STABILITY AND COMPACTNESS FOR COMPLETE f -MINIMAL SURFACES

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ABSTRACT. Let $(M, \bar{g}, e^{-f} d\mu)$ be a complete metric measure space with Bakry-Émery Ricci curvature bounded below by a positive constant. We prove that in M there is no complete two-sided L_f -stable immersed f -minimal hypersurface with finite weighted volume. Further, if M is a 3-manifold, we prove a smooth compactness theorem for the space of complete embedded f -minimal surfaces in M with the uniform upper bounds of genus and weighted volume, which generalizes the compactness theorem for complete self-shrinkers in $\mathbb{R}^3$ by Colding-Minicozzi.

1. INTRODUCTION

Recall that a self-shrinker (for mean curvature flow in $\mathbb{R}^{n+1}$) is a hypersurface Σ immersed in the Euclidean space $(\mathbb{R}^{n+1}, g_{can})$ satisfying that

$$H = \frac{1}{2} \langle x, \nu \rangle,$$

where x is the position vector in $\mathbb{R}^{n+1}$, ν is the unit normal at x , and H is the mean curvature of Σ at x . Self-shrinkers play an important role in the study of singularity of mean curvature flow and have been studied by many people in recent years. We refer to [4], [5] and the references therein. In particular, Colding-Minicozzi [4] proved the following compactness theorem for self-shrinkers in $\mathbb{R}^3$.

Theorem 1 ([4]). *Given an integer $g \geq 0$ and a constant $V > 0$, the space $\mathcal{S}(g, V)$ of smooth complete embedded self-shrinkers $\Sigma \subset \mathbb{R}^3$ with*

- *genus at most g ,*
- *$\partial\Sigma = \emptyset$,*
- *$\text{Area}(B_R(x_0) \cap \Sigma) \leq VR^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$*

is compact. Namely, any sequence of these has a subsequence that converges in the topology of C^m convergence on compact subsets for any $m \geq 2$.

In this paper, we extend Theorem 1 to the space of complete embedded f -minimal surfaces in a 3-manifold. A hypersurface Σ immersed in a Riemannian manifold $(M, \bar{g})$ is called an f -minimal hypersurface if its mean curvature H satisfies that, for any $p \in \Sigma$,

$$H = \langle \bar{\nabla} f, \nu \rangle,$$

Received by the editors March 6, 2013.

2010 *Mathematics Subject Classification.* Primary 58J50; Secondary 58E30.

The first and third authors were partially supported by CNPq and Faperj of Brazil.

The second author was supported by CNPq of Brazil.

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where f is a smooth function defined on M and $\overline{\nabla}f$ denotes the gradient of f on M . Here are some examples of f -minimal hypersurfaces:

- $f \equiv C$, an f -minimal hypersurface is just a minimal hypersurface.
- self-shrinker Σ in $\mathbb{R}^{n+1}$. $f = \frac{|x|^2}{4}$.
- Let $(M, \overline{g}, f)$ be a shrinking gradient Ricci solitons; i.e. after a normalization, $(M, \overline{g}, f)$ satisfies the equation $\overline{\text{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}\overline{g}$ or equivalently the Bakry-Émery Ricci curvature $\overline{\text{Ric}}_f := \overline{\text{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}$. We may consider f -minimal hypersurfaces in $(M, \overline{g}, f)$. In particular, the previous example: a self-shrinker Σ in $\mathbb{R}^{n+1}$ is f -minimal in Gauss shrinking soliton $(\mathbb{R}^{n+1}, g_{\text{can}}, \frac{|x|^2}{4})$.
- $M = \mathbb{H}^{n+1}(-1)$, the hyperbolic space. Let r denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Now $\overline{\text{Ric}}_f \geq n(2a - 1)$. The geodesic sphere of radius r centered at p is an f -minimal hypersurface if the radius r satisfies $2ar = \coth r$.

An f -minimal hypersurface Σ can be viewed in two ways. One is that Σ is f -minimal if and only if Σ is a critical point of the weighted volume functional $e^{-f}d\sigma$, where $d\sigma$ is the volume element of Σ . The other one is that Σ is f -minimal if and only if Σ is minimal in the new conformal metric $\tilde{g} = e^{-\frac{2f}{n}}\overline{g}$ (see Section 2 and Appendix). f -minimal hypersurfaces have been studied before as even more general stationary hypersurfaces for parametric elliptic functionals; see for instance the work of White [14] and Colding-Minicozzi [7].

We prove the following compactness result:

Theorem 2. *Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space and $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded f -minimal surfaces $\Sigma \subset M$ with*

- *genus at most g ,*
- $\partial\Sigma = \emptyset$,
- $\int_{\Sigma} e^{-f}d\sigma \leq V$

is compact in the C^m topology, for any $m \geq 2$. Namely, any sequence of $S_{g,V}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{g,V}$, for any $m \geq 2$.

Since the existence of the uniform scale-invariant area bound is equivalent to the existence of the uniform bound of the weighted area for self-shrinkers (see Remark 1 in Section 5), Theorem 2 implies Theorem 1. Also, in [2], we will apply Theorem 2 to obtain a compactness theorem for the space of closed embedded f -minimal surfaces with the upper bounds of genus and diameter.

To prove Theorem 2, we need to prove a nonexistence result on L_f -stable f -minimal hypersurfaces, which is of independent interest.

Theorem 3. *Let $(M^{n+1}, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is positive constant. Then there is no complete two-sided L_f -stable f -minimal hypersurface Σ immersed in $(M, \overline{g})$ without boundary and with finite weighted volume (i.e. $\int_{\Sigma} e^{-f}d\sigma < \infty$), where $d\sigma$ denotes the volume element on Σ determined by the induced metric from $(M, \overline{g})$.*

Here we explain briefly the meaning of L_f stability. For an f -minimal hypersurface Σ , the L_f operator is

$$L_f = \Delta_f + |A|^2 + \overline{\text{Ric}}_f(\nu, \nu),$$

where $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the weighted Laplacian on Σ . In particular, for self-shrinkers, it is the so-called L operator:

$$L = \Delta + |A|^2 - \frac{1}{2} \langle x, \nabla \cdot \rangle + \frac{1}{2}.$$

L_f -stability of Σ means that its weighted volume $\int_{\Sigma} e^{-f} d\sigma$ is locally minimal; that is, the second variation of its weighted volume is nonnegative for any compactly supported normal variation. We leave more details about the definition of L_f -stability and some of its properties to Section 2 and the Appendix.

For self-shrinkers in $\mathbb{R}^{n+1}$, Colding-Minicozzi [6] proved that

Theorem 4 ([6]). *There are no L -stable smooth complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.*

Since the first and third authors [3] of the present paper proved that for self-shrinkers, properness, the polynomial volume growth, and finite weighted volume are equivalent, Theorem 3 implies Theorem 4.

In this paper, we also discuss the relationship among the properness, polynomial volume growth and finite weighted volume of f -minimal submanifolds (Propositions 3, 4 and 5). We obtain their equivalence when the ambient space $(M, \bar{g}, f)$ is a shrinking gradient Ricci solitons, i.e. $\overline{\text{Ric}} + \bar{\nabla}^2 f = \frac{1}{2} \bar{g}$, with the condition that f is a convex function with $|\bar{\nabla} f|^2 \leq f$ (Corollary 1).

The rest of this paper is organized as follows: In Section 2 some definitions, notation and facts are given as preliminaries. In Section 3 we prove Propositions 3, 4 and 5. In Section 4 we prove Theorem 3. In Section 5 we prove Theorem 2. In the Appendix we calculate the second variation of the volume functional of f -minimal submanifolds and discuss some properties of L_f -stability for f -minimal submanifolds.

2. PRELIMINARIES

In general, a smooth metric measure space, denoted by $(M^m, \bar{g}, e^{-f} d\mu)$, is an m -dimensional Riemannian manifold $(M^m, \bar{g})$ together with a weighted volume form $e^{-f} d\mu$ on M , where f is a smooth function on M and $d\mu$ is the volume element induced by the metric $\bar{g}$. In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \bar{g})$, for instance by $\bar{\nabla}$ and $\overline{\text{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, \bar{g})$ respectively. For $(M, \bar{g}, e^{-f} d\mu)$, an important and natural tensor is the ∞ -Bakry-Émery Ricci curvature tensor $\overline{\text{Ric}}_f$ (for simplicity, Bakry-Émery Ricci curvature), which is defined by

$$\overline{\text{Ric}}_f := \overline{\text{Ric}} + \bar{\nabla}^2 f,$$

where $\bar{\nabla}^2 f$ is the Hessian of f on M . If f is constant, $\overline{\text{Ric}}_f$ is the Ricci curvature $\overline{\text{Ric}}$ on M respectively.

A Riemannian manifold with Bakry-Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. For instance, see the work of

Wei-Wylie [13], Munteanu-Wang [11, 12] and the references therein. In this paper, we will use the following proposition by Morgan [10] (see also its proof in [13]).

Proposition 1. *If a complete smooth metric measure space $(M, \bar{g}, e^{-f} du)$ has $\overline{\text{Ric}}_f \geq k$, where k is a positive constant, then M has finite weighted volume (i.e. $\int_M e^{-f} d\mu < \infty$) and finite fundamental group.*

Now, let $i : \Sigma^n \rightarrow M^m, n < m$, be an n -dimensional smooth immersion. Then $i : (\Sigma^n; i^* \bar{g}) \rightarrow (M^m, \bar{g})$ is an isometric immersion with the induced metric $i^* \bar{g}$. For simplicity, we still denote $i^* \bar{g}$ by $\bar{g}$ whenever there is no confusion. We will denote for instance by ∇ , Ric , Δ and $d\sigma$, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of $(\Sigma, \bar{g})$ respectively.

The function f induces a weighted measure $e^{-f} d\sigma$ on Σ . Thus we have an induced smooth metric measure space $(\Sigma^n, \bar{g}, e^{-f} d\sigma)$.

The associated weighted Laplacian Δ_f on $(\Sigma, \bar{g})$ is defined by

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle.$$

The second order operator Δ_f is a self-adjoint operator on the space of square integrable functions on Σ with respect to the measure $e^{-f} d\sigma$ (however the Laplacian operator in general does not have this property).

The second fundamental form A of $(\Sigma, \bar{g})$ is defined by

$$A(X, Y) = (\bar{\nabla}_X Y)^\perp, \quad X, Y \in T_p \Sigma, p \in \Sigma,$$

where $\perp$ denotes the projection to the normal bundle of Σ . The mean curvature vector $\mathbf{H}$ of Σ is defined by $\mathbf{H} = \text{tr} A = \sum_{i=1}^n (\bar{\nabla}_{e_i} e_i)^\perp$.

Definition 1. The weighted mean curvature vector of Σ with respect to the metric $\bar{g}$ is defined by

$$(1) \quad \mathbf{H}_f = \mathbf{H} + (\bar{\nabla} f)^\perp.$$

The immersed submanifold $(\Sigma, \bar{g})$ is called f -minimal if its weighted mean curvature vector $\mathbf{H}_f$ vanishes identically, or equivalently if its mean curvature vector satisfies

$$(2) \quad \mathbf{H} = -(\bar{\nabla} f)^\perp.$$

Definition 2. The weighted volume of $(\Sigma, \bar{g})$ is defined by

$$(3) \quad V_f(\Sigma) := \int_\Sigma e^{-f} d\sigma.$$

It is well known that Σ is f -minimal if and only if Σ is a critical point of the weighted volume functional. Namely, it holds that

Proposition 2. *If T is a compactly supported variational vector field on Σ , then the first variation formula of the weighted volume of $(\Sigma, \bar{g})$ is given by*

$$(4) \quad \left. \frac{d}{dt} V_f(\Sigma_t) \right|_{t=0} = - \int_\Sigma \langle T^\perp, \mathbf{H}_f \rangle_{\bar{g}} e^{-f} d\sigma.$$

On the other hand, an f -minimal submanifold can be viewed as a minimal submanifold under a conformal metric. Precisely, define the new metric $\tilde{g} = e^{-\frac{2}{n}f} \bar{g}$ on M , which is conformal to $\bar{g}$. Then the immersion $i : \Sigma \rightarrow M$ induces a metric $i^* \tilde{g}$

on Σ from $(M, \tilde{g})$. In the following, $i^*\tilde{g}$ is still denoted by $\tilde{g}$ for simplicity. The volume of $(\Sigma, \tilde{g})$ is

$$(5) \quad \tilde{V}(\Sigma) := \int_{\Sigma} d\tilde{\sigma} = \int_{\Sigma} e^{-f} d\sigma = V_f(\Sigma).$$

Hence Proposition 2 and (5) imply that

$$(6) \quad \int_{\Sigma} \langle T^{\perp}, \tilde{\mathbf{H}} \rangle_{\tilde{g}} d\tilde{\sigma} = \int_{\Sigma} \langle T^{\perp}, \mathbf{H}_f \rangle_{\tilde{g}} e^{-f} d\sigma,$$

where $d\tilde{\sigma} = e^{-f} d\sigma$ and $\tilde{\mathbf{H}}$ denote the volume element and the mean curvature vector of Σ with respect to the conformal metric $\tilde{g}$ respectively.

Identity (6) implies that $\tilde{\mathbf{H}} = e^{\frac{2f}{n}} \mathbf{H}_f$. Therefore $(\Sigma, \tilde{g})$ is f -minimal in $(M, \tilde{g})$ if and only if $(\Sigma, \tilde{g})$ is minimal in $(M, \tilde{g})$.

Now suppose that Σ^n is a hypersurface immersed in M^{n+1} . Let $p \in \Sigma$ and ν be a unit normal at p . The second fundamental form A and the mean curvature H of $(\Sigma, \tilde{g})$ are as follows:

$$A : T_p \Sigma \rightarrow T_p \Sigma, A(X) = \bar{\nabla}_X \nu, X \in T_p \Sigma,$$

$$H = \operatorname{tr} A = - \sum_{i=1}^n \langle \bar{\nabla}_{e_i} e_i, \nu \rangle.$$

Hence the mean curvature vector $\mathbf{H}$ of $(\Sigma, \tilde{g})$ satisfies $\mathbf{H} = -H\nu$. Define the weighted mean curvature H_f of $(\Sigma, \tilde{g})$ by $\mathbf{H}_f := -H_f \nu$. Then

$$H_f = H - \langle \bar{\nabla} f, \nu \rangle.$$

Definition 3. A hypersurface Σ immersed in $(M^{n+1}, \tilde{g}, e^{-f} d\mu)$ with the induced metric $\tilde{g}$ is called an f -minimal hypersurface if it satisfies

$$(7) \quad H = \langle \bar{\nabla} f, \nu \rangle.$$

For a hypersurface $(\Sigma, \tilde{g})$, the L_f operator is defined by

$$L_f := \Delta_f + |A|^2 + \bar{\operatorname{Ric}}_f(\nu, \nu),$$

where $|A|^2$ denotes the square of the norm of the second fundamental form A of Σ .

The L_f -stability of Σ is defined as follows:

Definition 4. A two-sided f -minimal hypersurface Σ is said to be L_f -stable if for any compactly supported smooth function $\varphi \in C_0^\infty(\Sigma)$, it holds that

$$(8) \quad - \int_{\Sigma} \varphi L_f \varphi e^{-f} d\sigma = \int_{\Sigma} [|\nabla \varphi|^2 - (|A|^2 + \bar{\operatorname{Ric}}_f(\nu, \nu)) \varphi^2] e^{-f} d\sigma \geq 0.$$

It is known that an f -minimal hypersurface $(\Sigma, \tilde{g})$ is L_f -stable if and only if $(\Sigma, \tilde{g})$ is stable as a minimal surface with respect to the conformal metric $\tilde{g} = e^{-f} \tilde{g}$. See more details in the Appendix of this paper.

In this paper, for closed hypersurfaces, we choose ν to be the outer unit normal.

3. PROPERNESS, POLYNOMIAL VOLUME GROWTH AND FINITE WEIGHTED VOLUME OF f -MINIMAL HYPERSURFACES

In [3], the first and third authors of the present paper proved that the finite weighted volume of a self-shrinker Σ^n immersed in $\mathbb{R}^n$ implies it is properly immersed. In [9], Ding-Xin proved that a properly immersed self-shrinker must have the Euclidean volume growth. Combining these two results, it was proved in [3] that for immersed self-shrinkers, properness, polynomial volume growth and finite weighted volume are equivalent.

In this section we study the relationship among the properness, polynomial volume growth and finite weighted volume of f -minimal submanifolds, some of which will be used later in this paper.

Let Σ be an n -dimensional submanifold in a complete manifold M^m , $n < m$. Σ is said to have polynomial volume growth if, for a $p \in M$ fixed, there exist constants C and d so that for all $r \geq 1$,

$$(9) \quad \text{Vol}(B_r^M(p) \cap \Sigma) \leq Cr^d,$$

where $B_r^M(p)$ is the extrinsic ball of radius r centered at p and $\text{Vol}(B_r^M(p))$ denotes the volume of $B_r^M(p) \cap \Sigma$. When $d = n$ in (9), Σ is said to be of Euclidean volume growth.

Before proving the following Proposition 3, we recall an estimate implied by the Hessian comparison theorem (cf., for instance, [6], Lemma 7.1).

Lemma 1. *Let $(M, \bar{g})$ be a complete Riemannian manifold with bounded geometry, that is, M has sectional curvature bounded by k ($|K_M| \leq k$), and injectivity radius bounded below by $i_0 > 0$. Then the distance function $r(x)$ satisfies*

$$|\bar{\nabla}^2 r(V, V) - \frac{1}{r}|V - \langle V, \bar{\nabla} r \rangle \bar{\nabla} r|^2| \leq \sqrt{k},$$

for $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and any unit vector $V \in T_x M$.

Using this estimate we will prove

Proposition 3. *Let Σ^n be a complete noncompact f -minimal submanifold immersed in a complete Riemannian manifold M^m . If Σ has finite weighted volume, then Σ is properly immersed.*

Proof. We argue by contradiction. Since the argument is local, we may assume that (M, g) has bounded geometry. Suppose that Σ is not properly immersed. Then there exist a number $2R < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and $o \in M$ so that $\bar{B}_R^M(o) \cap \Sigma$ is not compact in Σ , where $\bar{B}_R^M(o)$ denotes the closure of the (open) ball $B_R^M(o)$ in M of radius R centered at o . Then for any $a > 0$, there is a sequence $\{p_k\}$ of points in $B_R^M(o) \cap \Sigma$ with $\text{dist}_\Sigma(p_k, p_j) \geq a > 0$ for any $k \neq j$. So $B_{\frac{a}{2}}^\Sigma(p_k) \cap B_{\frac{a}{2}}^\Sigma(p_j) = \emptyset$ for any $k \neq j$, where $B_{\frac{a}{2}}^\Sigma(p_k)$ and $B_{\frac{a}{2}}^\Sigma(p_j)$ denote the intrinsic balls in Σ of the radius $\frac{a}{2}$, centered at p_k and p_j respectively. Choose $a < 2R$. Then $B_{\frac{a}{2}}^\Sigma(p_j) \subset B_{2R}^M(o)$. If

$p \in B_{\frac{\Sigma}{2}}(p_j)$, the extrinsic distance function $r_j(p) = \text{dist}_M(p, p_j)$ from p_j satisfies

$$\begin{aligned} \Delta r_j &= \sum_{i=1}^n \bar{\nabla}^2 r_j(e_i, e_i) + \langle \mathbf{H}, \bar{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j} |\nabla r_j|^2 - n\sqrt{k} - \langle \bar{\nabla} f^\perp, \bar{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j} |\nabla r_j|^2 - c, \end{aligned}$$

where $c = n\sqrt{k} + \sup_{B_{2R}^M(0)} |\bar{\nabla} f|$. Lemma 1 is used above. Hence

$$\Delta r_j^2 \geq 2n - 2cr_j.$$

Choosing $a \leq \min\{\frac{n}{2c}, 2R\}$, we have for $0 < \mu \leq \frac{a}{2}$,

$$\begin{aligned} (10) \quad \int_{B_\mu^\Sigma(p_j)} (2n - 2cr_j) d\sigma &\leq \int_{B_\mu^\Sigma(p_j)} \Delta r_j^2 d\sigma \\ &= \int_{\partial B_\mu^\Sigma(p_j)} \langle \nabla r_j^2, \nu \rangle d\sigma \\ &\leq 2\mu A(\mu), \end{aligned}$$

where ν denotes the outward unit normal vector of $\partial B_\mu^\Sigma(p_j)$ and $A(\mu)$ denotes the area of $\partial B_\mu^\Sigma(p_j)$. Using the co-area formula in (10), we have

$$(11) \quad \int_0^\mu (n - cs) A(s) ds \leq \int_0^\mu \int_{d_\Sigma(p, p_j)=s} (n - cr_j) d\sigma \leq \mu A(\mu).$$

This implies

$$(n - c\mu)V(\mu) \leq V'(\mu),$$

where $V(\mu)$ denotes the volume of $B_\mu^\Sigma(p_j)$. So

$$(12) \quad \frac{V'(\mu)}{V(\mu)} \geq \frac{n}{\mu} - c.$$

Integrating (12) from $\varepsilon > 0$ to μ , we have

$$\frac{V(\mu)}{V(\varepsilon)} \geq \left(\frac{\mu}{\varepsilon}\right)^n e^{-c(\mu-\varepsilon)}.$$

Since $\lim_{s \rightarrow 0^+} \frac{V(s)}{s^n} = \omega_n$,

$$(13) \quad V(\mu) \geq \omega_n \mu^n e^{-c\mu}.$$

Thus we conclude

$$\int_\Sigma e^{-f} d\sigma \geq \sum_{j=1}^\infty \int_{B_{\frac{\Sigma}{2}}^\Sigma(p_j)} e^{-f} d\sigma \geq \inf_{B_{2R}^M(o)} (e^{-f}) \sum_{j=1}^\infty \int_{B_{\frac{\Sigma}{2}}^\Sigma(p_j)} d\sigma = \infty.$$

This contradicts the assumption of the finite weighted volume of Σ . $\square$

Proposition 4. *Let $(M^m, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\text{Ric}}_f = k$, where k is a positive constant. Assume that f is a convex function. If Σ^n is a complete noncompact properly immersed f -minimal submanifold in M , then Σ has finite weighted volume and Euclidean (hence polynomial) volume growth.*

Proof. Since $(M, \bar{g}, f)$ is a gradient shrinking Ricci soliton, it is well known that, by a scaling of the metric $\bar{g}$ and a translating of f , still denoted by $\bar{g}$ and f respectively, we may normalize the metric so that $k = \frac{1}{2}$ and the following identities hold:

$$\begin{aligned}\bar{R} + |\bar{\nabla} f|^2 - f &= 0, \\ \bar{R} + \bar{\Delta} f &= \frac{m}{2}, \\ \bar{R} &\geq 0.\end{aligned}$$

From these equations, we have that

$$\bar{\Delta} f - |\bar{\nabla} f|^2 + f = \frac{m}{2} \quad \text{and} \quad |\bar{\nabla} f|^2 \leq f.$$

It was proved by Cao and the third author [1] that there is a positive constant c so that

$$(14) \quad \frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2$$

for any $x \in M$ with $r(x) = \text{dist}_M(p, x) \geq r_0$, where p is a fixed point in M and c, r_0 are positive constants that depend only on m and $f(p)$.

By (14), we know that f is a proper function on M . Since Σ is properly immersed in M and f is proper in M , $f|_\Sigma$ is also a proper smooth function on Σ . Note that with the scaling metric and translating f , Σ is still f -minimal. Hence

$$\begin{aligned}\Delta f - |\nabla f|^2 + f &= (\bar{\Delta} f - \sum_{\alpha=n+1}^m f_{\alpha\alpha} - |\bar{\nabla} f^\perp|^2) - |\bar{\nabla} f^\top|^2 + f \\ &= \bar{\Delta} f - |\bar{\nabla} f|^2 + f - \sum_{\alpha=n+1}^m f_{\alpha\alpha} \\ &\leq \frac{m}{2}.\end{aligned}$$

Also we have

$$|\nabla f|^2 = |\bar{\nabla} f^\top|^2 \leq |\bar{\nabla} f|^2 \leq f.$$

By Theorem 1.1 of [3], Σ has finite weighted volume and the Euclidean volume growth of the sub-level set of f with respect to the scaling metric and the translating f , and hence with respect to the original metric and f . Moreover, by the estimate (14), we have that Σ has the Euclidean volume growth. $\square$

Next we prove the following:

Proposition 5. *Let $(M^m, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\bar{\text{Ric}}_f \geq k$, where k is a positive constant. Assume that $|\bar{\nabla} f|^2 \leq 2kf$. If Σ^n is a complete submanifold (not necessarily f -minimal) with polynomial volume growth, then Σ has finite weighted volume.*

Proof. By a scaling of the metric, we may assume that $k = \frac{1}{2}$. The proof follows from an estimate of f . Munteanu-Wang [11] extended the estimate (14) to $(M^m, \bar{g}, e^{-f} d\mu)$ with $\bar{\text{Ric}}_f \geq \frac{1}{2}$ and $|\bar{\nabla} f|^2 \leq f$. Combining the assumption that Σ

has polynomial volume growth with the estimate (14), we have

$$\begin{aligned} \int_{\Sigma} e^{-f} d\sigma &= \int_{\Sigma \cap B_{r_0}^M(p)} e^{-f} d\sigma + \sum_{i=0}^{\infty} \int_{\Sigma \cap (B_{r_0+i+1}^M(p) \setminus B_{r_0+i}^M(p))} e^{-f} d\sigma \\ &\leq C_1 \text{Vol}(\Sigma \cap B_{r_0}^M(p)) + C \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} \text{Vol}(\Sigma \cap B_{r_0+i+1}^M(p)) \\ &\leq C \left[r_0^d + \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} (r_0+i+1)^d \right] \\ &< \infty. \end{aligned}$$

□

By Propositions 3, 4 and 5, we have the following.

Corollary 1. *Let $(M^m, \bar{g}, f)$ be a complete shrinking gradient Ricci soliton with $\overline{\text{Ric}}_f = \frac{1}{2}$. Assume that f is a convex function. If Σ is a complete f -minimal submanifold immersed in M , then for Σ the properness, polynomial volume growth, and finite weighted volume are equivalent.*

4. NONEXISTENCE OF L_f STABLE f -MINIMAL HYPERSURFACES

In this section, we prove Theorem 3, which is a key to proving the compactness theorem in Section 5.

Theorem 5 (Theorem 3). *Let $(M, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Then there is no two-sided L_f -stable complete f -minimal hypersurface Σ immersed in (M, g) without boundary and with finite weighted volume (i.e. $\int_{\Sigma} e^{-f} d\sigma < \infty$).*

Proof. We argue by contradiction. Suppose that Σ is an L_f -stable complete f -minimal hypersurface immersed in (M, g) without boundary and with finite weighted volume. Recall that a two-sided hypersurface Σ is L_f -stable if the following inequality holds, that is, for any compactly supported smooth function $\varphi \in \mathcal{C}_0^\infty(\Sigma)$,

$$(15) \quad \int_{\Sigma} \left[|\nabla \varphi|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi^2 \right] e^{-f} d\sigma \geq 0.$$

Observe that any closed hypersurface cannot be L_f -stable. This is because the assumption $\overline{\text{Ric}}_f \geq k > 0$ implies that (15) cannot hold for $\varphi \equiv c$ on Σ . Hence, Σ must be noncompact.

Let η be a nonnegative smooth function on $[0, \infty)$ satisfying

$$\eta(s) = \begin{cases} 1 & \text{if } s \in [0, 1) \\ 0 & \text{if } s \in [2, \infty) \end{cases}$$

and $|\eta'| \leq 2$.

Fix a point $p \in \Sigma$ and let $r(x) = \text{dist}_{\Sigma}(p, x)$ denote the (intrinsic) distance function on Σ . Define a sequence of functions $\varphi_j(x) = \eta(\frac{r(x)}{j})$, $j \geq 1$. Then

$|\nabla\varphi_j|^2 \leq 1$ for $j \geq 2$. Substituting $\varphi_j, j \geq 2$ for φ in (15):

$$\begin{aligned} & \int_{\Sigma} \left[|\nabla\varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu))\varphi_j^2 \right] e^{-f} d\sigma \\ & \leq \int_{\Sigma} (|\nabla\varphi_j|^2 - k\varphi_j^2) e^{-f} d\sigma \\ & = \int_{B_{2j}^{\Sigma}(p) \setminus B_j^{\Sigma}(p)} |\nabla\varphi_j|^2 e^{-f} d\sigma - \int_{B_{2j}^{\Sigma}(p)} k\varphi_j^2 e^{-f} d\sigma \\ & \leq \int_{B_{2j}^{\Sigma}(p) \setminus B_j^{\Sigma}(p)} e^{-f} d\sigma - k \int_{B_{2j}^{\Sigma}(p)} \varphi_j^2 e^{-f} d\sigma \\ & \leq \int_{B_{2j}^{\Sigma}(p) \setminus B_j^{\Sigma}(p)} e^{-f} d\sigma - k \int_{B_2^{\Sigma}(p)} e^{-f} d\sigma, \end{aligned}$$

where $B_j^{\Sigma}(p)$ is the intrinsic geodesic ball in M of radius j centered at p . Since Σ has finite weighted volume, we have, when $j \rightarrow \infty$,

$$\int_{B_{2j}^{\Sigma}(p) \setminus B_j^{\Sigma}(p)} e^{-f} d\sigma \rightarrow 0.$$

Choosing j large enough, we have that φ_j satisfies

$$\int_{\Sigma} (|\nabla\varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu))\varphi_j^2) e^{-f} d\sigma < -\frac{k}{2} \int_{B_2^{\Sigma}(p)} e^{-f} d\sigma < 0.$$

This contradicts the fact that Σ is L_f -stable. $\square$

5. COMPACTNESS OF COMPLETE f -MINIMAL SURFACES

Before proving Theorem 2, we give some facts.

Wei-Wylie ([13], Theorem 7.3) used the mean curvature comparison theorem to give a distance estimate for two compact hypersurfaces Σ_1 and Σ_2 in a smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$ with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. Observe that for two complete properly immersed hypersurfaces Σ_1 and Σ_2 , if at least one of them is compact, there is a minimizing geodesic segment joining Σ_1 and Σ_2 and realizing their distance. Hence the proof of Theorem 7.3 [13] can be applied to obtain the following.

Proposition 6. *Let $(M, \bar{g}, e^{-f} d\mu)$ be an $(n+1)$ -dimensional complete smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where k is a positive constant. If Σ_1 and Σ_2 are two complete properly immersed hypersurfaces, at least one of which is compact, then the distance $d(\Sigma_1, \Sigma_2)$ satisfies*

$$(16) \quad d(\Sigma_1, \Sigma_2) \leq \frac{1}{k} (\max_{x \in \Sigma_1} |H_f^{\Sigma_1}(x)| + \max_{x \in \Sigma_2} |H_f^{\Sigma_2}(x)|),$$

where $H_f^{\Sigma_i}, i = 1, 2$, denotes the weighted mean curvatures of Σ_i respectively.

Corollary 2. *Let $(M, \bar{g}, e^{-f} d\mu)$ be as in Proposition 6. Then there is a closed ball $\overline{B}^M$ of M satisfying that any complete properly immersed f -minimal hypersurface Σ must intersect it.*

Proof. Fix $p \in M$ and a geodesic sphere $S_r^M(p)$ of M . By Proposition 6,

$$d(S_r^M(p), \Sigma) \leq \frac{1}{k} \max_x |H_f^{S_r^M(p)}(x)| = C,$$

where C is independent of Σ . Therefore there is a closed ball $\overline{B}^M$ of M with radius big enough so that any Σ must intersect it. $\square$

We need the following fact:

Proposition 7. *Let M be a simply connected Riemannian manifold. If a hypersurface Σ is complete, not necessarily connected, properly embedded, and has no boundary, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

Proof. Suppose Σ_j is a component of Σ . By contrast, $M \setminus \Sigma_j$ has one component. Since Σ is a properly embedded f -minimal hypersurface, for any $p \in \Sigma_j$ there is a neighborhood W of p in M so that $W \cap \Sigma_j = W \cap \Sigma$ only has one piece (i.e. it is a graph above a connected domain in the tangent plane of p). Thus we have a simply closed curve γ passing p , transversal to Σ_j at p , and $\Sigma_j \cap \gamma = p$. Since M is simply connected, we have a disk D with the boundary γ . Again since Σ is proper, the intersection of Σ_j with $\partial D = \gamma$ cannot be one point, which is a contradiction. $\square$

Combining Proposition 3 in Section 3 with Proposition 7, we obtain

Proposition 8. *Let $(M, \overline{g}, e^{-f} d\mu)$ be a simply connected complete smooth metric measure space. If a complete f -minimal hypersurface has finite weighted volume, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

We will take the same approach as in Colding-Minicozzi's paper [4] to prove Theorem 2, a smooth compactness theorem for complete f -minimal surfaces. First we recall a well known local singular compactness theorem for embedded minimal surfaces in a Riemannian 3-manifold.

Proposition 9 (cf. [4], Proposition 2.1). *Given a point p in a Riemannian 3-manifold M , there exists an $R > 0$ such that the following holds: Let Σ_j be embedded minimal surfaces in $B_{2R}(p) \subset M$ with $\partial \Sigma_j \subset \partial B_{2R}(p)$. If each Σ_j has area at most V and genus at most g for some fixed V, g , then there exist a finite collection of points x_k , a smooth embedded minimal surface $\Sigma \subset B_R(p)$ with $\partial \Sigma \subset \partial B_R(p)$ and a subsequence of $\{\Sigma_j\}$ that converges in $B_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.*

Here and in the following, we denote by B_R the ball B_R^M in M for simplicity.

It is known that Σ is f -minimal with respect to metric $\overline{g}$ if and only if Σ is minimal with the conformal metric $\tilde{g} = e^{-f} g$ (see Appendix). Using this fact and applying Proposition 9, we may prove a global singular compactness theorem for f -minimal surfaces.

Proposition 10. *Let M be a complete 3-manifold and $(M, \overline{g}, e^{-f} d\mu)$ a smooth metric measure space. Suppose that $\Sigma_i \subset M$ is a sequence of smooth complete embedded f -minimal surfaces with genus at most g , without boundary, and with weighted area at most V , i.e.*

$$(17) \quad \int_{\Sigma_i} e^{-f} d\sigma \leq V < \infty.$$

Then there are a subsequence, still denoted by Σ_i , a smooth embedded complete non-trivial f -minimal surface $\Sigma \subset M$ without boundary, and a locally finite collection

of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with multiplicity) to Σ off of $\mathcal{S}$. Moreover, Σ satisfies $\int_{\Sigma} e^{-f} d\sigma \leq V$ and is properly embedded.

Here a set $\mathcal{S} \subset M$ is said to be locally finite if $B_R(p) \cap \mathcal{S}$ is finite for every $p \in M$ and for all $R > 0$.

Proof. Consider the conformal metric $\tilde{g} = e^{-f}\bar{g}$ on M . For a point $p \in M$, let $\tilde{B}_{2R}(p) \subset M$ denote the ball in $(M, \tilde{g})$ of radius $2R$ centered at p . Then the area of $\tilde{B}_{2R}(p) \cap \Sigma_j$ satisfies

$$(18) \quad \widetilde{\text{Area}}(\tilde{B}_{2R}(p) \cap \Sigma_j) \leq \int_{\Sigma_j} d\tilde{\sigma} = \int_{\Sigma_j} e^{-f} d\sigma \leq V.$$

Also, it is clear that the genus of $\tilde{B}_{2R}(p) \cap \Sigma_j$ remains at most g . Then by Proposition 9, there exist an $R > 0$ and a finite collection of points x_k , a smooth embedded minimal surface $\Sigma \subset \tilde{B}_R(p)$, with $\partial\Sigma \subset \partial\tilde{B}_R$ and a subsequence of $\{\Sigma_j\}$ that converges in $\tilde{B}_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.

Let $\{\tilde{B}_{R_i}(p_i)\}$ be a countable cover of $(M, \tilde{g})$ of small balls such that $\{\tilde{B}_{2R_i}(p_i)\}$ is still a cover of $(M, \tilde{g})$. On each $\tilde{B}_{2R_i}(p_i)$, applying the previous local convergence and then passing to a diagonal subsequence, we obtain that there are a subsequence of Σ_i , still denoted by Σ_i , a smooth embedded minimal surface Σ (with respect to the metric $\tilde{g}$) without boundary, and a locally finite collection of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with multiplicity) to Σ off of $\mathcal{S}$. Since Σ has no boundary, it is complete in the original metric $\bar{g}$. Thus we obtain the smooth convergence of the subsequence to the smooth embedded complete f -minimal surface Σ off of $\mathcal{S}$.

By Corollary 2, Σ is nontrivial. The convergence of Σ_i to Σ and (17) imply $\int_{\Sigma} e^{-f} d\sigma \leq V$. By Proposition 3, Σ is properly embedded. $\square$

We need to show that the convergence is smooth across the points in $\mathcal{S}$. To prove this, we need the following.

Proposition 11. *Assume that the ambient manifold M in Proposition 10 is simply connected. If the convergence of the sequence $\{\Sigma_i\}$ has multiplicity greater than one, then Σ is L_f -stable.*

Proof. By Proposition 8, we know that Σ_i and Σ are orientable. We may have two ways to prove the proposition. The first is to use the known fact on minimal surfaces. It is known that (cf. [6], Appendix A) if the multiplicity of the convergence of a sequence of embedded orientable minimal surfaces in a simply connected 3-manifold is not one, then the limit minimal surface is stable. Under the conformal metric $\tilde{g}$, a sequence $\{\Sigma_i\}$ of minimal surfaces converges to a smooth embedded orientable minimal surface Σ and thus Σ is stable. Also, the conclusion that Σ is stable with respect to the conformal metric $\tilde{g}$ is equivalent to saying that Σ is L_f -stable under the original metric $\bar{g}$ (see Appendix).

The second way is to prove it directly. We may prove that L_f is the linearization of the f -minimal equation by a proof similar to the one in [4], Appendix A. By arguing as in Proposition 3.2 in [4], we can find a smooth positive function u on Σ satisfying

$$(19) \quad L_f u = 0.$$

This implies that Σ is L_f -stable. $\square$

Proof of Theorem 2. By the assumption on $\overline{\text{Ric}}_f$ and Proposition 1, M has finite fundamental group. After passing to the universal covering, we may assume that M is simply connected. Given a sequence of smooth complete embedded f -minimal surfaces $\{\Sigma_i\}$ with genus g , $\partial\Sigma_i = \emptyset$, and the weighted area at most V , by Proposition 10 there is a subsequence, still denoted by $\{\Sigma_i\}$, that converges in the topology of smooth convergence on compact subsets to a smooth embedded complete f -minimal surface Σ away from a locally finite set $\mathcal{S} \subset \Sigma$ (possibly with multiplicity). Moreover, the limit surface $\Sigma \subset M$ is complete, properly embedded, $\int_{\Sigma} e^{-f} d\sigma \leq V$, has no boundary and has a well-defined unit normal ν . We also have the equivalent convergence under the conformal metric $\bar{g}$.

If $\mathcal{S}$ is not empty, Allard's regularity theorem implies that the convergence has multiplicity greater than one. Then by Proposition 11, we conclude that Σ is L_f -stable. But Proposition 5 says that there is no such Σ . This contradiction implies that $\mathcal{S}$ must be empty. We complete the proof of the theorem. $\square$

Remark 1. For self-shrinkers, the condition that the scale-invariant uniform area bound exists (i.e. there is a uniform bound V_1 : $\text{Area}(B_R(x_0) \cap \Sigma) \leq V_1 R^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$) implies that the uniform bound V of weighted area (i.e. $\int_{\Sigma} e^{-f} d\sigma < V$) exists (cf. the proof of Proposition 5). The converse is also true by the conclusion that the entropy of a self-shrinker can be achieved by $F_{0,1}$ for self-shrinkers with polynomial volume growth (see Section 7 of [5]). Therefore Theorem 2 generalizes the result of Colding-Minicozzi (Theorem 1) for self-shrinkers.

Remark 2. Combining Theorem 2 with the upper bound estimate of weighted area for closed embedded f -minimal surfaces of fixed genus in a complete 3-manifold with $\overline{\text{Ric}}_f \geq k > 0$, we may obtain the smooth compactness theorem for the space of closed embedded f -minimal surfaces of fixed topological type and with diameter bound. We discuss it in [2].

APPENDIX

In this appendix, we discuss the L_f -stability properties of f -submanifolds. With the same notation as in Section 2, let $(M^m, \bar{g})$ be an m -dimensional Riemannian manifold and $i : \Sigma^n \rightarrow M^m$, $n < m$, be an immersion. Let $\tilde{g} = e^{-\frac{2}{n}f} \bar{g}$ denote the new conformal metric on M . Therefore i may induce two isometric immersions of Σ : $(\Sigma, \bar{g}) \rightarrow (M, \bar{g})$ and $(\Sigma, \tilde{g}) \rightarrow (M, \tilde{g})$ respectively.

When $(\Sigma, \tilde{g})$ is minimal, it is well known that the second variation of the volume of $(\Sigma, \tilde{g})$ is given by

Proposition 12 (cf. [6]). *Let $(\Sigma, \tilde{g})$ be a minimal submanifold in $(M, \tilde{g})$. If T is a normal compactly supported variational vector field on Σ (that is, $T = T^{\perp}$), then the second variational formula of the volume $\tilde{V}$ of $(\Sigma, \tilde{g})$ is given by*

$$(20) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \langle T, JT \rangle_{\tilde{g}} d\tilde{\sigma},$$

where the stability operator (or Jacobi operator) J is defined on a normal vector field T to Σ by

$$(21) \quad JT = \Delta_{(\Sigma, \tilde{g})}^{\perp} T + \text{tr}_{(\Sigma, \tilde{g})} [\widetilde{Rm}(\cdot, T) \cdot]^{\perp} + \tilde{B}(T).$$

Here $\Delta_{(\Sigma, \bar{g})}^\perp T = \sum_{i=1}^n (\nabla_{\tilde{e}_i}^\perp \nabla_{\tilde{e}_i}^\perp T - \nabla_{\tilde{\nabla}_{\tilde{e}_i}^\perp \tilde{e}_i}^\perp T)$ is the Laplacian determined by the normal connection $\nabla^\perp$ of $(\Sigma, \bar{g})$, $\widetilde{Rm}$ is the curvature tensor on $(M, \bar{g})$, $\text{tr}_{(\Sigma, \bar{g})}[\widetilde{Rm}(\cdot, T)]^\perp = \sum_{i=1}^n [\widetilde{Rm}(\tilde{e}_i, T)\tilde{e}_i]^\perp$, $\tilde{A}$ denotes the second fundamental form of $(\Sigma, \bar{g})$, $\tilde{B}(T) = \sum_{i,j=1}^n \langle \tilde{A}(\tilde{e}_i, \tilde{e}_j), T \rangle \tilde{A}(\tilde{e}_i, \tilde{e}_j)$, and $\{\tilde{e}_i\}$, $i = 1, \dots, n$, is a local orthonormal base of $(\Sigma, \bar{g})$.

Recall that the weighted volume of $(\Sigma, \bar{g})$ is defined by

$$(22) \quad V_f(\Sigma) = \int_{\Sigma} e^{-f} d\sigma.$$

By a direct computation similar to that of (20), we may prove the second variation formula of the weighted volume of f -minimal submanifold $(\Sigma, \bar{g})$.

Definition 5. For any normal vector field T on $(\Sigma, \bar{g})$, the second order operator $\Delta_f^\perp$ is defined by

$$\begin{aligned} \Delta_f^\perp T &:= \Delta^\perp T - \text{tr}[\nabla f \otimes \nabla^\perp T(\cdot, \cdot)] \\ &= \sum_{i=1}^n (\nabla_{e_i}^\perp \nabla_{e_i}^\perp T - \nabla_{\tilde{\nabla}_{e_i}^\perp e_i}^\perp T) - \sum_{i=1}^n (e_i f)(\nabla_{e_i}^\perp T). \end{aligned}$$

The operator L_f on $(\Sigma, \bar{g})$ is defined by

$$(23) \quad L_f T = \Delta_f^\perp T + R(T) + B(T) + F(T).$$

In the above, $\nabla^\perp$ denotes the normal connection of $(\Sigma, \bar{g})$; $\{e_i\}$, $i = 1, \dots, n$, is a local orthonormal base of $(\Sigma, \bar{g})$; $B(T) = \sum_{i,j=1}^n \langle A(e_i, e_j), T \rangle A(e_i, e_j)$, where A denotes the second fundamental form of $(\Sigma, \bar{g})$; $R(T) = \text{tr}_{(\Sigma, \bar{g})}[\overline{Rm}(\cdot, T)]^\perp = \sum_{i=1}^n [\overline{Rm}(e_i, T)e_i]^\perp$, where $\overline{Rm}$ denotes the Riemannian curvature tensor of $(M, \bar{g})$; and $F(T) = [\tilde{\nabla}^2 f(T)]^\perp = \sum_{\alpha=n+1}^m \tilde{\nabla}^2 f(T, e_\alpha) e_\alpha$, where $\{e_\alpha\}$, $\alpha = n+1, \dots, m$, is a local orthonormal normal vector field on $(\Sigma, \bar{g})$.

Proposition 13. Let $(\Sigma, \bar{g})$ be an f -minimal submanifold in $(M, \bar{g})$. If T is a normal compactly supported variational vector field on Σ (that is, $T = T^\perp$), then the second variation of the weighted volume of $(\Sigma, \bar{g})$ is given by

$$(24) \quad \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

Proof. Let $\psi(\cdot, t)$, $t \in (-\varepsilon, \varepsilon)$ be a compactly supported variation of Σ so that $T = d\psi(\frac{\partial}{\partial t})$ is the variational vector field, $\Sigma_t = \psi(\Sigma, t)$, $\Sigma_0 = \Sigma$. Choose a normal coordinate system $\{x_1, \dots, x_n\}$ at a point $p \in \Sigma$. We can consider $\{x_1, \dots, x_n, t\}$ to be a coordinate system of $\Sigma \times (-\varepsilon, \varepsilon)$ near the point $(p, 0)$. Denote $e_i = d\psi(\frac{\partial}{\partial x_i})$ for $i = 1, \dots, n$. The induced metric on Σ_t from $(M, \bar{g})$ is given for $g_{ij} = \langle e_i, e_j \rangle$.

Hence $g_{ij}(p, 0) = \delta_{ij}$ and $\nabla_{e_i} e_j(p, 0) = 0$. Denote by $d\sigma_t$ the volume element of Σ_t . Then $d\sigma_t = J(x, t)d\sigma_0$, where $d\sigma_0 = d\sigma$ and the function $J(x, t)$ is given by

$$J(x, t) = \frac{\sqrt{G(x, t)}}{\sqrt{G(x, 0)}},$$

with $G(x, t) = \det(g_{ij}(x, t))$. Denote by $d(\sigma_f)_t$ the weighted volume element of Σ_t . Then $d(\sigma_f)_t = J_f(x, t)d\sigma_0$, where $J_f(x, t) = J(x, t)e^{-f(x, t)}$, $f(x, t) = f(\psi(x, t))$.

Since $\frac{\partial J}{\partial t} = \sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{e_i} T, e_j \rangle J$, $\frac{\partial J_f}{\partial t} = (\sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{e_i} T, e_j \rangle - \langle \bar{\nabla} f, T \rangle) J_f$. Note that T is a normal vector field. A direct computation gives, at $(p, 0)$,

$$\begin{aligned} \frac{\partial^2 J_f}{\partial t^2} \Big|_{t=0} &= \left[-2 \sum_{i,j=1}^n \langle A_{ij}, T \rangle^2 + \langle \bar{R}(e_i, T)T, e_i \rangle \right. \\ &\quad + \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla}_T T, e_i \rangle + \sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, \bar{\nabla}_{e_i} T \rangle \\ &\quad - \bar{\nabla}^2 f(T, T) - \langle \bar{\nabla} f, \bar{\nabla}_T T \rangle \\ &\quad \left. + \left(\sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, e_i \rangle - \langle \bar{\nabla} f, T \rangle \right) \left(\sum_{j=1}^n \langle \bar{\nabla}_{e_j} T, e_j \rangle - \langle \bar{\nabla} f, T \rangle \right) \right] J_f. \end{aligned}$$

By

$$\begin{aligned} \sum_{i=1}^n \langle \bar{\nabla}_{e_i} T, \bar{\nabla}_{e_i} T \rangle &= \sum_{i,j=1}^n \langle \bar{\nabla}_{e_i} T, e_j \rangle^2 + \sum_{i=1}^n \sum_{\alpha=n+1}^m \langle \bar{\nabla}_{e_i} T, e_\alpha \rangle^2 \\ &= \sum_{i,j=1}^n \langle A_{ij}, T \rangle^2 + \sum_{i=1}^n \langle \nabla_{e_i}^\perp T, \nabla_{e_i}^\perp T \rangle \\ &= |\langle A(\cdot, \cdot), T \rangle|^2 + |\nabla^\perp T|^2 \end{aligned}$$

and $\sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla}_T T, e_i \rangle = \operatorname{div}(\bar{\nabla}_T T)^\top - \langle (\bar{\nabla}_T T)^\perp, \mathbf{H} \rangle$ we have that, at p ,

$$\begin{aligned} \frac{\partial^2 J_f}{\partial t^2} \Big|_{t=0} &= \left[-|\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle \bar{R}(e_i, T)e_i, T \rangle + |\nabla^\perp T|^2 + \operatorname{div}(\bar{\nabla}_T T)^\top \right. \\ &\quad \left. - \langle (\bar{\nabla}_T T)^\perp, \vec{H} \rangle - \bar{\nabla}^2 f(T, T) - \langle \bar{\nabla} f, \bar{\nabla}_T T \rangle + \langle T, \mathbf{H}_f \rangle^2 \right] e^{-f}. \end{aligned}$$

Using $\operatorname{div}(e^{-f}(\bar{\nabla}_T T)^\top) = e^{-f} \operatorname{div}(\bar{\nabla}_T T)^\top - e^{-f} \langle (\bar{\nabla}_T T)^\top, \nabla f \rangle$, we have at p :

$$\begin{aligned} (25) \quad \frac{\partial^2 J_f}{\partial t^2} \Big|_{t=0} &= \left[|\nabla^\perp T|^2 - |\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle \bar{R}(e_i, T)e_i, T \rangle - \bar{\nabla}^2 f(T, T) \right. \\ &\quad \left. - \langle (\bar{\nabla}_T T)^\perp, \mathbf{H}_f \rangle + \langle T, \mathbf{H}_f \rangle^2 \right] e^{-f} + \operatorname{div}(e^{-f}(\bar{\nabla}_T T)^\top). \end{aligned}$$

Observe that the right-hand side of (25) is independent of the choice of coordinates. Hence (25) holds on Σ . By integrating (25) and using the fact that Σ is f -minimal (i.e. $\mathbf{H}_f = 0$), we obtain

$$\begin{aligned} \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} &= \int_{\Sigma} (|\nabla^\perp T|^2 - |\langle A(\cdot, \cdot), T \rangle|^2 - \langle R(T), T \rangle - \bar{\nabla}^2 f(T, T)) e^{-f} d\sigma \\ &= - \int_{\Sigma} \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma \\ &= - \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma. \end{aligned}$$

Substituting $e^{-f}T$ for T in the identity $\int_{\Sigma} |\nabla^\perp T|^2 d\sigma = - \int_{\Sigma} \langle T, \Delta^\perp T \rangle d\sigma$, we have

$$\int_{\Sigma} |\nabla^\perp T|^2 e^{-f} d\sigma = - \int_{\Sigma} \langle T, \Delta_f^\perp T \rangle e^{-f} d\sigma.$$

Thus we have the second variation formula of the weighted volume of Σ :

$$\begin{aligned} \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} &= - \int_{\Sigma} \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma \\ &= - \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma. \end{aligned}$$

□

Definition 6. An f -minimal submanifold $(\Sigma, \bar{g})$ is called L_f -stable if the second variation of the weighted volume of Σ given by (24) is nonnegative for any normal compactly supported variational vector field T on Σ .

Observe that for an f -minimal submanifold Σ and its normal compactly supported variation, it holds that $V_f(\Sigma_t) = \tilde{V}(\Sigma_t)$. Then

$$(26) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0}.$$

By (20), (24), and (26), we have

$$(27) \quad \int_{\Sigma} \langle T, JT \rangle_{\bar{g}} d\tilde{\sigma} = \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

This implies that

$$(28) \quad \int_{\Sigma} e^{-\frac{2f}{n}} \langle T, JT \rangle_{\bar{g}} e^{-f} d\sigma = \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

By (28), the following equality holds.

Corollary 3. For any normal vector field T on Σ ,

$$JT = e^{\frac{2f}{n}} L_f T.$$

The operator L_f corresponds to a symmetric bilinear form $B_f(T, T)$ for the space of normal compactly supported vector fields on Σ :

$$(29) \quad B_f(T, T) := - \int_{\Sigma} \langle T, L_f T \rangle_{\bar{g}} e^{-f} d\sigma.$$

We define the L_f -index, denoted by $L_f\text{-ind}$, of $(\Sigma, \bar{g})$ by the maximum of the dimensions of negative definite subspaces of B_f . Hence $(\Sigma, \bar{g})$ is L_f -stable if and only if its $L_f\text{-ind} = 0$.

On the other hand, for minimal $(\Sigma, \tilde{g})$, it is well known that the stability operator J also defines a symmetric bilinear form $\tilde{B}(T, T)$,

$$(30) \quad \tilde{B}(T, T) := - \int_{\Sigma} \langle T, JT \rangle_{\tilde{g}} d\tilde{\sigma}.$$

There are also the concepts of index and stability of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \tilde{g})$ is stable if and only if the index $\text{ind}(\Sigma, \tilde{g}) = 0$. Since $B_f(T, T) = \tilde{B}(T, T)$, it holds that

Proposition 14. *$L_f\text{-ind}$ of $(\Sigma, \bar{g})$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \bar{g})$ is L_f -stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.*

Now if Σ is a two-sided hypersurface, that is, if there is a globally-defined unit normal ν on $(\Sigma, \bar{g})$, take $T = \varphi\nu$. Then the second variation (24) implies that

Proposition 15. *Let Σ be a two-sided f -minimal hypersurface in $(M^{n+1}, \bar{g})$. If φ is a compactly supported smooth function on Σ , then the second variation of the weighted volume of $(\Sigma, \bar{g})$ is given by*

$$(31) \quad \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \varphi L_f(\varphi) e^{-f} d\sigma,$$

where ν denotes the unit normal of $(\Sigma, \bar{g})$ and the operator L_f is defined by $L_f = \Delta_f + |A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)$.

Definition 7. The operator $L_f = \Delta_f + |A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)$ is called the L_f -stability operator of hypersurface $(\Sigma, \bar{g})$.

A bilinear form on space $C_0^\infty(\Sigma)$ of compactly supported smooth functions on Σ is defined by

$$(32) \quad \begin{aligned} B_f(\varphi, \varphi) &:= - \int_{\Sigma} \varphi L_f \varphi e^{-f} d\sigma \\ &= \int_{\Sigma} [|\nabla \varphi|^2 - (|A|_{\bar{g}}^2 + \overline{\text{Ric}}_f(\nu, \nu)) \varphi^2] e^{-f} d\sigma. \end{aligned}$$

The L_f -index, denoted by $L_f\text{-ind}$, of $(\Sigma, \bar{g})$ is defined to be the maximum of the dimensions of negative definite subspaces of B_f . Hence $(\Sigma, \bar{g})$ is L_f -stable if and only if $L_f\text{-ind} = 0$. Clearly the definition of L_f -index is equivalent to the corresponding definition using the variational vector field T as before.

Also, for minimal hypersurface $i : (\Sigma, \tilde{g}) \rightarrow (M^{n+1}, \tilde{g})$, it is well known that if ψ is a compactly supported smooth function on Σ , then the second variation of the volume $\tilde{V}$ of $(\Sigma, i^*\tilde{g})$ is given by

$$(33) \quad \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0} = - \int_{\Sigma} \psi J(\psi) d\tilde{\sigma},$$

where $\tilde{A}$ denotes the second fundamental form of $(\Sigma, \tilde{g})$, $\tilde{\nu}$ denotes the unit normal of $(\Sigma, \tilde{g})$, and $J = \Delta_{\tilde{g}} + |\tilde{A}|_{\tilde{g}}^2 + \widetilde{\text{Ric}}(\tilde{\nu}, \tilde{\nu})$ is the stability operator (or the Jacobi operator) of $(\Sigma, \tilde{g})$.

The following holds, from (28).

Proposition 16. *Let (Σ^n, g) be an f -minimal hypersurface immersed in $(M, \bar{g})$. Then for all $\varphi \in C_o^\infty(\Sigma)$,*

$$(34) \quad \int_{\Sigma} (e^{-\frac{f}{n}} \varphi) J(e^{-\frac{f}{n}} \varphi) e^{-f} d\sigma = \int_{\Sigma} \varphi L_f(\varphi) e^{-f} d\sigma.$$

Corollary 4. *For $\varphi \in C^\infty(\Sigma)$, $J(e^{-\frac{f}{n}} \varphi) = e^{\frac{f}{n}} L_f(\varphi)$.*

Corollary 5. *L_f -ind of $(\Sigma, \bar{g})$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \bar{g})$ is L_f -stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.*

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