OPTIMAL REGULARITY OF THE CONVEX ENVELOPE

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ABSTRACT. We prove sharp regularity results for the convex envelope of a continuous function inside a convex domain.

1. INTRODUCTION

Given an open set $\Omega \subset \mathbb{R}^n$ and a continuous function $v : \overline{\Omega} \rightarrow \mathbb{R}$, we define its *convex envelope in Ω* as

$$(1.1) \quad \Gamma_v(x) := \sup\{\ell(x) : \ell \leq v \text{ in } \overline{\Omega}, \ell \text{ affine}\}.$$

We immediately see that $\Gamma_v : \overline{\Omega} \rightarrow \mathbb{R}$ is the largest convex function below v in Ω .

The issue of the regularity of the convex envelope (or inf-sup convolution) appears in several problems from analysis and geometry. To mention some: the Alexandrov-Bakelman-Pucci estimate for fully non-linear equations [4, 9], the Monge-Ampère equation [3, 7], the study of geometric flows [6, 12], the Hamilton-Jacobi equation [2, 8], etc. Moreover, as observed in [13], the convex envelope can also be seen as a solution of a non-linear obstacle problem.

Since convex functions are locally Lipschitz, one wants to understand under which assumptions the convex envelope is of class C^1 or better. Several results have been established in this direction [5, 10, 11, 14], but they all deal either with the Dirichlet problem (i.e., Γ_v is the convex envelope of $v|_{\partial\Omega}$), or they make some suitable global assumptions on v in order to “avoid” the influence of the boundary (see for instance Theorem 4.1 below). Our goal here is to investigate the regularity of Γ_v in full generality.

Before stating our results, we recall the following definition:

Definition 1.1. Given $\alpha \in (0, 1]$, a continuous function v is said to be $(1 + \alpha)$ -semiconcave in $\overline{\Omega}$ if for every $x_0 \in \overline{\Omega}$ there exists a slope $p_{x_0} \in \mathbb{R}^n$ such that

$$(1.2) \quad v(x) \leq v(x_0) + p_{x_0} \cdot (x - x_0) + C|x - x_0|^{1+\alpha} \quad \text{for every } x \in \overline{\Omega} \cap B(x_0, \varrho_0)$$

for some constants C and ϱ_0 independent of x_0 .

The following two theorems relate the regularity of Γ_v to the one of v and Ω .

Theorem 1.2. Let $\alpha, \beta \in (0, 1]$, Ω be a bounded convex domain of class $C^{1,\beta}$, and $v : \overline{\Omega} \rightarrow \mathbb{R}$ be a globally Lipschitz function which is $(1 + \alpha)$ -semiconcave in $\overline{\Omega}$. Then $\Gamma_v \in C_{\text{loc}}^{1,\min\{\alpha,\beta\}}(\Omega)$.

Theorem 1.3. Let Ω be a bounded uniformly convex domain of class $C^{3,1}$, and let $v \in C^{3,1}(\overline{\Omega})$. Then $\Gamma_v \in C^{1,1}(\overline{\Omega})$.

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FIGURE 2.1. If Ω is a C^∞ domain as above, then the function $|x|$ is of class C^∞ on $\partial\Omega$. This situation illustrates that convexity of Ω is necessary for regularity.

As we will show in the next section, all assumptions made in the above theorems are sharp.

The paper is structured as follows: in Section 2 we provide some examples showing that our results are optimal. Then in Section 3 we recall some preliminaries about convex sets and functions, and we prove some criteria to show that a function is $C^{1,\alpha}$. Sections 4 and 5 are devoted to the proof of Theorem 1.2 and Theorem 1.3, respectively. Moreover, at the end of Section 4 we also discuss some extension/variant of Theorem 1.2.

2. COUNTEREXAMPLES

Here we present some counterexamples showing the sharpness of our assumptions in Theorems 1.2 and 1.3. The construction of the counterexamples are mainly based on the following observation: If $v : \overline{\Omega} \rightarrow \mathbb{R}$ is harmonic in Ω , then (by the maximum principle)

$$(2.1) \quad \Gamma_v = \sup\{\ell : \ell \leq v \text{ in } \partial\Omega, \ell \text{ affine}\}.$$

All counterexamples are done in $\mathbb{R}^2$, and to simplify the notation, *only for this section* we will use (x, y) to denote the coordinates of a point in $\mathbb{R}^2$ (starting from the next section, x and y will be used to denote generic points in $\mathbb{R}^n$).

The set Ω has to be convex. Let Ω be a C^∞ domain as in Figure 2.1, with a flat part along the y axis. Then the function $|x|$ is of class C^∞ when restricted to $\partial\Omega$. (Indeed, if $x = f(y)$ locally parameterizes $\partial\Omega$ near the axis $\{x = 0\}$, then $y \mapsto |x| = |f(y)| \in C^\infty$.) So, by elliptic regularity, its harmonic extension v inside Ω belongs to $C^\infty(\overline{\Omega})$. However, because of (2.1), it is easy to check that $\Gamma_v(x, y) = |x|$, so Γ_v is not even C^1 .

The best regularity one can expect is $C^{1,1}$. Take $\Omega = [-2, 2] \subset \mathbb{R}$ and $v = (x^2 - 1)_+^2$. Then $\Gamma_v(x) = [(x^2 - 1)_+]^2$ is $C^{1,2}$ but not C^2 inside Ω .

In Theorem 1.2 the function v needs to be $(1 + \alpha)$ -semiconcave up to the boundary. Take $\Omega = B(0, 1)$ and $v(x, y)$ the harmonic extension of the boundary data $v(\cos \theta, \sin \theta) = |\cos \theta|$. Then by elliptic regularity v is of class C^∞ inside Ω , but $\Gamma_v(x, y) = |x|$ is not C^1 .

In Theorem 1.2 the boundary regularity of the domain does matter. Take $\Omega = \{(x, y) : |x|^p + |y|^p \leq 1\}$ for some $p \in (1, 2)$, and $v(x, y) = 1 - y^2$. Then v coincides on the boundary of Ω with $u(x, y) = f(x) = 1 - (1 - |x|^p)^{2/p}$. Hence, for $\varepsilon > 0$ small, the convex function

$$u_\varepsilon(x, y) = \begin{cases} f(-\varepsilon) + f'(-\varepsilon)(x + \varepsilon) & \text{if } x \leq -\varepsilon, \\ f(x) & \text{if } |x| \leq \varepsilon, \\ f(\varepsilon) + f'(\varepsilon)(x - \varepsilon) & \text{if } x \geq \varepsilon \end{cases}$$

is below v in Ω , and thus $\Gamma_v \geq u_\varepsilon$. Since $\Gamma_v(0, y) = 0$ and $f \simeq |x|^p$ near 0, this implies that the best regularity we can expect for Γ_v is $C^{1, p-1}$. (An easy modification of this example gives that in case $\partial\Omega$ is not of class C^1 , then the convex envelope may not be C^1 .)

In Theorem 1.2 v has to be globally Lipschitz. Take $\Omega = B(0, 1)$, and consider the concave function $v(x, y) = \sqrt{1 - |y|^2}$. Then v is $(1 + \alpha)$ -semiconcave for every $\alpha > 0$. However, since v is concave and $v = |x|$ on $\partial\Omega$, we get that $\Gamma_v(x, y) = |x| \notin C^1(\bar{\Omega})$.

In Theorem 1.3 $C^{3,1}$ regularity of v is necessary. Take $\Omega = B(0, 1)$ and $v(x, y)$ the harmonic extension of the $C^{3, 1-2\varepsilon}$ boundary data $v(\cos \theta, \sin \theta) = (1 + \cos \theta)^{2-\varepsilon}$. Then by elliptic regularity v is of class $C^{3, 1-2\varepsilon}$ up to the boundary, but $\Gamma_v(x, y) = (1 + x)^{2-\varepsilon} \notin C^{1,1}(\bar{\Omega})$.

In Theorem 1.3 uniform convexity of Ω is necessary. Take $\Omega = \{(x, y) : |x|^4 + |y|^4 \leq 1\}$, and v the harmonic extension of the $C^{7, 1-4\varepsilon}$ boundary data $v(x, y) = (1 + x)^{2-\varepsilon}$. Then by elliptic regularity v is of class $C^{7, 1-4\varepsilon}$ up to the boundary, but $\Gamma_v(x, y) = (1 + x)^{2-\varepsilon} \notin C^{1,1}(\bar{\Omega})$.

In Theorem 1.3 $C^{3,1}$ regularity of $\partial\Omega$ is necessary. Let Ω be a convex with $0 \in \partial\Omega$. We assume that $\partial\Omega \setminus \{0\}$ is of class C^∞ , that $\Omega \subset \{y \geq x^2 + x^{4-2\varepsilon}\}$, and that close to 0 the boundary of Ω is represented by the graph of $y = x^2 + x^{4-2\varepsilon}$. Then, near the origin we can write

$$\partial\Omega = \{(x, y) : y \geq 0, -x(y) \leq x \leq x(y)\}$$

where $x(y)$ is the inverse of $x^2 + x^{4-2\varepsilon}$ for $x \geq 0$. Now consider $v(x, y) = y - x^2$. Then

$$\begin{aligned} v(\pm x(y), y) &= y - x(y)^2 = x(y)^{4-2\varepsilon} = [x(y)^2]^{2-\varepsilon} \\ &= [y - x(y)^{4-2\varepsilon}]^{2-\varepsilon} = y^{2-\varepsilon}(1 + o(1)). \end{aligned}$$

Hence, since $v > 0$ on $\bar{\Omega} \setminus \{0\}$, it is easy to see that

$$\frac{y^{2-\varepsilon}}{M} \leq v(x, y) \leq My^{2-\varepsilon} \quad \text{on } \partial\Omega$$

for some constant M sufficiently large. Moreover, since the function $x \mapsto v(x, y)$ is concave, $\Gamma_v \geq y^{2-\varepsilon}/M$ inside Ω . We claim that $\partial_y \Gamma_v(0, 0) = 0$. In fact, if this were not the case, for $y > 0$ small we would have

$$\Gamma_v(0, y) \geq cy > My^{2-\varepsilon} \geq \Gamma_v(\pm x(y), y),$$

which is absurd since Γ_v is convex. It is thus clear that Γ_v cannot have second derivatives bounded up to the origin, as otherwise we would get

$$\frac{y^{2-\varepsilon}}{M} \leq \Gamma_v(0, y) \leq Cy^2,$$

which is clearly false for y small.

3. NOTATION AND PRELIMINARIES

From now on, Ω will denote a bounded open convex set. Given $k \in \mathbb{N}$ and $\alpha \in [0, 1]$, we say that Ω has boundary of class $C^{k, \alpha}$ if there exists a radius ϱ such that for every point $x \in \partial\Omega$, up to a rotation of coordinates, it holds

$$\Omega \cap B(x, \varrho) = \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \geq \varphi(x')\} \cap B(x, \varrho)$$

for some convex function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ of class $C^{k, \alpha}$. Notice that every convex set has boundary at least of class $C^{0, 1}$.

When Ω has boundary of class at least C^1 , we will denote by ν_x its outer normal at a point $x \in \partial\Omega$. Moreover, we say that a C^1 convex set is uniformly convex if there exists $R > 0$ such that

$$\Omega \subset B(x_0 - R\nu_{x_0}, R) \quad \forall x_0 \in \partial\Omega.$$

Given a set $\mathcal{C}$ we will denote by $\text{conv}(\mathcal{C})$ its closed convex hull. In case $\mathcal{C} = \{x_1, \dots, x_k\}$ is a finite set of points, we will simply write $\text{conv}(x_1, \dots, x_k)$. The segment between two points x_1 and x_2 will be indicated by $[x_1, x_2]$.

Given a continuous function $v : \overline{\Omega} \rightarrow \mathbb{R}$, we say that a “slope” $s_{x_0} \in \mathbb{R}^n$ is supporting v at $x_0 \in \overline{\Omega}$ if

$$(3.1) \quad v(x) \geq v(x_0) + s_{x_0} \cdot (x - x_0) \quad \text{for every } x \in \overline{\Omega}.$$

In case s_{x_0} supports v at x_0 , we will write $x_0 \in \text{Dom}(\partial v)$ and $s_{x_0} \in \partial v(x_0)$. Notice that the function $v(x) - v(x_0) + s_{x_0} \cdot (x - x_0)$ has a minimum at x_0 . We also recall that a convex function can be characterized by $\text{Dom}(\partial v) = \Omega$.

Given a $(1 + \alpha)$ -semiconcave function (see Definition 1.1), we observe that, for any $x_0 \in \overline{\Omega}$, the set of slopes p_{x_0} which satisfy (1.2) is a convex subset of $\mathbb{R}^n$. This allows us to choose an “optimal slope” $\bar{p}_{x_0}$ as the element of minimal norm inside such set. The reason for this choice is the following observation, whose elementary proof is left to the reader: if v is Lipschitz continuous, then

$$(3.2) \quad \sup_{x_0 \in \overline{\Omega}} |\bar{p}_{x_0}| \leq \text{Lip } v.$$

It is easy to see that if $x_0 \in \Omega \cap \text{Dom}(\partial v)$ and v is $(1 + \alpha)$ -semiconcave, then v is differentiable at x_0 with

$$\nabla v(x_0) = p_{x_0} = s_{x_0}.$$

In case $x_0 \in \partial\Omega$ and $\partial\Omega$ is at least of class C^1 , we have

$$(3.3) \quad p_{x_0} = s_{x_0} - \lambda \nu_{x_0}$$

where ν_{x_0} is the outer normal at x_0 and $\lambda \geq 0$.

Finally, given a closed set $\mathcal{K} \subset \mathbb{R}^n$, we say that a function $v : \mathcal{K} \rightarrow \mathbb{R}$ is $C^{k,\alpha}(\mathcal{K})$ if there exists a $C^{k,\alpha}$ extension of v in a neighborhood of $\mathcal{K}$.

We now give two criteria to show $C^{1,\alpha}$ regularity. The following lemmas are pretty standard (see for instance [5, Section 1]), but for the sake of completeness we provide the proofs.

Lemma 3.1. *Let Ω be a convex open set, C and $\bar{\varrho}$ be positive constants, and $\alpha \in (0, 1]$. Let $u : \Omega \rightarrow \mathbb{R}$ be a globally Lipschitz convex function such that for every $x \in \Omega$ there exists a supporting slope s_x satisfying*

$$(3.4) \quad u(y) - u(x) - s_x \cdot (y - x) \leq C|y - x|^{1+\alpha} \quad \text{for all } y \in \Omega \cap B(x, \bar{\varrho}).$$

Then $u \in C^{1,\alpha}(\bar{\Omega})$.

Proof. First of all notice that, by convexity and (3.4), u is differentiable at every point in Ω with $\nabla u(x) = s_x$, and

$$(3.5) \quad |u(y) - u(x) - \nabla u(x) \cdot (y - x)| \leq C|y - x|^{1+\alpha} \quad \text{for all } y \in \Omega \cap B(x, \bar{\varrho}).$$

We now show that there exists a constant $M > 0$ such that

$$(3.6) \quad |\nabla u(x) - \nabla u(y)| \leq M|x - y|^\alpha \quad \text{for all } x, y \in \Omega.$$

Since in the above estimate M is independent of x and y , this implies that u extends to a $C^{1,\alpha}$ function up to the boundary.

Because by assumption u is Lipschitz, it is enough to show (3.6) for $|x - y|$ small. Now recall that since Ω is convex, $\partial\Omega$ is Lipschitz. Hence there exist $\varrho_0 \in (0, \bar{\varrho}/2)$ and $\eta > 0$ such that, for any $x \in \Omega$, for any vector e , and for any $\varrho \leq \varrho_0$, there is a point $z \in B(x, \varrho_0) \cap \Omega$ satisfying

$$|e \cdot (z - x)| \geq \eta|z - x||e| \quad \text{and} \quad |z - x| = \varrho.$$

Up to subtracting a linear function, without loss of generality we can assume that $\nabla u(x) = 0$. We want to show that

$$(3.7) \quad |\nabla u(y)| \leq M|y - x|^\alpha \quad \text{for all } y \in B(x, \varrho_0)$$

for some constant $M > 0$ independent of x . Fix a point $z \in B(y, \varrho_0) \cap \Omega$ such that

$$|\nabla u(y) \cdot (z - y)| \geq \eta|z - y||\nabla u(y)| \quad \text{and} \quad |z - y| = |y - x|.$$

Thanks to (3.5) we have

$$\begin{aligned} \eta|z - y||\nabla u(y)| &\leq |\nabla u(y) \cdot (z - y)| \\ &\leq |u(z) - u(y) - \nabla u(y) \cdot (z - y)| \\ &\quad + |u(z) - u(x)| + |u(y) - u(x)| \\ &\leq C(|z - y|^{1+\alpha} + |z - x|^{1+\alpha} + |y - x|^{1+\alpha}). \end{aligned}$$

Since $|z - y| = |y - x|$, this gives (3.7) with $M = (2 + 2^{1+\alpha})C$. $\square$

In case one wants to prove $C^{1,1}$ regularity, the above lemma can be refined as follows (see [5, Section 1]).

Lemma 3.2. *Let Ω be a convex open set, $C > 0$, and $u : \Omega \rightarrow \mathbb{R}$ be a globally Lipschitz convex function such that for every $x \in \Omega$ there exist a supporting slope s_x and a radius $\varrho(x)$ satisfying*

$$(3.8) \quad u(y) - u(x) - s_x \cdot (y - x) \leq C|y - x|^2 \quad \text{for all } y \in \Omega \cap B(x, \varrho(x)).$$

Then $u \in C^{1,1}(\bar{\Omega})$.

Notice that in (3.8) the radius $\varrho(x)$ can depend on the point x , while in (3.4) the radius has to be uniform. It is indeed easy to see that the uniform assumption in (3.4) is necessary.

Proof. Thanks to Lemma 3.1, it suffices to show that (3.8) implies that

$$(3.9) \quad u(y) - u(x) - \nabla u(x) \cdot (y - x) \leq 2C|y - x|^2 \quad \text{for all } x, y \in \Omega.$$

In fact suppose there exists a point $\bar{y}$ where (3.9) fails, and along the segment $[x, \bar{y}]$ let us consider the following function:

$$f(t) = 2C|\bar{y} - x|^2 t^2 - u(x + t(\bar{y} - x)) + u(x) + t\nabla u(x) \cdot (\bar{y} - x).$$

Since $f(t) > 0$ for $t > 0$ small, $f(1) < 0$, and $f(0) = 0$, there exists a maximum point $\bar{t} \in (0, 1)$. Thus for any $h > 0$ small we have

$$\begin{aligned} 0 &\geq f(\bar{t} + h) + f(\bar{t} - h) - 2f(\bar{t}) \\ &= 2C|\bar{y} - x|^2 ((\bar{t} + h)^2 + (\bar{t} - h)^2 - 2\bar{t}^2) \\ &\quad - \left\{ u(x + (\bar{t} + h)(\bar{y} - x)) + u(x + (\bar{t} - h)(\bar{y} - x)) - 2u(x + \bar{t}(\bar{y} - x)) \right\}. \end{aligned}$$

Since for h small the term in curly brackets is bounded by $2C|x - y|^2 h^2$ (by (3.8)), we obtain

$$h^2 \geq (\bar{t} + h)^2 + (\bar{t} - h)^2 - 2\bar{t}^2 = 2h^2,$$

a contradiction. □

We close this section with the following well-known lemma. To make the presentation self-contained we provide its simple proof.

Lemma 3.3. *Let $v \in C(\overline{\Omega})$, and fix $x \in \Omega \setminus \{v = \Gamma_v\}$. Then there exist at most $n + 1$ points $x_1, \dots, x_{n+1}$ in $\overline{\Omega} \cap \{v = \Gamma_v\}$ such that*

$$x \in \text{conv}(x_1, \dots, x_{n+1}).$$

Moreover every supporting slope to Γ_v at x is a supporting slope for v at x_i .

Proof. Let s_x be a supporting slope to Γ_v at x and define

$$\mathcal{C} = \{y \in \overline{\Omega} : v(y) = \Gamma_v(x) + s_x(y - x)\}.$$

We claim that $x \in \text{conv}(\mathcal{C})$. Suppose it is not true, then there exists an affine function ℓ which strongly separates $\mathcal{C}$ and x : more precisely $\ell(x) > 0$ and $\ell(y) < 0$ for every y in $\mathcal{C}$. It is now easy to see that, for some small $\delta > 0$, the function

$$\tilde{\ell}(y) = \Gamma_v(x) + s_x(y - x) + \delta\ell(y)$$

is below v in $\overline{\Omega}$ but $\tilde{\ell}(x) > \Gamma_v(x)$, against the definition of Γ_v . The existence of at most $(n + 1)$ -points follows from the Carathéodory Theorem [15, Theorem 17.1]. Finally it is immediate to check that s_x is supporting v at x_i . □

4. PROOF OF THEOREM 1.2

Proof. Fix $\delta_0 > 0$. We want to show that Γ_v is of class $C^{1, \min\{\alpha, \beta\}}$ inside $\{x \in \Omega : \text{dist}(x_0, \partial\Omega) \geq \delta_0\}$.

Pick $x_0 \in \Omega$ with $\text{dist}(x_0, \partial\Omega) \geq \delta_0$, and let s_{x_0} be a supporting slope to Γ_v at x_0 . Up to subtracting an affine function we can assume that

$$(4.1) \quad |s_{x_0}| = \Gamma_v(x_0) = 0.$$

Then by Lemma 3.1 it suffices to prove that there exist positive constants M and ϱ_1 , depending only on δ_0 , such that

$$\Gamma_v(x_0 + h) \leq M|h|^{1+\min\{\alpha, \beta\}} \quad \text{for all } |h| \leq \varrho_1.$$

In the case $x_0 \in \{v = \Gamma_v\}$ we have $0 = s_{x_0} = \nabla v(x_0)$, so by the $(1 + \alpha)$ -semiconcavity

$$\Gamma_v(x_0 + h) \leq v(x_0) + C|h|^{1+\alpha} = C|h|^{1+\alpha}$$

for $|h| \leq \varrho_0$ and we are done.

Now suppose $x_0 \in \Omega \setminus \{v = \Gamma_v\}$. By Lemma 3.3 there exist $n + 1$ points $x_i \in \{v = \Gamma_v\} \cap \overline{\Omega}$ such that

$$x_0 = \sum_{j=1}^{n+1} \lambda_j x_j, \quad \lambda_j \geq 0, \quad \sum_{j=1}^{n+1} \lambda_j = 1,$$

and 0 is a supporting slope at x_i . Now take $\delta_1 \in (0, \delta_0/4)$ small (to be fixed later), and let us divide the set of indexes in two parts: $\{1, \dots, n+1\} = J_g \cup J_b$ with

$$(4.2) \quad J_g = \{j : \lambda_j \geq \delta_1\}, \quad J_b = \{1, \dots, n+1\} \setminus J_g.$$

Notice that, since $\sum_j \lambda_j = 1$, the set J_g of “good” indexes is non-empty provided we choose $\delta_1 \leq 1/(n+1)$. We now distinguish two cases.

Case 1: There exists $\bar{j} \in J_g$ such that $\text{dist}(x_{\bar{j}}, \partial\Omega) \geq \delta_0/4$. Up to reordering, we can assume $\bar{j} = 1$. In this case, thanks to (4.1) we have that

$$\Gamma_v(x_1 + h) \leq v(x_1 + h) \leq C|h|^{1+\alpha} \quad \forall |h| \leq \bar{\varrho} := \min\{\varrho_0, \delta_0/4\}.$$

Thus, using the convexity of Γ_v , for $|h| \leq \delta_1 \bar{\varrho}$ we get

$$\begin{aligned} \Gamma_v(x_0 + h) &= \Gamma_v\left(\lambda_1\left(x_1 + \frac{h}{\lambda_1}\right) + \sum_{j=2}^{n+1} \lambda_j x_j\right) \leq \lambda_1 \Gamma_v\left(x_1 + \frac{h}{\lambda_1}\right) \\ &\leq \frac{C}{\lambda_1^\alpha} |h|^{1+\alpha} \leq \frac{C}{\delta_1^\alpha} |h|^{1+\alpha}, \end{aligned}$$

where we have used that $\lambda_1 \geq \delta_1$, and that $\Gamma_v(x_i) = 0$ for $i = 1, \dots, n+1$.

Case 2: $d(x_j, \partial\Omega) \leq \delta_0/4$ for all $j \in J_g$. Let us define

$$\tilde{x}_0 = \frac{1}{\sum_{j \in J_g} \lambda_j} \sum_{j \in J_g} \lambda_j x_j$$

the “projection” of x_0 onto the simplex Σ generated by the $\{x_j\}_{j \in J_g}$ (see Figure 4.1).

FIGURE 4.1. The projection of x_0 onto the simplex Σ generated by $\{x_j\}_{j \in J_g}$.

Clearly $\tilde{x}_0$ belongs to Ω . Moreover, since for any $y \in \Omega$

$$\begin{aligned} \tilde{x}_0 - x_0 &= \frac{1}{\sum_{j \in J_g} \lambda_j} \sum_{j \in J_g} \lambda_j x_j - \sum_{j \in J_g} \lambda_j x_j - \sum_{j \in J_b} \lambda_j x_j \\ &= \frac{1}{\sum_{j \in J_g} \lambda_j} \sum_{j \in J_g} \lambda_j (x_j - y) - \sum_{j \in J_g} \lambda_j (x_j - y) - \sum_{j \in J_b} \lambda_j (x_j - y) \\ &= \left(\frac{1}{\sum_{j \in J_g} \lambda_j} - 1 \right) \sum_{j \in J_g} \lambda_j (x_j - y) - \sum_{j \in J_b} \lambda_j (x_j - y), \end{aligned}$$

we have

$$\begin{aligned} |x_0 - \tilde{x}_0| &\leq \frac{\sum_{j \in J_b} \lambda_j}{\sum_{j \in J_g} \lambda_j} \sum_{j \in J_g} \lambda_j \text{diam}(\Omega) + \sum_{j \in J_b} \lambda_j \text{diam}(\Omega) \\ &= 2 \text{diam}(\Omega) \sum_{j \in J_b} \lambda_j \leq 2(n+1) \text{diam}(\Omega) \delta_1 \leq \frac{\delta_0}{4}, \end{aligned}$$

provided δ_1 is small enough. Recalling that $\text{dist}(x_0, \partial\Omega) \geq \delta_0$, this implies in particular $\text{dist}(\tilde{x}_0, \partial\Omega) \geq 3\delta_0/4$.

The strategy is now the following: we first show that

$$(4.3) \quad \Gamma_v(\tilde{x}_0 + h) \leq M|h|^{1+\min\{\alpha, \beta\}} \quad \forall |h| \leq \varrho_1,$$

for some constants M and ϱ_1 depending only on δ_0 , and then we “propagate” this estimate to x_0 .

To prove (4.3), we need to distinguish between two further cases.

Case 2-a: There exists $\bar{j} \in J_g$ such that $x_{\bar{j}} \in \Omega$. Up to reordering, we can assume $\bar{j} = 1$. Hence $x_1 \in \Omega$, which implies in particular that $\nabla v(x_1) = 0$. Consider $\bar{x}$, the point of intersection of the boundary of Σ with the line through $\tilde{x}_0$ and x_1 (see Figure 4.2). More precisely, defining

$$\mu_j = \frac{\lambda_j}{\sum_{j \in J_g} \lambda_j} \quad \forall j \in J_g,$$

we have

$$\bar{x} = \frac{1}{1 - \mu_1} \sum_{j \in J_g \setminus \{1\}} \mu_j x_j$$

FIGURE 4.2. To estimate the value of Γ_v at $\tilde{x}_0 + h$, we use the value of v at y together with the convexity of Γ_v .

so that $\tilde{x}_0 = (1 - \mu_1)\bar{x} + \mu_1 x_1$. Notice that

$$(4.4) \quad \mu_1 = \frac{\lambda_1}{\sum_{j \in J_g} \lambda_j} \geq \frac{\delta_1}{1 - (n+1)\delta_1},$$

and $\Gamma_v(\bar{x}) = 0$.

Now choose $|h| \leq \bar{\varrho}$, where $\bar{\varrho}$ is small enough (to be fixed later). If h is parallel to the segment $[\bar{x}, x_1]$ we trivially have

$$\Gamma_v(\tilde{x}_0 + h) = 0 \quad \text{provided } \bar{\varrho} < \min\{|\bar{x} - \tilde{x}_0|, |x_1 - \tilde{x}_0|\}.$$

Notice that $\min\{|\bar{x} - \tilde{x}_0|, |x_1 - \tilde{x}_0|\}$ is bounded from below by a constant depending only on δ_0 , since so are $|x_1 - \tilde{x}_0|$ and μ_1 .

In case h is orthogonal to the segment $[\bar{x}, x_1]$, let us draw the half-line r from $\bar{x}$ through $\tilde{x}_0 + h$. Let us also consider the line s through x_1 parallel to h . Two things can happen: r meets s inside Ω or not.

In the first case call y the point of intersection (see Figure 4.2). By similarity and (4.4)

$$(4.5) \quad |x_1 - y| \leq K|h|, \quad \text{where } K = \frac{|\bar{x} - x_1|}{|\bar{x} - \tilde{x}_0|} = \frac{1}{\mu_1} \leq \frac{2}{\delta_1},$$

and

$$\tilde{x}_0 + h = \mu_1 y + (1 - \mu_1)\bar{x}.$$

Hence

$$\begin{aligned} \Gamma_v(\tilde{x}_0 + h) &= \Gamma_v(\mu_1 y + (1 - \mu_1)\bar{x}) \\ &\leq \mu_1 \Gamma_v(x_1 + (y - x_1)) \leq v(x_1 + (y - x_1)) \leq M|h|^{1+\alpha}, \end{aligned}$$

where we have used that $\Gamma_v(\bar{x}) = 0$.

In the second case let z be the intersection between r and $\partial\Omega$ (see Figure 4.3). The angle σ between $[x_1, z]$ and $[\bar{x}, x_1]$ is bounded from above by $\pi/2$ and from below by a universal constant. In fact, by convexity of Ω , the convex envelope of $\{x_1\} \cup B(\tilde{x}_0, \delta_0/2)$ is contained in Ω , thus

$$\sigma \geq \arctan\left(\frac{\delta_0}{2\text{diam}(\Omega)}\right).$$

FIGURE 4.3. Since the angle σ is uniformly bounded away from 0, $|z - x_1| \leq K'|h|$ for some constant $K' > 0$ depending only on δ_0 and $\text{diam}(\Omega)$, and we conclude as in the previous case.

FIGURE 4.4. When $x_1 \in \partial\Omega$, it is crucial to also exploit the regularity of $\partial\Omega$.

Hence $|x_1 - z| \leq K'|h|$ for some universal constant $K' > 0$. Since $\tilde{x}_0 + h = \gamma y + (1 - \gamma)\bar{x}$ for some $\gamma \in (0, 1)$, exactly as above we get

$$\Gamma_v(\tilde{x}_0 + h) \leq M|h|^{1+\alpha}.$$

This proves an estimate from above on $\Gamma_v(\tilde{x}_0 + h)$ whenever $|h| \leq \bar{\varrho}$ is either orthogonal or parallel to $[\bar{x}, x_1]$. Finally, for any $|h| \leq \bar{\varrho}/2$ write it as $h = h^p + h^o = \frac{2h^p + 2h^o}{2}$, with h^p perpendicular to $[x_1, x_2]$ and h^o parallel to $[\bar{x}, x_1]$. Then, by convexity of Γ_v we obtain

$$\begin{aligned} \Gamma_v(\tilde{x}_0 + h) &\leq \frac{1}{2}(\Gamma_v(\tilde{x}_0 + 2h^p) + \Gamma_v(\tilde{x}_0 + 2h^o)) \\ &\leq 2M(|h^p|^{1+\alpha} + |h^o|^{1+\alpha}) \leq 4M|h|^{1+\alpha}. \end{aligned}$$

Case 2-b: $x_j \in \partial\Omega$ for all $j \in J_g$. Pick any of these points, say (up to reordering) x_1 , and construct the point $\bar{x}$ as in Case 2-a. We again want to estimate $\Gamma_v(\tilde{x}_0 + h)$ for h orthogonal to $[\bar{x}, x_1]$ and small enough. As before we draw the half-line from $\bar{x}$ through $\tilde{x}_0 + h$. This half-line will intersect the boundary of Ω in a point z , and the tangent plane $T_{x_1}\partial\Omega$ in a point w (see Figure 4.4). Arguing as above we deduce that the angle between $T_{x_1}\partial\Omega$ and the segment through $[\bar{x}, x_1]$ is at least σ_0 , where $\sigma_0 > 0$ depends only on δ_0 and $\text{diam}(\Omega)$ (see Figure 4.3), so $|z - x_1|, |w - x_1| \leq K|h|$ for some universal constant K . In this case we cannot however say that $\nabla v(x_1) = 0$

(as we did in Case 2-a), and thus the above calculations give only

$$\Gamma_v(\tilde{x}_0 + h) \leq \bar{p}_{x_1} \cdot (z - x_1) + M|h|^{1+\alpha}.$$

However, since v is Lipschitz and $s_{x_1} = 0$ (recall (4.1)) we have

$$\bar{p}_{x_1} = -\lambda\nu_{x_1},$$

where ν_{x_1} is the outer normal to Ω at x_1 , and $0 \leq \lambda \leq \text{Lip } v$ (see (3.2) and (3.3)). We will thus prove (4.3) if we can show that

$$|\nu_{x_1} \cdot (z - x_1)| \leq C|h|^{1+\beta}.$$

This follows from the $C^{1,\beta}$ regularity of $\partial\Omega$. In fact, denoting with $\pi(z)$ the orthogonal projection of z on $T_{x_1}\partial\Omega$, we have

$$|\nu_{x_1} \cdot (z - x_1)| \leq C|\pi(z) - x_1|^{1+\beta} \leq C|w - x_1|^{1+\beta} \leq CK^{1+\beta}|h|^{1+\beta}.$$

This finally concludes the proof of (4.3).

We now want to prove (4.3) with $\tilde{x}_0$ replaced by x_0 . To this aim, let us define

$$\hat{x}_0 = \frac{1}{\sum_{j \in J_b} \lambda_j} \sum_{j \in J_b} \lambda_j x_j$$

so that

$$x_0 = (1 - \mu)\hat{x}_0 + \mu\tilde{x}_0, \quad \text{where } \mu = \sum_{j \in J_g} \lambda_j \geq \delta_1.$$

Then, by convexity of Γ_v and the fact that $\Gamma_v(\hat{x}_0) = 0$, we infer that

$$\Gamma_v(x_0 + h) \leq \mu\Gamma_v\left(\tilde{x}_0 + \frac{h}{\mu}\right) \leq \frac{M}{\delta_1}|h|^{1+\min\{\alpha,\beta\}} \quad \forall |h| \leq \delta_1\bar{\varrho}_1,$$

concluding the proof. $\square$

The above theorem can be extended to general moduli of continuity. More precisely, its very same proof shows that if v is ω_1 -semiconcave (i.e. (1.2) holds with $C|x - x_0|\omega_1(|x - x_0|)$ in place of $C|x - x_0|^{1+\alpha}$) and $\partial\Omega$ is C^1 with a modulus of continuity ω_2 , then for any $\Omega' \subset\subset \Omega$ there exists $M = M(\Omega', \Omega) > 0$ such that ∇u has modulus of continuity $t \mapsto M \max\{\omega_1(Mt), \omega_2(Mt)\}$ inside Ω' .

Furthermore, it is clear from the proof above (and from the counterexamples in Section 2) that the main difficulty when proving the regularity of Γ_v arises when the points $\{x_j\}_{j \in J_g}$ (see (4.2)) are close to the boundary (that is, Case 2 in the proof of Theorem 1.2). However, if for some special reason one is able to exclude that case, then the assumptions of Theorem 1.2 can be weakened, and in particular one does not need to assume any regularity on $\partial\Omega$. For example, one can prove the following classical result:

Theorem 4.1. *Let Ω be a convex bounded domain, and $v : \Omega \rightarrow \mathbb{R}$ be a continuous, locally $(1+\alpha)$ -semiconcave function. Assume that $v = 0$ on $\partial\Omega$, and that $\inf_{\Omega} v < 0$. Then $\Gamma_v \in C_{\text{loc}}^{1,\alpha}(\Omega)$.*

Proof. Let K_m define the increasing family of compact convex sets

$$K_m = \text{conv}(\{x \in \Omega : v(x) \leq -1/m\}), \quad m \in \mathbb{N}.$$

We observe that K_m is compactly supported inside Ω , and that $\bigcup_{m \in \mathbb{N}} K_m = \Omega$.

Fix $m \in \mathbb{N}$, and assume that $x_0 \in K_m \setminus \{\Gamma_v = v\}$. Using the same notation as in the proof of Theorem 1.2, we claim the following:

There exists $\delta_1 > 0$ small, depending on n , m , and $\inf_{\Omega} v$ only, such that $x_j \in K_{2m}$ for at least one $j \in J_g$ (see (4.2)).

Once this claim is proved, the $C^{1,\alpha}$ regularity of Γ_v inside K_m is obtained as in Case 1 in the proof of Theorem 1.2.

To prove the claim, we first observe that, by convexity of Γ_v , we have

$$(4.6) \quad \Gamma_v \leq -\frac{1}{m} \quad \text{inside } K_m,$$

which implies in particular

$$(4.7) \quad v \leq -\frac{1}{m} \quad \text{inside } K_m \cap \{\Gamma_v = v\}.$$

Now let s_{x_0} be a supporting slope at x_0 . We first claim that there exists at least one $j \in \{1, \dots, n+1\}$ such that $x_j \in K_{2m}$. In fact, if this were not the case, since s_{x_0} is supporting at x_i and $\sum_j \lambda_j = 1$, by (4.6) we get

$$-\frac{1}{m} \geq \Gamma_v(x_0) = \sum_{j=1}^{n+1} \lambda_j (\Gamma_v(x_j) + s_{x_0} \cdot (x_0 - x_j)) = \sum_{j=1}^{n+1} \lambda_j v(x_j) \geq -\frac{1}{2m},$$

a contradiction. Now suppose (again by contradiction) that for all j 's such that $x_j \in K_{2m}$, the corresponding λ_j satisfies $\lambda_j \leq \delta_1$. Then, by (4.7) and the same calculation as above,

$$\begin{aligned} -\frac{1}{m} &\geq \sum_{j=1}^{n+1} \lambda_j v(x_j) = \sum_{x_j \in K_{2m}} \lambda_j v(x_j) + \sum_{x_j \notin K_{2m}} \lambda_j v(x_j) \\ &\geq -(n+1)\delta_1 \sup_{\overline{\Omega}}(-v) - \frac{1}{2m}, \end{aligned}$$

which is impossible for δ_1 sufficiently small. This proves the claim and concludes the proof. $\square$

5. PROOF OF THEOREM 1.3

Proof. Thanks to Lemma 3.2 (notice that in this case the convex envelope is globally Lipschitz in Ω) and arguing as in Case 1 of the proof of Theorem 1.2, it is enough to show that for every $x_0 \in \Omega \setminus \{v = \Gamma_v\}$ there exists a radius $\varrho(x_0)$ such that

$$(5.1) \quad \Gamma_v(x) \leq \Gamma_v(x_0) + s_{x_0}(x - x_0) + C|x - x_0|^2 \quad \forall x \in B(x_0, \varrho(x_0)) \cap \overline{\Omega}$$

for some universal constant C independent of x_0 . Without loss of generality we can assume

$$(5.2) \quad \Gamma_v(x_0) = |s_{x_0}| = 0.$$

Then, using Lemma 3.3 we write x_0 as a convex combination of k points in $\{v = \Gamma_v\}$, with $k \leq n+1$.

We first show that we can reduce to the case $k = 2$. Indeed, suppose that (5.1) holds with a constant C_k for every point x_0 which is a convex combination of at most k points in $\{v = \Gamma_v\}$. We want to show it holds for all points which are a convex combination of $k+1$ points with constant $C_{k+1} = (k+1)C_k$. To do this, assume that

$$x_0 \in \text{conv}(x_1, \dots, x_{k+1})$$

and write it as

$$x_0 = (1 - \lambda)x_1 + \lambda\tilde{x}_0$$

where $\lambda \geq 1/(k+1)$ and

$$\tilde{x}_0 \in \text{conv}(x_1, \dots, x_k).$$

(Notice that, up to a relabeling, this is always possible.) Since $k \geq 2$, by uniform convexity of Ω both x_0 and $\tilde{x}_0$ are interior points. Hence there exists a radius $\varrho(x_0)$ such that $\lambda\varrho(x_0) \leq \varrho(\tilde{x}_0)$, and both balls $B(x_0, \varrho(x_0))$ and $B(\tilde{x}_0, \varrho(\tilde{x}_0))$ are contained in Ω . Recalling (5.2) we see that 0 supports Γ_v both at x_1 and $\tilde{x}_0$. Thus $\Gamma_v(x_1) = \Gamma_v(\tilde{x}_0) = 0$ and we get

$$\Gamma_v(x_0 + (x - x_0)) \leq \lambda \Gamma_v\left(\tilde{x}_0 + \frac{(x - x_0)}{\lambda}\right) \leq (k+1)C_k |x - x_0|^2 \quad \text{for } x \in B(x_0, r(x_0)).$$

This proves the validity of (5.1) when x_0 is the convex combination of at most $k+1$ points.

We are thus left to show (5.1) when x_0 is the convex combination of only two contact points:

$$x_0 = (1 - \lambda)x_1 + \lambda x_2.$$

By symmetry we can assume that $\lambda \geq 1/2$. In case x_2 is an interior point we can argue as above (with x_2 in place of $\tilde{x}_0$) to obtain (5.1), so we can assume that $x_2 \in \partial\Omega$. Up to a change of coordinates we assume that the inner normal to $\partial\Omega$ at x_2 is given by e_n (see Figure 5.1). Let θ denote the angle between the segment $[x_1, x_2]$ and the tangent plane to $\partial\Omega$ at x_2 . By the uniform convexity of Ω we have that

$$(5.3) \quad |x_1 - x_2| \leq 2R\theta,$$

where $R > 0$ is such that $\Omega \subset B(Re_n, R)$. Indeed if we draw the half-line from x_2 to x_1 , this meets the boundary of $B(Re_n, R)$ in a point x_3 which satisfies $|x_3 - x_2| = 2R\sin(\theta)$, thus

$$|x_1 - x_2| \leq |x_3 - x_2| \leq 2R\theta.$$

We now distinguish two cases, depending on whether θ is small or not.

Case 1: $\theta \leq \theta_0$, with θ_0 universally small. Up to a rotation of coordinate we can also assume that the segment $[x_1, x_2]$ lies in the plane generated by e_1, e_n (denoted by Π_{e_1, e_n}).

Let us consider a ball of radius $\varrho(x_0) \ll \theta$ around x_0 , and consider $x \in B(x_0, \varrho(x_0))$ such that $x - x_0$ is orthogonal to $[x_1, x_2]$.

In case $x - x_0$ is orthogonal to Π_{e_1, e_n} , let us draw the half-line from x_1 through x , and let y be its point of intersection with $\partial\Omega$. Then by similarity we get $|y - x_2| \leq 2|x - x_0|$ (recall that $\lambda \geq 1/2$), so by the very same arguments as in Case 2-b of Theorem 1.2 we get

$$\Gamma_v(x) \leq M|x - x_0|^2.$$

On the other hand, if $x - x_0$ belongs to the plane Π_{e_1, e_n} , then we need to do a more refined argument as outlined in Figure 5.1. First of all, we choose θ_0 so small that, if we call δ the distance between x_2 and the projection of x_1 on the tangent plane to $\partial\Omega$ at x_2 , then

$$\partial\Omega \cap B(x_2, 2\delta) = \{(x', \varphi(x')) : |x' - x'_2| \leq 2\delta\},$$

$\Gamma_v(x_1) = 0$ and $\partial_1 \tilde{v}(x'_2) = 0$)

$$\begin{aligned}\Gamma_v(x) &\leq \Gamma_v(y) \leq v(y', \psi(y')) = \tilde{v}(y') \\ &\leq \frac{1}{2} \partial_{11} \tilde{v}(x'_2) |y' - x'_2|^2 + C |y' - x'_2|^3 \\ &\leq K(\theta^2 |y' - x'_2|^2 + |y' - x'_2|^3),\end{aligned}$$

for some universal constant $K > 0$. We now observe that, if we call $\tilde{y}$ the intersection of r with the tangent plane at x_2 , then by choosing $\varrho(x_0)$ sufficiently small we get

$$|y' - x'_2| \leq 2|\tilde{y} - x_2| \leq \frac{4|x - x_0|}{\sin(\theta/2)} \leq \frac{10|x - x_0|}{\theta},$$

where in the last inequality we used that θ is small. Hence,

$$\Gamma_v(x) \leq K'|x - x_0|^2 + K' \frac{|x - x_0|^3}{\theta^3} \quad \forall x \in B(x_0, \varrho(x_0)),$$

and the desired estimate follows by choosing $\varrho(x_0) \leq \theta^3$.

This proves that

$$\Gamma_v(x) \leq M|x - x_0|^2$$

whenever $x - x_0 \in B(0, \varrho(x_0))$ is orthogonal to $[x_1, x_2]$, and either it belongs to Π_{e_1, e_n} or it is orthogonal to it. A simple argument based on the convexity of Γ_v (as the one used at the end of Case 2-a in the proof of Theorem 1.2) completes the proof.

Case 2: $\theta \geq \theta_0$. This case is much simpler: indeed, if $x \in B(x_0, \varrho(x_0))$ and we define y to be the intersection of the half-line from x_1 through x with $\partial\Omega$, then by choosing $\varrho(x_0)$ sufficiently small we get

$$|y - x_2| \leq \frac{10|x - x_0|}{\theta_0}.$$

Hence, by convexity of Γ_v and the fact that $\Gamma_v(x_1) = 0$, we obtain

$$\Gamma_v(x) \leq \Gamma_v(y) \leq C|y - x_2|^2 \leq \frac{100C|x - x_0|^2}{\theta_0^2} \quad \forall x \in B(x_0, \varrho(x_0)),$$

which concludes the proof. $\square$

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REFERENCES

- [1] J. Benoist and J.-B. Hiriart-Urruty, *What is the subdifferential of the closed convex hull of a function?*, SIAM J. Math. Anal. **27** (1996), no. 6, 1661–1679, DOI 10.1137/S0036141094265936. MR1416513 (97k:49026)
- [2] Patrick Bernard, *Lasry-Lions regularization and a lemma of Ilmanen*, Rend. Semin. Mat. Univ. Padova **124** (2010), 221–229, DOI 10.4171/RSMUP/124-15. MR2752687 (2012d:37191)
- [3] Luis A. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2) **131** (1990), no. 1, 135–150, DOI 10.2307/1971510. MR1038360 (91f:35059)
- [4] Luis A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR1351007 (96h:35046)

- [5] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for the degenerate Monge-Ampère equation*, Rev. Mat. Iberoamericana **2** (1986), no. 1-2, 19–27, DOI 10.4171/RMI/23. MR864651 (87m:35103)
- [6] Pierre Cardaliaguet, *Front propagation problems with nonlocal terms. II*, J. Math. Anal. Appl. **260** (2001), no. 2, 572–601, DOI 10.1006/jmaa.2001.7483. MR1845570 (2002f:35116)
- [7] Guido De Philippis and Alessio Figalli, *Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps*, Anal. PDE **6** (2013), no. 4, 993–1000, DOI 10.2140/apde.2013.6.993. MR3092736
- [8] A. Fathi and M. Zavidovique, *Ilmanen’s lemma on insertion of $C^{1,1}$ functions*, Rend. Semin. Mat. Univ. Padova **124** (2010), 203–219, DOI 10.4171/RSMUP/124-14. MR2752686 (2012i:58004)
- [9] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)
- [10] A. Griewank and P. J. Rabier, *On the smoothness of convex envelopes*, Trans. Amer. Math. Soc. **322** (1990), no. 2, 691–709, DOI 10.2307/2001721. MR986024 (91k:49021)
- [11] Bernd Kirchheim and Jan Kristensen, *Differentiability of convex envelopes* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 8, 725–728, DOI 10.1016/S0764-4442(01)02117-6. MR1868942 (2002g:49024)
- [12] Tom Ilmanen, *The level-set flow on a manifold*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 193–204. MR1216585 (94d:58040)
- [13] Adam M. Oberman, *The convex envelope is the solution of a nonlinear obstacle problem*, Proc. Amer. Math. Soc. **135** (2007), no. 6, 1689–1694 (electronic), DOI 10.1090/S0002-9939-07-08887-9. MR2286077 (2007k:35184)
- [14] Adam M. Oberman and Luis Silvestre, *The Dirichlet problem for the convex envelope*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5871–5886, DOI 10.1090/S0002-9947-2011-05240-2. MR2817413 (2012e:35064)
- [15] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original; Princeton Paperbacks. MR1451876 (97m:49001)

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