# STABILITY OF DIRICHLET HEAT KERNEL ESTIMATES FOR NON-LOCAL OPERATORS UNDER FEYNMAN-KAC PERTURBATION 

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#### Abstract

In this paper we show that Dirichlet heat kernel estimates for a class of (not necessarily symmetric) Markov processes are stable under nonlocal Feynman-Kac perturbations. This class of processes includes, among others, (reflected) symmetric stable-like processes in closed $d$-sets in $\mathbb{R}^{d}$, killed symmetric stable processes, censored stable processes in $C^{1,1}$ open sets, as well as stable processes with drifts in bounded $C^{1,1}$ open sets. These twosided estimates are explicit involving distance functions to the boundary.


## 1. Introduction

Suppose that $X$ is a Hunt process on a state space $E$ with transition semigroup $\left\{P_{t}: t \geq 0\right\}$. A Feynman-Kac transform of $X$ is given by

$$
T_{t} f(x)=\mathbb{E}_{x}\left[\exp \left(C_{t}\right) f\left(X_{t}\right)\right],
$$

where $C_{t}$ is an additive functional of $X$. When $C_{t}$ is a continuous additive functional of $X$, the transform above is called a local Feynman-Kac transform. When $X$ is discontinuous and $C_{t}$ is a discontinuous additive functional of $X$, the transform above is called a non-local Feynman-Kac transform. Feynman-Kac transforms play an important role in the probabilistic as well as analytic aspect of potential theory, and also in mathematical physics; see, for instance, [12 15, 26, 27] and the references therein. Most of the literature on Feynman-Kac semigroups are about local Feynman-Kac semigroups. We refer the reader to [15,27 for nice accounts on local Feynman-Kac semigroups of Brownian motion. Non-local Feynman-Kac transforms are also very important in various applications. For example, it is shown in 13 that the killed relativistic $\alpha$-stable process in any bounded $C^{1,1}$ open set $D$ can be obtained from the killed symmetric $\alpha$-stable process in $D$ via non-local FeynmanKac transforms. (See 19, 20 for some extension to more general open sets and more general processes.)

[^0]An important question related to Feynman-Kac transforms is the stability of various properties. This type of question has received intensive study in recent years. For instance, it is shown in [1,27] that under a certain Kato class condition, the integral kernel (also called the heat kernel) of a local Feynman-Kac semigroup of Brownian motion admits two-sided Gaussian bound estimates. In [22], sharp twosided estimates on the densities of (local) Feynman-Kac semigroups of killed Brownian motions in $C^{1,1}$ domains were established. Non-local Feynman-Kac semigroups for symmetric stable processes and their associated quadratic forms were studied in [28,29]. By combining some ideas from [32] with results from [11], it was proved in 30 that, under a certain Kato class condition, the heat kernel of the non-local Feynman-Kac semigroup of a symmetric stable-like process $X$ on $\mathbb{R}^{d}$ is comparable to that of $X$. The symmetry condition on $F(x, y)$ plays an essential role in the argument of [30]. The non-symmetric pure jump case for stable-like processes is dealt with in [31. For recent development in the study of non-local Feynman-Kac transforms for general symmetric Markov processes, we refer the reader to [4,5] and the references therein. We also mention that the stability of Martin boundary under non-local Feynman-Kac perturbation is addressed in 6].

Recently, sharp two-sided Dirichlet heat kernel estimates have been obtained for several classes of discontinuous processes (or non-local operators), including symmetric stable processes [7], censored stable processes [8], relativistic stable processes [9], and stable processes with drifts [10]. The main purpose of this paper is to study the stability of Dirichlet heat kernel estimates under the following non-local Feynman-Kac transform:

$$
T_{t} f(x)=\mathbb{E}_{x}\left[\exp \left(A_{t}+\sum_{s \leq t} F\left(X_{s-}, X_{s}\right)\right) f\left(X_{t}\right)\right]
$$

where $A$ is a continuous additive functional of $X$ having finite variations on each compact time interval and $F(x, y)$ is a measurable function that vanishes along the diagonal. The approach of this paper is quite robust so that it applies to a class of not necessarily symmetric Markov processes that includes all of the four families of processes mentioned above in bounded $C^{1,1}$ open sets.

To the best of the authors' knowledge, Dirichlet heat kernel estimates for (either local or non-local) Feynman-Kac semigroups of discontinuous processes are studied here for the first time. The main challenge in studying Dirichlet heat kernel estimates of Feynman-Kac semigroups is to get the exact boundary decay rate of the heat kernels. While our main interest is in the Dirichlet heat kernel estimates for Feynman-Kac semigroups, our theorem also covers the whole space case as well as "reflected" stable-like processes on subsets of $\mathbb{R}^{d}$. In particular, our result recovers and extends the main results of [30, 31 where $D=\mathbb{R}^{d}$. Even in the whole space case, our approach is different from those in [30, 31].
1.1. Setup and main result. In this paper we always assume that $\alpha \in(0,2)$, $d \geq 1, D$ is a Borel set in $\mathbb{R}^{d}$. For $x \in D, \delta_{D}(x)$ denotes the Euclidean distance between $x$ and $D^{c}$. We use " $:=$ " to denote a definition, which is read as "is defined to be". For $a, b \in \mathbb{R}, a \wedge b:=\min \{a, b\}$ and $a \vee b:=\max \{a, b\}$. The Euclidean distance between $x$ and $y$ is denoted as $|x-y|$.

For $\gamma \geq 0$, we define

$$
\begin{gather*}
q(t, x, y):=t^{-d / \alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \quad t>0, x, y \in \mathbb{R}^{d}  \tag{1.1}\\
\psi_{\gamma}(t, x):=\left(1 \wedge \frac{\delta_{D}(x)}{t^{1 / \alpha}}\right)^{\gamma}, \quad t>0, x \in D
\end{gather*}
$$

and

$$
\begin{equation*}
q_{\gamma}(t, x, y):=\psi_{\gamma}(t, x) \psi_{\gamma}(t, y) q(t, x, y), \quad t>0, x, y \in D \tag{1.2}
\end{equation*}
$$

Throughout this paper, $X$ is a Hunt process on $D$ with transition semigroup $\left\{P_{t}\right.$ : $t \geq 0\}$ that admits a jointly continuous transition density $p_{D}(t, x, y)$ with respect to the Lebesgue measure, and there exist $C_{0} \geq 1$ and $\gamma \in[0, \alpha \wedge d)$ such that

$$
\begin{equation*}
C_{0}^{-1} q_{\gamma}(t, x, y) \leq p_{D}(t, x, y) \leq C_{0} q_{\gamma}(t, x, y) \tag{1.3}
\end{equation*}
$$

for all $(t, x, y) \in(0,1] \times D \times D$. It is easy to see that under this assumption, $X$ is a Feller process satisfying the strong Feller property. Note that $q(t, x, y)$ is comparable to the transition density of symmetric $\alpha$-stable processes in $\mathbb{R}^{d}$. So by increasing the value of $C_{0}$ if necessary, we have

$$
\begin{equation*}
C_{0}^{-1} \leq \int_{\mathbb{R}^{d}} q(t, x, y) d y \leq C_{0} \quad \text { for all }(t, x) \in(0, \infty) \times \mathbb{R}^{d} \tag{1.4}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\int_{D} p_{D}(t, x, y) d y \leq C_{0}^{2} \psi_{\gamma}(t, x) \quad \text { for all }(t, x) \in(0,1] \times D . \tag{1.5}
\end{equation*}
$$

We remark that the process $X$ may be non-symmetric. We assume that $X$ has a Lévy system $(N, t)$, where $N=N(x, d y)$ is a kernel given by

$$
N(x, d y)=\frac{c(x, y)}{|x-y|^{d+\alpha}} d y
$$

with $c(x, y)$ a measurable function that is bounded between two positive constants on $D \times D$. That is, for any $x \in D$, any stopping time $T$ (with respect to the filtration of $X$ ) and any non-negative measurable function $f$ on $[0, \infty) \times D \times D$ with $f(s, y, y)=0$ for all $y \in D$ and $s \geq 0$ that is extended to be zero off $D \times D$,

$$
\begin{equation*}
\mathbb{E}_{x}\left[\sum_{s \leq T} f\left(s, X_{s-}, X_{s}\right)\right]=\mathbb{E}_{x}\left[\int_{0}^{T}\left(\int_{D} f\left(s, X_{s}, y\right) \frac{c\left(X_{s}, y\right)}{\left|X_{s}-y\right|^{d+\alpha}} d y\right) d s\right] . \tag{1.6}
\end{equation*}
$$

By increasing the value of $C_{0}$ if necessary, we may and do assume that

$$
\begin{equation*}
1 / C_{0} \leq c(x, y) \leq C_{0} \quad \text { for } x, y \in D \tag{1.7}
\end{equation*}
$$

Recall that an open set $D$ in $\mathbb{R}^{d}$ (when $d \geq 2$ ) is said to be a $C^{1,1}$ open set if there exist a localization radius $r_{0}>0$ and a constant $\Lambda_{0}>0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi=\phi_{z}: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\phi(0)=0, \nabla \phi(0)=(0, \ldots, 0),\|\nabla \phi\|_{\infty} \leq \Lambda_{0},|\nabla \phi(x)-\nabla \phi(w)| \leq \Lambda_{0}|x-w|$, and an orthonormal coordinate system $y=\left(y_{1}, \cdots, y_{d-1}, y_{d}\right):=\left(\widetilde{y}, y_{d}\right)$ such that $B\left(z, r_{0}\right) \cap D=B\left(z, r_{0}\right) \cap\left\{y: y_{d}>\phi(\widetilde{y})\right\}$. We call the pair $\left(r_{0}, \Lambda_{0}\right)$ the characteristics of the $C^{1,1}$ open set $D$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

It follows from [7,8, 10, 11] that the following are true:
(i) the (reflected) symmetric stable-like process in any closed $d$-subset $D$ in $\mathbb{R}^{d}$ (see Subsection 4.1 for the definition of $d$-set) satisfies the conditions (1.3) and (1.6) with $\gamma=0$ and $c(x, y)$ a symmetric measurable function that is bounded between two positive constants;
(ii) the killed symmetric $\alpha$-stable process in any $C^{1,1}$ open set $D$ satisfies the conditions (1.3) and (1.6) with $\gamma=\alpha / 2$ and $c(x, y) \equiv c$;
(iii) when $d \geq 2$ and $\alpha \in(1,2)$, the killed symmetric $\alpha$-stable process with drift in any bounded $C^{1,1}$ open set $D$ satisfies the conditions (1.3) and (1.6) with $\gamma=\alpha / 2$ and $c(x, y) \equiv c$; and
(iv) when $\alpha \in(1,2)$, the censored $\alpha$-stable process in any $C^{1,1}$ open set $D$ satisfies the conditions (1.3) and (1.6) with $\gamma=\alpha-1$ and $c(x, y) \equiv c$.
By a locally finite signed measure $\mu$ we mean in this paper the difference of two non-negative $\sigma$-finite measures $\mu_{1}$ and $\mu_{2}$ in $D$. We point out that $\mu=\mu_{1}-\mu_{2}$ may not be a signed measure in $D$ in the usual sense as both $\mu_{1}(D)$ and $\mu_{2}(D)$ may be infinite. However, there is an increasing sequence of subsets $\left\{F_{k}, k \geq 1\right\}$ whose union is $D$ so that $\mu_{1}\left(F_{k}\right)+\mu_{2}\left(F_{k}\right)<\infty$ for every $k \geq 1$. So when restricted to each $F_{k}, \mu$ is a finite signed measure. Consequently, the positive and negative parts of $\mu$ are well defined on each $F_{k}$ and hence on $D$, which will be denoted as $\mu^{+}$and $\mu^{-}$, respectively. We use $|\mu|=\mu^{+}+\mu^{-}$to denote the total variation measure of $\mu$.

For a locally finite signed measure $\mu$ on $D$ and $t>0$, we define

$$
N_{\mu}^{\alpha, \gamma}(t)=\sup _{x \in D} \int_{0}^{t} \int_{D} \psi_{\gamma}(s, y) q(s, x, y)|\mu|(d y) d s
$$

Definition 1.1. A locally finite signed measure $\mu$ on $D$ is said to be in the Kato class $\mathbf{K}_{\alpha, \gamma}$ if $\lim _{t \downarrow 0} N_{\mu}^{\alpha, \gamma}(t)=0$.

Note that if $N_{\mu}^{\alpha, \gamma}(t)<\infty$ for some $t>0$, then $|\mu|$ is a Radon measure on $D$. We say that a measurable function $g$ belongs to the Kato class $\mathbf{K}_{\alpha, \gamma}$ if $g(x) d x \in \mathbf{K}_{\alpha, \gamma}$ and we denote $N_{g(x) d x}^{\alpha, \gamma}$ by $N_{g}^{\alpha, \gamma}$. It is well-known that any $\mu \in \mathbf{K}_{\alpha, \gamma}$ is a smooth measure in the sense of [17]. Moreover, using the fact that $X$ has a transition density under each $\mathbb{P}_{x}$, one can show that the continuous additive functional $A_{t}^{\mu}$ of $X$ with Revuz measure $\mu \in \mathbf{K}_{\alpha, \gamma}$ can be defined without exceptional set; see [18, pp. 236237] for details. Concrete conditions for $\mu \in \mathbf{K}_{\alpha, \gamma}$ are given in Proposition 4.1

For any measurable function $F$ on $D \times D$ vanishing on the diagonal, we define

$$
\begin{aligned}
N_{F}^{\alpha, \gamma}(t):=\sup _{y \in D} \int_{0}^{t} \int_{D \times D} \psi_{\gamma}(s, z) q(s, y, z)(1 & \left.+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-z|}\right)^{\gamma} \\
& \times \frac{|F(z, w)|+|F(w, z)|}{|z-w|^{d+\alpha}} d w d z d s .
\end{aligned}
$$

Definition 1.2. Suppose that $F$ is a measurable function on $D \times D$ vanishing on the diagonal. We say that $F$ belongs to the Kato class $\mathbf{J}_{\alpha, \gamma}$ if $F$ is bounded and $\lim _{t \downarrow 0} N_{F}^{\alpha, \gamma}(t)=0$.

Observe that

$$
N_{F}^{\alpha, \gamma}(t) \geq \int_{0}^{t} \int_{D} \psi_{\gamma}(s, z) q(s, y, z)\left(\int_{D} \frac{|F(z, w)|+|F(w, z)|}{|z-w|^{d+\alpha}} d w\right) d z d s
$$

So if $F(x, y)$ belongs to $F \in \mathbf{J}_{\alpha, \gamma}$, then

$$
\int_{D} \frac{|F(x, y)|+|F(y, x)|}{|x-y|^{d+\alpha}} d y<\infty \quad \text { for a.e. } x \in D
$$

and, as a function of $x$, it belongs to $\mathbf{K}_{\alpha, \gamma}$.
On the other hand, according to Proposition 4.2, a sufficient condition for $F \in$ $\mathbf{J}_{\alpha, \gamma}$ is $|F|(z, w) \leq A\left(|z-w|^{\beta} \wedge 1\right)$ for some $A>0$ and $\beta>\alpha$. This sufficient condition is enough in our applications.

It is easy to check that if $F$ and $G$ belong to $\mathbf{J}_{\alpha, \gamma}$ and $c$ is a constant, then the functions $c F, e^{F}-1, F+G$ and $F G$ all belong to $\mathbf{J}_{\alpha, \gamma}$. Throughout this paper, we will use the following notation: For any given measurable function $F$ on $D \times D$,

$$
\begin{equation*}
F_{1}(x, y):=e^{F(x, y)}-1 \tag{1.8}
\end{equation*}
$$

For any locally finite signed measure $\mu$ on $D$ and any measurable function $F$ on $D \times D$ vanishing on the diagonal, we define

$$
N_{\mu, F}^{\alpha, \gamma}(t):=N_{\mu}^{\alpha, \gamma}(t)+N_{F}^{\alpha, \gamma}(t) .
$$

When $\mu \in \mathbf{K}_{\alpha, \gamma}$ and $F$ is a measurable function with $F_{1} \in \mathbf{J}_{\alpha, \gamma}$, we put

$$
A_{t}^{\mu, F}=A_{t}^{\mu}+\sum_{0<s \leq t} F\left(X_{s-}, X_{s}\right)
$$

Recall that for any non-negative Borel function $f$ on $D, P_{t} f(x)=\mathbb{E}_{x}\left[f\left(X_{t}\right)\right]$. For any non-negative Borel function $f$ on $D$, we define

$$
T_{t}^{\mu, F} f(x)=\mathbb{E}_{x}\left[\exp \left(A_{t}^{\mu, F}\right) f\left(X_{t}\right)\right], \quad t \geq 0, x \in D
$$

Then $\left(T_{t}^{\mu, F}: t \geq 0\right)$ is called the Feynman-Kac semigroup of $X$ corresponding to $\mu$ and $F$. It follows from [12, Remark 1] that, informally, the semigroup ( $T_{t}^{\mu, F}: t \geq 0$ ) has $L^{2}$-infinitesimal generator

$$
\mathcal{A} f(x)=(\mathcal{L}+\mu) f(x)+\int_{D}\left(e^{F(x, y)}-1\right) \frac{c(x, y)}{|x-y|^{d+\alpha}} f(y) d y
$$

where $\mathcal{L}$ is the $L^{2}$-infinitesimal generator of $X$.
The main purpose of this paper is to establish the following result. Recall that $\gamma \geq 0$ and $C_{0} \geq 1$ are the constants in (1.3) and (1.7). For any bounded measurable function $F$ on $D \times D$, we use $\|F\|_{\infty}$ to denote $\|F\|_{L^{\infty}(D \times D)}$.
Theorem 1.3. Let $d \geq 1, \alpha \in(0,2)$ and $\gamma \in[0, \alpha \wedge d)$. Suppose $X$ is a Hunt process in a Borel set $D \subset \mathbb{R}^{d}$ with a jointly continuous transition density $p_{D}(t, x, y)$ satisfying (1.3), (1.6) and (1.7). If $\mu$ is a locally finite signed measure in $\mathbf{K}_{\alpha, \gamma}$ and $F$ is a measurable function so that $F_{1} \in \mathbf{J}_{\alpha, \gamma}$, then the non-local Feynman-Kac semigroup $\left(T_{t}^{\mu, F}: t \geq 0\right)$ has a continuous density $q_{D}(t, x, y)$, and there exists a constant $C=C\left(d, \alpha, \gamma, C_{0}, N_{\mu, F_{1}}^{\alpha, \gamma},\left\|F_{1}\right\|_{\infty}\right)>0$ such that for all $(t, x, y) \in(0, \infty) \times$ $D \times D$,

$$
q_{D}(t, x, y) \leq e^{C t} q_{\gamma}(t, x, y)
$$

If $\mu \in \mathbf{K}_{\alpha, \gamma}$ and $F \in \mathbf{J}_{\alpha, \gamma}$, then there exists $\widetilde{C}=\widetilde{C}\left(d, \alpha, \gamma, C_{0}, N_{\mu, F}^{\alpha, \gamma},\|F\|_{\infty}, T\right) \geq 1$ for every $T>0$ such that for all $(t, x, y) \in(0, T] \times D \times D$,

$$
\widetilde{C}^{-1} q_{\gamma}(t, x, y) \leq q_{D}(t, x, y) \leq \widetilde{C} q_{\gamma}(t, x, y) .
$$

Here and in the sequel, the dependence of the constant $C$ (with or without subscripts) on $N_{\mu, F_{1}}^{\alpha, \gamma}$ and $\left\|F_{1}\right\|_{\infty}$ means that the value of the constant $C$ (with or without subscripts) depends only on the rate at which $N_{\mu, F_{1}}^{\alpha, \gamma}(t)$ goes to zero as $t \rightarrow 0$ and on a specific upper bound for $\left\|F_{1}\right\|_{\infty}$, so does the dependence of the constant on $N_{\mu, F}^{\alpha, \gamma}$ and $\|F\|_{\infty}$. When $D=\mathbb{R}^{d}$ and $\gamma=0$, Theorem 1.3 in particular recovers and extends the main results of 30, 31.
1.2. Approach. In this subsection, we outline the main ideas and the approach of this paper. Assuming the technical results in the next two sections, the contents of this subsection are rigorous.

We first recall the definition of the Stieltjes exponential. If $K_{t}$ is a right continuous function with left limits on $\mathbb{R}_{+}$with $K_{0}=1$ and $\Delta K_{t}:=K_{t}-K_{t-}>-1$ for every $t>0$, and if $K_{t}$ is of finite variation on each compact time interval, then the Stieltjes exponential $\operatorname{Exp}(K)_{t}$ of $K_{t}$ is the unique solution $Z_{t}$ of

$$
Z_{t}=1+\int_{(0, t]} Z_{s-} d K_{s}, \quad t>0
$$

By [24, IV 19] (or [26, (A4.17)]),

$$
\begin{equation*}
\operatorname{Exp}(K)_{t}=e^{K_{t}^{c}} \prod_{0<s \leq t}\left(1+\Delta K_{s}\right) \tag{1.9}
\end{equation*}
$$

where $K_{t}^{c}$ denotes the continuous part of $K_{t}$. Clearly $\exp \left(K_{t}\right) \geq \operatorname{Exp}(K)_{t}$ with the equality holds if and only if $K_{t}$ is continuous. The reason $\operatorname{Exp}(K)_{t}$ is called the Stieltjes exponential of $K_{t}$ is that, by [16, p. 184], $\operatorname{Exp}(K)_{t}$ can be expressed as the following infinite sum of Lebesgue-Stieltjes integrals (recall that $K_{t}$ is of finite variation on each compact time interval):

$$
\begin{equation*}
\operatorname{Exp}(K)_{t}=1+\sum_{n=1}^{\infty} \int_{[0, t]} d K_{s_{n}} \int_{\left[0, s_{n}\right)} d K_{s_{n-1}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}} . \tag{1.10}
\end{equation*}
$$

The advantage of using the Stieltjes exponential $\operatorname{Exp}(K)_{t}$ over the usual exponential $\exp \left(K_{t}\right)$ is the identity (1.10), which allows one to apply the Markov property of $X$.

Recall from (1.8) that $F_{1}(x, y)=e^{F(x, y)}-1$. In view of (1.9), we can express $\exp \left(A_{t}^{\mu, F}\right)$ in terms of the Stieltjes exponential:

$$
\exp \left(A_{t}^{\mu, F}\right)=\operatorname{Exp}\left(A^{\mu}+\sum_{s \leq \cdot} F_{1}\left(X_{s-}, X_{s}\right)\right)_{t} \quad \text { for } t \geq 0
$$

Applying (1.10) with $K_{t}:=A_{t}^{\mu}+\sum_{s \leq t} F_{1}\left(X_{s-}, X_{s}\right)$ and using the Markov property of $X$, we have for any bounded measurable function $f \geq 0$ on $D$,

$$
\begin{align*}
T_{t}^{\mu, F} f(x) & =\mathbb{E}_{x}\left[\exp \left(A_{t}^{\mu, F}\right) f\left(X_{t}\right)\right]=\mathbb{E}_{x}\left[f\left(X_{t}\right) \operatorname{Exp}\left(A^{\mu}+\sum_{s \leq} F_{1}\left(X_{s-}, X_{s}\right)\right)_{t}\right]  \tag{1.11}\\
& =P_{t} f(x)+\mathbb{E}_{x}\left[f\left(X_{t}\right) \sum_{n=1}^{\infty} \int_{[0, t]} d K_{s_{n}} \int_{\left[0, s_{n}\right)} d K_{s_{n-1}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}}\right]
\end{align*}
$$

Using our integral 3P inequalities (Lemma 2.4 and Theorem 2.7 below), we will show in the proof of Theorem 3.4 that for any $\mu \in \mathbf{K}_{\alpha, \gamma}$ and any measurable
function $F$ with $F_{1} \in \mathbf{J}_{\alpha, \gamma}$, we can change the order of the expectation and the infinite sum on the right hand side of (1.11). Hence using the Markov property of $X$ at $s_{n}$, we have for every $t>0$ and bounded measurable function $f \geq 0$ on $D$ that

$$
\begin{align*}
T_{t}^{\mu, F} f(x) & =P_{t} f(x)+\sum_{n=1}^{\infty} \mathbb{E}_{x}\left[f\left(X_{t}\right) \int_{[0, t]} d K_{s_{n}} \int_{\left[0, s_{n}\right)} d K_{s_{n-1}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}}\right]  \tag{1.12}\\
& =P_{t} f(x)+\sum_{n=1}^{\infty} \mathbb{E}_{x}\left[\int_{[0, t]} P_{t-s_{n}} f\left(X_{s_{n}}\right) d K_{s_{n}} \int_{\left[0, s_{n}\right)} d K_{s_{n-1}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}}\right] .
\end{align*}
$$

Let $h_{1}(s)=1$ and $h_{n-1}(s)=\int_{[0, s)} d K_{s_{n-2}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}}$ for $n \geq 3$. When $n \geq 2$, using the Markov property of $X$ we have

$$
\begin{align*}
& \mathbb{E}_{x}\left[\int_{[0, t]} P_{t-s_{n}} f\left(X_{s_{n}}\right) d K_{s_{n}} \int_{\left[0, s_{n}\right)} d K_{s_{n-1}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}}\right]  \tag{1.13}\\
= & \mathbb{E}_{x}\left[\int_{[0, t]}\left(\int_{\left[0, s_{n}\right)} P_{t-s_{n}} f\left(X_{s_{n}}\right) h_{n-1}\left(s_{n-1}\right) d K_{s_{n-1}}\right) d K_{s_{n}}\right] \\
= & \mathbb{E}_{x}\left[\int_{[0, t)}\left(\int_{\left(s_{n-1}, t\right]} P_{t-s_{n}} f\left(X_{s_{n}}\right) d K_{s_{n}}\right) h_{n-1}\left(s_{n-1}\right) d K_{s_{n-1}}\right] \\
= & \mathbb{E}_{x}\left[\int_{[0, t)} \mathbb{E}_{x}\left[\int_{\left(s_{n-1}, t\right]} P_{t-s_{n}} f\left(X_{s_{n}}\right) d K_{s_{n}} \mid \mathcal{F}_{s_{n-1}}\right] h_{n-1}\left(s_{n-1}\right) d K_{s_{n-1}}\right] \\
= & \mathbb{E}_{x}\left[\int_{[0, t)} \mathbb{E}_{X_{s_{n-1}}}\left[\int_{\left(0, t-s_{n-1}\right]} P_{t-s_{n-1}-r} f\left(X_{r}\right) d K_{r}\right] h_{n-1}\left(s_{n-1}\right) d K_{s_{n-1}}\right] \\
= & \mathbb{E}_{x}\left[\int_{[0, t)} \mathbb{E}_{X_{s_{n-1}}}\left[\int_{\left(0, t-s_{n-1}\right]} P_{t-s_{n-1}-r} f\left(X_{r}\right) d K_{r}\right]\right. \\
& \left.\times \int_{\left[0, s_{n-1}\right)} d K_{s_{n-2}} \cdots \int_{\left[0, s_{2}\right)} d K_{s_{1}} d K_{s_{n-1}}\right] .
\end{align*}
$$

Furthermore by (1.6), for any bounded measurable function $g \geq 0$ on $[0, \infty) \times D \times D$,

$$
\begin{align*}
\mathbb{E}_{x} & {\left[\int_{(0, s]} g\left(s-r, X_{r}\right) d K_{r}\right] }  \tag{1.14}\\
= & \mathbb{E}_{x}\left[\int_{(0, s]} g\left(s-r, X_{r}\right) d A_{r}^{\mu}+\sum_{r \leq s} g\left(s-r, X_{r}\right) F_{1}\left(X_{r-}, X_{r}\right)\right] \\
= & \int_{0}^{s} \int_{D} p_{D}(r, x, y) g(s-r, y) \mu(d y) d r \\
& +\mathbb{E}_{x}\left[\int_{0}^{s}\left(\int_{D} F_{1}\left(X_{r}, y\right) g(s-r, y) \frac{c\left(X_{r}, y\right)}{\left|X_{r}-y\right|^{d+\alpha}} d y\right) d r\right] \\
= & \int_{0}^{s} \int_{D} p_{D}(r, x, y) g(s-r, y) \mu(d y) d r \\
& +\int_{0}^{s} \int_{D} p_{D}(r, x, z)\left(\int_{D} F_{1}(z, y) g(s-r, y) \frac{c(z, y)}{|y-z|^{d+\alpha}} d y\right) d z d r
\end{align*}
$$

Equations (1.11)-(1.14) motivate us to define $p^{0}(t, x, y):=p_{D}(t, x, y)$ and, for $k \geq 1$, (1.15)

$$
\begin{aligned}
p^{k}(t, x, y)= & \int_{0}^{t}\left(\int_{D} p_{D}(s, x, z) p^{k-1}(t-s, z, y) \mu(d z)\right) d s \\
& +\int_{0}^{t}\left(\int_{D \times D} p_{D}(s, x, z) \frac{c(z, w) F_{1}(z, w)}{|z-w|^{d+\alpha}} p^{k-1}(t-s, w, y) d z d w\right) d s
\end{aligned}
$$

One then concludes from (1.12) and (1.13) (see the proof of Theorem (3.4) that

$$
T_{t}^{\mu, F} f(x)=\int_{D} q_{D}(t, x, y) f(y) d y, \quad(t, x) \in(0, \infty) \times D
$$

where

$$
\begin{equation*}
q_{D}(t, x, y):=\sum_{k=0}^{\infty} p^{k}(t, x, y), \quad(t, x, y) \in(0, \infty) \times D \times D \tag{1.16}
\end{equation*}
$$

Moreover (see (3.10)), there exist constants $t_{1}>0, c>0$ and $0<\lambda<1$ such that (1.17) $\left|p^{k}(t, x, y)\right| \leq\left(\lambda^{k}+c k \lambda^{k-1}\right) p_{D}(t, x, y) \quad$ on $\left(0, t_{1}\right] \times D \times D$ for every $k \geq 1$.

From this we deduce that for every $t \in\left(0, t_{1}\right]$,

$$
\begin{equation*}
q_{D}(t, x, y)=\sum_{k=0}^{\infty} p^{k}(t, x, y) \leq\left(\frac{1}{1-\lambda}+\frac{c}{(1-\lambda)^{2}}\right) p_{D}(t, x, y) \tag{1.18}
\end{equation*}
$$

For the lower bound estimate under the assumption $F \in \mathbf{J}_{\alpha, \gamma}$, we use (1.17) for $k=1$ and deduce that

$$
q_{D}(t, x, y) \geq 2^{-2(\lambda+c)} p_{D}(t, x, y) .
$$

(See Theorem 3.5 and its proof below.) This and (1.18) establish Theorem 1.3 for $t \leq t_{1}$. The general case of $t \leq T$ follows from an application of the ChapmanKolmogorov equation.

The keys to establish the estimate (1.17) are two integral forms of the 3 P inequality given in Lemma 2.4 and Theorem [2.7below. For a killed Brownian motion in a smooth domain, the following form of 3P inequality is known (see [23, Lemma 3.1]): for any $0<c<a \wedge(b-a)$, there exists $N=N(a, b, c)>0$ such that for every $0<s<t$ and $x, y, z \in D$,

$$
\begin{equation*}
\frac{p_{b}^{W}(s, x, z) p_{a}^{W}(t-s, z, y)}{p_{a}^{W}(t, x, y)} \leq N \frac{\delta_{D}(z)}{\delta_{D}(x)} p_{c}^{W}(s, x, z)+N \frac{\delta_{D}(z)}{\delta_{D}(y)} p_{c}^{W}(t-s, z, y) \tag{1.19}
\end{equation*}
$$

where $p_{c}^{W}(t, x, y):=\psi_{1}(t, x) \psi_{1}(t, y) t^{-d / 2} e^{-c|x-y|^{2} / t}$. Recall that when $D$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^{d}$, the transition density $p_{D}(t, x, y)$ of the killed Brownian motion in $D$ has the following two-sided estimates:

$$
\begin{equation*}
c_{1} p_{c_{2}}^{W}(t, x, y) \leq p_{D}(t, x, y) \leq c_{3} p_{c_{4}}^{W}(t, x, y) \quad \text { for } t \in(0,1] \text { and } x, y \in D \tag{1.20}
\end{equation*}
$$

For symmetric $\alpha$-stable processes in $\mathbb{R}^{d}$, its transition density $p(t, x, y)$ is comparable to $q(t, x, y)$; that is, $c_{1} q(t, x, y) \leq p(t, x, y) \leq c_{2} q(t, x, y)$ for all $t>0$ and $x, y \in \mathbb{R}^{d}$. One has the following form of 3 P inequality (see [3] and (2.11) below):

$$
\begin{equation*}
\frac{q(s, x, z) q(t-s, z, y)}{q(t, x, y)} \leq c(q(s, x, z)+q(t-s, z, y)) \tag{1.21}
\end{equation*}
$$

for every $0<s<t$ and $x, y, z \in \mathbb{R}^{d}$. The above two inequalities (1.19) and (1.21) are called 3P inequalities because they involve three probability transition
density functions $(q(s, x, z), q(t-s, z, y)$ and $q(t, x, y)$ for (1.21)) or their estimates $\left(p_{D}(s, x, z), p_{D}(t-s, z, y)\right.$ and $p_{D}(t, x, y)$ in the case of (1.19) in view of (1.20)). They are coined as analogs to the 3 G inequality for Green functions in the literature; see [15. Observe that by the elementary inequalities

$$
\frac{1}{2}(a \wedge b)(a+b) \leq(a \wedge b)(a \vee b)=a b \leq(a \wedge b)(a+b), \quad a, b \geq 0
$$

(1.21) is equivalent to

$$
q(s, x, z) \wedge q(t-s, z, y) \leq c q(t, x, y) \quad \text { for every } 0<s<t \text { and } x, y, z \in \mathbb{R}^{d} .
$$

The above inequality is called a 3 P inequality in [3]. The 3 P type inequalities (1.19) and (1.21) played essential roles in establishing the heat kernel estimates in [3, 21, 23. However, for processes we are dealing with in this paper, the above two types of 3 P inequalities are no longer true in general (see Remark 2.3 below). Nevertheless, we will show in Lemma 2.4 that an integral version of the 3 P inequality holds. Moreover we need an estimate on $p_{D}(t-s, x, z) p_{D}(s, w, y) / p_{D}(t, x, y)$, where $z \neq w$. The desired inequalities are established in Theorem 2.7below. We call these inequalities integral 3P inequalities because the left hand sides of these inequalities contain integrals of $3 q_{\gamma}$ 's in the form $q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) / q_{\gamma}(t, x, y)$.

Assumption (1.3) plays a crucial role in this paper. It is worthwhile to study stability of Dirichlet heat kernel estimates without this condition.

The rest of the paper is organized as follows. In Section 2 we prove some key inequalities, including two forms of the integral 3P inequality. The main estimates (1.17) and Theorem 1.3 will be established in Section 3 In the last section, we give some applications of our main results.

In this paper, we will use capital letters $\widetilde{C}, C, C_{0}, C_{1}, C_{2}, \ldots$ to denote constants in the statements of results, and their values will be fixed. The lowercase letters $c_{1}, c_{2}, \ldots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lowercase constants starts anew in each proof. For two positive functions $f$ and $g$, we use the notation $f \asymp g$, which means that there are two positive constants $c_{1}$ and $c_{2}$ whose values depend only on $d, \alpha$ and $\gamma$ so that $c_{1} g \leq f \leq c_{2} g$.

## 2. Integral 3P inequalities

In this section we will establish some key inequalities which will be essential in proving Theorem 1.3. The main results of this section are Lemma 2.2. Theorem 2.5. Lemma 2.6 and Theorem 2.7. Throughout this section, $D$ is a Borel set in $\mathbb{R}^{d}$.

We start with two lemmas that will be used several times in this section.
Lemma 2.1. For any $s, t>0$ and $(y, z) \in D \times D$, we have

$$
\begin{equation*}
1 \wedge \frac{\delta_{D}(z)}{t^{1 / \alpha}}=\frac{\delta_{D}(y)}{t^{1 / \alpha}}\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{\delta_{D}(y)}\right) \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1 \wedge \frac{\delta_{D}(y)}{s^{1 / \alpha}}\right)\left(1 \wedge \frac{\delta_{D}(z)}{t^{1 / \alpha}}\right) \leq 2\left(1+\frac{|y-z|}{s^{1 / \alpha}+\delta_{D}(y)}\right)\left(1 \wedge \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right) . \tag{2.2}
\end{equation*}
$$

Proof. The identity (2.1) is clear, so we only need to prove (2.2). Since $\delta_{D}(z) \leq$ $|y-z|+\delta_{D}(y)$, we see that

$$
1 \wedge \frac{\delta_{D}(z)}{t^{1 / \alpha}} \leq 1 \wedge\left(\left(\frac{|y-z|+\delta_{D}(y)}{\delta_{D}(y)}\right) \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right) \leq\left(1+\frac{|y-z|}{\delta_{D}(y)}\right)\left(1 \wedge \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right) .
$$

Thus, applying the elementary inequality

$$
\begin{equation*}
\frac{a}{a+b} \leq 1 \wedge \frac{a}{b} \leq \frac{2 a}{a+b}, \quad a, b>0 \tag{2.3}
\end{equation*}
$$

we get

$$
\begin{aligned}
\left(1 \wedge \frac{\delta_{D}(y)}{s^{1 / \alpha}}\right)\left(1 \wedge \frac{\delta_{D}(z)}{t^{1 / \alpha}}\right) & \leq\left(1 \wedge \frac{\delta_{D}(y)}{s^{1 / \alpha}}\right)\left(1+\frac{|y-z|}{\delta_{D}(y)}\right)\left(1 \wedge \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right) \\
& \leq 2\left(1+\frac{|y-z|}{s^{1 / \alpha}+\delta_{D}(y)}\right)\left(1 \wedge \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right)
\end{aligned}
$$

Using (1.1) and (2.3), we get that

$$
\begin{equation*}
\frac{t}{\left(t^{1 / \alpha}+|x-y|\right)^{d+\alpha}} \leq q(t, x, y) \leq 2^{d+\alpha} \frac{t}{\left(t^{1 / \alpha}+|x-y|\right)^{d+\alpha}} \tag{2.4}
\end{equation*}
$$

Lemma 2.2. For any $\gamma \in[0,2 \alpha)$, there exists a constant $C_{1}:=C_{1}(d, \alpha, \gamma) \geq 1$ such that for all $(t, y, z) \in(0, \infty) \times D \times D$,

$$
\begin{equation*}
\psi_{\gamma}(t, z) \int_{0}^{t / 2} q_{\gamma}(s, z, y) d s \leq C_{1} \psi_{\gamma}(t, y) \int_{0}^{t / 2} \psi_{\gamma}(s, z) q(s, z, y) d s \tag{2.5}
\end{equation*}
$$

Proof. The inequality holds trivially when $\gamma=0$ with $C_{1}=1$, so for the rest of the proof we assume $\gamma \in(0,2 \alpha)$. The inequality (2.5) is obvious if $\delta_{D}(y) \geq t^{1 / \alpha}$ or $\delta_{D}(z) \leq 2 \delta_{D}(y)$. So we will assume $\delta_{D}(y)<t^{1 / \alpha} \wedge\left(\delta_{D}(z) / 2\right)$ throughout this proof. Note that in this case,

$$
\begin{equation*}
|z-y| \geq \delta_{D}(z)-\delta_{D}(y) \geq \frac{\delta_{D}(z)}{2} \geq \delta_{D}(y) \tag{2.6}
\end{equation*}
$$

Let

$$
I=\psi_{\gamma}(t, z) \int_{(t / 2) \wedge|z-y|^{\alpha}}^{t / 2} q_{\gamma}(s, z, y) d s, \quad I I=\psi_{\gamma}(t, z) \int_{0}^{(t / 2) \wedge|z-y|^{\alpha}} q_{\gamma}(s, z, y) d s
$$

By (2.2), we have

$$
\begin{equation*}
I \leq 2^{2 \gamma} \psi_{\gamma}(t, y) \int_{(t / 2) \wedge|z-y|^{\alpha}}^{t / 2} \psi_{\gamma}(s, z) q(s, z, y) d s \tag{2.7}
\end{equation*}
$$

while by (2.1),

$$
\begin{equation*}
I I \leq\left(\frac{\delta_{D}(y)}{t^{1 / \alpha}}\right)^{\gamma}\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{\delta_{D}(y)}\right)^{\gamma} \int_{0}^{(t / 2) \wedge|z-y|^{\alpha}} \psi_{\gamma}(s, y) q(s, z, y) d s \tag{2.8}
\end{equation*}
$$

In view of (2.4) and (2.6),

$$
\begin{align*}
\int_{0}^{(t / 2) \wedge|z-y|^{\alpha}} \psi_{\gamma}(s, y) q(s, z, y) d s  \tag{2.9}\\
\begin{aligned}
\asymp & \int_{0}^{(t / 2) \wedge \delta_{D}(y)^{\alpha}} \frac{s}{|z-y|^{d+\alpha}} d s+\int_{(t / 2) \wedge \delta_{D}(y)^{\alpha}}^{(t / 2) \wedge|z-y|^{\alpha}}\left(\frac{\delta_{D}(y)}{s^{1 / \alpha}}\right)^{\gamma} \frac{s}{|z-y|^{d+\alpha}} d s \\
\asymp & \frac{1}{|z-y|^{d+\alpha}}\left(\left((t / 2) \wedge \delta_{D}(y)^{\alpha}\right)^{2}+\delta_{D}(y)^{\gamma}\right. \\
& \left.\quad \times\left(\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}-\left((t / 2) \wedge \delta_{D}(y)^{\alpha}\right)^{2-\gamma / \alpha}\right)\right) \\
\asymp & \frac{1}{|z-y|^{d+\alpha}}\left(\left((t / 2) \wedge \delta_{D}(y)^{\alpha}\right)^{2}\right. \\
& \left.+\left(\delta_{D}(y) \wedge(t / 2)^{1 / \alpha}\right)^{\gamma}\left(\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}-\left((t / 2) \wedge \delta_{D}(y)^{\alpha}\right)^{2-\gamma / \alpha}\right)\right) \\
\asymp & \frac{\left(\delta_{D}(y) \wedge(t / 2)^{1 / \alpha}\right)^{\gamma}\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}}{|z-y|^{d+\alpha}} \\
\asymp & \frac{\delta_{D}(y)^{\gamma}\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}}{|z-y|^{d+\alpha}} .
\end{aligned} .
\end{align*}
$$

On the other hand, considering the right hand side of (2.5) and using (2.6) we have

$$
\begin{align*}
& \int_{0}^{(t / 2) \wedge|z-y|^{\alpha}} \psi_{\gamma}(s, z) q(s, z, y) d s  \tag{2.10}\\
& \asymp \int_{0}^{(t / 2) \wedge\left(\delta_{D}(z) / 2\right)^{\alpha}} \frac{s}{|z-y|^{d+\alpha}} d s+\int_{(t / 2) \wedge\left(\delta_{D}(z) / 2\right)^{\alpha}}^{(t / 2) \wedge|z-y|^{\alpha}}\left(\frac{\delta_{D}(z)}{s^{1 / \alpha}}\right)^{\gamma} \frac{s}{|z-y|^{d+\alpha}} d s \\
& \asymp \frac{1}{|z-y|^{d+\alpha}}\left(\left(\frac{t}{2} \wedge\left(\frac{\delta_{D}(z)}{2}\right)^{\alpha}\right)^{2}\right. \\
& \left.+\delta_{D}(z)^{\gamma}\left(\left(\frac{t}{2} \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}-\left(\frac{t}{2} \wedge\left(\frac{\delta_{D}(z)}{2}\right)^{\alpha}\right)^{2-\gamma / \alpha}\right)\right) \\
& \geq \frac{1}{|z-y|^{d+\alpha}}\left(\left(\frac{t}{2} \wedge\left(\frac{\delta_{D}(z)}{2}\right)^{\alpha}\right)^{2}+\left(\frac{\delta_{D}(z)}{2} \wedge\left(\frac{t}{2}\right)^{1 / \alpha}\right)^{\gamma}\right. \\
& \left.\times\left(\left(\frac{t}{2} \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}-\left(\frac{t}{2} \wedge\left(\frac{\delta_{D}(z)}{2}\right)^{\alpha}\right)^{2-\gamma / \alpha}\right)\right) \\
& \asymp \frac{\left(\delta_{D}(z) \wedge t^{1 / \alpha}\right)^{\gamma}\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}}{|z-y|^{d+\alpha}} \text {. }
\end{align*}
$$

One then deduces from (2.8)-(2.10) and the assumption $\delta_{D}(y) \leq t^{1 / \alpha}$ that

$$
\begin{aligned}
I I & \leq c_{1} \psi_{\gamma}(t, y)\left(\delta_{D}(z) \wedge t^{1 / \alpha}\right)^{\gamma} \frac{\left((t / 2) \wedge|z-y|^{\alpha}\right)^{2-\gamma / \alpha}}{|z-y|^{d+\alpha}} \\
& \leq c_{2} \int_{0}^{(t / 2) \wedge|z-y|^{\alpha}} \psi_{\gamma}(s, z) q(s, z, y) d s .
\end{aligned}
$$

This combined with (2.7) establishes the inequality (2.5).
We now give a proof of (1.21) (see also 3). It follows from (2.3) and (2.4) that for every $0<s<t$ and $x, y, z \in \mathbb{R}^{d}$,

$$
\begin{align*}
& \frac{q(s, x, z) q(t-s, z, y)}{} \quad q(t, x, y)  \tag{2.11}\\
& \quad \leq 4^{d+\alpha} \frac{s(t-s)}{t}\left(\frac{t^{1 / \alpha}+|x-y|}{\left(s^{1 / \alpha}+|x-z|\right)\left((t-s)^{1 / \alpha}+|y-z|\right)}\right)^{d+\alpha} \\
& \quad \leq 4^{d+\alpha}(s \wedge(t-s))\left(\frac{(s+(t-s))^{1 / \alpha}+|x-z|+|y-z|}{\left(s^{1 / \alpha}+|x-z|\right)\left((t-s)^{1 / \alpha}+|y-z|\right)}\right)^{d+\alpha} \\
& \quad \leq 2^{(d+\alpha)(3+1 / \alpha)}(s \wedge(t-s))\left(\frac{1}{\left(s^{1 / \alpha}+|x-z|\right)^{d+\alpha}}+\frac{1}{\left((t-s)^{1 / \alpha}+|y-z|\right)^{d+\alpha}}\right) \\
& \quad \leq 2^{(d+\alpha)(3+1 / \alpha)}(q(s, x, z)+q(t-s, z, y)) .
\end{align*}
$$

Remark 2.3. It follows from (2.3) and (2.4) that for every $\frac{t}{4}<s<\frac{3 t}{4}$ and $x, y, z \in$ $\mathbb{R}^{d}$ with $2|x-y| \geq|x-z|+|y-z|$,

$$
\begin{align*}
& \frac{q(s, x, z) q(t-s, z, y)}{q(t, x, y)} \geq 2^{-d-\alpha} \frac{s(t-s)}{t}\left(\frac{t^{1 / \alpha}+|x-y|}{\left(s^{1 / \alpha}+|x-z|\right)\left((t-s)^{1 / \alpha}+|y-z|\right)}\right)^{d+\alpha}  \tag{2.12}\\
& \quad \geq 2^{-d-\alpha} \frac{s(t-s)}{t}\left(\frac{(s+(t-s))^{1 / \alpha}+(|x-z|+|y-z|) / 2}{\left(s^{1 / \alpha}+|x-z|\right)\left((t-s)^{1 / \alpha}+|y-z|\right)}\right)^{d+\alpha} \\
& \quad \asymp(s \wedge(t-s))\left(\frac{1}{\left(s^{1 / \alpha}+|x-z|\right)^{d+\alpha}}+\frac{1}{\left((t-s)^{1 / \alpha}+|y-z|\right)^{d+\alpha}}\right) \\
& \quad \asymp q(s, x, z)+q(t-s, z, y) .
\end{align*}
$$

Thus using (2.3), (2.11) and (2.12) we have that for every $\frac{t}{4}<s<\frac{3 t}{4}$ and $x, y, z \in D$ with $2|x-y| \geq|x-z|+|y-z|$,

$$
\begin{aligned}
& \frac{q_{\gamma}(t, x, y)\left(q_{\gamma}(s, x, z)+q_{\gamma}(t-s, z, y)\right)}{q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y)} \\
& \quad \leq c_{1}\left(\frac{\delta_{D}(x)\left(\delta_{D}(z)+t^{1 / \alpha}\right)}{\delta_{D}(z)\left(\delta_{D}(x)+t^{1 / \alpha}\right)}\right)^{\gamma}+\left(\frac{\delta_{D}(y)\left(\delta_{D}(z)+t^{1 / \alpha}\right)}{\delta_{D}(z)\left(\delta_{D}(y)+t^{1 / \alpha}\right)}\right)^{\gamma}
\end{aligned}
$$

which goes to zero if $\delta_{D}(y)=\delta_{D}(x) \rightarrow 0$ with $2|x-y| \geq|x-z|+|y-z|$. This shows that the inequality

$$
\frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y)}{q_{\gamma}(t, x, y)} \leq c\left(q_{\gamma}(s, x, z)+q_{\gamma}(t-s, z, y)\right)
$$

for every $0<s<t$ and $x, y, z \in D$ cannot be true, even for balls.

We record here a simple inequality which we will use several times in this paper. For $\beta \in(0,1)$ and $a>0$,

$$
\begin{equation*}
\int_{0}^{t}\left(1 \wedge \frac{a}{s^{\beta}}\right) d s \leq \frac{t}{1-\beta}\left(1 \wedge \frac{a}{t^{\beta}}\right) . \tag{2.13}
\end{equation*}
$$

We are now ready to prove one form of the integral 3P inequality. Note that the right hand side of the integral 3 P inequality below has the term $q(s, x, z)+q(s, z, y)$ rather than $q(t-s, x, z)+q(s, z, y)$.

Lemma 2.4 (Integral 3P inequality). For every $\gamma \in[0, \alpha)$, there exists $C_{2}:=$ $C_{2}(d, \alpha, \gamma)>0$ such that for all $(t, x, y, z) \in(0, \infty) \times D \times D \times D$,

$$
\int_{0}^{t} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y)}{q_{\gamma}(t, x, y)} d s \leq C_{2} \int_{0}^{t} \psi_{\gamma}(s, z)(q(s, x, z)+q(s, z, y)) d s
$$

Proof. When $\gamma=0$, the desired inequality follows from (2.11) with $C_{2}=$ $2^{(d+\alpha)(3+1 / \alpha)}$. So for the rest of the proof, we assume $\gamma \in(0, \alpha)$. Let

$$
J(t, x, y, z):=\int_{0}^{t} q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y) d s
$$

Since

$$
J(t, x, y, z) \leq c_{1} q_{\gamma}(t, x, z) \int_{0}^{t / 2} q_{\gamma}(s, z, y) d s+c_{1} q_{\gamma}(t, z, y) \int_{t / 2}^{t} q_{\gamma}(t-s, x, z) d s
$$

we have by Lemma 2.2 that

$$
\begin{aligned}
J(t, x, y, z) \leq & c_{2} \psi_{\gamma}(t, x) \psi_{\gamma}(t, y) \int_{0}^{t / 2} \psi_{\gamma}(s, z) q(t-s, x, z) q(s, z, y) d s \\
& +c_{2} \psi_{\gamma}(t, x) \psi_{\gamma}(t, y) \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(t-s, x, z) q(s, z, y) d s
\end{aligned}
$$

It then follows from (2.11) that

$$
\begin{aligned}
J(t, x, y, z) \leq & c_{3} q_{\gamma}(t, x, y) \int_{0}^{t / 2} \psi_{\gamma}(s, z)(q(t-s, x, z)+q(s, z, y)) d s \\
& +c_{3} q_{\gamma}(t, x, y) \int_{t / 2}^{t} \psi_{\gamma}(t-s, z)(q(t-s, x, z)+q(s, z, y)) d s \\
\leq & c_{4} q_{\gamma}(t, x, y) \int_{0}^{t} \psi_{\gamma}(s, z)(q(s, z, y)+q(s, x, z)) d s
\end{aligned}
$$

Here in the last inequality, we used the fact that

$$
\int_{0}^{t / 2} \psi_{\gamma}(s, z) q(t-s, x, z) d s \leq c_{5} \int_{t / 2}^{t} \psi_{\gamma}(s, z) q(s, x, z) d s
$$

and

$$
\int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(s, z, y) d s \leq c_{5} \int_{t / 2}^{t} \psi_{\gamma}(s, z) q(s, z, y) d s
$$

The above two inequalities follow easily from the fact that $q(s, x, y) \asymp q(t, x, y)$ for $s \in[t / 2, t]$ and

$$
\int_{0}^{t / 2} \psi_{\gamma}(s, z) d s \leq \frac{\alpha}{\alpha-\gamma}(t / 2) \psi_{\gamma}(t / 2, z) \leq c_{6} \int_{t / 2}^{t} \psi_{\gamma}(s, z) d s
$$

where we have used (2.13) for the first inequality. This completes the proof of the lemma.

The above integral 3P inequality immediately implies the following theorem, which will be used later.

Theorem 2.5. For every $\gamma \in[0, \alpha)$, there exists a constant $C_{3}=C_{3}(d, \alpha, \gamma)>0$ such that for any measure $\mu$ on $D$ and any $(t, x, y) \in(0, \infty) \times D \times D$,

$$
\begin{aligned}
& \int_{0}^{t} \int_{D} q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y) \mu(d z) d s \\
& \quad \leq C_{3} q_{\gamma}(t, x, y) \sup _{u \in D} \int_{0}^{t} \int_{D} \psi_{\gamma}(s, z) q(s, u, z) \mu(d z) d s
\end{aligned}
$$

The results of the remainder of this section are geared towards dealing with the discontinuous part of $A^{\mu, F}$.
Lemma 2.6. For every $\gamma \in[0,2 \alpha)$, there exists a constant $C_{4}:=C_{4}(d, \alpha, \gamma) \geq 1$ such that for all $(t, y, z, w) \in(0, \infty) \times D \times D \times D$,

$$
\begin{align*}
& \psi_{\gamma}(t, z) \int_{0}^{t / 2} q_{\gamma}(s, w, y) d s  \tag{2.14}\\
\leq & C_{4} \psi_{\gamma}(t, y)\left(1+\frac{|y-z| \wedge|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \int_{0}^{t / 2} \psi_{\gamma}(s, w) q(s, w, y) d s
\end{align*}
$$

Proof. The desired inequality holds trivially for $\gamma=0$ with $C_{4}=1$, so for the rest of the proof we assume $\gamma \in(0,2 \alpha)$. The inequality (2.14) is obvious if $\delta_{D}(y) \geq t^{1 / \alpha}$ or $\delta_{D}(z) \leq 2 \delta_{D}(y)$, so we will assume $\delta_{D}(y)<t^{1 / \alpha} \wedge\left(\delta_{D}(z) / 2\right)$ in the remainder of this proof. Note that in this case

$$
\begin{equation*}
|y-z| \geq \delta_{D}(z)-\delta_{D}(y) \geq \frac{\delta_{D}(z)}{2} \geq \delta_{D}(y) \tag{2.15}
\end{equation*}
$$

By (2.1), (2.3) and our assumption $\delta_{D}(y)<t^{1 / \alpha}$, we have that

$$
\begin{align*}
\left(1 \wedge \frac{\delta_{D}(z)}{t^{1 / \alpha}}\right)\left(1 \wedge \frac{\delta_{D}(y)}{s^{1 / \alpha}}\right) & \leq 2 \frac{\delta_{D}(y)}{t^{1 / \alpha}}\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{s^{1 / \alpha}+\delta_{D}(y)}\right)  \tag{2.16}\\
& =2\left(1 \wedge \frac{\delta_{D}(y)}{t^{1 / \alpha}}\right)\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{s^{1 / \alpha}+\delta_{D}(y)}\right)
\end{align*}
$$

When $s \geq|y-w|^{\alpha}$, by (2.15),

$$
\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{s^{1 / \alpha}+\delta_{D}(y)} \leq 2 \frac{|y-z| \wedge t^{1 / \alpha}}{|y-w|} \leq 2\left(1+\frac{|y-z| \wedge|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right),
$$

where the last inequality is due to the fact that $|y-z| \leq|y-w|+(|y-z| \wedge|z-w|)$. This together with (2.16) implies that

$$
\begin{align*}
& \psi_{\gamma}(t, z) \int_{(t / 2) \wedge|y-w|^{\alpha}}^{t / 2} q_{\gamma}(s, w, y) d s  \tag{2.17}\\
\leq & 4^{\gamma} \psi_{\gamma}(t, y)\left(1+\frac{|y-z| \wedge|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \int_{(t / 2) \wedge|y-w|^{\alpha}}^{t / 2} \psi_{\gamma}(s, w) q(s, w, y) d s
\end{align*}
$$

On the other hand, by (2.15),

$$
\begin{align*}
& \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}}\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{s^{1 / \alpha}+\delta_{D}(y)}\right)^{\gamma} \psi_{\gamma}(s, w) q(s, w, y) d s  \tag{2.18}\\
\leq & 2^{\gamma} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}}\left(\frac{|y-z| \wedge t^{1 / \alpha}}{s^{1 / \alpha}}\right)^{\gamma} \psi_{\gamma}(s, w) \frac{s}{|y-w|^{d+\alpha}} d s \\
= & c_{1} \frac{\left(|y-z| \wedge t^{1 / \alpha}\right)^{\gamma}}{|y-w|^{d+\alpha}} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} s^{1-\gamma / \alpha} \psi_{\gamma}(s, w) d s .
\end{align*}
$$

We claim that

$$
\begin{equation*}
\int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} s^{1-\gamma / \alpha} \psi_{\gamma}(s, w) d s \asymp\left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{-\gamma / \alpha} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} s \psi_{\gamma}(s, w) d s \tag{2.19}
\end{equation*}
$$

The case $\delta_{D}(w)>(t / 2)^{1 / \alpha}$ is clear. If $\delta_{D}(w) \leq|y-w| \wedge(t / 2)^{1 / \alpha}$,

$$
\begin{aligned}
& \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} s^{1-\gamma / \alpha} \psi_{\gamma}(s, w) d s \\
= & \int_{0}^{\delta_{D}(w)^{\alpha}} s^{1-\gamma / \alpha} d s+\delta_{D}(w)^{\gamma} \int_{\delta_{D}(w)^{\alpha}}^{(t / 2) \wedge|y-w|^{\alpha}} s^{1-2 \gamma / \alpha} d s \\
\asymp & \delta_{D}(w)^{2 \alpha-\gamma}+\delta_{D}(w)^{\gamma}\left(\left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{2-2 \gamma / \alpha}-\delta_{D}(w)^{2(\alpha-\gamma)}\right) \\
\asymp & \delta_{D}(w)^{\gamma}\left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{2-2 \gamma / \alpha} \\
= & \left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{-\gamma / \alpha}\left(\delta_{D}(w)\right)^{\gamma}\left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{2-\gamma / \alpha} \\
\asymp & \left(\frac{t}{2} \wedge|y-w|^{\alpha}\right)^{-\gamma / \alpha} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} s \psi_{\gamma}(s, w) d s .
\end{aligned}
$$

The remaining case $|y-w|<\delta_{D}(w) \leq(t / 2)^{1 / \alpha}$ is simpler and is left to the reader. Thus we have proved the claim (2.19). Now by (2.18) and (2.19),

$$
\begin{aligned}
& \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}}\left(\frac{\delta_{D}(z) \wedge t^{1 / \alpha}}{s^{1 / \alpha}+\delta_{D}(y)}\right)^{\gamma} \psi_{\gamma}(s, w) q(s, w, y) d s \\
\leq & c_{2}\left(\frac{|y-z| \wedge t^{1 / \alpha}}{|y-w| \wedge t^{1 / \alpha}}\right)^{\gamma} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} \psi_{\gamma}(s, w) \frac{s}{|y-w|^{d+\alpha}} d s \\
\leq & c_{2}\left(1+\frac{|y-z| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} \psi_{\gamma}(s, w) q(s, w, y) d s \\
\leq & 2 c_{2}\left(1+\frac{|y-z| \wedge|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \int_{0}^{(t / 2) \wedge|y-w|^{\alpha}} \psi_{\gamma}(s, w) q(s, w, y) d s
\end{aligned}
$$

Here again the last inequality is due to the fact that $|y-z| \leq|y-w|+(|y-z| \wedge|z-w|)$. This together with (2.16) and (2.17) establishes the inequality (2.14).

In the remainder of this section, we use the following notation: For any $(x, y) \in$ $D \times D$,

$$
\begin{aligned}
V_{x, y} & :=\{(z, w) \in D \times D:|x-y| \geq 4(|y-w| \wedge|x-z|)\} \\
U_{x, y} & :=(D \times D) \backslash V_{x, y}
\end{aligned}
$$

Recall that for any bounded measurable function $F$ on $D \times D$ we use $\|F\|_{\infty}$ to denote $\|F\|_{L^{\infty}(D \times D)}$.

Now we are ready to prove the following generalized integral 3P inequality.
Theorem 2.7 (Generalized integral 3P inequality). For every $\gamma \in[0, \alpha \wedge d)$, there exists a constant $C_{5}:=C_{5}(\alpha, \gamma, d)>0$ such that for any non-negative bounded function $F(x, y)$ on $D \times D$, the following are true for $(t, x, y) \in(0, \infty) \times D \times D$.
(a) If $|x-y| \leq t^{1 / \alpha}$, then

$$
\begin{aligned}
& \int_{0}^{t} \int_{D \times D} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{q_{\gamma}(t, x, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
\leq & C_{5} \int_{0}^{t} \int_{D \times D} \psi_{\gamma}(s, z) q(s, x, z)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
& +C_{5} \int_{0}^{t} \int_{D \times D} \psi_{\gamma}(s, w) q(s, y, w)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s .
\end{aligned}
$$

(b) If $|x-y|>t^{1 / \alpha}$, then

$$
\begin{aligned}
& \int_{0}^{t} \int_{U_{x, y}} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{q_{\gamma}(t, x, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
\leq & C_{5} \int_{0}^{t} \int_{U_{x, y}} \psi_{\gamma}(s, z) q(s, x, z)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
& +C_{5} \int_{0}^{t} \int_{U_{x, y}} \psi_{\gamma}(s, w) q(s, y, w)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s .
\end{aligned}
$$

(c) If $|x-y|>t^{1 / \alpha}$, then

$$
\int_{0}^{t} \int_{V_{x, y}} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{q_{\gamma}(t, x, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \leq C_{5}\|F\|_{\infty}
$$

Proof. By Lemma 2.6 we get that

$$
\begin{gather*}
\quad \int_{0}^{t} \int_{D \times D} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{\psi_{\gamma}(t, x) \psi_{\gamma}(t, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s  \tag{2.20}\\
\leq c_{1} \int_{D \times D} \int_{0}^{t / 2} \psi_{\gamma}(s, w) q(s, w, y) q(t-s, x, z) \\
\quad \times\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w \\
\quad+c_{1} \int_{D \times D} \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(s, w, y) q(t-s, x, z) \\
\quad \times\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w
\end{gather*}
$$

If $|x-y| \leq t^{1 / \alpha}$ and $s \in(0, t / 2]$, we have $q(t-s, x, z) \leq 2^{d / \alpha} q(t, x, y)$; and if $|x-y| \leq t^{1 / \alpha}$ and $s \in(t / 2, t]$, we have $q(s, w, y) \leq 2^{d / \alpha} q(t, x, y)$. Thus (a) follows immediately from (2.20).

In the remainder of this proof, we fix $(t, x, y) \in(0, \infty) \times D \times D$ with $|x-y|>t^{1 / \alpha}$. Let

$$
\begin{aligned}
& U_{1}:=\left\{(z, w) \in D \times D:|y-w|>4^{-1}|x-y|,|y-w| \geq|x-z|\right\}, \\
& U_{2}:=\left\{(z, w) \in D \times D:|x-z|>4^{-1}|x-y|\right\} .
\end{aligned}
$$

Since $q(t-s, x, z) \leq 4^{d+\alpha} q(t, x, y)$ for $(s, z, w) \in(0, t) \times U_{2}$, by Lemma 2.6, we have (2.21)

$$
\begin{aligned}
& \int_{0}^{t / 2} \int_{U_{2}} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{\psi_{\gamma}(t, x) \psi_{\gamma}(t, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
\leq & c_{2} \int_{U_{2}} \int_{0}^{t / 2} \psi_{\gamma}(s, w) q(s, w, y) q(t-s, x, z)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w \\
\leq & c_{3} q(t, x, y) \int_{U_{2}} \int_{0}^{t / 2} \psi_{\gamma}(s, w) q(s, w, y)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w
\end{aligned}
$$

and, similarly,

$$
\begin{align*}
& \quad \int_{t / 2}^{t} \int_{U_{1}} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{\psi_{\gamma}(t, x) \psi_{\gamma}(t, y)} \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s  \tag{2.22}\\
& \leq c_{4} \int_{U_{1}} \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(s, w, y) q(t-s, x, z) \\
& \quad \times\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w \\
& \leq c_{5} q(t, x, y) \int_{U_{1}} \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(t-s, x, z) \\
& \quad \times\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w .
\end{align*}
$$

On the other hand, we observe that, since $q(s, w, y) \leq 4^{d+\alpha} q(t, x, y)$ for $(s, z, w) \in$ $(0, t / 2] \times U_{1}$,

$$
\begin{aligned}
& \int_{0}^{t / 2} \int_{U_{1}} q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s \\
\leq & c_{6} \psi_{\gamma}(t, x) \psi_{\gamma}(t, z) q(t, x, y) \int_{U_{1}} q(t, x, z) \int_{0}^{t / 2} \psi_{\gamma}(s, w) \psi_{\gamma}(s, y) d s \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w
\end{aligned}
$$

Now, applying the inequality (using (2.13))

$$
\int_{0}^{t / 2} \psi_{\gamma}(s, w) \psi_{\gamma}(s, y) d s \leq \int_{0}^{t / 2} \psi_{\gamma}(s, y) d s \leq \frac{\alpha}{\alpha-\gamma} 2^{\gamma / \alpha-1} t \psi_{\gamma}(t, y)
$$

we get

$$
\begin{align*}
& \int_{0}^{t / 2} \int_{U_{1}} q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s  \tag{2.23}\\
\leq & c_{7} q_{\gamma}(t, x, y) \int_{U_{1}} q(t, x, z) t \psi_{\gamma}(t, z) \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w \\
\leq & c_{8} q_{\gamma}(t, x, y) \int_{U_{1}} \int_{0}^{t / 2} q(t-s, x, z) \psi_{\gamma}(t-s, z) \frac{F(z, w)}{|z-w|^{d+\alpha}} d s d z d w .
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \int_{t / 2}^{t} \int_{U_{2}} q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) \frac{F(z, w)}{|z-w|^{d+\alpha}} d z d w d s  \tag{2.24}\\
\leq & c_{9} q_{\gamma}(t, x, y) \int_{U_{2}} \int_{t / 2}^{t} \psi_{\gamma}(s, w) q(s, w, y) \frac{F(z, w)}{|z-w|^{d+\alpha}} d s d z d w .
\end{align*}
$$

Since $U_{x, y}=U_{1} \cup U_{2}$, from (2.21)-(2.24), we know that (b) is true.
Note that for $(z, w) \in V_{x, y}$, we have $|z-w| \geq|x-y|-(|x-z|+|y-w|) \geq 2^{-1}|x-y|$. Thus, by Lemma 2.6 and (1.4), it is easy to see that

$$
\begin{aligned}
& \quad \int_{0}^{t} \int_{V_{x, y}} \frac{q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y)}{\psi_{\gamma}(t, x) \psi_{\gamma}(t, y)} \frac{F(z, w)}{q(t, x, y)|z-w|^{d+\alpha}} d z d w d s \\
& \leq c_{10}\|F\|_{\infty} t^{-1} \int_{V_{x, y}} \int_{0}^{t / 2} q(s, w, y) q(t-s, x, z)\left(1+\frac{t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d s d z d w \\
& \quad+c_{10}\|F\|_{\infty} t^{-1} \int_{V_{x, y}} \int_{t / 2}^{t} q(s, w, y) q(t-s, x, z)\left(1+\frac{t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d s d z d w \\
& \leq c_{11}\|F\|_{\infty} t^{-1} \int_{0}^{t}\left(\int_{D} q(s, w, y)\left(1+\frac{t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d w\right. \\
& \left.\quad+\int_{D} q(s, x, z)\left(1+\frac{t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d z\right) d s
\end{aligned}
$$

Since, using $\gamma \in(0, \alpha \wedge d)$,

$$
\begin{aligned}
& \int_{0}^{t}\left(\int_{D} q(s, w, y)\left(1+\frac{t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d w+\int_{D} q(s, x, z)\left(1+\frac{t^{1 / \alpha}}{|x-z|}\right)^{\gamma} d z\right) d s \\
\leq & 2^{d+\alpha+1} \int_{0}^{t} \int_{\mathbb{R}^{d}} \frac{s}{\left(s^{1 / \alpha}+|w|\right)^{d+\alpha}}\left(\frac{t^{1 / \alpha}}{|w|}\right)^{\gamma} d w d s \\
= & c_{12}\left(\int_{0}^{\infty} \frac{u^{d-1-\gamma} d u}{(1+u)^{d+\alpha}}\right) t^{\gamma / \alpha} \int_{0}^{t} s^{-\gamma / \alpha} d s \leq c_{13} t
\end{aligned}
$$

(c) follows immediately.

## 3. Heat kernel estimates

In this section we give the proof of our main result, Theorem 1.3 Throughout this section, we fix $\gamma \in[0, \alpha \wedge d)$. Recall the definition of $p^{k}(t, x, y)$ given by (1.15).

Using (1.3), (1.7), Theorems 2.5 and 2.7 we can choose a constant

$$
\begin{equation*}
M=M\left(\alpha, \gamma, d, C_{0}\right)>\frac{\alpha}{\alpha-\gamma} 2^{2 \gamma / \alpha+d+\alpha+1} C_{0}^{4}\left(C_{1} \vee C_{4}\right) \tag{3.1}
\end{equation*}
$$

such that for any $\mu$ in $\mathbf{K}_{\alpha, \gamma}$, any measurable function $F$ with $F_{1}=e^{F}-1 \in \mathbf{J}_{\alpha, \gamma}$ and any $(t, x, y) \in(0,1] \times D \times D$,

$$
\begin{align*}
& \int_{0}^{t} \int_{D} q_{\gamma}(t-s, x, z) q_{\gamma}(s, z, y)|\mu|(d z) d s \leq M p_{D}(t, x, y) N_{\mu}^{\alpha, \gamma}(t),  \tag{3.2}\\
& \int_{0}^{t} \int_{D \times D} q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) \frac{c(z, w)\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w d s  \tag{3.3}\\
& \leq M p_{D}(t, x, y)\left(N_{F_{1}}^{\alpha, \gamma}(t)+\left\|F_{1}\right\|_{\infty} \mathbf{1}_{\left\{|x-y|>t^{1 / \alpha}\right\}}\right)
\end{align*}
$$

and
(3.4)

$$
\int_{0}^{t} \int_{U_{x, y}} q_{\gamma}(t-s, x, z) q_{\gamma}(s, w, y) \frac{c(z, w)\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w d s \leq M p_{D}(t, x, y) N_{F_{1}}^{\alpha, \gamma}(t)
$$

In the remainder of this section, we fix a locally finite signed measure $\mu \in \mathbf{K}_{\alpha, \gamma}$, a measurable function $F$ with $F_{1}=e^{F}-1 \in \mathbf{J}_{\alpha, \gamma}$ and the constant $M>0$ in (3.1).

Lemma 3.1. For every $k \geq 0$ and $(t, x) \in(0,1] \times D$,

$$
\begin{equation*}
\int_{D}\left|p^{k}(t, x, y)\right| d y \leq C_{0}^{2} M^{k} \psi_{\gamma}(t, x)\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k} \tag{3.5}
\end{equation*}
$$

Proof. We use induction on $k \geq 0$. By (1.5), (3.5) is clear when $k=0$. Suppose (3.5) is true for $k-1 \geq 0$. Then by (1.15) we have

$$
\begin{aligned}
& \int_{D} p^{k}(t, x, y) d y=\int_{0}^{t / 2}\left(\int_{D} p^{0}(t-s, x, z)\left(\int_{D} p^{k-1}(s, z, y) d y\right) \mu(d z)\right) d s \\
& \quad+\int_{0}^{t / 2}\left(\int_{D} \int_{D} p^{0}(t-s, x, z) \frac{c(z, w) F_{1}(z, w)}{|z-w|^{d+\alpha}}\left(\int_{D} p^{k-1}(s, w, y) d y\right) d z d w\right) d s \\
& \quad+\int_{t / 2}^{t}\left(\int_{D} p^{0}(t-s, x, z)\left(\int_{D} p^{k-1}(s, z, y) d y\right) \mu(d z)\right) d s \\
& \quad+\int_{t / 2}^{t}\left(\int_{D} \int_{D} p^{0}(t-s, x, z) \frac{c(z, w) F_{1}(z, w)}{|z-w|^{d+\alpha}}\left(\int_{D} p^{k-1}(s, w, y) d y\right) d z d w\right) d s
\end{aligned}
$$

Thus using (1.3) and our induction hypothesis, we have

$$
\begin{aligned}
& \int_{D}\left|p^{k}(t, x, y)\right| d y \\
\leq & 2^{\gamma / \alpha} C_{0}^{3} M^{k-1}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\left(\psi_{\gamma}(t, x) \int_{0}^{t / 2}\left(\int_{D} \psi_{\gamma}(t-s, z) q(t-s, x, z)|\mu|(d z)\right) d s\right. \\
& +C_{0} \psi_{\gamma}(t, x) \int_{0}^{t / 2}\left(\int_{D} \int_{D} \psi_{\gamma}(t-s, z) q(t-s, x, z) \frac{\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w\right) d s \\
& +\int_{D} \psi_{\gamma}(t, z) \int_{t / 2}^{t} q_{\gamma}(t-s, x, z) d s|\mu|(d z) \\
& \left.+C_{0} \int_{D} \int_{D} \psi_{\gamma}(t, w) \int_{t / 2}^{t} q_{\gamma}(t-s, x, z) d s \frac{\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w\right) .
\end{aligned}
$$

Applying (3.1), Lemmas 2.2 and 2.6, the above is no larger than

$$
\begin{aligned}
& 4^{-1} C_{0} M^{k}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\left(\psi_{\gamma}(t, x) \int_{0}^{t / 2}\left(\int_{D} \psi_{\gamma}(t-s, z) q(t-s, x, z)|\mu|(d z)\right) d s\right. \\
& +C_{0} \psi_{\gamma}(t, x) \int_{0}^{t / 2}\left(\int_{D} \int_{D} \psi_{\gamma}(t-s, z) q(t-s, x, z) \frac{\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w\right) d s \\
& +\int_{D} \psi_{\gamma}(t, x) \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(t-s, x, z) d s|\mu|(d z) \\
& +C_{0} \int_{D} \int_{D} \psi_{\gamma}(t, x)\left(1+\frac{|x-w| \wedge|z-w| \wedge t^{1 / \alpha}}{|x-z|}\right)^{\gamma} \\
& \left.\quad \times \int_{t / 2}^{t} \psi_{\gamma}(t-s, z) q(t-s, x, z) \frac{\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} d z d w\right) \\
& \leq C_{0}^{2} M^{k} \psi_{\gamma}(t, x)\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k} .
\end{aligned}
$$

Lemma 3.2. For every $k \geq 0$ and $(t, x, y) \in(0,1] \times D \times D$,

$$
\begin{aligned}
& \int_{0}^{t} \int_{D} p_{D}(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s \\
& \quad \leq t \frac{\alpha}{\alpha-\gamma} 2^{2 \gamma / \alpha} C_{0}^{4} M^{k} \psi_{\gamma}(t, x) \psi_{\gamma}(t, y)\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k}
\end{aligned}
$$

Proof. By (1.3) and Lemma 3.1

$$
\begin{aligned}
& \int_{0}^{t} \int_{D} p_{D}(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s \\
= & \int_{0}^{t / 2} \int_{D} p_{D}(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s \\
& +\int_{t / 2}^{t} \int_{D} p_{D}(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s
\end{aligned}
$$

$$
\begin{aligned}
\leq & C_{0} \int_{0}^{t / 2} \int_{D} \psi_{\gamma}(t-s, x) q(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s \\
& +C_{0} \int_{t / 2}^{t} \int_{D} \psi_{\gamma}(t-s, x) q(t-s, x, z) d z \int_{D}\left|p^{k}(s, w, y)\right| d w d s \\
\leq & C_{0}^{3} M^{k} \int_{0}^{t / 2} \int_{D} \psi_{\gamma}(t-s, x) q(t-s, x, z) d z \psi_{\gamma}(s, y)\left(N_{\mu, F_{1}}(s)\right)^{k} d s \\
& +C_{0}^{3} M^{k} \int_{t / 2}^{t} \int_{D} \psi_{\gamma}(t-s, x) q(t-s, x, z) d z \psi_{\gamma}(s, y)\left(N_{\mu, F_{1}}(s)\right)^{k} d s \\
\leq & 2^{\gamma / \alpha} C_{0}^{3} M^{k}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k} \psi_{\gamma}(t, x)\left(\int_{0}^{t / 2} \psi_{\gamma}(s, y) d s\right) \int_{D} q(t-s, x, z) d z \\
& +2^{\gamma / \alpha} C_{0}^{3} M^{k}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k} \psi_{\gamma}(t, y) \int_{D}\left(\int_{t / 2}^{t} \psi_{\gamma}(t-s, x) d s\right) q(t-s, x, z) d z
\end{aligned}
$$

Using (2.13) on both $\int_{0}^{t / 2} \psi_{\gamma}(s, y) d s$ and $\int_{t / 2}^{t} \psi_{\gamma}(t-s, x) d s$ we get that

$$
\begin{aligned}
& \int_{0}^{t} \int_{D} p_{D}(t-s, x, z) d z \int_{D} p^{k}(s, w, y) d w d s \\
& \leq \frac{\alpha}{\alpha-\gamma} 2^{2 \gamma / \alpha} C_{0}^{3} M^{k}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k} \psi_{\gamma}(t, x) \psi_{\gamma}(t, y) \int_{D} q(t-s, x, z) d z
\end{aligned}
$$

Applying (1.4), we have proved the lemma.
Lemma 3.3. For $k \geq 0$ and $(t, x, y) \in(0,1] \times D \times D$ we have

$$
\begin{equation*}
\left|p^{k}(t, x, y)\right| \leq p^{0}(t, x, y)\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k}+k\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\right) \tag{3.6}
\end{equation*}
$$

Proof. We use induction on $k \geq 0$. The $k=0$ case is obvious. Suppose that (3.6) is true for $k-1 \geq 0$. Recall that
$V_{x, y}=\{(z, w) \in D \times D:|x-y| \geq 4(|y-w| \vee|x-z|)\}, \quad U_{x, y}=(D \times D) \backslash V_{x, y}$.
Applying (1.15), (1.3), (3.2) and (3.4), we have by our induction hypothesis,

$$
\begin{aligned}
& \quad\left|p^{k}(t, x, y)\right| \leq \int_{0}^{t}\left(\int_{D} p^{0}(t-s, x, z)\left|p^{k-1}(s, z, y) \| \mu\right|(d z)\right) d s \\
& +\int_{0}^{t}\left(\int_{U_{x, y}} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}(z, w)\right|}{|z-w|^{d+\alpha}}\left|p^{k-1}(s, w, y)\right| d z d w\right) d s \\
& \quad+\int_{0}^{t}\left(\int_{V_{x, y}} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}(z, w)\right|}{|z-w|^{d+\alpha}}\left|p^{k-1}(s, w, y)\right| d z d w\right) d s \\
& \leq\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right) \\
& \quad \times \int_{0}^{t}\left(\int_{D} p^{0}(t-s, x, z) p^{0}(s, z, y)|\mu|(d z)\right) d s \\
& +\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right)
\end{aligned}
$$

$$
\begin{align*}
& \quad \times \int_{0}^{t}\left(\int_{U_{x, y}} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} p^{0}(s, w, y) d z d w\right) d s \\
& +\int_{0}^{t}\left(\int_{V_{x, y}} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}}\left|p^{k-1}(s, w, y)\right| d z d w\right) d s \\
& \leq \quad p^{0}(t, x, y)\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\right. \\
& \left.\quad+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right) C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)  \tag{t}\\
& + \\
& \quad C_{0} \frac{2^{d+\alpha}\left\|F_{1}\right\|_{\infty}}{|x-y|^{d+\alpha}} \int_{0}^{t}\left(\int_{D \times D} p^{0}(t-s, x, z)\left|p^{k-1}(s, w, y)\right| d z d w\right) d s .
\end{align*}
$$

Applying Lemma 3.2 and using (3.1), we get that if $|x-y|^{\alpha} \geq t$,

$$
\begin{aligned}
& C_{0} \frac{2^{d+\alpha}\left\|F_{1}\right\|_{\infty}}{|x-y|^{d+\alpha}} \int_{0}^{t}\left(\int_{D \times D} p^{0}(t-s, x, z)\left|p^{k-1}(s, w, y)\right| d z d w\right) d s \\
\leq & \psi_{\gamma}(t, x) \psi_{\gamma}(t, y) \frac{t}{|x-y|^{d+\alpha}}\left\|F_{1}\right\|_{\infty} C_{0}^{5} \frac{\alpha}{\alpha-\gamma} 2^{d+\alpha+\gamma / \alpha} M^{k-1}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1} \\
\leq & p^{0}(t, x, y)\left\|F_{1}\right\|_{\infty} C_{0}^{6} \frac{\alpha}{\alpha-\gamma} 2^{d+\alpha+\gamma / \alpha} M^{k-1}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1} \\
\leq & p^{0}(t, x, y)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M^{k}\left(N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1} \\
\leq & p^{0}(t, x, y)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1} .
\end{aligned}
$$

Thus (3.6) is true for $k$ when $|x-y|^{\alpha} \geq t$.
If $|x-y|^{\alpha} \leq t$, using (1.3), (1.15), (3.2) and (3.3), we have by our induction hypothesis,

$$
\begin{gathered}
\left|p^{k}(t, x, y)\right| \leq \int_{0}^{t}\left(\int_{D} p^{0}(t-s, x, z)\left|p^{k-1}(s, z, y) \| \mu\right|(d z)\right) d s \\
+\int_{0}^{t}\left(\int_{D \times D} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}(z, w)\right|}{|z-w|^{d+\alpha}}\left|p^{k-1}(s, w, y)\right| d z d w\right) d s \\
\leq\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right) \\
\quad \times \int_{0}^{t}\left(\int_{D} p^{0}(t-s, x, z) p^{0}(s, z, y)|\mu|(d z)\right) d s \\
+\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right) \\
\quad \times \int_{0}^{t}\left(\int_{D \times D} p^{0}(t-s, x, z) \frac{c(z, w)\left|F_{1}\right|(z, w)}{|z-w|^{d+\alpha}} p^{0}(s, w, y) d z d w\right) d s \\
\leq \quad p^{0}(t, x, y)\left(\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\right. \\
\left.\quad+(k-1)\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-2}\right) C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t) .
\end{gathered}
$$

The proof is now complete.

Theorem 3.4. The series $\sum_{k=0}^{\infty} p^{k}(t, x, y)$ converges absolutely to a jointly continuous function $q_{D}(t, x, y)$ on $(0, \infty) \times D \times D$. Moreover, $q_{D}(t, x, y)$ is the transition density of the Feynman-Kac semigroup $\left(T_{t}^{\mu, F} ; t \geq 0\right)$, and there exists a positive constant $C_{6}:=C_{6}\left(d, \alpha, \gamma, C_{0}, M,\left\|F_{1}\right\|_{\infty}, T\right)$ such that

$$
\begin{equation*}
q_{D}(t, x, y) \leq e^{C_{6} t} q_{\gamma}(t, x, y) \tag{3.7}
\end{equation*}
$$

for every $(t, x, y) \in(0, \infty) \times D \times D$.
Proof. Let $\widetilde{p}^{k}(t, x, y)$ be defined as in (1.15) with $|\mu|$ and $\left|F_{1}\right|$ in place of $\mu$ and $F_{1}$. Clearly

$$
\begin{equation*}
\left|p^{k}(t, x, y)\right| \leq \tilde{p}^{k}(t, x, y) \quad \text { for every } k \geq 0 \tag{3.8}
\end{equation*}
$$

Set $\widetilde{q}_{D}(t, x, y)=\sum_{k=0}^{\infty} \widetilde{p}^{k}(t, x, y)$. One has from (1.11)-(1.14) with $|\mu|$ and $\left|F_{1}\right|$ in place of $\mu$ and $F_{1}$, where each term is non-negative, and Fubini's theorem that for every bounded $f \geq 0$ on $D$,

$$
\begin{equation*}
\mathbb{E}_{x}\left[f\left(X_{t}\right) \operatorname{Exp}\left(A^{|\mu|}+\sum_{s \leq \cdot}\left|F_{1}\right|\left(X_{s-}, X_{s}\right)\right)_{t}\right]=\int_{D} \widetilde{q}_{D}(t, x, y) f(y) d y \tag{3.9}
\end{equation*}
$$

Since $F_{1} \in \mathbf{J}_{\alpha, \gamma}$, there is $t_{1}:=t_{1}\left(d, \alpha, \gamma, C_{0}, M, N_{\mu, F_{1}}^{\alpha, \gamma},\left\|F_{1}\right\|_{\infty}\right) \in(0,1)$ so that

$$
N_{\mu, F_{1}}^{\alpha, \gamma}\left(t_{1}\right) \leq\left(3 C_{0}^{2} M\right)^{-1} \wedge\left(9\left(C_{0}^{2} M\right)^{2}\left\|F_{1}\right\|_{\infty}\right)^{-1}
$$

It follows from Lemma 3.3 that for every $(t, x, y) \in\left(0, t_{1}\right] \times D \times D$,

$$
\begin{align*}
& \widetilde{q}_{D}(t, x, y)=p^{0}(t, x, y)+\sum_{k=1}^{\infty} \widetilde{p}^{k}(t, x, y)  \tag{3.10}\\
& \begin{array}{l}
\leq p^{0}(t, x, y)+p^{0}(t, x, y)\left(\sum_{k=1}^{\infty}\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k}\right. \\
\left.\quad+\left\|F_{1}\right\|_{\infty} C_{0}^{2} M \sum_{k=1}^{\infty} k\left(C_{0}^{2} M N_{\mu, F_{1}}^{\alpha, \gamma}(t)\right)^{k-1}\right) \\
\leq p_{D}(t, x, y)+p_{D}(t, x, y)\left(\frac{1}{2}+\frac{9}{4}\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\right) \\
\leq c_{1} q_{\gamma}(t, x, y)
\end{array}
\end{align*}
$$

and $\widetilde{q}_{D}(t, x, y)$ is jointly continuous on $\left(0, t_{1}\right] \times D \times D$. Here

$$
c_{1}=C_{0}\left(\frac{3}{2}+\frac{9}{4}\left\|F_{1}\right\|_{\infty} C_{0}^{2} M\right)
$$

which depends only on $d, \alpha, \gamma, C_{0}, M$ and $\left\|F_{1}\right\|_{\infty}$.
Denote the left hand side of (3.9) by $T_{t} f$. Using the semigroup property of $T_{t}$, we see that

$$
\widetilde{q}_{D}(t, x, y)=\int_{D} \widetilde{q}_{D}(t / 2, x, z) \widetilde{q}_{D}(t / 2, z, y) d z, \quad \text { for all }(t, x, y) \in\left[0,2 t_{1}\right] \times D \times D
$$

is jointly continuous on $\left[0,2 t_{1}\right] \times D \times D$. Thus inductively we conclude that $\widetilde{q}_{D}(t, x, y)$ is continuous in $(t, x, y) \in(0, \infty) \times D \times D$. Moreover, by the two-sided
estimates of the rotationally symmetric stable process in $\mathbb{R}^{d}$, we have

$$
\int_{\mathbb{R}^{d}} q(t / 2, x, z) q(t / 2, z, y) d z \leq c_{2} q(t, x, y) \quad \text { for all }(t, x, y) \in[0, \infty) \times \mathbb{R}^{d} \times \mathbb{R}^{d}
$$

Thus, by the upper bound of $q_{D}$ on $\left[0, t_{1}\right] \times D \times D$ we have for $(t, x, y) \in\left(t_{1}, 2 t_{1}\right] \times$ $D \times D$,

$$
\begin{aligned}
\widetilde{q}_{D}(t, x, y) & \leq c_{1}^{2} \int_{D} q_{\gamma}(t / 2, x, z) q_{\gamma}(t / 2, z, y) d z \\
& \leq c_{1}^{2} \psi_{\gamma}(t, x) \psi_{\gamma}(t, y) \int_{\mathbb{R}^{d}} q(t / 2, x, z) q(t / 2, z, y) d z \leq c_{1}^{2} c_{2} q_{\gamma}(t, x, y)
\end{aligned}
$$

Iterating the above argument one can deduce that there is a constant $c_{3}=$ $c_{3}\left(d, \alpha, \gamma, C_{0}, M,\left\|F_{1}\right\|_{\infty}\right) \geq 0$ so that

$$
\widetilde{q}_{D}(t, x, y) \leq e^{c_{3} t} q_{\gamma}(t, x, y) \quad \text { for every } t>0 \text { and } x, y \in D .
$$

The conclusion of the theorem now follows from the fact that the total variational process of $A_{t}^{\mu}+\sum_{s \leq t} F_{1}\left(X_{s-}, X_{s}\right)$ is $A_{t}^{|\mu|}+\sum_{s \leq t}\left|F_{1}\right|\left(X_{s-}, X_{s}\right)$, (3.8), (1.11) (1.14) and Fubini's theorem.

For the lower bound estimate, we need to assume that $F$ is a function in $\mathbf{J}_{\alpha, \gamma}$.
Theorem 3.5. Suppose that $\mu \in \mathbf{K}_{\alpha, \gamma}$ and $F$ is a function in $\mathbf{J}_{\alpha, \gamma}$. Then for every $T>0$ there exists a positive constant $C_{7}:=C_{7}\left(\alpha, \gamma, C_{0}, M, N_{\mu, F}^{\alpha, \gamma},\|F\|_{\infty}, T\right) \geq 1$ such that

$$
\begin{equation*}
C_{7}^{-1} q_{\gamma}(t, x, y) \leq q_{D}(t, x, y) \leq C_{7} q_{\gamma}(t, x, y) \tag{3.11}
\end{equation*}
$$

for every $(t, x, y) \in(0, T] \times D \times D$.
Proof. Since $F$ is a bounded function in $\mathbf{J}_{\alpha, \gamma}$, so is $F_{1}$ with $\left|F_{1}(x, y)\right| \leq$ $e^{\|F\|_{\infty}}|F|(x, y)$ and $N_{F_{1}}^{\alpha, \gamma} \leq e^{\|F\|_{\infty}} N_{F}^{\alpha, \gamma}$. Thus the upper bound estimate in (3.11) follows directly from Theorem [3.4 To establish the lower bound, we define for $(t, x, y) \in(0, \infty) \times D \times D$,

$$
\begin{aligned}
\hat{p}^{1}(t, x, y)= & \int_{0}^{t}\left(\int_{D} p^{0}(t-s, x, z) p^{0}(s, z, y)|\mu|(d z)\right) d s \\
& +\int_{0}^{t}\left(\int_{D} \int_{D} p^{0}(t-s, x, z) \frac{c(z, w)|F|(z, w)}{|z-w|^{d+\alpha}} p^{0}(s, w, y) d z d w\right) d s
\end{aligned}
$$

Then for any bounded Borel function $f$ on $D$ and any $(t, x) \in(0, \infty) \times D$, we have

$$
\mathbb{E}_{x}\left[A_{t}^{|\mu|,|F|} f\left(X_{t}\right)\right]=\int_{D} \hat{p}^{1}(t, x, y) f(y) d y
$$

Applying Lemma3.3 with $|\mu|$ and $|F|$ in place of $\mu$ and $F_{1}$, we have, for all $(t, x, y) \in$ $(0,1] \times D \times D$,

$$
\hat{p}^{1}(t, x, y) \leq\left(C_{0}^{2} M N_{\mu, F}^{\alpha, \gamma}(1)+C_{0}^{2} M\|F\|_{\infty}\right) p^{0}(t, x, y) \leq(k / 2) p^{0}(t, x, y)
$$

where $k \geq 2$ is an integer so that $k \geq 2\left(C_{0}^{2} M N_{\mu, F}^{\alpha, \gamma}(1)+C_{0}^{2} M\|F\|_{\infty}\right)$. Hence we have for all $(t, x, y) \in(0,1] \times D \times D$,

$$
\begin{equation*}
p^{0}(t, x, y)-\frac{1}{k} \hat{p}^{1}(t, x, y) \geq \frac{1}{2} p^{0}(t, x, y) \tag{3.12}
\end{equation*}
$$

Using the elementary fact that

$$
1-A_{t}^{|\mu|,|F|} / k \leq \exp \left(-A_{t}^{|\mu|,|F|} / k\right) \leq \exp \left(A_{t}^{\mu, F} / k\right)
$$

we get that for any $B(x, r) \subset D$ and any $(t, y) \in(0,1] \times D$,
$\frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\left(1-A_{t}^{|\mu|,|F|} / k\right) \mathbf{1}_{B(x, r)}\left(X_{t}\right)\right] \leq \frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\exp \left(A_{t}^{\mu, F} / k\right) \mathbf{1}_{B(x, r)}\left(X_{t}\right)\right]$.
Thus, by (3.12) and Hölder's inequality, we have

$$
\begin{aligned}
& \frac{1}{2} \frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\mathbf{1}_{B(x, r)}\left(X_{t}\right)\right] \leq \frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\exp \left(A_{t}^{\mu, F} / k\right) \mathbf{1}_{B(x, r)}\left(X_{t}\right)\right] \\
& \leq\left(\frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\exp \left(A_{t}^{\mu, F}\right) \mathbf{1}_{B(x, r)}\left(X_{t}\right)\right]\right)^{1 / k}\left(\frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\mathbf{1}_{B(x, r)}\left(X_{t}\right)\right]\right)^{1-1 / k}
\end{aligned}
$$

Therefore,

$$
\frac{1}{2^{k}} \frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\mathbf{1}_{B(x, r)}\left(X_{t}\right)\right] \leq \frac{1}{|B(x, r)|} \mathbb{E}_{y}\left[\exp \left(A_{t}^{\mu, F}\right) \mathbf{1}_{B(x, r)}\left(X_{t}\right)\right]
$$

We conclude by sending $r \downarrow 0$ such that for every $(t, x, y) \in(0,1] \times D \times D$,

$$
q_{D}(t, x, y) \geq 2^{-k} p^{0}(t, x, y)
$$

By the semigroup property of $q_{D}(t, x, y)$, for every $(t, x, y) \in(0, n] \times D \times D$,

$$
\begin{aligned}
q_{D}(t, x, y) & =\int_{D} \cdots \int_{D} q_{D}\left(t / n, x, z_{1}\right) \cdots q_{D}\left(t / n, z_{n-1}, y\right) d z_{1} \cdots d z_{n-1} \\
& \geq 2^{-n k} \int_{D} \cdots \int_{D} p_{D}\left(t / n, x, z_{1}\right) \cdots p_{D}\left(t / n, z_{n-1}, y\right) d z_{1} \cdots d z_{n-1}
\end{aligned}
$$

Since for each fixed $n \geq 1$,

$$
\begin{aligned}
p_{D}(t / n, z, w) & \asymp q_{\gamma}(t / n, z, w) \asymp q_{\gamma}\left(t / n^{2}, z, w\right) \\
& \asymp p_{D}\left(t / n^{2}, z, w\right), \quad(t, z, w) \in(0, n] \times D \times D,
\end{aligned}
$$

by (1.3), from the semigroup property of $p_{D}(t, x, y)$, we have for each $(t, x, y) \in$ $(0, n] \times D \times D$,

$$
\begin{aligned}
& q_{D}(t, x, y) \geq c_{1} \int_{D} \cdots \int_{D} p_{D}\left(t / n^{2}, x, z_{1}\right) \cdots p_{D}\left(t / n^{2}, z_{n-1}, y\right) d z_{1} \cdots d z_{n-1} \\
& =c_{1} p_{D}(t / n, x, y) \geq c_{2} q_{\gamma}(t / n, x, y) \geq c_{3} q_{\gamma}(t, x, y)
\end{aligned}
$$

This proves the theorem.
Combining the two theorems above, we immediately get the main result of this paper, Theorem 1.3 .

## 4. Applications

In this section, we will apply our main result to (reflected) symmetric stablelike processes, killed symmetric $\alpha$-stable processes, censored $\alpha$-stable processes and stable processes with drifts. We first record the following two facts.

Suppose that $d \geq 2$ and $\alpha \in(0,2)$. A locally finite signed measure $\mu$ on $\mathbb{R}^{d}$ is said to be in Kato class $\mathbb{K}_{d, \alpha}$ if

$$
\lim _{r \rightarrow 0} \sup _{x \in \mathbb{R}^{d}} \int_{B(x, r)} \frac{1}{|x-y|^{d-\alpha}}|\mu|(d y)=0 .
$$

A measurable function $g$ on $\mathbb{R}^{d}$ is said to be in $\mathbb{K}_{d, \alpha}$ if $g(x) d x \in \mathbb{K}_{d, \alpha}$.
Proposition 4.1. Suppose that $d \geq 2$ and $\alpha \in(0,2)$.
(i) Let $D$ be an arbitrary Borel subset of $\mathbb{R}^{d} . \mu \in \mathbf{K}_{\alpha, 0}$ if and only if $\mathbf{1}_{D} \mu \in$ $\mathbb{K}_{d, \alpha}$. Hence for every $\mu \in \mathbb{K}_{d, \alpha},\left.\mu\right|_{D} \in \mathbf{K}_{\alpha, \gamma}$ for every $\gamma \geq 0$. In particular, $L^{\infty}(D ; d x) \subset \mathbf{K}_{\alpha, \gamma}$ and $L^{p}(D ; d x) \subset \mathbf{K}_{\alpha, \gamma}$ for every $p>d / \alpha$ and $\gamma \geq 0$.
(ii) Suppose that $D$ is a bounded Lipschitz open set in $\mathbb{R}^{d}$ and $\gamma \in(0, \alpha)$. Let $g$ be a measurable function defined on $D$. If there exist constants $c>0$, $\beta \in(0, \gamma+(\alpha-\gamma) / d)$ and a compact subset $K$ of $D$ such that $\mathbf{1}_{K}(x) g(x) \in$ $\mathbb{K}_{d, \alpha}$ and

$$
|g(x)| \leq c \delta_{D}(x)^{-\beta} \quad \text { for } x \in D \backslash K
$$

then $g \in \mathbf{K}_{\alpha, \gamma}$.
Proof. (i) By the proof of [33, Theorem 2], we have that $\mu \in \mathbb{K}_{d, \alpha}$ if and only if

$$
\lim _{t \rightarrow 0} \sup _{x \in \mathbb{R}^{d}} \int_{0}^{t} \int_{\mathbb{R}^{d}} q(s, x, y) \mu(d y) d s=0
$$

This implies that $\mu \in \mathbf{K}_{\alpha, 0}$ if and only if $\mathbf{1}_{D} \mu \in \mathbb{K}_{d, \alpha}$. In particular we have for every $\mu \in \mathbb{K}_{d, \alpha},\left.\mu\right|_{D} \in \mathbf{K}_{\alpha, \gamma}$ for every $\gamma \geq 0$. Clearly $L^{\infty}(D ; d x) \subset \mathbb{K}_{d, \alpha}$. Using Hölder's inequality, one concludes that $L^{p}\left(\mathbb{R}^{d} ; d x\right) \subset \mathbb{K}_{d, \alpha}$ for every $p>d / \alpha$.
(ii) Let $g$ be a measurable function defined on $D$ such that there exist constants $c_{1}>0, \beta \in(0, \gamma+(\alpha-\gamma) / d)$ and a compact subset $K$ of $D$ so that $\mathbf{1}_{K}(x) g(x) \in \mathbb{K}_{d, \alpha}$ and $|g(x)| \leq c_{1} \delta_{D}(x)^{-\beta}$ for $x \in D \backslash K$. In view of (i), it suffices to show that $\mathbf{1}_{D \backslash K} g \in \mathbf{K}_{\alpha, \gamma}$. Note that

$$
\begin{align*}
\sup _{x \in D} & \int_{0}^{t} \int_{D \backslash K} \psi_{\gamma}(s, y) q(s, x, y)|g(y)| d y d s  \tag{4.1}\\
\leq & c_{1} \sup _{x \in D} \int_{0}^{t} \int_{D \backslash K} \psi_{\gamma}(s, y) \delta_{D}(y)^{-\beta} q(s, x, y) d y d s \\
\leq & c_{1} \sup _{x \in D} \int_{D \backslash K}\left(\int_{0}^{\delta_{D}(y)^{\alpha} \wedge t}\left(s^{-d / \alpha} \wedge \frac{s}{|x-y|^{d+\alpha}}\right) d s\right) \delta_{D}(y)^{-\beta} d y \\
& +c_{1} \sup _{x \in D} \int_{D \backslash K}\left(\int_{\delta_{D}(y)^{\alpha} \wedge t}^{t} s^{-\gamma / \alpha}\left(s^{-d / \alpha} \wedge \frac{s}{|x-y|^{d+\alpha}}\right) d s\right) \delta_{D}(y)^{\gamma-\beta} d y \\
= & I+I I
\end{align*}
$$

Here
(4.2) $I \leq c_{1} \sup _{x \in D}\left(\int_{D} \int_{0}^{\delta_{D}(y)^{\alpha} \wedge|x-y|^{\alpha} \wedge t} \frac{s}{|x-y|^{d+\alpha}} d s \delta_{D}(y)^{-\beta} d y\right.$

$$
\left.+\int_{D} \int_{\delta_{D}(y)^{\alpha} \wedge|x-y|^{\alpha} \wedge t}^{\delta_{D}(y)^{\alpha} \wedge t} s^{-d / \alpha} d s \delta_{D}(y)^{-\beta} d y\right)
$$

$$
\leq c_{2} \sup _{x \in D} \int_{D}\left(\frac{\left(\delta_{D}(y)^{\alpha} \wedge|x-y|^{\alpha} \wedge t\right)^{2} \delta_{D}(y)^{-\beta}}{|x-y|^{d+\alpha}}\right.
$$

$$
\left.+\mathbf{1}_{\left\{|x-y|<\delta_{D}(y) \wedge t^{1 / \alpha}\right\}} \frac{\delta_{D}(y)^{-\beta}}{|x-y|^{d-\alpha}}\right)
$$

$$
\leq c_{2} \sup _{x \in D} \int_{D}\left(\frac{\left(|x-y| \wedge t^{1 / \alpha}\right)^{2 \alpha-\beta}}{|x-y|^{d}}+\mathbf{1}_{\left\{|x-y|<\delta_{D}(y) \wedge t^{1 / \alpha}\right\}} \frac{1}{|x-y|^{d-\alpha+\beta}}\right) d y
$$

$$
\leq 2 c_{2} t^{(\alpha-\beta) /(2 \alpha)} \sup _{x \in D} \int_{D} \frac{1}{|x-y|^{d-(\alpha-\beta) / 2}} d y=c_{3} t^{(\alpha-\beta) /(2 \alpha)}
$$

while

$$
\begin{align*}
I I \leq & c_{1} \sup _{x \in D} \int_{D}\left(\int_{\delta_{D}(y)^{\alpha} \wedge t}^{t} \mathbf{1}_{\left\{s<|x-y|^{\alpha}\right\}} \frac{s^{1-\gamma / \alpha}}{|x-y|^{d+\alpha}} d s\right) \delta_{D}(y)^{\gamma-\beta} d y  \tag{4.3}\\
& +c_{1} \sup _{x \in D} \int_{D}\left(\int_{\delta_{D}(y)^{\alpha} \wedge t}^{t} \mathbf{1}_{\left\{s \geq|x-y|^{\alpha}\right\}} s^{-(d+\gamma) / \alpha} d s\right) \delta_{D}(y)^{\gamma-\beta} d y \\
\leq & c_{4} \sup _{x \in D} \int_{D} \frac{\left(|x-y| \wedge t^{1 / \alpha}\right)^{2 \alpha-\gamma}}{|x-y|^{d+\alpha}} \mathbf{1}_{\left\{\delta_{D}(y)<|x-y| \wedge t^{1 / \alpha}\right\}} \delta_{D}(y)^{\gamma-\beta} d y \\
& +c_{4} \sup _{x \in D} \int_{D}|x-y|^{\alpha-d-\gamma} \mathbf{1}_{\left\{|x-y| \leq t^{1 / \alpha}\right\}} \delta_{D}(y)^{\gamma-\beta} d y \\
\leq & c_{4} t^{\delta / \alpha} \sup _{x \in D} \int_{D} \frac{1}{|x-y|^{d-\alpha+\varepsilon} \delta_{D}(y)^{\beta-\gamma}} d y
\end{align*}
$$

where $\delta:=(\alpha-\gamma-d(\beta-\gamma)) / 2>0$ and $\varepsilon:=(\alpha+\gamma-d(\beta-\gamma)) / 2>0$. Note that $\varepsilon+\delta=\alpha-d(\beta-\gamma)$ and $\varepsilon-\delta=\gamma$. Let $p=d /(d-\alpha+\varepsilon+\delta / 2)$ and $q=d /(\alpha-(\varepsilon+\delta / 2))$ so that $1 / p+1 / q=1$. Since $D$ is a bounded Lipschitz open set, $p(d-\alpha+\varepsilon)<d$ and $q(\beta-\gamma)<1$, we have by Young's inequality:

$$
\begin{aligned}
& \sup _{x \in D} \int_{D} \frac{1}{|x-y|^{d-\alpha+\varepsilon} \delta_{D}(y)^{\beta-\gamma}} d y \\
& \quad \leq \sup _{x \in D} \int_{D}\left(\frac{1}{p} \frac{1}{|x-y|^{p(d-\alpha+\varepsilon)}}+\frac{1}{q} \frac{1}{\delta_{D}(y)^{q(\beta-\gamma)}}\right) d y<\infty .
\end{aligned}
$$

This together with (4.1) -(4.3) implies that $\lim _{t \rightarrow 0} N_{g \mathbf{1}_{D \backslash K}}^{\alpha, \gamma}(t)=0$; that is, $g \mathbf{1}_{D \backslash K} \in$ $\mathbf{K}_{\alpha, \gamma}$. This completes the proof of the proposition.

Proposition 4.2. Suppose $\gamma \in[0, \alpha \wedge d)$ and $|F|(z, w) \leq A\left(|z-w|^{\beta} \wedge 1\right)$ for some $A>0$ and $\beta>\alpha$. Then there exists $C_{8}=C_{8}(\beta, d, \alpha, \gamma)>0$ such that for every arbitrary Borel subset $D$ of $\mathbb{R}^{d}$,

$$
\begin{equation*}
N_{F}^{\alpha, \gamma}(t) \leq C_{8} A t \tag{4.4}
\end{equation*}
$$

This in particular implies that $F \in \mathbf{J}_{\alpha, \gamma}$.

Proof. By (2.4), we have that

$$
\begin{aligned}
& \int_{0}^{t} \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} q(s, y, w)\left(1+\frac{|z-w| \wedge t^{1 / \alpha}}{|y-w|}\right)^{\gamma} \frac{|F|(z, w)+|F|(w, z)}{|z-w|^{d+\alpha}} d z d w d s \\
\leq & 2 A\left(\int_{\mathbb{R}^{d}}\left(|z|^{\beta} \wedge 1\right)|z|^{-d-\alpha} d z\right) \int_{0}^{t} \int_{\mathbb{R}^{d}} q(s, y, w)\left(1+\frac{t^{1 / \alpha}}{|y-w|}\right)^{\gamma} d w d s \\
\leq & c_{1} A\left(\int_{B(0,1)} \frac{d z}{|z|^{d+\alpha-\beta}}+\int_{B(0,1)^{c}} \frac{d z}{|z|^{d+\alpha}}\right) \\
& \times \int_{0}^{t}\left(1+\int_{D} \frac{s t^{\gamma / \alpha}}{\left(s^{1 / \alpha}+|y-w|\right)^{d+\alpha}|y-w|^{\gamma}}\right) d w d s \\
\leq & c_{2} A \int_{0}^{t}\left(1+t^{\gamma / \alpha} s \int_{0}^{\infty} \frac{r^{d-1}}{r^{\gamma}\left(s^{1 / \alpha}+r\right)^{d+\alpha}} d r\right) d s \\
\leq & c_{2} A t+c_{3} A\left(\int_{0}^{\infty} \frac{u^{d-1-\gamma}}{(1+u)^{d+\alpha}} d u\right) t^{\gamma / \alpha} \int_{0}^{t} s^{-\gamma / \alpha} d s \leq c_{4} A t
\end{aligned}
$$

where the assumption $\gamma \in[0, \alpha \wedge d)$ is used in the last inequality. This establishes (4.4).
4.1. Stable-like processes on closed $d$-sets. A Borel subset $D$ in $\mathbb{R}^{d}$ with $d \geq 1$ is said to be a $d$-set if there exist constants $r_{0}>0, \mathcal{C}_{2}>\mathcal{C}_{1}>0$ so that

$$
\begin{equation*}
\mathcal{C}_{1} r^{d} \leq|B(x, r) \cap D| \leq \mathcal{C}_{2} r^{d} \quad \text { for all } x \in D \text { and } 0<r \leq r_{0} \tag{4.5}
\end{equation*}
$$

where for a Borel set $A \subset \mathbb{R}^{d}$, we use $|A|$ to denote its Lebesgue measure. The notion of a $d$-set arises both in the theory of function spaces and in fractal geometry. It is known that if $D$ is a $d$-set, then so is its Euclidean closure $\bar{D}$. Every uniformly Lipschitz open set in $\mathbb{R}^{d}$ is a $d$-set, so is its Euclidean closure. It is easy to check that the classical von Koch snowflake domain in $\mathbb{R}^{2}$ is an open 2 -set. A $d$-set can have very rough boundary since every $d$-set with a subset of zero Lebesgue measure removed is still a $d$-set.

Suppose that $D$ is a closed $d$-set in $\mathbb{R}^{d}$ and $c(x, y)$ is a symmetric function on $D \times D$ that is bounded between two strictly positive constants $\mathcal{C}_{4}>\mathcal{C}_{3}>0$, that is,

$$
\begin{equation*}
\mathcal{C}_{3} \leq c(x, y) \leq \mathcal{C}_{4} \quad \text { for a.e. } x, y \in D . \tag{4.6}
\end{equation*}
$$

For $\alpha \in(0,2)$, we define

$$
\begin{align*}
\mathcal{F} & =\left\{u \in L^{2}(D ; d x): \int_{D \times D} \frac{(u(x)-u(y))^{2}}{|x-y|^{d+\alpha}} d x d y<\infty\right\}  \tag{4.7}\\
\mathcal{E}(u, v) & =\frac{1}{2} \int_{D \times D}(u(x)-u(y))(v(x)-v(y)) \frac{c(x, y)}{|x-y|^{d+\alpha}} d x d y, \quad u, v \in \mathcal{F} . \tag{4.8}
\end{align*}
$$

It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^{2}(D, d x)$ and therefore there is an associated symmetric Hunt process $X$ on $D$ starting from every point in $D$ except for an exceptional set that has zero capacity. The process $X$ is called a symmetric $\alpha$-stable-like process on $D$ in [11. When $c(x, y)$ is a constant function, $X$ is the reflected $\alpha$-stable process that appeared in [2]. Note that when $D=\mathbb{R}^{d}$ and $c(x, y)$ is a constant function, then $X$ is nothing but a symmetric $\alpha$-stable process on $\mathbb{R}^{d}$.

It follows as a special case from [11, Theorem 1.1] that the symmetric stablelike process $X$ on a closed $d$-set in $\mathbb{R}^{d}$ has a Hölder continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on $D$ that satisfies the estimate (1.3) with $\gamma=0$ and the comparison constant $C_{0}$ depending only on $d, \alpha$, $r_{0}$ and the constants $\mathcal{C}_{k}, k=1, \cdots, 4$, in (4.5) and (4.6). In particular, this implies that the process $X$ can be refined so it can start from every point in $D$. Thus as a special case of Theorem 1.3 we have the following.

Theorem 4.3. Suppose that $X$ is a symmetric $\alpha$-stable-like process on a closed $d$-set $D$ in $\mathbb{R}^{d}$. Assume $\mu \in \mathbf{K}_{\alpha, 0}$ and $F \in \mathbf{J}_{\alpha, 0}$. Let $q$ be the density of the Feynman-Kac semigroup of $X$ corresponding to $A^{\mu, F}$. For any $T>0$, there exists a constant $C_{9} \geq 1$ such that for all $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{9}^{-1} q(t, x, y) \leq q(t, x, y) \leq C_{9} q(t, x, y)
$$

Remark 4.4. Let $n \geq 1$ be an integer and $d \in(0, n]$. In general, a Borel subset $D$ in $\mathbb{R}^{n}$ is said to be a $d$-set if there exist a measure $\mu$ and constants $r_{0}>0, \mathcal{C}_{2}>\mathcal{C}_{1}>0$ so that

$$
\begin{equation*}
\mathcal{C}_{1} r^{d} \leq \mu(B(x, r) \cap D) \leq \mathcal{C}_{2} r^{d} \quad \text { for all } x \in D \text { and } 0<r \leq r_{0} \tag{4.9}
\end{equation*}
$$

It is established in 11 that for every $\alpha \in(0,2)$, a symmetric $\alpha$-stable-like process $X$ can always be constructed on any closed $d$-set $D$ in $\mathbb{R}^{n}$ via the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^{2}(D ; \mu)$ defined by (4.7) (4.8) but with the $d$-measure $\mu(d x)$ in place of the Lebesgue measure $d x$ there. Moreover by [11, Theorem 1.1], the process $X$ has a jointly Hölder continuous transition density function $p(t, x, y)$ with respect to the $d$-measure $\mu$ on $D$ that satisfies the estimate (1.3) with $\gamma=0$. The proof of Theorem 1.3 also works for such process $X$; in other words, Theorem 4.3 continues to hold for such kinds of symmetric stable-like processes.
4.2. Killed symmetric $\alpha$-stable processes. A symmetric $\alpha$-stable process $X$ in $\mathbb{R}^{d}$ is a Lévy process whose characteristic function is given by $\mathbb{E}_{0}\left[\exp \left(i \xi \cdot X_{t}\right)\right]=$ $e^{-t|\xi|^{\alpha}}$. It is well-known that the process $X$ has a Lévy intensity function $J(x, y)=$ $\mathcal{A}(d,-\alpha)|x-y|^{-(d+\alpha)}$, where

$$
\begin{equation*}
\mathcal{A}(d,-\alpha)=\alpha 2^{-1+\alpha} \Gamma\left(\frac{d+\alpha}{2}\right) \pi^{-d / 2}\left(\Gamma\left(1-\frac{\alpha}{2}\right)\right)^{-1} \tag{4.10}
\end{equation*}
$$

Here $\Gamma$ is the Gamma function defined by $\Gamma(\lambda):=\int_{0}^{\infty} t^{\lambda-1} e^{-t} d t$ for every $\lambda>0$. Let $X^{D}$ be the killed symmetric $\alpha$-stable process $X^{D}$ in a $C^{1,1}$ open set $D$. It follows from [7] that $X^{D}$ satisfies the assumption of Section 1 with $\gamma=\alpha / 2$. Thus as a special case of Theorem 1.3, we have the following.

Theorem 4.5. Suppose that $X$ is a killed symmetric $\alpha$-stable process in a $C^{1,1}$ open set $D$. Assume $\mu \in \mathbf{K}_{\alpha, \alpha / 2}$ and $F \in \mathbf{J}_{\alpha, \alpha / 2}$. Let $q_{D}$ be the density of the Feynman-Kac semigroup of $X$ corresponding to $A^{\mu, F}$. For any $T>0$, there exists a constant $C_{10} \geq 1$ such that for all $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{10}^{-1} q_{\alpha / 2}(t, x, y) \leq q_{D}(t, x, y) \leq C_{10} q_{\alpha / 2}(t, x, y)
$$

Let $X^{m}$ be a relativistic $\alpha$-stable process in $\mathbb{R}^{d}$ with mass $m>0$; i.e., $X^{m}$ is a Lévy process in $\mathbb{R}^{d}$ with

$$
\mathbb{E}_{0}\left[\exp \left(i \xi \cdot X_{t}^{m}\right)\right]=\exp \left(t\left(m-\left(|\xi|^{2}+m^{2 / \alpha}\right)^{\alpha / 2}\right)\right)
$$

$X^{m}$ has a Lévy intensity function $J^{m}(x, y)=\mathcal{A}(d,-\alpha) \varphi\left(m^{1 / \alpha}|x-y|\right)|x-y|^{-d-\alpha}$ where

$$
\begin{equation*}
\varphi(r):=2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_{0}^{\infty} s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^{2}}{s}} d s \tag{4.11}
\end{equation*}
$$

which is decreasing and is a smooth function of $r^{2}$ satisfying $\varphi(0)=1$ and

$$
\begin{equation*}
\varphi(r) \asymp e^{-r}\left(1+r^{(d+\alpha-1) / 2}\right) \quad \text { on }[0, \infty) \tag{4.12}
\end{equation*}
$$

(see [13, pp. 276-277] for details).
Let $X^{m, D}$ be a killed relativistic $\alpha$-stable process in a bounded $C^{1,1}$ open set. Define

$$
K_{t}^{m}:=\exp \left(\sum_{0<s \leq t} \ln \left(\varphi\left(m^{1 / \alpha}\left|X_{s}^{D}-X_{s-}^{D}\right|\right)\right)+m\left(t \wedge \tau_{D}\right)\right)
$$

Since $\int_{\mathbb{R}^{d}} J(x, y)-J^{m}(x, y) d y=m$ for all $x \in \mathbb{R}^{d}$ (see [25), it follows from [13, p. 279] that $X^{m, D}$ can be obtained from the killed symmetric $\alpha$-stable process $X^{D}$ in $D$ through the non-local Feynman-Kac transform $K_{t}^{m}$. That is, $\mathbb{E}_{x}\left[f\left(X_{t}^{m, D}\right)\right]:=$ $\mathbb{E}_{x}\left[K_{t}^{m} f\left(X_{t}^{D}\right)\right]$. By (4.11), for any $M>0$, there exists a constant $c=c(d, \alpha, M$, $\operatorname{diam}(D))>0$ such that for all $m \in(0, M],\left|\ln \left(\varphi\left(m^{1 / \alpha}|x-y|\right)\right)\right| \leq c\left(|x-y|^{2} \wedge 1\right)$, and so, by Proposition 4.2, $F_{m}(x, y):=\ln \left(\varphi\left(m^{1 / \alpha}|x-y|\right)\right) \in \mathbf{J}_{\alpha, \alpha / 2}$. The constant function $m$ is in $\mathbf{K}_{\alpha, \alpha / 2}$, and so $N_{m, F_{m}}^{\alpha, \alpha / 2}(t)$ goes to zero as $t$ goes to zero uniformly on $m \in(0, M]$. Thus, as an application of Theorem [1.3, we arrive at the following result, which is the bounded open set case of a more general result recently obtained in 9 by a different method.
Theorem 4.6. Suppose that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^{d}$. For any $m>0$, let $p_{D}^{m}$ be the transition density of the killed relativistic $\alpha$-stable process with weight $m$ in $D$. For any $M>0$ and $T>0$, there exists a constant $C_{11} \geq 1$ such that for all $m \in(0, M]$ and $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{11}^{-1} q_{\alpha / 2}(t, x, y) \leq p_{D}^{m}(t, x, y) \leq C_{11} q_{\alpha / 2}(t, x, y)
$$

4.3. Censored stable processes. Fix an open set $D$ in $\mathbb{R}^{d}$ with $d \geq 1$. Recall that $\mathcal{A}(d,-\alpha)$ is the constant defined in (4.10). Define a bilinear form $\mathcal{E}$ on $C_{c}^{\infty}(D)$ by

$$
\begin{equation*}
\mathcal{E}(u, v):=\frac{1}{2} \int_{D} \int_{D}(u(x)-u(y))(v(x)-v(y)) \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} d x d y, \quad u, v \in C_{c}^{\infty}(D) \tag{4.13}
\end{equation*}
$$

Using Fatou's lemma, it is easy to check that the bilinear form $\left(\mathcal{E}, C_{c}^{\infty}(D)\right)$ is closable in $L^{2}(D, d x)$. Let $\mathcal{F}$ be the closure of $C_{c}^{\infty}(D)$ under the Hilbert inner product $\mathcal{E}_{1}:=\mathcal{E}+(\cdot, \cdot)_{L^{2}(D, d x)}$. As noted in [2] $(\mathcal{E}, \mathcal{F})$ is Markovian and hence a regular symmetric Dirichlet form on $L^{2}(D, d x)$, and therefore there is an associated symmetric Hunt process $Y=\left\{Y_{t}, t \geq 0, \mathbb{P}_{x}, x \in D\right\}$ taking values in $D$ (cf. Theorem 3.1.1 of [18]). The process $Y$ is the censored $\alpha$-stable process in $D$ that is studied
in [2]. By (4.13), the jumping kernel $J(x, y)$ of the censored $\alpha$-stable process $Y$ is given by

$$
J(x, y)=\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} \quad \text { for } x, y \in D
$$

As a particular case of a more general result established in [8, Theorem 1.1], when $\alpha \in(1,2)$ and $D$ is a $C^{1,1}$ open subset of $\mathbb{R}^{d}$, the censored $\alpha$-stable process on $D$ satisfies the assumption of Section $\mathbb{1}$ with $\gamma=\alpha-1$. Thus as a special case of Theorem 1.3, we have the following:

Theorem 4.7. Suppose that $\alpha \in(1,2)$ and that $Y$ is a censored stable process in a $C^{1,1}$ open set $D$. Assume $\mu \in \mathbf{K}_{\alpha, \alpha-1}$ and $F \in \mathbf{J}_{\alpha, \alpha-1}$. Let $q_{D}$ be the density of the Feynman-Kac semigroup of $Y$ corresponding to $A^{\mu, F}$. For any $T>0$, there exists a constant $C_{12} \geq 1$ such that for all $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{12}^{-1} q_{\alpha-1}(t, x, y) \leq q_{D}(t, x, y) \leq C_{12} q_{\alpha-1}(t, x, y)
$$

Similarly to [2, we can define a censored relativistic $\alpha$-stable process in $D$. Alternatively, with

$$
\begin{aligned}
& K_{t}:=\exp \left(\sum _ { 0 < s \leq t } \operatorname { l n } \left(\varphi\left(m^{1 / \alpha}\left(\left|Y_{s-}-Y_{s}\right|\right)\right)\right.\right. \\
&\left.+\mathcal{A}(d,-\alpha) \int_{0}^{t} \int_{D} \frac{1-\varphi\left(m^{1 / \alpha}\left|Y_{s}-y\right|\right)}{\left|Y_{s}-y\right|^{\alpha+d}} d y d s\right)
\end{aligned}
$$

if $D$ is a bounded $C^{1,1}$ open set, a censored relativistic stable process $Y^{m}$ can also be obtained from the censored stable process $Y$ through the Feynman-Kac transform $K_{t}$. That is, $\mathbb{E}_{x}\left[f\left(Y_{t}^{m}\right)\right]=\mathbb{E}_{x}\left[K_{t} f\left(Y_{t}\right)\right]$ (see [6, 13]). By an argument similar to that of Subsection 4.2, one can see that $F_{m}:=\ln \left(\varphi\left(m^{1 / \alpha}|x-y|\right)\right) \in \mathbf{J}_{\alpha, \alpha / 2}$. Moreover, since

$$
\begin{aligned}
g_{m}(x) & :=\int_{D}\left(1-\varphi\left(m^{1 / \alpha}|x-y|\right)\right)|x-y|^{-\alpha-d} d y \\
& \leq \int_{\mathbb{R}^{d}}\left(1-\varphi\left(m^{1 / \alpha}|x-y|\right)\right)|x-y|^{-\alpha-d} d y=m
\end{aligned}
$$

$g_{m} \in \mathbf{K}_{\alpha, \alpha / 2}$ and $N_{g_{m}, F_{m}}^{\alpha, \alpha / 2}(t)$ goes to zero as $t$ goes to zero uniformly on $m \in(0, M]$. Thus as a particular case of Theorem 4.7 we have the following.

Theorem 4.8. Suppose that $\alpha \in(1,2)$ and that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^{d}$. For any $m>0$, let $q_{D}^{m}$ be the transition density of the censored relativistic $\alpha$-stable process with weight $m$ in $D$. For any $M>0$ and $T>0$, there exists a constant $C_{13} \geq 1$ such that for all $m \in(0, M]$ and $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{13}^{-1} q_{\alpha-1}(t, x, y) \leq q_{D}^{m}(t, x, y) \leq C_{13} q_{\alpha-1}(t, x, y)
$$

In fact, Theorems 4.7 and 4.8 are applicable to certain classes of censored stablelike processes whose Dirichlet heat kernel estimates are given in [8].
4.4. Stable processes with drifts. Let $\alpha \in(1,2)$ and $d \geq 2$. In this subsection, we apply our main result to a non-symmetric process.

For $b=\left(b_{1}, \ldots, b_{d}\right)$ with $b_{i} \in \mathbb{K}_{d, \alpha-1}$, a Feller process $Z$ on $\mathbb{R}^{d}$ with infinitesimal generator $\mathcal{L}^{b}:=\Delta^{\alpha / 2}+b(x) \cdot \nabla$ is constructed in [3] through the fundamental solution of $\mathcal{L}^{b}$. Let $Z^{D}$ be the subprocess of $Z$ killed upon leaving $D$. The following result is established in [10.

Theorem 4.9. If $\alpha \in(1,2), d \geq 2$ and $D$ is a bounded $C^{1,1}$ open set, then $Z^{D}$ has a jointly continuous transition density function $p_{D}(t, x, y)$ that satisfies (1.3) with $\gamma=\alpha / 2$.

Thus as a special case of Theorem 1.3, we also have the following.
Theorem 4.10. Suppose that $\alpha \in(1,2), d \geq 2$, that $D$ is a bounded $C^{1,1}$ open set and that $Z^{D}$ is the subprocess of $Z$ killed upon leaving $D$. Assume $\mu \in \mathbf{K}_{\alpha, \alpha / 2}$ and $F \in \mathbf{J}_{\alpha, \alpha / 2}$. Let $q_{D}$ be the density of the Feynman-Kac semigroup of $Z^{D}$ corresponding to $A^{\mu, F}$. For any $T>0$, there exists a constant $C_{14} \geq 1$ such that for all $(t, x, y) \in(0, T] \times D \times D$,

$$
C_{14}^{-1} q_{\alpha / 2}(t, x, y) \leq q_{D}(t, x, y) \leq C_{14} q_{\alpha / 2}(t, x, y)
$$

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