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L_p-BLASCHKE VALUATIONS

JIN LI, SHUFENG YUAN, AND GANGSONG LENG

Abstract. In this article, a classification of continuous, linearly intertwining, symmetric L_p-Blaschke ($p > 1$) valuations is established as an extension of Haberl's work on Blaschke valuations. More precisely, we show that for dimensions $n \geq 3$, the only continuous, linearly intertwining, normalized symmetric L_p-Blaschke valuation is the normalized L_p-curvature image operator, while for dimension $n = 2$, a rotated normalized L_p-curvature image operator is the only additional one. One of the advantages of our approach is that we deal with normalized symmetric L_p-Blaschke valuations, which makes it possible to handle the case $p = n$. The cases where $p \neq n$ are also discussed by studying the relations between symmetric L_p-Blaschke valuations and normalized ones.

1. Introduction

A *valuation* is a function $Z : \mathcal{Q} \to \langle \mathcal{G}, + \rangle$ defined on a class of subsets of $\mathbb{R}^n$ with values in an Abelian semigroup $\langle \mathcal{G}, + \rangle$ which satisfies

$$Z(K \cup L) + Z(K \cap L) = ZK + ZL, \tag{1.1}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{Q}$. In recent years, important new results on the classification of valuations on the space of convex bodies have been obtained. The starting point for a systematic investigation of general valuations was Hadwiger's [11] fundamental characterization of the linear combinations of intrinsic volumes as the continuous valuations that are rigid motion invariant (see [1–3, 22] for recent important variants). Its beautiful applications in integral geometry and geometric probability are described in Hadwiger's book [10] and Klain and Rota's recent book [12].

Excellent surveys on the history of valuations from Dehn's solution of Hilbert's third problem up to approximately 1990 are in McMullen and Schneider [32] or McMullen [31].

First results on convex body valued valuations were obtained by Schneider [39] in the 1970s, where the addition of convex bodies in (1.1) is the Minkowski sum. In recent years, the investigations of convex and star body valued valuations gained momentum through a series of articles by Ludwig [18–21] (see also [4–8, 34, 35, 41, 43, 44]). A very recent development in this area explores the connections between these valuations and the theory of isoperimetric inequalities (see, e.g., [9, 36, 42]).

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Assuming compatibility with the general linear group, Ludwig [20] obtained a complete classification of L_p-*Minkowski valuations*, i.e., valuations where the addition in (1.1) is the L_p-Minkowski sum. Her results establish simple characterizations of fundamental operators like the projection or centroid body operator. Haberl [6] established a classification of all continuous symmetric Blaschke valuations, where addition in (1.1) is the Blaschke sum "#", compatible with the general linear group. For $n \geq 3$, the only two examples of such valuations are a scalar multiple of the curvature image operator and the Blaschke symmetrical $ZK = K\#(-K)$. For $n = 2$, Blaschke sum coincides with Minkowski sum; a classification is provided by Ludwig's results [20].

In this paper, we extend Haberl's [6] results in the context of the L_p-Brunn-Minkowski theory when $p > 1$ for $n \geq 2$. To treat the case that $p = n$ when n is not even at the same time as the case for general $p > 1$, we deal with normalized symmetric L_p-Blaschke valuations (that is, the addition in (1.1) is the normalized L_p-Blaschke sum). For $n \geq 3$, the only example (up to a dilation) of a continuous, linearly intertwining, normalized symmetric L_p-Blaschke valuation is the normalized L_p-curvature image operator. For $n = 2$, the rotation of the normalized L_p-curvature image operator by an angle $\pi/2$ is the only additional example. As by-products, by the relationship between symmetric L_p-Blaschke valuations and the corresponding normalized case, we also classify continuous, linearly intertwining, symmetric L_p-Blaschke valuations for $p \neq n$.

Since the classification of L_p-Blaschke valuations is based on Ludwig's results [20], some other classifications of Minkowski valuations should be remarked upon here. Schneider and Schuster [41] and Schuster [43] classified some rotation covariant Minkowski valuations. Schuster and Wannerer [44] classified $GL(n)$ contravariant Minkowski valuations without any restrictions on their domain. Very recently, Haberl [7] showed that the homogeneity assumptions of $p = 1$ in Ludwig [20] are not necessary, and Parapatits [34, 35] showed that the homogeneity assumptions of $p > 1$ in Ludwig [20] are also not necessary. But the homogeneity assumptions are still needed in this paper.

In order to state the main result, we collect some notation. Let $\mathcal{K}^n$ be the space of *convex bodies*, i.e., nonempty, compact, convex subsets of $\mathbb{R}^n$, endowed with Hausdorff metric. We denote by $\mathcal{K}_o^n$ the set of n-dimensional convex bodies which contain the origin, and by $\overline{\mathcal{K}}_o^n$ the set of convex bodies which contain the origin. The set of n-dimensional origin-symmetric convex bodies is denoted by $\mathcal{K}_c^n$.

We will always assume that $p \in \mathbb{R}$ and $p > 1$ in this paper, unless noted otherwise.

In [26], Lutwak introduced the notion of the L_p-surface area measure $S_p(K, \cdot)$ and posed the even L_p-Minkowski problem: given an even Borel measure μ on the unit sphere S^{n-1}, does there exist an n-dimensional convex body K such that $\mu = S_p(K, \cdot)$? An affirmative answer was given, if $p \neq n$ and if μ is not concentrated on any great subsphere. For $p \neq n$, using the uniqueness of the even L_p-Minkowski problem on $\mathcal{K}_c^n$, the L_p-*Blaschke sum* $K\#_pL \in \mathcal{K}_c^n$ of $K, L \in \mathcal{K}_c^n$ was defined by $S_p(K\#_pL, \cdot) = S_p(K, \cdot) + S_p(L, \cdot)$. Thus $\mathcal{K}_c^n$ endowed with the L_p-Blaschke sum is an Abelian semigroup which we denote by $\langle \mathcal{K}_c^n, \#_p \rangle$.

The volume-normalized even L_p-Minkowski problem, for which the case $p = n$ can be handled as well, was introduced and solved by Lutwak, Yang, and Zhang [30]. If μ is an even Borel measure on the unit sphere S^{n-1}, then there exists a unique n-dimensional origin-symmetric convex body $\widetilde{K}$ such that

$$\frac{S_p(\widetilde{K},\cdot)}{V(\widetilde{K})}=\mu \tag{1.2}$$

if and only if μ is not concentrated on any great subsphere, where $V(\widetilde{K})$ is the volume of $\widetilde{K}$.

The volume-normalized even L_p-Minkowski problem also suggests the following composition of bodies in $\mathcal{K}_c^n$. For $K,L\in\mathcal{K}_c^n$, we define the *normalized L_p-Blaschke sum* $K\widetilde{\#}_pL\in\mathcal{K}_c^n$ by

$$\frac{S_p(K\widetilde{\#}_pL,\cdot)}{V(K\widetilde{\#}_pL)}=\frac{S_p(K,\cdot)}{V(K)}+\frac{S_p(L,\cdot)}{V(L)}.$$

Obviously the existence and uniqueness of $K\widetilde{\#}_pL$ are guaranteed by relation (1.2). Also $\mathcal{K}_c^n$ endowed with the normalized L_p-Blaschke sum is an Abelian semigroup which we denote by $\langle\mathcal{K}_c^n,\widetilde{\#}_p\rangle$.

We call a valuation $Z:\mathcal{K}_o^n\to\langle\mathcal{K}_c^n,\#_p\rangle$ a *symmetric L_p-Blaschke valuation*, and a valuation $Z:\mathcal{K}_o^n\to\langle\mathcal{K}_c^n,\widetilde{\#}_p\rangle$ a *normalized symmetric L_p-Blaschke valuation*.

A convex body K, which contains the origin in its interior, is said to have an *L_p-curvature function* $f_p(K,\cdot):S^{n-1}\to\mathbb{R}$ if $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure σ, and

$$\frac{dS_p(K,\cdot)}{d\sigma(\cdot)}=f_p(K,\cdot)$$

almost everywhere with respect to σ.

For $p\geq 1$ and $p\neq n$, the *symmetric L_p-curvature image* Λ_c^pK of $K\in\mathcal{K}_o^n$ is defined as the unique body in $\mathcal{K}_c^n$ such that

$$f_p(\Lambda_c^pK,\cdot)=\frac{1}{2}\rho(K,\cdot)^{n+p}+\frac{1}{2}\rho(-K,\cdot)^{n+p},$$

where $\rho_K(\cdot)=\rho(K,\cdot):S^{n-1}\to\mathbb{R}$ is the radial function of K, i.e., $\rho(K,u)=\max\{\lambda>0:\lambda u\in K\}$. When $p=1$, this is the classical curvature image operator, a central notion in the affine geometry of convex bodies; see e.g., [15, 16, 23–25, 27]. When $p>1$, it should be noticed that the definition of the L_p-curvature image operator here differs from the definition of Lutwak [28].

For $p\geq 1$, the *normalized symmetric L_p-curvature image* $\widetilde{\Lambda}_c^pK$ of $K\in\mathcal{K}_o^n$ is defined as the unique body in $\mathcal{K}_c^n$ such that

$$\frac{f_p(\widetilde{\Lambda}_c^pK,\cdot)}{V(\widetilde{\Lambda}_c^pK)}=(\frac{1}{2}\rho(K,\cdot)^{n+p}+\frac{1}{2}\rho(-K,\cdot)^{n+p}).$$

Remark. By the uniqueness of the even L_p-Minkowski problem and the volume-normalized even L_p-Minkowski problem, if $p\geq 1$ and $p\neq n$, it follows that

$$V(\widetilde{\Lambda}_c^pK)^{1/(p-n)}\widetilde{\Lambda}_c^pK=\Lambda_c^pK.$$

An operator $Z:\mathcal{Q}\to\langle\mathfrak{P}(\mathbb{R}^n),+\rangle$, where $\mathfrak{P}(\mathbb{R}^n)$ denotes the power set of $\mathbb{R}^n$, is called *$SL(n)$ covariant* if

$$Z(\phi K)=\phi ZK$$

for every $K \in \mathcal{Q}$ and $\phi \in SL(n)$. It is called $SL(n)$ *contravariant* if

$$Z(\phi K) = \phi^{-t} ZK$$

for every $K \in \mathcal{Q}$ and $\phi \in SL(n)$. Here, ϕ^{-t} denotes the inverse of the transpose of ϕ. We call Z *homogeneous of degree* $q \in \mathbb{R}$ if

$$Z(\lambda K) = \lambda^q ZK$$

for every $K \in \mathcal{Q}$ and $\lambda > 0$, and we call Z homogeneous if it is homogeneous of some degree $q \in \mathbb{R}$. If Z is homogeneous and $SL(n)$ covariant or contravariant, then we call it *linearly intertwining*.

Our main results are the following two theorems.

Theorem 1.1. *Let $n \geq 2$. For $p > 1$ and p not an even integer, the operator $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a continuous, homogeneous, $SL(n)$ contravariant valuation if and only if there exists a constant $c > 0$ such that*

$$ZK = c\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^n$.

Theorem 1.2. *Let $n \geq 3$. For $p > 1$ and p not an even integer, there are no continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$.*

For $p > 1$ and p not an even integer, the operator $Z : \mathcal{K}_o^2 \to \langle \mathcal{K}_c^2, \widetilde{\#}_p \rangle$ is a continuous, homogeneous, $SL(2)$ covariant valuation if and only if there exists a constant $c > 0$ such that

$$ZK = c\psi_{\pi/2}\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^2$. Here $\psi_{\pi/2}$ is the rotation by an angle $\pi/2$.

Theorems 1.1 and 1.2 establish a classification of continuous, linearly intertwining, normalized symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$ when $p > 1$ and p is not an even integer. For $p = 1$, Haberl [6] obtained a complete classification of continuous, linearly intertwining symmetric Blaschke valuations and we can easily get the corresponding results in the normalized case by reversing the process of Theorem 5.3 and Theorem 5.4. Therefore we state the results here only for $p > 1$.

In Section 2, some preliminaries are given. The aim of Section 3 is to derive the characterizing properties (stated in Theorem 1.1) of the normalized symmetric L_p-curvature image operator $\widetilde{\Lambda}_c^p$. In Section 4, Lemma 4.1 - Lemma 4.5 generate a homogeneous, $SL(n)$ covariant L_p-Minkowski valuation on $\overline{\mathcal{K}}_o^n$ by a continuous, homogeneous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuation on $\mathcal{K}_o^n$. Using properties of the support set of the L_p-projection bodies established in Lemma 4.6 and characterization theorems of L_p-Minkowski valuations [20], we classify continuous, homogeneous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuations. In a similar way, we also classify continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations. In Section 5, from the relationship between normalized symmetric L_p-Blaschke valuations and symmetric L_p-Blaschke valuations (Lemma 5.1 and Lemma 5.2), we also classify continuous, linearly intertwining, symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$ for $p \neq n$ (see Theorem 5.3 and Theorem 5.4).

2. Preliminaries

We work in Euclidean n-space $\mathbb{R}^n$ with $n \geq 2$. Let $\{e_i\}, i = 1, \cdots, n$, be the standard basis of $\mathbb{R}^n$. The usual scalar product of two vectors x and $y \in \mathbb{R}^n$ shall be denoted by $x \cdot y$. For $u \in S^{n-1}$, $u^- = \{x \in \mathbb{R}^n : x \cdot u \leq 0\}$, $u^+ = \{x \in \mathbb{R}^n : x \cdot u \geq 0\}$ and $u^\perp = \{x \in \mathbb{R}^n : x \cdot u = 0\}$. The convex hull of a set $A \subset \mathbb{R}^n$ will be denoted by $[A]$. To shorten the notation we write $[A, \pm x_1, \cdots, \pm x_m]$ instead of $[A \cup \{x_1, -x_1, \cdots, x_m, -x_m\}]$ for $A \subset \mathbb{R}^n$, $m \in \mathbb{N}$, and $x_1, \cdots, x_m \in \mathbb{R}^n$. In $\mathbb{R}^2$, we write $\psi_{\pi/2}$ for the rotation by an angle $\pi/2$.

The *Hausdorff distance* of two convex bodies K, L is defined as $d(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|$, where $h_K : \mathbb{R}^n \to \mathbb{R}$ is the *support function* of $K \in \mathcal{K}^n$, i.e., $h_K(x) = \max\{x \cdot y : y \in K\}$. Sometimes we also write $h_K(\cdot)$ as $h(K, \cdot)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function (i.e., $f(\lambda x) = \lambda f(x)$ for every $\lambda \geq 0$ and $x \in \mathbb{R}^n$; $f(x+y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^n$), then there exists a unique convex body K such that $f = h_K$.

Let $S(K, \cdot)$ be the classical surface area measure of a convex body K. If K contains the origin in its interior, the Borel measure $S_p(K, \cdot) = h_K(\cdot)^{1-p} S(K, \cdot)$ on S^{n-1} is the *L_p-surface area measure* of K.

For $K, L \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$ (not both 0), the *Minkowski linear combination* $\alpha K + \beta L$ is defined by $\alpha K + \beta L = \{\alpha x + \beta y : x \in K, y \in L\}$. For $K, L \in \overline{\mathcal{K}_o^n}$ and $\alpha, \beta \geq 0$, the *L_p-Minkowski linear combination* $\alpha \cdot K +_p \beta \cdot L$ (not both 0) is defined by $h(\alpha \cdot K +_p \beta \cdot L, u)^p = \alpha h(K, u)^p + \beta h(L, u)^p$ for every $u \in S^{n-1}$. Note that "$\cdot$" rather than "$\cdot_p$" is written for L_p-Minkowski scalar multiplication. This should create no confusion. Also note that the relationship between L_p-Minkowski and Minkowski scalar multiplication is $\alpha \cdot K = \alpha^{1/p} K$.

For $p \geq 1$, the L_p-mixed volume $V_p(K, L)$ of the convex bodies K, L containing the origin in their interiors was defined in [26] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon},$$

where the existence of this limit was demonstrated in [26]. Obviously, for each K, $V_p(K, K) = V(K)$. It was also shown in [26] that the L_p-mixed volume V_p has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u).$$

For $p \geq 1$, the *L_p-cosine transform* of a finite, signed Borel measure μ on S^{n-1} is defined by

$$C_p \mu(x) = \int_{S^{n-1}} |x \cdot v|^p d\mu(v), \ x \in \mathbb{R}^n.$$

Similarly, the *L_p-cosine transform* of a Borel measurable function f on S^{n-1} is defined by

$$(C_p f)(x) = \int_{S^{n-1}} |x \cdot v|^p f(v) d\sigma(v), \ x \in \mathbb{R}^n,$$

where σ is the spherical Lebesgue measure. An important property of this integral transform is the following injectivity behavior. If p is not an even integer and μ is a signed finite even Borel measure, then

$$\int_{S^{n-1}} |u \cdot v|^p d\mu(v) = 0 \text{ for all } u \in S^{n-1} \Rightarrow \mu = 0 \tag{2.1}$$

(see, e.g., Koldobsky [13, 14], Lonke [17], Neyman [33], and Rubin [37, 38]).

For $p \geq 1$, the *L_p-projection body*, $\Pi_p K$, of a convex body K containing the origin in its interior is the origin-symmetric convex body whose support function is defined by

$$h(\Pi_p K, u)^p = \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v)$$

for every $u \in S^{n-1}$. The notion of the L_p-projection body (with a different normalization) was introduced by Lutwak, Yang, and Zhang [29].

It is proved in [29] that

$$\Pi_p \phi K = |\det \phi|^{1/p} \phi^{-t} \Pi_p K$$

for every $\phi \in GL(n)$. Then we immediately get

$$C_p S_p(\phi K, \cdot)(x) = |\det \phi| C_p S_p(K, \cdot)(\phi^{-1} x) \tag{2.2}$$

and

$$C_p \frac{S_p(\phi K, \cdot)}{V(\phi K)}(x) = C_p \frac{S_p(K, \cdot)}{V(K)}(\phi^{-1} x). \tag{2.3}$$

The notion of the *L_p-centroid body* was introduced by Lutwak, Yang, and Zhang [29]: For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and for $p \geq 1$, the L_p-centroid body $\Gamma_p K$ is defined by

$$h(\Gamma_p K, u) = (\frac{1}{c_{n,p} V(K)} \int_K |x \cdot u|^p dx)^{1/p} \tag{2.4}$$

for every $u \in S^{n-1}$, where the constant $c_{n,p}$ is chosen so that $\Gamma_p B = B$. For $p = 2$, the Γ_2-centroid body is the Legendre ellipsoid of classical mechanics. It is easy to see that

$$\Gamma_p \phi K = \phi \Gamma_p K \tag{2.5}$$

for every $\phi \in GL(n)$. We also can rewrite relation (2.4) for the L_p-cosine transform:

$$\begin{aligned} h(\Gamma_p K, u)^p &= \frac{1}{(n+p) c_{n,p} V(K)} (C_p \rho_K^{n+p})(u) \\ &= \frac{1}{(n+p) c_{n,p} V(K)} (C_p (\frac{1}{2} \rho_K^{n+p} + \frac{1}{2} \rho_{-K}^{n+p}))(u). \end{aligned} \tag{2.6}$$

3. Normalized symmetric L_p-curvature images

In this section, we will show that the normalized symmetric L_p-curvature image operator $\widetilde{\Lambda}_c^p$ is a continuous, homogeneous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuation.

We remark that a valuation $Z : \mathcal{Q} \to \langle \mathfrak{P}(\mathbb{R}^n), + \rangle$ is $SL(n)$ covariant and homogeneous of degree q if and only if it satisfies

$$Z(\phi K) = (\det \phi)^{\frac{q-1}{n}} \phi Z K \tag{3.1}$$

for every $K \in \mathcal{Q}$ and $\phi \in GL(n)$ with positive determinant. Similarly, a valuation Z is $SL(n)$ contravariant and homogeneous of degree q if and only if it satisfies

$$Z(\phi K) = (\det \phi)^{\frac{q+1}{n}} \phi^{-t} Z K \tag{3.2}$$

for every $K \in \mathcal{Q}$ and $\phi \in GL(n)$ with positive determinant.

To prove that $\widetilde{\Lambda}_c^p$ is a continuous valuation, we will first show the following lemma.

Lemma 3.1. *If $K_i, K \in \mathcal{K}_c^n$, $i = 1, 2, \cdots$, such that $\frac{S_p(K_i,\cdot)}{V(K_i)} \to \frac{S_p(K,\cdot)}{V(K)}$ weakly, then $K_i \to K$.*

Proof. First, we want to show that $\{K_i\}$ has a subsequence, $\{K_{i_j}\}$, converging to an origin-symmetric convex body containing the origin in its interior (the proof is similar to [30, Theorem 2]).

Define $f_K(u)$ by

$$f_K(u)^p = \frac{1}{n}\int_{S^{n-1}} |u \cdot v|^p \frac{dS_p(K,v)}{V(K)}.$$

Thus $f_K(u)$ is a support function of some convex body. Since $\frac{S_p(K,\cdot)}{V(K)}$ is not concentrated on any great subsphere, $f_K(u) > 0$ for every $u \in S^{n-1}$. By the continuity of $f_K(u)$ on the compact set S^{n-1}, there exist two constants $a, b > 0$, such that $\frac{1}{2}a \geq f_K(u) \geq 2b$ for every $u \in S^{n-1}$. Since $\frac{S_p(K_i,\cdot)}{V(K_i)} \to \frac{S_p(K,\cdot)}{V(K)}$ weakly, we get $f_{K_i}(u) \to f_K(u)$. The convergence is uniform in $u \in S^{n-1}$ by [40, Theorem 1.8.12]. Hence $a \geq f_{K_i} \geq b$ for sufficiently large i uniformly.

In order to show that K_i is uniformly bounded, define real numbers M_i and vectors $u_i \in S^{n-1}$ by

$$M_i = \max_{u \in S^{n-1}} h(K_i, u) = h(K_i, u_i).$$

Now, $M_i[-u_i, u_i] \subset K_i$. Hence $M_i|u_i \cdot v| \leq h(K_i, v)$ for every $v \in S^{n-1}$. Thus,

$$\begin{aligned} M_i^p b^p &\leq M_i^p \frac{1}{n}\int_{S^{n-1}} |u_i \cdot v|^p \frac{dS_p(K_i,v)}{V(K_i)} \\ &\leq \frac{1}{n}\int_{S^{n-1}} h(K_i,v)^p \frac{dS_p(K_i,v)}{V(K_i)} = \frac{V_p(K_i,K_i)}{V(K_i)} = 1 \end{aligned}$$

for sufficiently large i. Hence K_i is uniformly bounded. By the Blaschke selection theorem, there exists a subsequence $\{K_{i_j}\}$ converging to a convex body, say K'. Since K_{i_j} are origin-symmetric, K' is origin-symmetric. Define real numbers m_i and vectors $u_i' \in S^{n-1}$ by

$$m_i = \min_{u \in S^{n-1}} h(K_i, u) = h(K_i, u_i').$$

The property $a \geq f_{K_i}$ for sufficiently large i uniformly, together with Jensen's inequality, shows that

$$\begin{aligned} a &\geq \Big(\frac{1}{n}\int_{S^{n-1}} |u_i' \cdot v|^p \frac{dS_p(K_i,v)}{V(K_i)}\Big)^{\frac{1}{p}} = \Big(\frac{1}{n}\int_{S^{n-1}} \Big(\frac{|u_i' \cdot v|}{h(K_i,v)}\Big)^p \frac{h(K_i,v)dS(K_i,v)}{V(K_i)}\Big)^{\frac{1}{p}} \\ &\geq \frac{1}{n}\int_{S^{n-1}} \frac{|u_i' \cdot v|}{h(K_i,v)} \frac{h(K_i,v)dS(K_i,v)}{V(K_i)} = \frac{2V(K_i|(u_i')^{\perp})}{nV(K_i)}. \end{aligned}$$

Since K_i is contained in the right cylinder $K_i|(u_i')^{\perp} \times m_i[-u_i', u_i']$, we have $2m_i V(K_i|(u_i')^{\perp}) \geq V(K_i)$. Thus,

$$a \geq \frac{2V(K_i|(u_i')^{\perp})}{nV(K_i)} \geq \frac{1}{nm_i},$$

which shows $m_i \geq \frac{1}{na}$ for sufficiently large i. Hence

$$\frac{1}{na}B \subseteq K',$$

where B is the unit ball in $\mathbb{R}^n$. Thus, K' contains the origin in its interior. The first step is complete.

Next, we argue the assertion by contradiction. Assume $K_i \nrightarrow K$; then there exists a subsequence, $\{K_{i_j}\}$, such that $d(K_{i_j}, K) \geq \varepsilon$ for a suitable $\varepsilon > 0$. Since $\{K_{i_j}\}$ also satisfies the condition of this lemma, from the conclusion above, there exists a subsequence of $\{K_{i_j}\}$, say $\{K_{i_{j_k}}\}$, converging to an origin-symmetric convex body, say K', containing the origin in its interior. Thus, $\frac{S_p(K_{i_{j_k}},\cdot)}{V(K_{i_{j_k}})} \to \frac{S_p(K',\cdot)}{V(K')}$ weakly. By the uniqueness of weak convergence and the normalized even L_p-Minkowski problem, we get $K_{i_{j_k}} \to K' = K$, which is a contradiction. □

Theorem 3.2. *The normalized symmetric L_p-curvature image operator $\widetilde{\Lambda}_c^p : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a continuous, $SL(n)$ contravariant valuation which is homogeneous of degree $-\frac{n}{p}-1$. Moreover, $\psi_{\pi/2}\widetilde{\Lambda}_c^p : \mathcal{K}_o^2 \to \langle \mathcal{K}_c^2, \widetilde{\#}_p \rangle$ is a continuous, $SL(2)$ covariant valuation which is homogeneous of degree $-\frac{2}{p}-1$.*

Proof. To prove that $\widetilde{\Lambda}_c^p$ is a normalized symmetric L_p-Blaschke valuation, we just need to show

$$\frac{S_p(\widetilde{\Lambda}_c^p(K \cup L),\cdot)}{V(\widetilde{\Lambda}_c^p(K \cup L))} + \frac{S_p(\widetilde{\Lambda}_c^p(K \cap L),\cdot)}{V(\widetilde{\Lambda}_c^p(K \cap L))} = \frac{S_p(\widetilde{\Lambda}_c^p K,\cdot)}{V(\widetilde{\Lambda}_c^p K)} + \frac{S_p(\widetilde{\Lambda}_c^p L,\cdot)}{V(\widetilde{\Lambda}_c^p L)} \tag{3.3}$$

for every $K, L, K \cup L, K \cap L \in \mathcal{K}_o^n$. Since

$$\rho(K \cup L,\cdot)^{n+p} + \rho(K \cap L,\cdot)^{n+p} = \rho(K,\cdot)^{n+p} + \rho(L,\cdot)^{n+p},$$
$$\rho(-(K \cup L),\cdot)^{n+p} + \rho(-(K \cap L),\cdot)^{n+p} = \rho(-K,\cdot)^{n+p} + \rho(-L,\cdot)^{n+p}$$

for every $K, L, K \cup L, K \cap L \in \mathcal{K}_o^n$, it follows from the definition of $\widetilde{\Lambda}_c^p$, that the relation (3.3) is true. Hence the valuation property is established.

To prove homogeneity and $SL(n)$ contravariance of $\widetilde{\Lambda}_c^p$, by relation (3.2), we need to show

$$\widetilde{\Lambda}_c^p \phi K = (\det \phi)^{-1/p} \phi^{-t} \widetilde{\Lambda}_c^p K \tag{3.4}$$

for every $\phi \in GL(n)$ with positive determinant. Indeed, the definition of $\widetilde{\Lambda}_c^p$, the relations (2.5) and (2.6), together with (2.3), imply that

$$\begin{aligned}
C_p \frac{S_p(\widetilde{\Lambda}_c^p \phi K,\cdot)}{V(\widetilde{\Lambda}_c^p \phi K)}(u) &= (C_p(\frac{1}{2}\rho_{\phi K}^{n+p} + \frac{1}{2}\rho_{-\phi K}^{n+p}))(u) \\
&= (n+p)c_{n,p} V(\phi K) h(\Gamma_p \phi K, u)^p \\
&= |\det \phi|(n+p)c_{n,p} V(K) h(\Gamma_p K, \phi^t u)^p \\
&= |\det \phi| C_p \frac{S_p(\widetilde{\Lambda}_c^p K,\cdot)}{V(\widetilde{\Lambda}_c^p K)}(\phi^t u) \\
&= C_p \frac{S_p(|\det \phi|^{-1/p}\phi^{-t}\widetilde{\Lambda}_c^p K,\cdot)}{V(|\det \phi|^{-1/p}\phi^{-t}\widetilde{\Lambda}_c^p K)}(u).
\end{aligned}$$

The injectivity property (2.1) and the uniqueness of the volume-normalized even L_p-Minkowski problem now imply relation (3.4).

If $K_i \to K$, then $\rho(K_i, \cdot) \to \rho(K, \cdot)$ almost everywhere with respect to spherical Lebesgue measure (see [6, Lemma 1]). Hence

$$(\frac{1}{2}\rho(K_i, \cdot)^{n+p} + \frac{1}{2}\rho(-K_i, \cdot)^{n+p}) \to (\frac{1}{2}\rho(K, \cdot)^{n+p} + \frac{1}{2}\rho(-K, \cdot)^{n+p})$$

almost everywhere. Since $(\frac{1}{2}\rho(K_i, \cdot)^{n+p} + \frac{1}{2}\rho(-K_i, \cdot)^{n+p})$ are uniformly bounded, $\frac{S_p(\widetilde{\Lambda}_c^p K_i, \cdot)}{V(\widetilde{\Lambda}_c^p K_i)} \to \frac{S_p(\widetilde{\Lambda}_c^p K, \cdot)}{V(\widetilde{\Lambda}_c^p K)}$ weakly. Hence, by Lemma 3.1, we get $\widetilde{\Lambda}_c^p K_i \to \widetilde{\Lambda}_c^p K$. Thus, $\widetilde{\Lambda}_c^p K$ is a continuous valuation.

If $\phi \in SL(2)$, we have $\psi_{\pi/2}\phi^{-t}\psi_{-\pi/2} = \phi$. Then we get

$$\psi_{\pi/2}\widetilde{\Lambda}_c^p \phi K = \psi_{\pi/2}\phi^{-t}\widetilde{\Lambda}_c^p K = \psi_{\pi/2}\phi^{-t}\psi_{-\pi/2}\psi_{\pi/2}\widetilde{\Lambda}_c^p K = \phi\psi_{\pi/2}\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^n$. Since the operator $\psi_{\pi/2}$ is continuous, we obtain that $\psi_{\pi/2}\widetilde{\Lambda}_c^p$ is continuous. Moreover, it is easy to verify that $\psi_{\pi/2}\widetilde{\Lambda}_c^p$ is a normalized symmetric L_p-Blaschke valuation which is homogeneous of degree $-\frac{2}{p} - 1$. Hence, $\psi_{\pi/2}\widetilde{\Lambda}_c^p$ is a continuous, $SL(2)$ covariant normalized symmetric L_p-Blaschke valuation which is homogeneous of degree $-\frac{2}{p} - 1$. □

4. Normalized L_p-Blaschke valuations

In this section, for the contravariant and covariant case, respectively, we establish our classification results for continuous, linearly intertwining, normalized symmetric L_p-Blaschke valuations.

We remark first the fact that the $SL(n)$ covariance (or contravariance) and homogeneity of a valuation $Z : \overline{\mathcal{K}}_o^n \to \langle \mathfrak{P}(\mathbb{R}^n), + \rangle$ are completely determined by the restriction of Z to n-dimensional convex bodies if the Abelian semigroup $\langle \mathfrak{P}(\mathbb{R}^n), + \rangle$ has the cancellation property. (Actually this property is generalized from Lemma 4 and Lemma 9 of Haberl [6], and the proof of this property is almost the same as Haberl's.)

Lemma 4.1. *If $Z : \overline{\mathcal{K}}_o^n \to \langle \mathfrak{P}(\mathbb{R}^n), + \rangle$ is a valuation which is $SL(n)$ covariant (or contravariant) and homogeneous of degree q on n-dimensional convex bodies, and $\langle \mathfrak{P}(\mathbb{R}^n), + \rangle$ has the cancellation property, then Z is $SL(n)$ covariant (or contravariant respectively) and homogeneous of degree q on $\overline{\mathcal{K}}_o^n$.*

Proof. In the covariant case, we have to show that

$$Z\phi K = (\det \phi)^{\frac{q-1}{n}} \phi Z K \tag{4.1}$$

for every $K \in \overline{\mathcal{K}}_o^n$ and $\phi \in GL(n)$ with positive determinant. Let $\dim K = n - k$, where $0 \le k \le n$. We prove our assertion by induction on k. Indeed, (4.1) is true for $k = 0$ by assumption. Assume that (4.1) holds for $(n - k)$-dimensional convex bodies and $\dim K = n - (k + 1)$. Choose $u \notin \text{lin } K$, where lin K denotes the linear hull of K. Clearly $[K, u]$, $[K, -u]$, $[K, u, -u]$, $\phi[K, u]$, $\phi[K, -u]$, $\phi[K, u, -u]$ are of dimension $n - k$, and

$$[K, u] \cup [K, -u] = [K, u, -u],\ [K, u] \cap [K, -u] = K,$$
$$\phi[K, u] \cup \phi[K, -u] = \phi[K, u, -u],\ \phi[K, u] \cap \phi[K, -u] = \phi K.$$

Since Z is a valuation,

$$Z\phi K + Z\phi[K, u, -u] = Z\phi[K, u] + Z\phi[K, -u].$$

With the induction assumption, we get

$$Z\phi K + (\det\phi)^{\frac{q-1}{n}}\phi Z[K, u, -u] = (\det\phi)^{\frac{q-1}{n}}\phi Z[K, u] + (\det\phi)^{\frac{q-1}{n}}\phi Z[K, -u].$$

So,

$$(\det\phi)^{-\frac{q-1}{n}}\phi^{-1}Z\phi K + Z[K, u, -u] = Z[K, u] + Z[K, -u].$$

By the cancellation property of $\langle\mathfrak{P}(\mathbb{R}^n), +\rangle$, combined with the relation

$$ZK + Z[K, u, -u] = Z[K, u] + Z[K, -u],$$

we have

$$(\det\phi)^{-\frac{q-1}{n}}\phi^{-1}Z\phi K = ZK. \tag{4.2}$$

This immediately proves that (4.1) holds for bodies of dimension $n-k-1$.

The contravariant case is proved similarly to the covariant case. $\square$

Since $\mathcal{K}_o^n$ endowed with L_p-Minkowski sum is an Abelian semigroup which has the cancellation property, we immediately get the following.

Lemma 4.2. *If $Z : \overline{\mathcal{K}}_o^n \to \langle\mathcal{K}_o^n, +_p\rangle$ is a L_p-Minkowski valuation which is $SL(n)$ covariant (or contravariant) and homogeneous of degree q on n-dimensional convex bodies, then Z is $SL(n)$ covariant (or contravariant respectively) and homogeneous of degree q on $\overline{\mathcal{K}}_o^n$.*

4.1. **The contravariant case.** First, we reduce the possible degrees of homogeneity of continuous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuations.

Lemma 4.3. *If $Z : \mathcal{K}_o^n \to \langle\mathcal{K}_c^n, \widetilde{\#}_p\rangle$ is a continuous, $SL(n)$ contravariant valuation which is homogeneous of degree q, then $q \le -1$.*

Proof. Suppose $K \in \mathcal{K}_o^n$ is an arbitrary convex body and that $K \cap e_n^+$ and $K \cap e_n^-$ are n-dimensional. For every positive s we have

$$[K \cap e_n^+, \pm se_n] \cup [K \cap e_n^-, \pm se_n] = [K, \pm se_n],$$

$$[K \cap e_n^+, \pm se_n] \cap [K \cap e_n^-, \pm se_n] = [K \cap e_n^\perp, \pm se_n].$$

Since Z is a normalized symmetric L_p-Blaschke valuation, we have

$$\begin{aligned} &C_p\frac{S_p(Z[K \cap e_n^\perp, \pm se_n], \cdot)}{V(Z[K \cap e_n^\perp, \pm se_n])}(e_1) \\ =&C_p\frac{S_p(Z[K \cap e_n^+, \pm se_n], \cdot)}{V(Z[K \cap e_n^+, \pm se_n])}(e_1) + C_p\frac{S_p(Z[K \cap e_n^-, \pm se_n], \cdot)}{V(Z[K \cap e_n^-, \pm se_n])}(e_1) \\ &- C_p\frac{S_p(Z[K, \pm se_n], \cdot)}{V(Z[K, \pm se_n])}(e_1). \end{aligned} \tag{4.3}$$

Define a linear map ϕ by

$$\phi e_i = e_i, i = 1, \cdots, n-1, \phi e_n = se_n.$$

From the $SL(n)$ contravariance and homogeneity of Z as well as relations (3.2) and (2.3), we get

$$C_p \frac{S_p(Z[K\cap e_n^\perp, \pm s e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm s e_n])}(e_1) = C_p \frac{S_p(s^{\frac{q+1}{n}}\phi^{-t} Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(s^{\frac{q+1}{n}}\phi^{-t} Z[K\cap e_n^\perp, \pm e_n])}(e_1)$$
$$= s^{\frac{-(q+1)p}{n}} C_p \frac{S_p(Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm e_n])}(\phi^t e_1).$$

Since $|e_1\cdot u| > 0$ for all $u\in S^{n-1}\setminus e_1^\perp$, and the L_p-surface area measure of n-dimensional bodies is not concentrated on any great sphere, we conclude that

$$C_p \frac{S_p(Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm e_n])}(\phi^t e_1)$$
$$= \frac{1}{V(Z[K\cap e_n^\perp, \pm e_n])}\int_{S^{n-1}} |e_1\cdot u|^p dS_p(Z[K\cap e_n^\perp, \pm e_n], u) > 0.$$

Moreover, we have

$$\begin{aligned} \lim_{s\to 0^+}[K\cap e_n^+, \pm s e_n] &= K\cap e_n^+,\\ \lim_{s\to 0^+}[K\cap e_n^-, \pm s e_n] &= K\cap e_n^-,\\ \lim_{s\to 0^+}[K, \pm s e_n] &= K. \end{aligned}$$

Hence the continuity of Z and volume, together with the weak continuity of L_p-surface area measures, imply that the right side of (4.3) converges to a finite number as $s\to 0^+$. This implies $\frac{-(q+1)p}{n}\ge 0$, so $q\le -1$. □

In the next two lemmas, we will show how to generate a homogeneous, $SL(n)$ covariant L_p-Minkowski valuation on $\overline{\mathcal{K}}_o^n$ by a continuous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuation which is homogeneous of degree q on $\mathcal{K}_o^n$, where $q\le -1$.

Lemma 4.4. *Let $Z:\mathcal{K}_o^n\to\langle\mathcal{K}_c^n, \widetilde{\#}_p\rangle$ be a continuous, $SL(n)$ contravariant valuation which is homogeneous of degree $q=-1$. Define the map $\overline{Z}_1:\ \overline{\mathcal{K}}_o^n\to\langle\overline{\mathcal{K}}_o^n, +_p\rangle$ by*

$$h(\overline{Z}_1K, x)^p = \begin{cases} C_p\frac{S_p(ZK,\cdot)}{V(ZK)}(x), & \dim K = n,\\ C_p\frac{S_p(Z[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K,\pm b_{k+1},\cdots,\pm b_n])}(\pi_K x), & \dim K = k < n,\end{cases}$$

for every $x\in\mathbb{R}^n$, where the $b_{k+1},\cdots,b_n$ are an orthonormal basis of the orthogonal complement of lin K *and π_K is the orthogonal projection onto* lin K. *Then $\overline{Z}_1$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree* 1.

Proof. In order to show that $\overline{Z}_1$ is well defined, suppose that $\dim K = k < n$ and $b_{k+1},\cdots,b_n$ as well as $c_{k+1},\cdots,c_n$ are two different orthonormal bases of $(\text{lin } K)^\perp$. Fix an orthonormal basis $b_1,\cdots,b_k$ of lin K. Denote by θ a proper rotation with $\theta b_i = b_i,\ i=1,\cdots,k$, and $\theta b_i\in\{\pm c_i\},\ i=k+1,\cdots,n$. Then the contravariance of Z and relation (2.3) induce that

$$\begin{aligned}
C_p\frac{S_p(Z[K,\pm c_{k+1},\cdots,\pm c_n],\cdot)}{V(Z[K,\pm c_{k+1},\cdots,\pm c_n])}(\pi_K x) &= C_p\frac{S_p(Z\theta[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z\theta[K,\pm b_{k+1},\cdots,\pm b_n])}(\pi_K x)\\
&= C_p\frac{S_p(\theta Z[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(\theta Z[K,\pm b_{k+1},\cdots,\pm b_n])}(\pi_K x)\\
&= C_p\frac{S_p(Z[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K,\pm b_{k+1},\cdots,\pm b_n])}(\theta^{-1}\pi_K x)\\
&= C_p\frac{S_p(Z[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K,\pm b_{k+1},\cdots,\pm b_n])}(\pi_K x).
\end{aligned}$$

Thus, $\overline{Z}_1$ is well defined.

Next, we show that $\overline{Z}_1$ is an L_p-Minkowski valuation. Suppose that $K, L \in \overline{\mathcal{K}}_o^n$ such that $K\cup L \in \overline{\mathcal{K}}_o^n$ and let k be an integer not larger than n. If $\dim(K\cup L)=k$, then one of the following four cases is valid:
(1_k) $\dim K = k, \dim L = k, \dim K\cap L = k,\ 0\le k\le n$,
(2_k) $\dim K = k, \dim L = k, \dim K\cap L = k-1,\ 1\le k\le n$,
(3_k) $\dim K = k, \dim L = k-1,\ 1\le k\le n$,
(4_k) $\dim K = k-1, \dim L = k,\ 1\le k\le n$.
The valuation property trivially holds true for the cases (3_k) and (4_k), since we have $L\subset K$ and $K\subset L$ respectively in these situations. Therefore it suffices to prove

$$h^p_{\overline{Z}_1(K\cup L)} + h^p_{\overline{Z}_1(K\cap L)} = h^p_{\overline{Z}_1 K} + h^p_{\overline{Z}_1 L}$$

for the cases $(1_k), 0\le k\le n$, and $(2_k), 1\le k\le n$.

Let us start with the easy case (1_n). The valuation property of Z implies

$$\frac{S_p(Z(K\cup L),\cdot)}{V(Z(K\cup L))}+\frac{S_p(Z(K\cap L),\cdot)}{V(Z(K\cap L))}=\frac{S_p(ZK,\cdot)}{V(ZK)}+\frac{S_p(ZL,\cdot)}{V(ZL)},$$

and thus

$$C_p\frac{S_p(Z(K\cup L),\cdot)}{V(Z(K\cup L))}+C_p\frac{S_p(Z(K\cap L),\cdot)}{V(Z(K\cap L))}=C_p\frac{S_p(ZK,\cdot)}{V(ZK)}+C_p\frac{S_p(ZL,\cdot)}{V(ZL)}.$$

Hence the definition of $\overline{Z}_1$ immediately proves the assertion. Next we deal with the case (1_k), $0\le k<n$. Note that

$$\begin{aligned}
[K,\pm b_{k+1},\cdots,\pm b_n]\cup[L,\pm b_{k+1},\cdots,\pm b_n] &= [K\cup L,\pm b_{k+1},\cdots,\pm b_n],\\
[K,\pm b_{k+1},\cdots,\pm b_n]\cap[L,\pm b_{k+1},\cdots,\pm b_n] &= [K\cap L,\pm b_{k+1},\cdots,\pm b_n].
\end{aligned}$$

Since lin $K=$ lin $L=$ lin $(K\cup L)=$ lin $(K\cap L)$, we have $\pi_K x=\pi_L x=\pi_{(K\cup L)}x=\pi_{(K\cap L)}x$. With the valuation property of case (1_n) proved above, we get

$$\begin{aligned}
&C_p\frac{S_p(Z[K\cup L,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K\cup L,\pm b_{k+1},\cdots,\pm b_n])}(x)+C_p\frac{S_p(Z[K\cap L,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K\cap L,\pm b_{k+1},\cdots,\pm b_n])}(x)\\
&= C_p\frac{S_p(Z[K,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[K,\pm b_{k+1},\cdots,\pm b_n])}(x)+C_p\frac{S_p(Z[L,\pm b_{k+1},\cdots,\pm b_n],\cdot)}{V(Z[L,\pm b_{k+1},\cdots,\pm b_n])}(x)
\end{aligned}$$

for every $x\in\mathbb{R}^n$. Changing x to $\pi_K x$, we get the positive assertion of the case (1_k).

Now we consider the case (2_k), $1 \le k \le n$. It is enough to show

$$h^p_{\overline{Z}_1 K} + h^p_{\overline{Z}_1 (K\cap u^\perp)} = h^p_{\overline{Z}_1 (K\cap u^+)} + h^p_{\overline{Z}_1 (K\cap u^-)} \tag{4.4}$$

for $\dim K = k$ and a unit vector $u \in \text{lin } K$ such that $K \cap u^+, K \cap u^-$ are both k-dimensional. Notice that if $k = n$, then $\pi_K x = x$. So we will prove the case (2_k) without distinguishing between $k = n$ and $k < n$. Let $b_1, \cdots, b_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $\text{lin } K = \text{lin }\{b_1, \cdots, b_k\}$, and $u = b_k$. With the valuation property of case (1_k) proved above, we have

$$\begin{aligned}
C_p &\frac{S_p(Z[K, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
&\quad + C_p \frac{S_p(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
&= C_p \frac{S_p(Z[K \cap b_k^+, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^+, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
&\quad + C_p \frac{S_p(Z[K \cap b_k^-, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^-, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x)
\end{aligned} \tag{4.5}$$

for sufficiently small $s > 0$. Define a linear map ϕ by

$$\phi b_k = s b_k, \phi b_i = b_i,\ i = 1, \cdots, k-1, k+1, \cdots, n.$$

Note that $\det \phi$ is independent of the choice of orthonormal basis of $\mathbb{R}^n$, so $\det \phi = s$. The contravariance of Z and relations (3.2) as well as (2.3) give

$$\begin{aligned}
C_p &\frac{S_p(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
=& C_p \frac{S_p(Z\phi[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z\phi[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
=& C_p \frac{S_p(s^{\frac{q+1}{n}}\phi^{-t} Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(s^{\frac{q+1}{n}}\phi^{-t} Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
=& s^{\frac{-(q+1)p}{n}} C_p \frac{S_p(Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\phi^t \pi_K x).
\end{aligned} \tag{4.6}$$

Note that $\lim\limits_{s\to 0^+} \phi^t \pi_K x = \pi_{K\cap b_k^\perp} x$. Since $q = -1$,

$$\begin{aligned}
\lim_{s\to 0^+} C_p &\frac{S_p(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^\perp, \pm s b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_K x) \\
&= C_p \frac{S_p(Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n], \cdot)}{V(Z[K \cap b_k^\perp, \pm b_k, \pm b_{k+1}, \cdots, \pm b_n])}(\pi_{K\cap b_k^\perp} x).
\end{aligned}$$

So if s tends to zero in (4.5), then we immediately obtain (4.4). Hence we have proved that $\overline{Z}_1$ is an L_p-Minkowski valuation. Moreover, it is easy to calculate that $\overline{Z}_1$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree 1 on n-dimensional convex bodies. Lemma 4.2 implies that $\overline{Z}_1$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree 1 on $\overline{\mathcal{K}}_o^n$. $\square$

Lemma 4.5. *Let $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ be a continuous, $SL(n)$ contravariant valuation which is homogeneous of degree $q < -1$. Define the map $\overline{Z}_2 : \overline{\mathcal{K}}_o^n \to \langle \overline{\mathcal{K}}_o^n, +_p \rangle$ by*

$$h(\overline{Z}_2 K, x)^p = \begin{cases} C_p \frac{S_p(ZK,\cdot)}{V(ZK)}(x), & \dim K = n, \\ 0, & \dim K = k < n, \end{cases}$$

for every $x \in \mathbb{R}^n$. Then $\overline{Z}_2$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree $r = -q$.

Proof. We use the notation of Lemma 4.4. Since the case (1_n) is the same as in Lemma 4.4, and the cases (1_k), $0 \le k < n$, (2_k), $1 \le k < n$, are trivially true, we just need to consider the case (2_n).

Hence we need to show

$$h^p_{\overline{Z}_2 K} + h^p_{\overline{Z}_2 (K \cap u^\perp)} = h^p_{\overline{Z}_2 (K \cap u^+)} + h^p_{\overline{Z}_2 (K \cap u^-)} \tag{4.7}$$

for $\dim K = n$ and a unit vector $u \in \mathbb{R}^n$ such that $K \cap u^+, K \cap u^-$ are both n-dimensional. Let $b_1, \cdots, b_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $u = b_n$. Comparing with the proof of Lemma 4.4, we just need to show the relation (4.6) of the case $k = n$ tends to zero for $q < -1$ when s tends to zero. Actually, the relation (4.6) of the case $k = n$ is

$$C_p \frac{S_p(Z[K \cap b_n^\perp, \pm s b_n], \cdot)}{V(Z[K \cap b_n^\perp, \pm s b_n])}(x) = s^{\frac{-(q+1)p}{n}} C_p \frac{S_p(Z[K \cap b_n^\perp, \pm b_n], \cdot)}{V(Z[K \cap b_n^\perp, \pm b_n])}(\phi^t x),$$

where ϕ is a linear map defined by $\phi b_n = s b_n, \phi b_i = b_i,\ i = 1, \cdots, n-1$. Since $q < -1$,

$$\lim_{s \to 0^+} C_p \frac{S_p(Z[K \cap b_n^\perp, \pm s b_n], \cdot)}{V(Z[K \cap b_n^\perp, \pm s b_n])}(x) = 0.$$

Hence $\overline{Z}_2$ is an L_p-Minkowski valuation. Moreover, it is easy to calculate that $\overline{Z}_2$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree $r = -q$. □

For $p > 1$, the following lemma shows that every support set of an L_p-projection body consists of precisely one point. It will help to rule out the existence of continuous, normalized symmetric L_p-Blaschke valuations which are homogeneous of degree -1 (see Theorem 4.8 and Theorem 4.13 for more details). A similar result for $p = 1$ can be found in Schneider [40, Lemma 3.5.5].

For $K \in \mathcal{K}^n$, $e \in S^{n-1}$, write $K_e := \{x \in K | x \cdot e = h(K, e)\}$.

Lemma 4.6. *For $p > 1$, if the support function of the convex body $K \in \mathcal{K}^n$ is given by*

$$h(K, u) = (\int_{S^{n-1}} |u \cdot v|^p d\mu(v))^{1/p}$$

for $u \in S^{n-1}$, with an even signed measure μ, then, for $e \in S^{n-1}$,

$$h(K_e, u) = v_e \cdot u$$

for $u \in S^{n-1}$, where $v_e = 2(\int_{S^{n-1}} |e \cdot v|^p d\mu(v))^{\frac{1}{p}-1} \int_{e^+} (e \cdot v)^{p-1} v d\mu(v)$.

Proof. The assertion of the lemma is true for $u = \pm e$, since $h(K_e, \pm e) = \pm h(K, e)$. Hence we may assume that u and e are linearly independent. Note that $h(K_e, u) = \lim\limits_{s\to 0^+} \frac{h(K,e+su)-h(K,e)}{s}$ (see Schneider [40, Theorem 1.7.2]). Put

$$\begin{aligned} A_s &:= \{v \in S^{n-1} | e\cdot v > 0, (e+su)\cdot v > 0\},\\ B_s &:= \{v \in S^{n-1} | e\cdot v \le 0, (e+su)\cdot v > 0\},\\ C_s &:= \{v \in S^{n-1} | e\cdot v > 0, (e+su)\cdot v \le 0\}. \end{aligned}$$

We obtain

$$\begin{aligned} h(K_e,u) &= \lim_{s\to 0^+} \frac{h(K,e+su)-h(K,e)}{s}\\ &= \frac{1}{p}\Big(\int_{S^{n-1}} |e\cdot v|^p d\mu(v)\Big)^{\frac{1}{p}-1} \lim_{s\to 0^+} \frac{1}{s}\Big(\int_{S^{n-1}} |(e+su)\cdot v|^p d\mu(v) - \int_{S^{n-1}} |e\cdot v|^p d\mu(v)\Big) \end{aligned}$$

and

$$\begin{aligned} &\lim_{s\to 0^+} \frac{1}{s}\Big(\int_{S^{n-1}} |(e+su)\cdot v|^p d\mu(v) - \int_{S^{n-1}} |e\cdot v|^p d\mu(v)\Big)\\ &= 2\lim_{s\to 0^+} \frac{1}{s}\Big(\int_{A_s\cup B_s} ((e+su)\cdot v)^p d\mu(v) - \int_{A_s\cup C_s} (e\cdot v)^p d\mu(v)\Big)\\ &= 2p\lim_{s\to 0^+} \int_{A_s\cup B_s} (e\cdot v)^{p-1}(u\cdot v) d\mu(v)\\ &\quad + \lim_{s\to 0^+} \int_{A_s\cup B_s} p(p-1)(e\cdot v)^{p-2}(u\cdot v)^2 s + o(s) d\mu(v)\\ &\quad + 2\lim_{s\to 0^+} \frac{1}{s}\int_{B_s} (e\cdot v)^p d\mu(v) - 2\lim_{s\to 0^+} \frac{1}{s}\int_{C_s} (e\cdot v)^p d\mu(v). \end{aligned}$$

Let

$$\begin{aligned} \mu_+(E) &= \sup\{\mu(A) | A \subset E \text{ and } A \text{ is a Borel set of } S^{n-1}\},\\ \mu_-(E) &= -\inf\{\mu(A) | A \subset E \text{ and } A \text{ is a Borel set of } S^{n-1}\},\\ \mu'(E) &= \mu_+(E) + \mu_-(E) \end{aligned}$$

for every Borel set E of S^{n-1}. We get

$$\begin{aligned} &\Big|\int_{A_s\cup B_s} p(p-1)(e\cdot v)^{p-2}(u\cdot v)^2 s + o(s) d\mu(v)\Big|\\ &\qquad \le \int_{S^{n-1}} \big|p(p-1)(e\cdot v)^{p-2}(u\cdot v)^2 s + o(s)\big| d\mu'(v) \xrightarrow{s\to 0^+} 0. \end{aligned}$$

For $v \in B_s$, we have $|e\cdot v| \le cs$ with a constant c independent of s. Writing

$$B'_s := \{v \in S^{n-1} | e\cdot v < 0, (e+su)\cdot v > 0\},$$

we obtain

$$\Big|\frac{1}{s}\int_{B_s} (e\cdot v)^p d\mu(v)\Big| = \Big|\frac{1}{s}\int_{B'_s} (e\cdot v)^p d\mu(v)\Big| \le c^p s^{p-1}\mu(B'_s).$$

Since (in the set-theoretic sense) $\lim\limits_{s\to 0^+} B'_s = \emptyset$, we have $\lim\limits_{s\to 0^+} \mu'(B'_s) = 0$. With $p > 1$, we get

$$\lim_{s\to 0^+} \frac{1}{s}\int_{B_s} (e\cdot v)^p d\mu(v) = 0.$$

From $\lim_{s\to 0^+} C_s = \emptyset$, we similarly find

$$\lim_{s\to 0^+} \frac{1}{s}\int_{C_s} (e\cdot v)^p d\mu(v) = 0.$$

Further, if $\lim_{s\to 0^+} A_s = e^+\backslash e^\perp$, $\lim_{s\to 0^+} B_s = \{v \in S^{n-1} | e\cdot v = 0, u\cdot v > 0\}$, and $p > 1$, we get

$$\lim_{s\to 0^+} \int_{A_s\cup B_s} (e\cdot v)^{p-1}(u\cdot v)d\mu(v) = \int_{e^+} (e\cdot v)^{p-1}(u\cdot v)d\mu(v).$$

Finally, we get

$$\begin{aligned} h(K_e, u) &= 2(\int_{S^{n-1}} |e\cdot v|^p d\mu(v))^{\frac{1}{p}-1} \int_{e^+} (e\cdot v)^{p-1}(u\cdot v)d\mu(v) \\ &= \big(2(\int_{S^{n-1}} |e\cdot v|^p d\mu(v))^{\frac{1}{p}-1} \int_{e^+} (e\cdot v)^{p-1} v d\mu(v)\big)\cdot u, \end{aligned}$$

which completes the proof of the lemma. □

To classify continuous, homogeneous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuations, we need the following results from Ludwig [20].

For $-1 \le \tau \le 1$, define $M_p^\tau : \overline{\mathcal{K}}_o^n \to \overline{\mathcal{K}}_o^n$ by

$$h^p(M_p^\tau K, v) = \int_K (|v\cdot x| + \tau(v\cdot x))^p dx$$

for $v \in \mathbb{R}^n$. In particular, $M_p^0 K$ is a dilate of the L_p-centroid body, if $V(K) > 0$.

A polytope is the convex hull of finitely many points in $\mathbb{R}^n$. Let $\mathcal{P}_o^n$ be the set of n-dimensional polytopes which contain the origin and $\overline{\mathcal{P}}_o^n$ the set of polytopes which contain the origin. Let $\xi_o(P)$ denote the set of edges of a polytope P which contain the origin.

Lemma 4.7 ([20])**.** *Let $Z : \overline{\mathcal{P}}_o^n \to \langle \overline{\mathcal{K}}_o^n, +_p\rangle$, $n \ge 3$, be an L_p-Minkowski valuation, $p > 1$, which is $SL(n)$ covariant and homogeneous of degree r. If $r = n/p + 1$, then there are constants $a \ge 0$ and $-1 \le \tau \le 1$ such that*

$$ZP = aM_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^n$. If $r = 1$, then there are constants $a, b \ge 0$ such that

$$ZP = aP +_p b(-P)$$

for every $P \in \overline{\mathcal{P}}_o^n$. In all other cases, $ZP = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^n$.

Let $Z : \overline{\mathcal{P}}_o^2 \to \langle \overline{\mathcal{K}}_o^2, +_p\rangle$ be an L_p-Minkowski valuation, $p > 1$, which is $SL(2)$ covariant and homogeneous of degree r. If $r = 2/p + 1$, then there are constants $a \ge 0$ and $-1 \le \tau \le 1$ such that

$$ZP = aM_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^2$. If $r = 1$, then there are constants $a_0, b_0 \ge 0$ and $a_i, b_i \in \mathbb{R}$ with $a_0 + a_i,\ b_0 + b_i \ge 0,\ i = 1, 2$, such that

$$ZP = a_0P +_p b_0(-P) +_p \sum\nolimits^p (a_iE_i +_p b_i(-E_i))$$

for every $P \in \overline{\mathcal{P}}_o^2$, where $\sum^p$ denotes the L_p-Minkowski sum, and the sum is taken over $E_i \in \xi_o(P)$. If $r = 2/p - 1$, then there are constants $a \geq 0$ and $-1 \leq \tau \leq 1$ such that

$$ZP = a\psi_{\pi/2}\hat{\Pi}_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^2$. Here $\hat{\Pi}_p^\tau P$ is defined by the relation (4.16). *In all other cases, $ZP = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^2$.*

Now we can classify continuous, homogeneous, $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuations.

Theorem 4.8. *Let $n \geq 2$, $p > 1$ and p not an even integer. If $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a continuous, homogeneous, $SL(n)$ contravariant valuation, then there exists a constant $c > 0$ such that*

$$ZK = c\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^n$.

Proof. Let q be the degree of homogeneity of Z. Lemma 4.3 shows that $q \leq -1$.

If $q = -1$, then $\overline{Z}_1$, introduced in Lemma 4.4, is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree 1. If $n \geq 3$, from Lemma 4.7, we derive that there are constants $a, b \geq 0$ such that

$$\overline{Z}_1 P = aP +_p b(-P)$$

for every $P \in \overline{\mathcal{P}}_o^n$. If $n = 2$, from Lemma 4.7, we derive that there are constants $a_0, b_0 \geq 0$ and $a_i, b_i \in \mathbb{R}$ with $a_0 + a_i,\ b_0 + b_i \geq 0,\ i = 1, 2$, such that

$$\overline{Z}_1 P = a_0 P +_p b_0(-P) +_p \sum\nolimits^p (a_i E_i +_p b_i(-E_i))$$

for every $P \in \overline{\mathcal{P}}_o^n$, where the sum is taken over $E_i \in \xi_o(P)$. For $P_0 = [\pm e_1, \cdots, \pm e_n]$, we have

$$\frac{\Pi_p Z P_0}{V(ZP_0)^{1/p}} = cP_0,$$

for a suitable $c \geq 0$ when $n \geq 2$. The assumption that Z does not contain $\{o\}$ in its range gives $c > 0$. For $p > 1$, every support set of an L_p-projection body consists of precisely one point (Lemma 4.6). However, P_0 has the support set $[e_1, \cdots, e_n]$ which does not consist of precisely one point. This is a contradiction.

If $q = -n/p - 1$, then $\overline{Z}_2$, introduced in Lemma 4.5, is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree $n/p + 1$. For $n \geq 2$, from Lemma 4.7, we infer the existence of constants $a \geq 0$ and $-1 \leq \tau \leq 1$ such that

$$\overline{Z}_2 P = aM_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^n$. The assumption that Z does not contain $\{o\}$ in its range gives $a > 0$. Since $\overline{Z}_2 P$ is origin-symmetric, we deduce that $\tau = 0$. Thus, $\frac{\Pi_p ZP}{V(ZP)^{1/p}} = aM_p^0 P$ for every $P \in \mathcal{P}_o^n$. Since the operators $\frac{\Pi_p Z}{V^{1/p}}$ and Γ_p are continuous on $\mathcal{K}_o^n$, and $\mathcal{P}_o^n$ is dense in $\mathcal{K}_o^n$, we obtain

$$\frac{\Pi_p ZK}{V(ZK)^{1/p}} = aM_p^0 K$$

for every $K \in \mathcal{K}_o^n$. By rewriting this in terms of the L_p-cosine transforms (via relation (2.6) and $(c_{n,p}V(K))^{\frac{1}{p}}\Gamma_p K = M_p^0 K$), we get

$$C_p \frac{S_p(ZK,\cdot)}{V(ZK)} = bC_p(\rho_K(\cdot)^{n+p}) = bC_p(\frac{1}{2}\rho_K(\cdot)^{n+p} + \frac{1}{2}\rho_{-K}(\cdot)^{n+p})$$

for a suitable constant $b > 0$. Since $S_p(ZK,\cdot)$ is an even measure, the injectivity property (2.1) and the definition of the normalized symmetric L_p-curvature image operator finally shows

$$ZK = c\widetilde{\Lambda}_c^p K \tag{4.8}$$

for a suitable constant $c > 0$.

If $q = -2/p + 1$ and $n = 2$, then $\overline{Z}_2$, introduced in Lemma 4.5, is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree $2/p - 1$. By Lemma 4.7, there are constants $a \geq 0$ and $-1 \leq \tau \leq 1$ such that

$$\overline{Z}_2 P = a\hat{\Pi}_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^n$. $\hat{\Pi}_p^\tau$ is not continuous on $\mathcal{P}_o^n$, while $\frac{\Pi_p Z}{V^{1/p}}$ is continuous on $\mathcal{P}_o^n$. Thus, this is a contradiction.

In all other cases, $\overline{Z}_2$, introduced in Lemma 4.5, is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree r, where $r \neq 1$, $r \neq n/p + 1$ for $n \geq 2$, and $r \neq 2/p - 1$ as an addition for $n = 2$. By Lemma 4.7, $\overline{Z}_2 P = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^n$. So

$$h_{\overline{Z}_2 P}(x)^p = C_p \frac{S_p(ZP,\cdot)}{V(ZP)}(x) = 0 \tag{4.9}$$

for every $P \in \mathcal{P}_o^n$. $S_p(ZP,\cdot)$ is an even measure since ZP is an origin-symmetric convex body. Thus, by relation (2.1), we have $S_p(ZP,\cdot) = 0$. This is a contradiction. □

Hence Theorem 3.2 and Theorem 4.8 directly imply Theorem 1.1.

4.2. **The covariant case.** The following Lemma 4.9, Lemma 4.10, and Lemma 4.11 are the counterparts of Lemma 4.3, Lemma 4.4, and Lemma 4.5, respectively.

Lemma 4.9. *If $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a continuous, $SL(n)$ covariant valuation which is homogeneous of degree q, then $q \leq -n + 1$.*

Proof. Suppose $K \in \mathcal{K}_o^n$ and $s > 0$. As in the proof of Lemma 4.3, we get that

$$\begin{aligned} &C_p \frac{S_p(Z[K\cap e_n^\perp, \pm se_n],\cdot)}{V(Z[K\cap e_n^\perp, \pm se_n])}(e_n) \\ &= C_p \frac{S_p(Z[K\cap e_n^+, \pm se_n],\cdot)}{V(Z[K\cap e_n^+, \pm se_n])}(e_n) + C_p \frac{S_p(Z[K\cap e_n^-, \pm se_n],\cdot)}{V(Z[K\cap e_n^-, \pm se_n])}(e_n) \\ &\quad - C_p \frac{S_p(Z[K, \pm se_n],\cdot)}{V(Z[K, \pm se_n])}(e_n), \end{aligned} \tag{4.10}$$

and thus $C_p \frac{S_p(Z[K\cap e_n^\perp, \pm se_n],\cdot)}{V(Z[K\cap e_n^\perp, \pm se_n])}(e_n)$ must converge to a finite number as $s \to 0^+$. (The difference between relation (4.3) and relation (4.10) is that the independent variable of the L_p-cosine transform is changed from e_1 to e_n.) Define the linear map ϕ as before by

$$\phi e_i = e_i, i = 1, \cdots, n-1, \phi e_n = se_n.$$

From the $SL(n)$ covariance and homogeneity of Z as well as relations (3.1) and (2.3), we get

$$C_p \frac{S_p(Z[K\cap e_n^\perp, \pm s e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm s e_n])}(e_n) = C_p \frac{S_p(s^{\frac{q-1}{n}}\phi Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(s^{\frac{q-1}{n}}\phi Z[K\cap e_n^\perp, \pm e_n])}(e_n)$$
$$= s^{\frac{-(q-1)p}{n}} C_p \frac{S_p(Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm e_n])}(\phi^{-1}e_n).$$

Since $|e_n\cdot u|>0$ for all $u\in S^{n-1}\setminus e_n^\perp$ and the L_p-surface area measure of n-dimensional bodies is not concentrated on any great sphere, we conclude that

$$s^p C_p \frac{S_p(Z[K\cap e_n^\perp, \pm e_n], \cdot)}{V(Z[K\cap e_n^\perp, \pm e_n])}(\phi^{-1}e_n)$$
$$= \frac{1}{V(Z[K\cap e_n^\perp, \pm e_n])}\int_{S^{n-1}} |e_n\cdot u|^p dS_p(Z[K\cap e_n^\perp, \pm e_n], u) > 0.$$

Thus, $\frac{-(q-1)p}{n} - p \geq 0$, so $q\leq -n+1$. ◻

Lemma 4.10. *Let $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p\rangle$ be a continuous, $SL(n)$ covariant valuation which is homogeneous of degree $q=-n+1$. Define the map $\overline{Z}_1 : \overline{\mathcal{K}}_o^n \to \langle \mathcal{K}_o^n, +_p\rangle$ by*

$$h(\overline{Z}_1 K, x)^p = \begin{cases} C_p \frac{S_p(ZK,\cdot)}{V(ZK)}(x), & \dim K = n,\\ C_p \frac{S_p(Z[K,\pm v],\cdot)}{V(Z[K,\pm v])}((x\cdot v)v), & \dim K = n-1,\\ 0, & \dim K\leq n-2,\end{cases}$$

for every $x\in\mathbb{R}^n$, where v is a unit vector perpendicular to lin K. *Then $\overline{Z}_1$ is a $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $n-1$.*

Proof. Obviously, the definition of $\overline{Z}_1$ is independent of the choice of v, so it is well defined. Next, we show that $\overline{Z}_1$ is an L_p-Minkowski valuation. We still use the notation of the proof of Lemma 4.4. The case (1_n) is the same as and the case (1_{n-1}) is similar to (change $\pi_K x$ to $(x\cdot v)v$) the corresponding parts in the proof of Lemma 4.4. The cases $(1_k), 0\leq k\leq n-2$, and $(2_k), 1\leq k\leq n-2$, are trivial.

Now we consider the case (2_n). It is enough to show

$$h^p_{\overline{Z}_1 K} + h^p_{\overline{Z}_1 (K\cap u^\perp)} = h^p_{\overline{Z}_1 (K\cap u^+)} + h^p_{\overline{Z}_1 (K\cap u^-)} \tag{4.11}$$

for $\dim K = n$ and a unit vector $u\in$ lin K such that $K\cap u^+, K\cap u^-$ are both n-dimensional. Let $b_1,\cdots,b_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $u=b_n$. With the valuation property of case (1_n), we have

$$\begin{aligned} &C_p \frac{S_p(Z[K, \pm s b_n], \cdot)}{V(Z[K, \pm s b_n])}(x) + C_p \frac{S_p(Z[K\cap b_n^\perp, \pm s b_n], \cdot)}{V(Z[K\cap b_n^\perp, \pm s b_n])}(x)\\ &= C_p \frac{S_p(Z[K\cap b_n^+, \pm s b_n], \cdot)}{V(Z[K\cap b_n^+, \pm s b_n])}(x) + C_p \frac{S_p(Z[K\cap b_n^-, \pm s b_n], \cdot)}{V(Z[K\cap b_n^-, \pm s b_n])}(x) \end{aligned} \tag{4.12}$$

for sufficiently small $s>0$. Define a linear map ϕ by

$$\phi b_n = s b_n, \phi b_i = b_i,\ i=1,\cdots,n-1.$$

The covariance of Z and relations (3.1) as well as (2.3) give

$$C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm s b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm s b_n])}(x) = s^{\frac{-(q-1)p}{n}} C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm b_n])}(\phi^{-1}x)$$

$$= s^{\frac{-(q-1)p}{n}-p} C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm b_n])}(s\phi^{-1}x). \tag{4.13}$$

Note that $\lim_{s\to 0^+} s\phi^{-1}x = (x\cdot b_n)b_n$. Since $q=-n+1$,

$$\lim_{s\to 0^+} C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm s b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm s b_n])}(x) = C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm b_n])}((x\cdot b_n)b_n).$$

So if s tends to zero in (4.12), then we immediately obtain (4.11).

The case (2_{n-1}) is similar to the case (2_n). We will show the relation (4.11) is still true for $\dim K = n-1$ and a unit vector $u \in \operatorname{lin} K$ such that $K\cap u^+, K\cap u^-$ are both $(n-1)$-dimensional. Let $b_1, \cdots, b_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $\operatorname{lin} K = \operatorname{lin}\{b_1, \cdots, b_{n-1}\}$ and $u = b_{n-1}$. Thus, choose $v = b_n$. With the valuation property of case (1_{n-1}), we have

$$\begin{aligned}
&C_p \frac{S_p(Z[K, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n)\\
&\qquad + C_p \frac{S_p(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n)\\
=&C_p \frac{S_p(Z[K\cap b_{n-1}^{+}, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K\cap b_{n-1}^{+}, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n)\\
&\qquad + C_p \frac{S_p(Z[K\cap b_{n-1}^{-}, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K\cap b_{n-1}^{-}, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n)
\end{aligned} \tag{4.14}$$

for sufficiently small $s>0$. Define a linear map ϕ by

$$\phi b_{n-1} = s b_{n-1}, \phi b_i = b_i,\ i \neq n-1.$$

The covariance of Z and relations (3.1) as well as (2.3) give

$$\begin{aligned}
&\frac{S_p(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n)\\
&= s^{\frac{-(q-1)p}{n}} C_p \frac{S_p(Z[K\cap b_n^{\perp}, \pm b_n], \cdot)}{V(Z[K\cap b_n^{\perp}, \pm b_n])}(\phi^{-1}(x\cdot b_n)b_n).
\end{aligned}$$

Note that $\lim_{s\to 0^+} \phi^{-1}(x\cdot b_n)b_n = (x\cdot b_n)b_n$. Since $q=-n+1$,

$$\lim_{s\to 0^+} C_p \frac{S_p(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n], \cdot)}{V(Z[K\cap b_{n-1}^{\perp}, \pm s b_{n-1}, \pm b_n])}((x\cdot b_n)b_n) = 0.$$

So if s tends to zero in (4.14), then we immediately obtain (4.11). Hence we have proved that $\overline{Z}_1$ is an L_p-Minkowski valuation.

Moreover, it is easy to calculate that $\overline{Z}_1$ is an $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $n-1$ on n-dimensional convex bodies. Lemma 4.2 implies that $\overline{Z}_1$ is an $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $n-1$. □

Lemma 4.11. *Let $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ be a continuous, $SL(n)$ covariant valuation which is homogeneous of degree $q < -n+1$. Define the map $\overline{Z}_2 : \overline{\mathcal{K}}_o^n \to \langle \mathcal{K}_o^n, +_p \rangle$ by*

$$h(\overline{Z}_2 K, x)^p = \begin{cases} C_p \frac{S_p(ZK,\cdot)}{V(ZK)}(x), & \dim K = n, \\ 0, & \dim K = k < n, \end{cases}$$

for every $x \in \mathbb{R}^n$. Then $\overline{Z}_2$ is an $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $r = -q$.

Proof. To prove that $\overline{Z}_2$ is an L_p-Minkowski valuation, as in the proof of Lemma 4.5, we just need to show

$$\lim_{s \to 0^+} C_p \frac{S_p(Z[K \cap b_n^\perp, \pm s b_n], \cdot)}{V(Z[K \cap b_n^\perp, \pm s b_n])}(x) = 0. \tag{4.15}$$

Actually, since $q < -n+1$, by the relation (4.13), we immediately get the conclusion.

Moreover, it is easy to calculate that $\overline{Z}_2$ is an $SL(n)$ covariant L_p-Minkowski valuation which is homogeneous of degree $r = -q$. □

As in the contravariant case, we also need the following results from [20] to classify $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations.

For $-1 \le \tau \le 1$, define Π_p^τ on the set of all convex bodies containing the origin in their interiors by

$$h(\Pi_p^\tau K, v)^p = \int_{S^{n-1}} (|v \cdot u| + \tau(v \cdot u))^p dS_p(K, u)$$

for $v \in \mathbb{R}^n$. In particular, $\Pi_p^0 K$ is the L_p-projection body of K. To extend the operator Π_p^τ to polytopes that contain the origin in their boundaries, for $P \in \overline{\mathcal{P}}_o^n$, the set of polytopes which contain the origin, define $\hat{\Pi}_p^\tau P$ by

$$h(\hat{\Pi}_p^\tau P, v)^p = \int_{S^{n-1} \setminus \omega_o(P)} (|v \cdot u| + \tau(v \cdot u))^p dS_p(P, u), \tag{4.16}$$

where $\omega_o(P)$ is the set of outer unit normal vectors to facets of P that contain the origin.

Lemma 4.12 ([20])**.** *Let $Z : \overline{\mathcal{P}}_o^n \to \langle \mathcal{K}_o^n, +_p \rangle$ be an L_p-Minkowski valuation, $p > 1, n \ge 3$, which is $SL(n)$ contravariant and homogeneous of degree r. If $r = n/p - 1$, then there are constants $a \ge 0$ and $-1 \le \tau \le 1$ such that*

$$ZP = a \hat{\Pi}_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^n$. In all other cases, $ZP = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^n$.

Let $Z : \overline{\mathcal{P}}_o^2 \to \langle \mathcal{K}_o^2, +_p \rangle$ be an L_p-Minkowski valuation, $p > 1$, which is $SL(2)$ contravariant and homogeneous of degree r. If $r = 2/p + 1$, then there are constants $a \ge 0$ and $-1 \le \tau \le 1$ such that

$$ZP = a \psi_{\pi/2} M_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^2$. If $r = 1$, then there are constants $a_0, b_0 \ge 0$ and $a_i, b_i \in \mathbb{R}$ with $a_0 + a_i,\ b_0 + b_i \ge 0,\ i = 1, 2$, such that

$$ZP = \psi_{\pi/2}(a_0 P +_p b_0(-P) +_p \sum\nolimits^p (a_i E_i +_p b_i(-E_i)))$$

for every $P \in \overline{\mathcal{P}}_o^2$, where $\sum^p$ denotes the L_p-Minkowski sum which is taken over $E_i \in \xi_o(P)$. If $r = 2/p - 1$, then there are constants $a \geq 0$ and $-1 \leq \tau \leq 1$ such that

$$ZP = a\hat{\Pi}_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^2$. In all other cases, $ZP = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^2$.

Now we classify continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations.

Theorem 4.13. *Let $n \geq 3$, $p > 1$ and p not an even integer. Then there exist no continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$.*

Let $p > 1$ and p not an even integer. If $Z : \mathcal{K}_o^2 \to \langle \mathcal{K}_c^2, \widetilde{\#}_p \rangle$ is a continuous, homogeneous, $SL(2)$ covariant valuation, then there exists a constant $c > 0$ such that

$$ZK = c\psi_{\pi/2}\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^2$.

Proof. Assume that $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a continuous, $SL(n)$ covariant valuation which is homogeneous of degree q. Lemma 4.9 shows that $q \leq -n + 1$.

We first consider the cases $n \geq 3$. If $q < -n + 1$, then $\overline{Z}_2$, introduced in Lemma 4.11, is an $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $r > n - 1$. By Lemma 4.12, we have $\overline{Z}_2 P = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^n$. If $q = -n+1$, $\overline{Z}_1$, introduced in Lemma 4.10, is an $SL(n)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $n - 1$. By Lemma 4.12, $\overline{Z}_1 P = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^n$.

Combined with the injectivity relation of the L_p-cosine transform (2.1), all cases $q \leq -n + 1$ imply that

$$\frac{S_p(ZP, \cdot)}{V(ZP)} = 0$$

for every $P \in \overline{\mathcal{P}}_o^n$. This is a contradiction to the existence of continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$.

Next we consider the case $n = 2$. If $q < -1$, $q \neq -2/p - 1$, then $\overline{Z}_2$, introduced in Lemma 4.11, is an $SL(2)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $r > 1$, $r \neq 2/p + 1$. By Lemma 4.12, we have $\overline{Z}_2 P = \{o\}$ for every $P \in \overline{\mathcal{P}}_o^2$. Combined with the injectivity relation of the L_p-cosine transform (2.1), we get $\frac{S_p(ZP,\cdot)}{V(ZP)} = 0$. This is a contradiction.

If $q = -2/p - 1$, then $\overline{Z}_2$, introduced in Lemma 4.11, is an $SL(2)$ contravariant L_p-Minkowski valuation which is homogeneous of degree $2/p + 1$. By Lemma 4.12, there are constants $a \geq 0$ and $-1 \leq \tau \leq 1$ such that

$$\overline{Z}_2 P = a\psi_{\pi/2} M_p^\tau P$$

for every $P \in \overline{\mathcal{P}}_o^2$. Thus, $\psi_{-\pi/2}\overline{Z}_2 P = aM_p^\tau P$ for every $P \in \mathcal{P}_o^2$. The assumption that Z does not contain $\{o\}$ in its range gives $a > 0$. Since $\overline{Z}_2 P$ is origin-symmetric, we get $\tau = 0$. Thus, $\psi_{-\pi/2}(\frac{\Pi_p ZP}{V(ZP)^{1/p}}) = aM_p^0 P$ for every $P \in \mathcal{P}_o^2$. Since the operators $\psi_{-\pi/2}$, $\frac{\Pi_p Z}{V^{1/p}}$ and Γ_p are continuous on $\mathcal{K}_o^2$, and $\mathcal{P}_o^2$ is dense in $\mathcal{K}_o^2$, we obtain

$$\psi_{-\pi/2}(\frac{\Pi_p ZK}{V(ZK)^{1/p}}) = aM_p^0 K$$

for every $K \in \mathcal{K}_o^2$. By rewriting this in terms of the L_p-cosine transforms (via relation (2.6) and $(c_{n,p}V(K))^{\frac{1}{p}}\Gamma_p K = M_p^0 K$), we get

$$C_p \frac{S_p(ZK,\cdot)}{V(ZK)}(\psi_{\pi/2}x) = bC_p(\frac{1}{2}\rho_K(\cdot)^{n+p} + \frac{1}{2}\rho_{-K}(\cdot)^{n+p})(x)$$

for a suitable constant $b > 0$. Since

$$C_p \frac{S_p(\psi_{-\pi/2}ZK,\cdot)}{V(\psi_{-\pi/2}ZK)}(x) = C_p \frac{S_p(ZK,\cdot)}{V(ZK)}(\psi_{\pi/2}x)$$

(by relation (2.3)), the injectivity property (2.1) and the definition of the normalized symmetric L_p-curvature image operator finally show

$$\psi_{-\pi/2}ZK = c\widetilde{\Lambda}_c^p K$$

for a suitable constant $c > 0$. Hence

$$ZK = c\psi_{\pi/2}\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^2$

If $q = -1$, $\overline{Z}_1$, introduced in Lemma 4.10, is an $SL(2)$ contravariant L_p-Minkowski valuation which is homogeneous of degree 1. By Lemma 4.12, there are constants $a_0, b_0 \geq 0$ and $a_i, b_i \in \mathbb{R}$ with $a_0 + a_i,\ b_0 + b_i \geq 0,\ i = 1, 2$, such that

$$\overline{Z}_1 P = \psi_{\pi/2}(a_0 P +_p b_0(-P) +_p \sum\nolimits^p (a_i E_i +_p b_i(-E_i)))$$

for every $P \in \overline{\mathcal{P}}_o^2$, where $\sum^p$ denotes the L_p-Minkowski sum, and the sum is taken over $E_i \in \xi_o(P)$. For $P_0 = [\pm e_1, \pm e_2]$, we have

$$\frac{\Pi_p Z P_0}{V(ZP_0)^{1/p}} = c\psi_{\pi/2}P_0$$

for a suitable $c \geq 0$. The assumption that Z does not contain $\{o\}$ in its range gives $c > 0$. For $p > 1$, every support set of an L_p-projection body consists of precisely one point (Lemma 4.6). However, $\psi_{\pi/2}P_0$ has a support set $[e_1, e_2]$ which does not consist of precisely one point. This is a contradiction. □

Theorem 3.2 and Theorem 4.13 now directly imply Theorem 1.2.

5. L_p-Blaschke valuations

We first give the relationship between normalized symmetric L_p-Blaschke valuations and symmetric L_p-Blaschke valuations.

Lemma 5.1. *If $Z : \mathcal{Q} \to \langle \mathcal{K}_c^n, \#_p\rangle$ is a symmetric L_p-Blaschke valuation, then $\widetilde{Z} : \mathcal{Q} \to \langle \mathcal{K}_c^n, \widetilde{\#}_p\rangle$, defined by*

$$\frac{S_p(\widetilde{Z}K,\cdot)}{V(\widetilde{Z}K)} = S_p(ZK) \tag{5.1}$$

for every $K \in \mathcal{Q}$, is a normalized symmetric L_p-Blaschke valuation. Moreover, $\widetilde{Z}$ is continuous if Z is continuous, $\widetilde{Z}$ is $SL(n)$ covariant (or contravariant) if Z is

$SL(n)$ covariant (or contravariant respectively), and $\widetilde{Z}$ is homogeneous of degree $q(p-n)/p$ if Z is homogeneous of degree q.

Proof. Since Z is a symmetric L_p-Blaschke valuation,

$$S_p(Z(K \cup L), \cdot) + S_p(Z(K \cap L), \cdot) = S_p(ZK, \cdot) + S_p(ZL, \cdot),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{Q}$. By the definition of $\widetilde{Z}$ and the normalized L_p-Blaschke sum, $\widetilde{Z}$ is a normalized symmetric L_p-Blaschke valuation.

We can prove continuity of $\widetilde{Z}$ in a similar way to showing continuity of the normalized symmetric L_p-curvature image. But because of the existence of ZK, we can prove it in an easier way (without using Lemma 3.1).

By the uniqueness of the volume-normalized even L_p-Minkowski problem, we can rewrite relation (5.1) as

$$\widetilde{Z}K = V(ZK)^{-1/p} ZK \tag{5.2}$$

for every $K \in \mathcal{K}^n$. Since $V(ZK) > 0$, if $ZK_i \to ZK$,

$$\widetilde{Z}K_i = V(ZK_i)^{-1/p} ZK_i \to V(ZK)^{-1/p} ZK = \widetilde{Z}K.$$

Thus, if Z is continuous, then $\widetilde{Z}$ is continuous.

If $Z(\lambda K) = \lambda^q ZK$, for every $\lambda > 0$, then

$$\widetilde{Z}(\lambda K) = V(Z\lambda K)^{-1/p} Z\lambda K = \lambda^{q(p-n)/p} V(ZK)^{-1/p} ZK = \lambda^{q(p-n)/p} \widetilde{Z}K.$$

Thus, if Z is homogeneous of degree q, $\widetilde{Z}$ is homogeneous of degree $q(p-n)/p$.

The proof of covariance or contravariance of $\widetilde{Z}$ is similar to the proof of homogeneity. □

Lemma 5.1 introduces a map from the space of symmetric L_p-Blaschke valuations to the space of normalized symmetric L_p-Blaschke valuations, and the continuity, homogeneity, or $SL(n)$ covariance (or contravariance) of symmetric L_p-Blaschke valuations are inherited by the corresponding normalized cases. For $p \neq n$, the relation (5.1) can also be rewritten as

$$V(\widetilde{Z}K)^{1/(p-n)} \widetilde{Z}K = ZK \tag{5.3}$$

for every $K \in \mathcal{Q}$. Then we get the following lemma in a similar way. Hence the map is a bijection and these properties are also transferred by the inverse map.

Lemma 5.2. *If $\widetilde{Z} : \mathcal{Q} \to \langle \mathcal{K}_c^n, \widetilde{\#}_p \rangle$ is a normalized symmetric L_p-Blaschke valuation, $p \neq n$, then $Z : \mathcal{Q} \to \langle \mathcal{K}_c^n, \#_p \rangle$, defined by*

$$ZK = V(\widetilde{Z}K)^{1/(p-n)} \widetilde{Z}K \tag{5.4}$$

for every $K \in \mathcal{Q}$, is a symmetric L_p-Blaschke valuation. Moreover, Z is continuous if $\widetilde{Z}$ is continuous, Z is $SL(n)$ covariant (or contravariant) if $\widetilde{Z}$ is $SL(n)$ covariant (or contravariant respectively), and Z is homogeneous of degree $qp/(p-n)$ if $\widetilde{Z}$ is homogeneous of degree q.

Lemma 5.1 and Lemma 5.2 together with Theorem 1.1 (or Theorem 3.2 as well as Theorem 4.8) provide a classification of continuous, homogeneous $SL(n)$ contravariant symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$.

Theorem 5.3. *For $n \geq 2$, $p > 1$, $p \neq n$ and p not an even integer, a map $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \#_p \rangle$ is a continuous, homogeneous, $SL(n)$ contravariant symmetric L_p-Blaschke valuation if and only if there exists a constant $c > 0$ such that*

$$ZK = c\Lambda_c^p K$$

for every $K \in \mathcal{K}_o^n$.

Proof. Since Z is a continuous, homogeneous $SL(n)$ contravariant symmetric L_p-Blaschke valuation, $\widetilde{Z}$ defined in Lemma 5.1 is a continuous, homogeneous $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuation. Theorem 4.8 implies that there exists a constant $c > 0$ such that

$$\widetilde{Z}K = c\widetilde{\Lambda}_c^p K$$

for every $K \in \mathcal{K}_o^n$. Note that $\Lambda_c^p K = V(\widetilde{\Lambda}_c^p K)^{1/(p-n)}\widetilde{\Lambda}_c^p K$. By relation (5.3),

$$ZK = V(\widetilde{Z}K)^{1/(p-n)}\widetilde{Z}K = V(c\widetilde{\Lambda}_c^p K)^{1/(p-n)}c\widetilde{\Lambda}_c^p K = c^{p/(p-n)}\Lambda_c^p K \tag{5.5}$$

for every $K \in \mathcal{K}_o^n$.

On the other hand, Theorem 3.2 implies that $\widetilde{\Lambda}_c^p K$ is a continuous, homogeneous $SL(n)$ contravariant normalized symmetric L_p-Blaschke valuation. Then, $\Lambda_c^p K$ is a continuous, homogeneous, $SL(n)$ contravariant symmetric L_p-Blaschke valuation by Lemma 5.2. □

Lemma 5.1 and Lemma 5.2 together with Theorem 1.2 (or Theorem 3.2 as well as Theorem 4.13) imply the following theorem.

Theorem 5.4. *Let $n \geq 3$, $p > 1$ and p not an even integer. Then there exist no continuous, homogeneous, $SL(n)$ covariant symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$.*

Let $p > 1$ and p not an even integer. If $Z : \mathcal{K}_o^2 \to \langle \mathcal{K}_c^2, \#_p \rangle$ is a continuous, homogeneous, $SL(2)$ covariant symmetric L_p-Blaschke valuation, then there exists a constant $c > 0$ such that

$$ZK = c\psi_{\pi/2}\Lambda_c^p K$$

for every $K \in \mathcal{K}_o^2$.

Proof. For $n \geq 3$, we argue by contradiction. Assume that Z is a continuous, homogeneous, $SL(n)$ covariant symmetric L_p-Blaschke valuation and $\widetilde{Z}$ defined in Lemma 5.1 is a continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuation. But Theorem 4.13 implies that there are no continuous, homogeneous, $SL(n)$ covariant normalized symmetric L_p-Blaschke valuations on $\mathcal{K}_o^n$. This is a contradiction.

For $n = 2$, the proof is almost the same as in Theorem 5.3. □

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Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

E-mail address: `lijin2955@gmail.com`

Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

E-mail address: `yuanshufeng2003@163.com`

Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

E-mail address: `gleng@staff.shu.edu.cn`