.29}$$

For example, using the first case in (2.14) of Lemma 1, we can write (4.29) as follows:

$$f^{*\sharp B(i)}(\lambda) = C_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)\frac{n}{l_i} - n},\ C_{\sharp B(i)}(\lambda) = B_{\sharp B(i)}(\lambda)|\lambda|^{\sharp B(i)(\kappa - \frac{n}{l_i})}, \tag{4.30}$$

where $\lim_{|\lambda|\to 0} C_{\sharp B(i)}(\lambda) = 0$ because $\kappa > n/m \ge n/l_i$.

Thus,

$$(4.28) \leq \varepsilon^{-\frac{pn\chi}{2}} o(\varepsilon^{\chi n(\sum \frac{\sharp B(i)}{l_i})}) Q_\varepsilon, \tag{4.31}$$

where

$$Q_\varepsilon = \int_{\mathbb{R}^{np}} \Big\{ \prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \Big\} \\ \times \Big[\prod_{i; B(i) \neq \emptyset} \int e^{i\langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)} \rangle} |\lambda_{i,j(i)}|^{n(\frac{\sharp B(i)}{l_i} - 1)} d\lambda_{i,j(i)} \Big] d\mathbf{y}_1 \cdots d\mathbf{y}_p,$$

which is bounded in $0 < \epsilon \ll 1$, because, firstly, for each $\lambda_{i,j(i)}$, by (4.2) the following is bounded in $0 < \epsilon \ll 1$:

$$\int_{\mathbb{R}^{np}} \Big\{ \prod_{i=1}^{p} \varepsilon^{\frac{n}{\alpha}} G(\varepsilon, \varepsilon^{\frac{1}{\alpha}} \mathbf{y}_i) \Big\} \Big\{ \prod_{i; B(i) \neq \emptyset} e^{i\langle \mathbf{y}_i - \mathbf{y}_{j(i)}, \lambda_{i,j(i)} \rangle} \Big\} d\mathbf{y}_1 \cdots d\mathbf{y}_p, \tag{4.32}$$

and moreover

$$\prod_{i; B(i) \neq \emptyset} |\lambda_{i,j(i)}|^{n(\frac{\sharp B(i)}{l_i} - 1)}$$

is integrable with respect to $\prod_{i; B(i) \neq \emptyset} d\lambda_{i,j(i)}$ near the origin. Finally, the convergence of (4.31) to zero is followed by the inequality cited above, i.e. $\sum_{i=1}^{p} \frac{\sharp B(i)}{l_i} \geq \frac{p}{2}$. □

Proof of Theorem 3. By the solution form (2.2) and $\int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(t, \mathbf{x}) d\mathbf{x} = 1$,

$$\begin{aligned} & T^{\frac{m\kappa}{4}} \Big\{ u(Tt, \sqrt{T}\mathbf{x}; h(\zeta(\cdot))) - C_0 \Big\} \\ =& T^{\frac{m\kappa}{4}} \Big\{ \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) \Big[C_0 + \sum_{k=m}^{\infty} C_k \frac{H_k(\zeta(\mathbf{y}))}{\sqrt{k!}} \Big] d\mathbf{y} - C_0 \Big\} \\ =& \sum_{k=m}^{\infty} T^{\frac{m\kappa}{4}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(Tt, \sqrt{T}\mathbf{x} - \mathbf{y}) H_k(\zeta(\mathbf{y})) d\mathbf{y} =: \sum_{k=m}^{\infty} u_{k,T}(t, \mathbf{x}). \end{aligned} \tag{4.33}$$

By the Slutsky argument (see, for example, [19, p. 6]), Theorem 3 will be proved if we can show that

$$\begin{cases} \text{(i) } u_{m,T}(t, \mathbf{x}) \Rightarrow U_m(t, \mathbf{x}), \\ \text{(ii) } \sum_{k=m+1}^{\infty} u_{k,T}(t, \mathbf{x}) \to 0 \text{ in probability,} \end{cases} \quad \text{as } T \to \infty. \tag{4.34}$$

Proof of (i). Replacing the component $H_m(\zeta(\mathbf{y}))$ in the expression of $u_{m,T}(t, \mathbf{x})$ with its Itô-Wiener expansion (2.10) and using the Fourier transform $\widehat{G}_{\alpha,\mathfrak{m}}(t, \lambda)$ of

$G_{\alpha,\mathfrak{m}}(t,\mathbf{x})$ in (2.1), we have

$$u_{m,T}(t,\mathbf{x}) = T^{\frac{m\kappa}{4}} \frac{C_m}{\sqrt{m!}} \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(Tt, \sqrt{T}\mathbf{x}-\mathbf{y}) \Big\{ \int'_{\mathbb{R}^{n\times m}} e^{i\langle \mathbf{y}, \lambda_1+\cdots+\lambda_m\rangle} \prod_{\sigma=1}^{m} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) \Big\} d\mathbf{y}$$
$$= T^{\frac{m\kappa}{4}} \frac{C_m}{\sqrt{m!}} \int'_{\mathbb{R}^{n\times m}} e^{i\langle \sqrt{T}\mathbf{x}, \lambda_1+\cdots+\lambda_m\rangle} \widehat{G}_{\alpha,\mathfrak{m}}(Tt, \lambda_1+\cdots+\lambda_m) \prod_{\sigma=1}^{m} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma). \tag{4.35}$$

By the definition of $\int'_{\mathbb{R}^{n\times m}}$ in (2.10) and the self-similarity property $W(T^{-\frac{1}{2}}d\lambda) \overset{d}{=} T^{-\frac{n}{4}}W(d\lambda)$, $u_{m,T}$ has the same finite-dimensional distributions ($\overset{d}{=}$) as $\widetilde{u}_{m,T}$, where

$$\widetilde{u}_{m,T}(t,\mathbf{x}) = \frac{C_m}{\sqrt{m!}} T^{\frac{m(\kappa-n)}{4}} \int'_{\mathbb{R}^{n\times m}} e^{i\langle \mathbf{x}, \lambda_1+\cdots+\lambda_m\rangle} \widehat{G}_{\alpha,\mathfrak{m}}(Tt, T^{-\frac{1}{2}}(\lambda_1+\cdots+\lambda_m)) \times \prod_{\sigma=1}^{m} \sqrt{f(T^{-\frac{1}{2}}\lambda_\sigma)} W(d\lambda_\sigma). \tag{4.36}$$

From the isometry property of the multiple Wiener integrals and the integral representation of the limiting field $U_m(t,\mathbf{x})$ in (3.5),

$$\mathbb{E}|\widetilde{u}_{m,T}(t,\mathbf{x}) - U_m(t,\mathbf{x})|^2 = C_m^2 \int_{\mathbb{R}^{nm}} \Big| T^{\frac{m(\kappa-n)}{4}} \widehat{G}_{\alpha,\mathfrak{m}}(Tt, T^{-\frac{1}{2}}(\lambda_1+\cdots+\lambda_m)) \prod_{\sigma=1}^{m} \sqrt{f(T^{-\frac{1}{2}}\lambda_\sigma)} - B(\mathbf{0})^{\frac{m}{2}} \frac{\exp(-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda_1+\cdots+\lambda_m|^2)}{(|\lambda_1|\cdots|\lambda_m|)^{\frac{n-\kappa}{2}}} \Big|^2 \prod_{\sigma=1}^{m} d\lambda_\sigma. \tag{4.37}$$

Condition C and (4.1) allow us to apply the dominated convergence theorem to show that (4.37) will converge to zero when $T\to\infty$. We note that the convergence in (4.1) can be shown to be monotone decreasing when $T\uparrow\infty$ for each $t>0$ and $\lambda\in\mathbb{R}^n$.

Thus, we get

$$\lim_{T\to\infty} \mathbb{E}|\widetilde{u}_{m,T}(t,\mathbf{x}) - U_m(t,\mathbf{x})|^2 = 0, \tag{4.38}$$

and claim (i) is concluded by $u_{m,T} \overset{d}{=} \widetilde{u}_{m,T}$ and the Cramer-Wold theorem.

Proof of (ii). By the orthogonal property (2.9), the semigroup property of $G_{\alpha,\mathfrak{m}}(t,\mathbf{x})$, and (2.13),

(4.39)

$$\begin{aligned}
&\mathbb{E}\big[(\sum_{k=m+1}^{\infty} u_{k,T}(t,\mathbf{x}))^2\big]\\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(Tt,\sqrt{T}\mathbf{x}-\mathbf{y})G_{\alpha,\mathfrak{m}}(Tt,\sqrt{T}\mathbf{x}-\mathbf{y}')R^k(\mathbf{y}-\mathbf{y}')d\mathbf{y}\ d\mathbf{y}'\\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(2Tt,\mathbf{z})R^k(\mathbf{z})d\mathbf{z}\\
&= T^{\frac{m\kappa}{2}} \sum_{k=m+1}^{\infty} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha,\mathfrak{m}}(2Tt,\lambda)f^{*k}(\lambda)d\lambda \quad \text{(by (2.13))}\\
&= T^{\frac{m\kappa-n}{2}}\Big(\sum_{k=m+1}^{k^*}+\sum_{k=k^*+1}^{\infty}\Big)C_k^2\int_{\mathbb{R}^n}\widehat{G}_{\alpha,\mathfrak{m}}(2Tt,T^{-\frac12}\lambda)f^{*k}(T^{-\frac12}\lambda)d\lambda =: (I)+(II),
\end{aligned}$$

where $k^* = \max\{k\in\mathbb{N}|\ k\ge m+1,\ k\kappa\le n\}$.

For the case $k^*\kappa<n$, by Lemma 1 and (4.1),

$$\begin{aligned}
\lim_{T\to\infty}(I) &= \lim_{T\to\infty} T^{\frac{m\kappa-n}{2}}\sum_{k=m+1}^{k^*} C_k^2\int_{\mathbb{R}^n}\widehat{G}_{\alpha,\mathfrak{m}}(2Tt,T^{-\frac12}\lambda)B_k(T^{-\frac12}\lambda)|T^{-\frac12}\lambda|^{k\kappa-n}d\lambda\\
&\le \lim_{T\to\infty}\sum_{k=m+1}^{k^*} T^{\frac{m\kappa-k\kappa}{2}}C_k^2\ \|B_k\|_\infty\int_{\mathbb{R}^n} e^{-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2}|\lambda|^{k\kappa-n}d\lambda\\
&\le \lim_{T\to\infty} T^{-\frac{\kappa}{2}}\sum_{k=m+1}^{k^*} C_k^2\ \|B_k\|_\infty\int_{\mathbb{R}^n} e^{-t\frac{\alpha}{2}\mathfrak{m}^{1-\frac{2}{\alpha}}|\lambda|^2}|\lambda|^{k\kappa-n}d\lambda = 0.
\end{aligned}$$

For the case $k^*\kappa=n$, we still have $\lim_{T\to\infty}(I)=0$ because

$$\lim_{T\to\infty} T^{\frac{m\kappa-n}{2}}C_{k^*}^2\int_{\mathbb{R}^n}\widehat{G}_{\alpha,\mathfrak{m}}(2Tt,T^{-\frac12}\lambda)B_{k^*}(T^{-\frac12}\lambda)\ln(2+T^{\frac12}|\lambda|^{-1})d\lambda = 0.$$

On the other hand, by the assumption $m\kappa<n$ in Condition C and Lemma 1, for any $k>k^*+1$ we have $\|f^{*k}\|_\infty\le\|f^{*(k^*+1)}\|_\infty$, so

$$\lim_{T\to\infty}(II)\le\lim_{T\to\infty}T^{\frac{m\kappa-n}{2}}\sum_{k=k^*+1}^{\infty}C_k^2\ \|f^{*(k^*+1)}\|_\infty\int_{\mathbb{R}^n}\widehat{G}_{\alpha,\mathfrak{m}}(2Tt,T^{-\frac12}\lambda)d\lambda = 0.$$

Hence $\lim_{T\to\infty}\mathbb{E}\big[(\sum_{k=m+1}^{\infty}u_{k,T}(t,\mathbf{x}))^2\big]=0$, and the claim (ii) is proved by the Markov inequality. □

Proof of Theorem 4. The following proof is a hybrid of the proofs of Theorems 2 and 3; we give a full presentation mainly to see how the re-scaling of the initial data proceeds. By the Hermite expansion and the solution form (2.2) we can rewrite

(4.40)
$$u^\varepsilon(t,\mathbf{x}) = \sum_{k=m}^{\infty} \varepsilon^{-\frac{\chi m\kappa}{2}} \frac{C_k}{\sqrt{k!}} \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(\varepsilon t,\mathbf{y}) H_k(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y})))d\mathbf{y} =: \sum_{k=m}^{\infty} I_k^\varepsilon(t,\mathbf{x}).$$

By the Slutsky argument again, we show that

$$\text{(4.41)} \qquad \begin{cases} \text{(i) } I_m^\varepsilon(t,\mathbf{x}) \Rightarrow V_m(t,\mathbf{x}), \\ \text{(ii) } \sum_{k=m+1}^{\infty} I_k^\varepsilon(t,\mathbf{x}) \to 0 \text{ in probability,} \end{cases} \text{ as } \varepsilon \to 0.$$

Proof of (i). By substituting the Itô-Wiener expansion (2.10) for the random field $H_m(\zeta(\cdot))$ into $I_m^\varepsilon(t,\mathbf{x})$ and exchanging the order of integration:

(4.42)
$$\begin{aligned}
&I_m^\varepsilon(t,\mathbf{x}) \\
&= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m\kappa}{2}} \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(\varepsilon t,\mathbf{y}) H_m(\zeta(\varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y})))d\mathbf{y} \\
&= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m\kappa}{2}} \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(\varepsilon t,\mathbf{y}) \int'_{\mathbb{R}^{n\times m}} e^{i\langle \varepsilon^{-\frac{1}{\alpha}-\chi}(\varepsilon^{\frac{1}{\alpha}}\mathbf{x}-\mathbf{y}),\lambda_1+\cdots+\lambda_m\rangle} \prod_{\sigma=1}^{m} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) d\mathbf{y} \\
&= \frac{C_m}{\sqrt{m!}} \varepsilon^{-\frac{\chi m\kappa}{2}} \int'_{\mathbb{R}^{n\times m}} e^{i\langle \varepsilon^{-\chi}\mathbf{x},\lambda_1+\cdots+\lambda_m\rangle} \widehat{G}_{\alpha,\mathfrak{m}}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}-\chi}(\lambda_1+\cdots+\lambda_m)) \\
&\qquad\qquad \times \prod_{\sigma=1}^{m} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) d\mathbf{y} \\
&\stackrel{d}{=} \frac{C_m}{\sqrt{m!}} \varepsilon^{\frac{\chi m(n-\kappa)}{2}} \int'_{\mathbb{R}^{n\times m}} e^{i\langle \mathbf{x},\lambda'_1+\cdots+\lambda'_m\rangle} \widehat{G}_{\alpha,\mathfrak{m}}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}(\lambda'_1+\cdots+\lambda'_m)) \\
&\qquad\qquad \times \prod_{\sigma=1}^{m} \sqrt{f(\varepsilon^\chi \lambda'_\sigma)} W(d\lambda'_\sigma) \\
&=: \widetilde{I}_m^\varepsilon(t,\mathbf{x}),
\end{aligned}$$

where we have used the self-similarity property $W(\varepsilon^\chi d\lambda) \stackrel{d}{=} \varepsilon^{\frac{n\chi}{2}} W(d\lambda)$ in the last equality.

Now, applying the isometry property of the multiple Wiener integrals to the difference of $\widetilde{I}_m^\varepsilon(t,\mathbf{x})$ and the random field $V_m(t,\mathbf{x})$ in (3.7), we have

$$\begin{aligned}
&\mathbb{E}|\widetilde{I}_m^\varepsilon(t,\mathbf{x}) - V_m(t,\mathbf{x})|^2 \\
&\quad = C_m^2 \int_{\mathbb{R}^{nm}} \Big|\varepsilon^{\frac{\chi m(n-\kappa)}{2}} \widehat{G}_{\alpha,\mathfrak{m}}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}(\lambda_1+\cdots+\lambda_m)) \prod_{\sigma=1}^{m} \sqrt{f(\varepsilon^\chi\lambda_\sigma)} \\
&\qquad - B(\mathbf{0})^{\frac{m}{2}} e^{-t|\lambda_1+\cdots+\lambda_m|^\alpha} (|\lambda_1|\cdots|\lambda_m|)^{\frac{\kappa-n}{2}}\Big|^2 \prod_{\sigma=1}^{m} d\lambda_\sigma \to 0 \qquad (4.43)
\end{aligned}$$

when $\varepsilon \to 0$, by Condition C and (4.2).

By the Markov inequality, (4.43) implies $\widetilde{I}_m^\varepsilon(t,\mathbf{x}) \to V_m(t,\mathbf{x})$ in probability. However, because $I_m^\varepsilon(t,\mathbf{x}) \overset{d}{=} \widetilde{I}_m^\varepsilon(t,\mathbf{x})$, the claim (i) is concluded by the Cramer-Wold argument.

Proof of (ii). From (4.40), by the orthogonal property (2.9) and the semigroup property of $G_{\alpha,\mathfrak{m}}(t,\mathbf{x})$,

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k=m+1}^{\infty} I_k^\varepsilon(t,\mathbf{x})\Big)^2 &= \sum_{k=m+1}^{\infty} \mathbb{E}(I_k^\varepsilon(t,\mathbf{x}))^2 \\
&= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m\kappa} C_k^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(\varepsilon t,\mathbf{y}) G_{\alpha,\mathfrak{m}}(\varepsilon t,\mathbf{y}') R^k(\varepsilon^{-\frac{1}{\alpha}-\chi}(\mathbf{y}-\mathbf{y}'))d\mathbf{y}d\mathbf{y}' \\
&= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m\kappa} C_\rho^2 \int_{\mathbb{R}^n} G_{\alpha,\mathfrak{m}}(2\varepsilon t,\mathbf{z}) R^k(\varepsilon^{-\frac{1}{\alpha}-\chi}\mathbf{z})d\mathbf{z} \\
&= \sum_{k=m+1}^{\infty} \varepsilon^{-\chi m\kappa} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha,\mathfrak{m}}(2\varepsilon t,\varepsilon^{-\frac{1}{\alpha}-\chi}\lambda) f^{*k}(\lambda)d\lambda \\
&= \Big(\sum_{k=m+1}^{k^*} + \sum_{k=k^*+1}^{\infty}\Big)\varepsilon^{\chi(n-m\kappa)} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha,\mathfrak{m}}(2\varepsilon t,\varepsilon^{-\frac{1}{\alpha}}\lambda) f^{*k}(\varepsilon^\chi\lambda)d\lambda =: (I)+(II),
\end{aligned}$$

where $k^* = \max\{k\in\mathbb{N}|\ k\geq m+1,\ k\kappa\leq n\}$.

For the case $k^*\kappa < n$, by Lemma 1,

$$\begin{aligned}
\lim_{\varepsilon\to 0}(I) =& \lim_{\varepsilon\to 0}\sum_{k=m+1}^{k^*} \varepsilon^{\chi(n-m\kappa)} C_k^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha,\mathfrak{m}}(2\varepsilon t,\varepsilon^{-\frac{1}{\alpha}}\lambda) B_k(\varepsilon^\chi\lambda)|\varepsilon^\chi\lambda|^{k\kappa-n}d\lambda \\
\leq& \lim_{\varepsilon\to 0}\sum_{k=m+1}^{k^*} \varepsilon^{\chi\kappa(k-m)} C_k^2\ \|B_k\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^\alpha}|\lambda|^{k\kappa-n}d\lambda = 0.
\end{aligned}$$

For the case $k^*\kappa = n$, we still have $\lim_{\varepsilon\to 0}(I) = 0$ because

$$\lim_{\varepsilon\to 0}\varepsilon^{\chi(n-m\kappa)} C_{k^*}^2 \int_{\mathbb{R}^n} \widehat{G}_{\alpha,\mathfrak{m}}(2\varepsilon t,\varepsilon^{-\frac{1}{\alpha}}\lambda) B_{k^*}(\varepsilon^\chi\lambda)\ln(2+|\varepsilon^\chi\lambda|^{-1})d\lambda = 0.$$

On the other hand, by the assumption $\kappa < n/m$ in Condition C and Lemma 1, for any $k > k^*+1$ we have $\|f^{*k}\|_\infty \leq \|f^{*(k^*+1)}\|_\infty$, so

$$\lim_{\varepsilon\to 0}\ (II) \leq \lim_{\varepsilon\to 0}\sum_{k=k^*+1}^{\infty} \varepsilon^{\chi(n-m\kappa)} C_k^2\ \|f^{*(k^*+1)}\|_\infty \int_{\mathbb{R}^n} e^{-2t|\lambda|^\alpha}d\lambda = 0.$$

Hence $\lim_{\varepsilon\to 0}\ \mathbb{E}\big[(\sum_{k=m+1}^{\infty} I_k^\varepsilon(t,\mathbf{x}))^2\big] = 0$, and the claim (ii) is proved by the Markov inequality. □

Appendix: Proof of Lemma 1

The idea of the following proofs comes from [32, p. 115, Theorem 3] and [17, p. 160, Theorem 8.8]. We only consider their results for the density functions on the whole space. Suppose that two spectral density functions f_1 and f_2 are in the form

$$0 \le f_j(\lambda) = \frac{K_j(\lambda)}{|\lambda|^{n-\kappa_j}},\ \kappa_j > 0,\ j = 1, 2, \tag{A.1}$$

where $K_1(\lambda)$ and $K_2(\lambda)$ are non-negative functions belonging to $\mathrm{C}(\mathbb{R}^n \backslash \{\mathbf{0}\})$.

Let $g(\lambda) = \int_{\mathbb{R}^n} f_1(\lambda - \eta) f_2(\eta) d\eta,\ \lambda \in \mathbb{R}^n$. To prove Lemma 1, we show that g can be written as

$$g(\lambda) = \begin{cases} B(\lambda)|\lambda|^{\kappa_1+\kappa_2-n}, & \text{for } \kappa_1 + \kappa_2 < n, \\ B(\lambda)\ln(2 + \frac{1}{|\lambda|}), & \text{for } \kappa_1 + \kappa_2 = n, \\ B(\lambda) \in \mathrm{C}(\mathbb{R}^n), & \text{for } \kappa_1 + \kappa_2 > n, \end{cases}$$

for some bounded function $B(\lambda) \in \mathrm{C}(\mathbb{R}^n \backslash \{\mathbf{0}\})$.

Case 1 ($\kappa_1 + \kappa_2 < n$). For any $\lambda_0 \neq \mathbf{0}$, we divide $\mathbb{R}^n$ into four parts: $\mathbb{R}^n = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned} D_1 &= \{\eta \in \mathbb{R}^n |\ |\eta - \lambda_0| < |\lambda_0|/2\}, \\ D_2 &= \{\eta \in \mathbb{R}^n |\ |\eta| < |\lambda_0|/2\}, \\ D_3 &= \{\eta \in (D_1 \cup D_2)^c |\ |\eta - \lambda_0| < |\eta|\}, \\ D_4 &= \{\eta \in (D_1 \cup D_2)^c |\ |\eta - \lambda_0| > |\eta|\}. \end{aligned}$$

Therefore,

$$g(\lambda_0) = \sum_{j=1}^{4} \int_{D_j} f_1(\lambda_0 - \eta) f_2(\eta) d\eta =: I_1 + I_2 + I_3 + I_4,$$

$$\begin{aligned} I_1(\lambda_0) \le& \big(\sup_{\eta' \in D_1} f_2(\eta')\big) \int_{D_j} f_1(\lambda_0 - \eta) d\eta \\ \le& \big(\sup_{\eta \in D_1} K_2(\eta)\big) \big(\frac{|\lambda_0|}{2}\big)^{\kappa_2 - n} \big(\sup_{\eta \in D_2} K_1(\eta)\big) c_n \int_0^{\frac{|\lambda_0|}{2}} r^{\kappa_1 - 1} dr = C|\lambda_0|^{\kappa_1+\kappa_2-n}, \end{aligned}$$

where c_n is the surface area of the unit sphere on $\mathbb{R}^n$ and C is a constant independent of λ_0. Similarly, $I_2(\lambda_0) \le C|\lambda_0|^{\kappa_1+\kappa_2-n}$.

By the fact that $I_3 \in \mathrm{C}(\mathbb{R}^n \backslash \{\mathbf{0}\}) \cap L^1(\mathbb{R}^n)$, we know that $\sup_{|\lambda_0| \ge 1} I_3(\lambda_0) < \infty$. So we suffice to study the behavior of $I_3(\cdot)$ on the domain $\{\lambda_0 |\ |\lambda_0| < 1\}$.

By the requirement (A.1), $\lim_{|\eta| \to \infty} K_j(\eta)|\eta|^{\kappa_j} = 0$; that is, for any $\varepsilon > 0$, there exists a constant $M = M(\varepsilon) > 0$ such that

$$K_j(\eta) \le \varepsilon |\eta|^{-\kappa_j} \quad \text{for all } |\eta| > M. \tag{A.2}$$

Because $|\eta - \lambda_0| < |\eta|$ for $\eta \in D_3$,

$$\begin{aligned} I_3(\lambda_0) \ \le& \ \Big(\int_{D_3 \cap \{|\eta - \lambda_0| > M+1\}} + \int_{D_3 \cap \{|\eta - \lambda_0| < M+1\}}\Big) \frac{K_1(\lambda_0 - \eta) K_2(\eta)}{|\lambda_0 - \eta|^{2n-\kappa_1-\kappa_2}} d\eta \\ =:& \ I_{3,1}(\lambda_0) + I_{3,2}(\lambda_0). \end{aligned}$$

By using (A.2) and $|\eta-\lambda_0|<|\eta|$ again,

$$
\begin{aligned}
I_{3,1}(\lambda_0) \le\ & \varepsilon \int_{D_3\cap\{|\eta-\lambda_0|>M+1\}} \frac{K_1(\lambda_0-\eta)|\eta|^{-\kappa_2}}{|\lambda_0-\eta|^{2n-\kappa_1-\kappa_2}}d\eta \\
\le\ & \varepsilon \int_{\{|\eta-\lambda_0|>M+1\}} \frac{K_1(\lambda_0-\eta)}{|\lambda_0-\eta|^{2n-\kappa_1}}d\eta \\
\le\ & \varepsilon(M+1)^{-n}\int_{\mathbb{R}^n}\frac{K_1(\eta)}{|\eta|^{n-\kappa_1}}d\eta=\varepsilon(M+1)^{-n},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
I_{3,2}(\lambda_0) &\le \|K_1\|_\infty\|K_2\|_\infty\ c_n\int_{\frac{\lambda_0}{2}}^{M+1}\frac{r^{n-1}}{r^{2n-\kappa_1-\kappa_2}}d\eta \\
&< \frac{\|K_1\|_\infty\|K_2\|_\infty\ c_n}{n-\kappa_1-\kappa_2}(\frac{|\lambda_0|}{2})^{\kappa_1+\kappa_2-n}.
\end{aligned}
\tag{A.4}
$$

Combining (A.3) and (A.4), we get $I_3(\lambda)=B(\lambda)|\lambda|^{\kappa_1+\kappa_2-n}$ for some bounded function B. This observation still holds for I_4. Therefore, the proof for the case $\kappa_1+\kappa_2<n$ is finished. □

Case 2 ($\kappa_1+\kappa_2=n,\ \kappa_1,\ \kappa_2>0$). Let $\hat\lambda_0=\lambda_0/|\lambda_0|$,

$$
\begin{aligned}
g(\lambda_0) &= \int_{\mathbb{R}^n}\frac{K_1(\lambda_0-\eta)K_2(\eta)}{|\lambda_0-\eta|^{n-\kappa_1}|\eta|^{n-\kappa_2}}d\eta \\
&= \Big(\int_{\{|\eta|<2\}}+\int_{|\eta|>2}\Big)\frac{K_1(|\lambda_0|(\widehat{\lambda}_0-\eta))K_2(|\lambda_0|\eta)}{|\widehat{\lambda}_0-\eta|^{n-\kappa_1}|\eta|^{n-\kappa_2}}d\eta \\
&=: J_1(\lambda_0)+J_2(\lambda_0),
\end{aligned}
$$

$$
J_1(\lambda_0)\le\int_{\{|\eta|<2\}}\frac{\|K_1\|_\infty\|K_2\|_\infty\ d\eta}{|\widehat{\lambda}_0-\eta|^{n-\kappa_1}|\eta|^{n-\kappa_2}}=\int_{\{|\eta|<2\}}\frac{\|K_1\|_\infty\|K_2\|_\infty\ d\eta}{|\widehat{\mathbf{x}}-\eta|^{n-\kappa_1}|\eta|^{n-\kappa_2}}<\infty,
\tag{A.5}
$$

where the last equality holds for any unit vector $\widehat{\mathbf{x}}$. Moreover, we also have that

$$
\begin{aligned}
J_2(\lambda_0) &= \Big(\int_{\{2<|\eta|<2(2+\frac{1}{|\lambda_0|})\}}+\int_{\{|\eta|>2(2+\frac{1}{|\lambda_0|})\}}\Big)\frac{K_1(|\lambda_0|(\widehat{\lambda}_0-\eta))K_2(|\lambda_0|\eta)}{|\widehat{\lambda}_0-\eta|^{n-\kappa_1}|\eta|^{n-\kappa_2}}d\eta \\
&=: J_{2,1}(\lambda_0)+J_{2,2}(\lambda_0).
\end{aligned}
$$

Because $|\widehat{\lambda}_0-\eta|\ge|\eta|-1$,

(A.6)

$$
\begin{aligned}
J_{2,1}(\lambda_0) &\le \int_{\{2<|\eta|<2(2+\frac{1}{|\lambda_0|})\}}\frac{\|K_1\|_\infty\|K_2\|_\infty\ d\eta}{(|\eta|-1)^{n-\kappa_1}|\eta_2|^{n-\kappa_2}}=c_n\int_2^{2(2+\frac{1}{|\lambda_0|})}\frac{\|K_1\|_\infty\|K_2\|_\infty\ dr}{(r-1)^{n-\kappa_1}r^{1-\kappa_2}} \\
&= c_n\ \|K_1\|_\infty\|K_2\|_\infty\int_2^{2(2+\frac{1}{|\lambda_0|})}\frac{dr}{(1-\frac{1}{r})^{n-\kappa_1}r}\le 2^{n-\kappa_1}c_n\ \|K_1\|_\infty\|K_2\|_\infty\ \ln(2+\frac{1}{|\lambda_0|}).
\end{aligned}
$$

Changing from variable η to $\frac{\tau}{|\lambda_0|}$ and using the inequality $|\lambda_0-\tau| \geq |\tau|-|\lambda_0| \geq 2(1+2|\lambda_0|)-|\lambda_0| \geq 2$ for $|\tau|>2(1+|\lambda_0|)$,

$$J_{2,2}(\lambda_0) = \int\limits_{\{|\tau|>2(1+2|\lambda_0|)\}} \frac{K_1(\lambda_0-\tau)K_2(\eta)}{|\lambda_0-\tau|^{n-\kappa_1}|\tau|^{n-\kappa_2}} d\tau$$
$$\leq \frac{\|K_1\|_\infty}{2^{n-\kappa_1}} \int\limits_{\{|\tau|>2(1+2|\lambda_0|)\}} \frac{K_2(\tau)}{|\tau|^{n-\kappa_2}} d\eta \leq \frac{\|K_1\|_\infty}{2^{n-\kappa_1}}.$$

The last estimation, together with (A.5) and (A.6), implies that there exist bounded and positive functions $\widetilde{B}(\lambda)$ and $C(\lambda)$ such that $g(\lambda)=\widetilde{B}(\lambda)\ln(2+\frac{1}{|\lambda|})+C(\lambda)=B(\lambda)\ln(2+\frac{1}{|\lambda|})$, where $B(\lambda)=\widetilde{B}(\lambda)+\frac{C(\lambda)}{\ln(2+|\lambda|^{-1})}$ is also a bounded function.

Case 3 $(\kappa_1+\kappa_2>n,\ \kappa_1,\ \kappa_2>0)$. Because $\kappa_1+\kappa_2>n$ implies that there exist $p,\ p'\in(1,\infty)$ such that $p(n-\kappa_1),\ p'(n-\kappa_2)<n$ and $\frac{1}{p}+\frac{1}{p'}=1$. For any $\lambda\in\mathbb{R}^n$, by Hölder's inequality, $g(\lambda)\leq\|f_1\|_p\|f_2\|_{p'}$. Meanwhile, it also implies the continuity of g as follows:

$$|g(\lambda)-g(\lambda_0)| = |\int_{\mathbb{R}^n}\big(f_1(\lambda-\eta)-f_1(\lambda_0-\eta)\big)f_2(\eta)d\eta|$$
$$\leq \|f_1(\lambda-\cdot)-f_1(\lambda_0-\cdot)\|_p\|f_2\|_{p'}\to 0$$

when $\lambda\to\lambda_0$ for any $\lambda_0\in\mathbb{R}^n$.

Finally, by taking successive convolutions and using the result of Case 1, for any $k_1>k_2>n/\kappa$, $f^{*k_1}(\lambda)$ and $f^{*k_2}(\lambda)$ are bounded functions, which implies

$$f^{*k_1}(\lambda)=\int_{\mathbb{R}^n} f^{*k_2}(\lambda-\eta)f^{*(k_1-k_2)}(\eta)d\eta \leq \|f^{*k_2}\|_\infty \int_{\mathbb{R}^n} f^{*(k_1-k_2)}(\eta)d\eta = \|f^{*k_2}\|_\infty .$$

Acknowledgements

This article was mainly worked on while the second author visited York University (Canada) in Fall 2011 and Chinese University of Hong Kong in Springs 2012 and 2013; their hospitality and financial support are appreciated. The contents of this article were reported by the second author at the probability scientific session of the Canadian Mathematical Society 2011 Winter Meeting.

References

[1] V. V. Anh and N. N. Leonenko, *Non-Gaussian scenarios for the heat equation with singular initial conditions*, Stochastic Process. Appl. **84** (1999), no. 1, 91–114, DOI 10.1016/S0304-4149(99)00053-8. MR1720100 (2001c:60083)

[2] V. V. Anh and N. N. Leonenko, *Renormalization and homogenization of fractional diffusion equations with random data*, Probab. Theory Related Fields **124** (2002), no. 3, 381–408, DOI 10.1007/s004400200217. MR1939652 (2003m:60129)

[3] V. V. Anh, N. N. Leonenko, and L. M. Sakhno, *Higher-order spectral densities of fractional random fields*, J. Statist. Phys. **111** (2003), no. 3-4, 789–814, DOI 10.1023/A:1022898131682. MR1972127 (2004a:62175)

[4] B. Baeumer, M. M. Meerschaert, and M. Naber, Stochastic models for relativistic diffusion, *Phys. Rev. E* **82** (2010), 1132-1136.

[5] O. E. Barndorff-Nielsen and N. N. Leonenko, *Burgers' turbulence problem with linear or quadratic external potential*, J. Appl. Probab. **42** (2005), no. 2, 550–565. MR2145493 (2006a:60112)

[6] Jean Bertoin, *Subordinators: examples and applications*, Lectures on probability theory and statistics (Saint-Flour, 1997), Lecture Notes in Math., vol. 1717, Springer, Berlin, 1999, pp. 1–91, DOI 10.1007/978-3-540-48115-7_1. MR1746300 (2002a:60001)

[7] Peter Breuer and Péter Major, *Central limit theorems for nonlinear functionals of Gaussian fields*, J. Multivariate Anal. **13** (1983), no. 3, 425–441, DOI 10.1016/0047-259X(83)90019-2. MR716933 (85d:60042)

[8] René Carmona, Wen Chen Masters, and Barry Simon, *Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions*, J. Funct. Anal. **91** (1990), no. 1, 117–142, DOI 10.1016/0022-1236(90)90049-Q. MR1054115 (91i:35139)

[9] Zhen-Qing Chen, Panki Kim, and Renming Song, *Sharp heat kernel estimates for relativistic stable processes in open sets*, Ann. Probab. **40** (2012), no. 1, 213–244, DOI 10.1214/10-AOP611. MR2917772 (2012m:60183)

[10] R. L. Dobrushin, *Gaussian and their subordinated self-similar random generalized fields*, Ann. Probab. **7** (1979), no. 1, 1–28. MR515810 (80e:60069)

[11] R. L. Dobrushin and P. Major, *Non-central limit theorems for nonlinear functionals of Gaussian fields*, Z. Wahrsch. Verw. Gebiete **50** (1979), no. 1, 27–52, DOI 10.1007/BF00535673. MR550122 (81i:60019)

[12] Paul Doukhan, George Oppenheim, and Murad S. Taqqu, *Theory and applications of long-range dependence*, Birkhäuser Boston, Inc., Boston, MA, 2003. MR1956041 (2003h:60004)

[13] A. V. Ivanov and N. N. Leonenko, *Statistical analysis of random fields*, Mathematics and its Applications (Soviet Series), vol. 28, Kluwer Academic Publishers Group, Dordrecht, 1989. With a preface by A. V. Skorokhod; Translated from the Russian by A. I. Kochubinskiĭ. MR1009786 (90g:62235)

[14] M. Ya. Kelbert, N. N. Leonenko, and M. D. Ruiz-Medina, *Fractional random fields associated with stochastic fractional heat equations*, Adv. in Appl. Probab. **37** (2005), no. 1, 108–133, DOI 10.1239/aap/1113402402. MR2135156 (2005m:60109)

[15] J. Kampé de Fériet, *Random solutions of partial differential equations*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, University of California Press, Berkeley and Los Angeles, 1956, pp. 199–208. MR0084927 (18,949d)

[16] R. Kimmel, N. Sochen, and J. Weickert, *Scale-Space and PDE Methods in Computer Vision*, Lecture Notes in Computer Science V. 3459. Springer, 2005.

[17] M. A. Krasnosel′skiĭ, P. P. Zabreĭko, E. I. Pustyl′nik, and P. E. Sobolevskiĭ, *Integral operators in spaces of summable functions*, Noordhoff International Publishing, Leiden, 1976. Translated from the Russian by T. Ando; Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. MR0385645 (52 #6505)

[18] A. Kumar, Mark M. Meerschaert, and P. Vellaisamy, *Fractional normal inverse Gaussian diffusion*, Statist. Probab. Lett. **81** (2011), no. 1, 146–152, DOI 10.1016/j.spl.2010.10.007. MR2740078 (2012e:60113)

[19] N. N. Leonenko, *Limit theorems for random fields with singular spectrum*, Mathematics and its Applications, vol. 465, Kluwer Academic Publishers, Dordrecht, 1999. MR1687092 (2000k:60102)

[20] N. N. Leonenko and O. O. Mel′nikova, *Rescaling and homogenization of solutions of the heat equation with a linear potential and of the corresponding Burgers equation with random data* (Ukrainian, with Ukrainian summary), Teor. Ĭmovīr. Mat. Stat. **62** (2000), 72–82; English transl., Theory Probab. Math. Statist. **62** (2001), 77–88. MR1871510 (2002k:60135)

[21] N. N. Leonenko and Wojbor A. Woyczynski, *Scaling limits of solutions of the heat equation for singular non-Gaussian data*, J. Statist. Phys. **91** (1998), no. 1-2, 423–438, DOI 10.1023/A:1023060625577. MR1632518 (99f:60101)

[22] N. N. Leonenko and W. A. Woyczynski, *Exact parabolic asymptotics for singular n-D Burgers' random fields: Gaussian approximation*, Stochastic Process. Appl. **76** (1998), no. 2, 141–165, DOI 10.1016/S0304-4149(98)00031-3. MR1642664 (99h:60101)

[23] Elliott H. Lieb, *The stability of matter*, Rev. Modern Phys. **48** (1976), no. 4, 553–569. MR0456083 (56 #14314)

[24] Gi-Ren Liu and Narn-Rueih Shieh, *Homogenization of fractional kinetic equations with random initial data*, Electron. J. Probab. **16** (2011), no. 32, 962–980, DOI 10.1214/EJP.v16-896. MR2801457 (2012g:60168)

[25] Péter Major, *Multiple Wiener-Itô integrals*, Lecture Notes in Mathematics, vol. 849, Springer, Berlin, 1981. With applications to limit theorems. MR611334 (82i:60099)

[26] M. Rosenblatt, *Remarks on the Burgers equation*, J. Mathematical Phys. **9** (1968), 1129–1136. MR0264252 (41 #8848)

[27] Jan Rosiński, *Tempering stable processes*, Stochastic Process. Appl. **117** (2007), no. 6, 677–707, DOI 10.1016/j.spa.2006.10.003. MR2327834 (2008g:60146)

[28] M. D. Ruiz-Medina, J. M. Angulo, and V. V. Anh, *Scaling limit solution of a fractional Burgers equation*, Stochastic Process. Appl. **93** (2001), no. 2, 285–300, DOI 10.1016/S0304-4149(00)00106-X. MR1828776 (2002d:60057)

[29] Michał Ryznar, *Estimates of Green function for relativistic α-stable process*, Potential Anal. **17** (2002), no. 1, 1–23, DOI 10.1023/A:1015231913916. MR1906405 (2003f:60087)

[30] Narn-Rueih Shieh, *On time-fractional relativistic diffusion equations*, J. Pseudo-Differ. Oper. Appl. **3** (2012), no. 2, 229–237, DOI 10.1007/s11868-012-0049-6. MR2925182

[31] Narn-Rueih Shieh, *Free fields associated with the relativistic operator* $-(m - \sqrt{m^2 - \Delta})$, J. Pseudo-Differ. Oper. Appl. **3** (2012), no. 3, 309–319, DOI 10.1007/s11868-012-0053-x. MR2964808

[32] V. I. Smirnov, *A course of higher mathematics. Vol. V [Integration and functional analysis]*, Translated by D. E. Brown; translation edited by I.N. Sneddon. ADIWES International Series in Mathematics, Pergamon Press, Oxford-New York; Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. MR0168707 (29 #5964)

[33] Murad S. Taqqu, *Convergence of integrated processes of arbitrary Hermite rank*, Z. Wahrsch. Verw. Gebiete **50** (1979), no. 1, 53–83, DOI 10.1007/BF00535674. MR550123 (81i:60020)

[34] M. W. Wong, *A contraction semigroup generated by a pseudo-differential operator*, Differential Integral Equations **5** (1992), no. 1, 193–200. MR1141736 (93b:47082)

[35] M. W. Wong, *An introduction to pseudo-differential operators*, 2nd ed., World Scientific Publishing Co., Inc., River Edge, NJ, 1999. MR1698573 (2000c:35002)

[36] Wojbor A. Woyczyński, *Burgers-KPZ turbulence*, Lecture Notes in Mathematics, vol. 1700, Springer-Verlag, Berlin, 1998. Göttingen lectures. MR1732301 (2000j:60077)

Ph.D. Class in Mathematics, National Taiwan University, Taipei 10617, Taiwan

Department of Mathematics, Honorary Faculty, National Taiwan University, Taipei 10617, Taiwan

E-mail address: shiehnr@ntu.edu.tw

URL: www.math.ntu.edu.tw/~shiehnr/