MEAN CURVATURE FLOW OF KILLING GRAPHS

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ABSTRACT. We study a Neumann problem related to the evolution of graphs under mean curvature flow in Riemannian manifolds endowed with a Killing vector field. We prove that in a particular case these graphs converge to a trivial minimal graph which contacts the cylinder over the domain orthogonally along its boundary.

1. INTRODUCTION

Let M be an $(n + 1)$ -dimensional Riemannian manifold endowed with a Killing vector field Y . Suppose that the distribution orthogonal to Y is of constant rank and integrable. Given an integral leaf P of that distribution, let $\Omega \subset P$ be a bounded domain with regular boundary $\Gamma = \partial\Omega$. Let $\vartheta : \mathbb{I} \times \bar{\Omega} \rightarrow M$ be the flow generated by Y with initial values in M , where $\mathbb{I}$ is a maximal interval of definition. In geometric terms, the ambient manifold is a warped product $M = P \times_{1/\sqrt{\gamma}} \mathbb{I}$ where $\gamma = 1/|Y|^2$.

Given $T \in [0, +\infty)$, let $u : \bar{\Omega} \times [0, T) \rightarrow \mathbb{I}$ be a smooth function. Fixing this notation, the Killing graph of $u(\cdot, t)$, $t \in [0, T)$, is the hypersurface $\Sigma_t \subset M$ parametrized by the map

$$X(t, x) = \vartheta(u(x, t), x), \quad x \in \bar{\Omega}.$$

Notice that this definition could be slightly more general if we suppose that the coordinates of $x \in \bar{\Omega}$ change with the parameter $t \in [0, T)$. To abolish this possibility is equivalent to ruling out tangential diffeomorphisms of Ω .

The Killing cylinder K over Γ is in turn defined by

$$(1.1) \quad K = \{\vartheta(s, x) : s \in \mathbb{I}, x \in \Gamma\}.$$

Let N be a unit normal vector field along Σ_t . In what follows, we denote by H the mean curvature of Σ_t with respect to the orientation given by N . We are then concerned with establishing conditions for longtime existence of a prescribed mean curvature flow of the form

$$(1.2) \quad \frac{\partial X}{\partial t} = (nH - \mathcal{H})N,$$

$$(1.3) \quad X(0, \cdot) = \vartheta(u_0(\cdot), \cdot),$$

for given functions $u_0 : \bar{\Omega} \rightarrow \mathbb{R}$ and $\mathcal{H} : \bar{\Omega} \rightarrow \mathbb{R}$. In order to define boundary conditions for the evolution problem (1.2) we consider a function $\phi \in C^\infty(\Gamma)$ such that $|\phi| \leq \phi_0 < 1$ for some positive constant ϕ_0 . Let ν be the inward unit normal

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vector field along K . We impose the following Neumann condition associated to (1.2):

$$(1.4) \quad \langle N, \nu \rangle|_{\partial \Sigma_t} = \phi,$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric in M .

The main result in this paper may be stated as follows.

Theorem 1. *There exists a unique solution $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{I}$ to the problem (1.2)-(1.4). Moreover, if $\phi = 0$ and $\mathcal{H} = 0$, the functions $u(\cdot, t)$ converge to a constant function as $t \rightarrow \infty$.*

Theorem 1 extends Theorem 1.1 in [3] as well as Theorem 2.4 in [2] and Theorem 2.4 in [1] in a twofold way. The corresponding theorems in [3] and [2] concern evolution of graphs in Euclidean space whereas [1] deals with the case of graphs in Riemannian product spaces of the form $P \times \mathbb{R}$. Moreover those earlier results hold only for the case when the prescribed mean curvature is $\mathcal{H} = 0$. An existence result for evolution of graphs in Euclidean space by the Gauss-Kronecker curvature under Neumann boundary conditions is proved in [8]. We also mention that the Dirichlet problem for the evolution of graphs in warped spaces is extensively studied in [7].

The paper is organized as follows. Section 2 describes the evolution problem in nonparametric terms. Height and boundary gradient *a priori* estimates for (1.2)-(1.4) are presented respectively in Sections 3 and 4. Interior gradient estimates are obtained in Section 5. Some technical computations needed in the body of the proofs are compiled in an appendix. In Section 6 we prove the convergence of the mean curvature flow (1.2) for $\phi = 0$ and $\mathcal{H} = 0$ to slices of the form $\vartheta(\{s\} \times \Omega)$ for some $s \in \mathbb{I}$. This generalizes the corresponding convergence statement of Theorem 1.1 in [3].

2. FUNDAMENTAL EQUATIONS

Since we will consider the mean curvature flow in nonparametric terms it seems adequate to describe all geometric invariants as well as their evolution equations in terms of graphical coordinates.

Let $x^1, \dots, x^n$ be local coordinates in P . This system is augmented to be a coordinate system in M by setting $x^0 = s$, the flow parameter of Y . The tangent space of Σ_t at a point $X(t, x)$, $x \in \bar{\Omega}$, is spanned by the coordinate vector fields

$$(2.1) \quad X_* \frac{\partial}{\partial x^i} = \vartheta_* \frac{\partial}{\partial x^i} + u_i \vartheta_* \frac{\partial}{\partial x^0} = \frac{\partial}{\partial x^i} \Big|_X + u_i \frac{\partial}{\partial x^0} \Big|_X.$$

In terms of these coordinates the induced metric in Σ_t is expressed in local components by

$$(2.2) \quad g_{ij} = \sigma_{ij} + \frac{1}{\gamma} u_i u_j,$$

where $\gamma = \frac{1}{|Y|^2}$ and σ_{ij} are the local components of the metric in P .

In order to compute the mean curvature of Σ_t , we fix N as the vector field

$$(2.3) \quad N = \frac{1}{W} (\gamma Y - \vartheta_* \nabla u),$$

where ∇u is the gradient of u in P and

$$(2.4) \quad W = \sqrt{\gamma + |\nabla u|^2}.$$

The second fundamental form of Σ_t calculated with respect to this choice of normal vector field has local components

$$(2.5) \quad a_{ij} = \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} X_* \frac{\partial}{\partial x^j}, N \rangle,$$

where $\bar{\nabla}$ denotes the covariant derivative in M . We then compute

$$\begin{aligned} a_{ij} &= \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} u_j \vartheta_* \frac{\partial}{\partial x^0}, N \rangle \\ &= \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + u_i \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^0}} \vartheta_* \frac{\partial}{\partial x^j}, N \rangle + u_j \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^i}} \vartheta_* \frac{\partial}{\partial x^0}, N \rangle \\ &\quad + u_{i,j} \langle \vartheta_* \frac{\partial}{\partial x^0}, N \rangle + u_i u_j \langle \bar{\nabla}_{\vartheta_* \frac{\partial}{\partial x^0}} \vartheta_* \frac{\partial}{\partial x^0}, N \rangle. \end{aligned}$$

Hence, using the fact that the maps $x \mapsto \vartheta(s, x)$ are isometries and that the hypersurfaces defined by $\{\vartheta(s, x) : x \in P\}$, $s \in \mathbb{I}$, are totally geodesic, one concludes that

$$\begin{aligned} a_{ij} &= \langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, -\frac{1}{W} \nabla u \rangle + u_i \langle \bar{\nabla}_{\frac{\partial}{\partial x^j}} Y, \frac{1}{W} \gamma Y \rangle + u_j \langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y, \frac{1}{W} \gamma Y \rangle \\ &\quad + u_{i,j} \langle Y, \frac{1}{W} \gamma Y \rangle + u_i u_j \langle \bar{\nabla}_Y Y, -\frac{1}{W} \nabla u \rangle. \end{aligned}$$

It follows from Killing's equation that

$$(2.6) \quad a_{ij} = \frac{u_{i,j}}{W} - \frac{u_i}{W} \frac{\gamma_j}{2\gamma} - \frac{u_j}{W} \frac{\gamma_i}{2\gamma} - \frac{u_i u_j}{2W} u^k \frac{\gamma_k}{\gamma^2}.$$

It turns out that a_{ij} could also be expressed by

$$(2.7) \quad a_{ij} = \frac{u_{i,j}}{W} - \frac{u_i}{W} \gamma \langle \bar{\nabla}_Y Y, \frac{\partial}{\partial x^j} \rangle - \frac{u_j}{W} \gamma \langle \bar{\nabla}_Y Y, \frac{\partial}{\partial x^i} \rangle - \frac{u_i u_j}{W} \langle \bar{\nabla}_Y Y, \nabla u \rangle.$$

Taking traces with respect to the induced metric, one obtains the following expression for the mean curvature H of the hypersurface Σ_t :

$$(2.8) \quad nH = \left(\sigma^{ij} - \frac{u^i u^j}{W} \right) \frac{u_{i,j}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \langle \frac{\bar{\nabla} \gamma}{2\gamma}, \nabla u \rangle.$$

Alternatively one has

$$(2.9) \quad nH = \left(\sigma^{ij} - \frac{u^i u^j}{W} \right) \frac{u_{i,j}}{W} - \frac{2\gamma + |\nabla u|^2}{W^3} \gamma \langle \bar{\nabla}_Y Y, \nabla u \rangle.$$

At this point we recall that

$$(2.10) \quad \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y = -\frac{1}{2} \frac{\gamma_i}{\gamma} Y$$

and

$$(2.11) \quad \bar{\nabla}_Y Y = \frac{1}{2} \frac{\nabla \gamma}{\gamma^2},$$

which implies that

$$(2.12) \quad \langle \bar{\nabla}_Y Y, \nabla u \rangle = -\langle \bar{\nabla}_{\nabla u} Y, Y \rangle = \frac{1}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle.$$

Using this one easily verifies that (2.8) may be written in divergence form as

$$(2.13) \quad \operatorname{div} \frac{\nabla u}{W} - \frac{1}{2\gamma W} \langle \nabla \gamma, \nabla u \rangle = nH.$$

In fact we have

$$\left(\frac{u^i}{W}\right)_{;i} = \frac{1}{W}u^i_{;i} - \frac{1}{W^3}u^iu^ju_{i;j} - \frac{1}{2W^3}u^i\gamma_i.$$

It is worth pointing out that (2.13) is equivalent to

$$(2.14) \quad \operatorname{div} \frac{\nabla u}{W} - \frac{\gamma}{W} \langle \bar{\nabla}_Y Y, \nabla u \rangle = nH.$$

We conclude that (1.2) may be written nonparametrically as

$$(2.15) \quad \frac{\partial u}{\partial t} = W \operatorname{div} \frac{\nabla u}{W} - W\mathcal{H} - \gamma \langle \bar{\nabla}_Y Y, \nabla u \rangle.$$

Indeed it holds that

$$nH - \mathcal{H} = \left\langle \frac{\partial X}{\partial t}, N \right\rangle = \left\langle \frac{\partial u}{\partial t} \vartheta_* \frac{\partial}{\partial x^0}, \frac{\gamma}{W} \vartheta_* \frac{\partial}{\partial x^0} \right\rangle = \frac{1}{W} \frac{\partial u}{\partial t}.$$

Using (2.8) one verifies that (2.15) is equivalent to

$$(2.16) \quad \frac{\partial u}{\partial t} = \left(\sigma^{ij} - \frac{u^i}{W} \frac{u^j}{W} \right) u_{i;j} - \frac{2\gamma + |\nabla u|^2}{W^2} \left\langle \frac{\bar{\nabla} \gamma}{2\gamma}, \nabla u \right\rangle - W\mathcal{H}.$$

We conclude that the Neumann problem (1.2)-(1.4) has the nonparametric form

$$(2.17) \quad u_t = \left(\sigma^{ij} - \frac{u^i}{W} \frac{u^j}{W} \right) u_{i;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \gamma^i u_i - W\mathcal{H} \quad \text{in } \Omega \times [0, T),$$

$$(2.18) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega \times \{0\},$$

with boundary condition

$$(2.19) \quad \langle N, \nu \rangle = \phi \quad \text{on } \partial\Omega \times [0, T).$$

This boundary value problem describes the evolution of the Killing graph of the function $u(\cdot, t)$ by its mean curvature in the direction of the unit normal N with prescribed contact angle at the boundary.

The standard theory for quasilinear parabolic equations [5] guarantees that the problem of solving (1.2)-(1.4) is reduced to obtaining *a priori* height and gradient estimates for solutions to (2.17)-(2.19).

3. HEIGHT ESTIMATES

From now on, we consider the parabolic linear operator given by

$$(3.1) \quad \mathcal{L}v = g^{ij}v_{i;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \gamma^i v_i - \mathcal{H} \frac{u^i}{W} v_i - v_t,$$

where $v \in C^\infty(\Omega \times [0, T))$.

Proposition 1. *For a solution $u \in C^\infty(\bar{\Omega} \times [0, T^*])$, $T^* < T$, of (2.17)-(2.19), it holds that*

$$\max_{\bar{\Omega} \times [0, T^*]} |u_t| = \max_{\bar{\Omega}} |u_t(0, \cdot)|.$$

Then it follows that

$$\max_{\bar{\Omega} \times [0, T^*]} |u| \leq CT^*$$

for a given constant $C > 0$ which depends on T^ .*

Proof. First of all we verify that u_t is a solution for a linear parabolic equation. Indeed one has

$$\begin{aligned}\mathcal{L}u_t &= g^{ij}u_{ti;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt} \\ &= (g^{ij}u_{i;j})_t - g_{;t}^{ij}u_{i;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle\nabla\gamma, \nabla u_t\rangle - u_{tt} \\ &= -g_{;t}^{ij}u_{i;j} + \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)_t\langle\nabla\gamma, \nabla u\rangle + \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle\nabla\gamma_t, \nabla u\rangle + W_t\mathcal{H}.\end{aligned}$$

However since $\gamma = \gamma(x)$ in (2.16) and x is independent of t it follows that

$$\left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)_t = \left(\frac{1}{2\gamma}\right)_t - \frac{1}{W^3}W_t = -\frac{1}{2W^4}(\gamma_t + 2u^k u_{k;t}) = -\frac{1}{W^4}u^k u_{t;k}.$$

In the same way we have

$$(3.2) \quad W_t = \frac{1}{2W}(\gamma_t + 2u^k u_{k;t}) = \frac{1}{W}u^k u_{t;k}.$$

We conclude that

$$\mathcal{L}u_t = -g_{;t}^{ij}u_{i;j} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.$$

Now using the fact that $\sigma_{;t}^{ij} = 0$ and $\gamma_t = 0$ we have

$$\begin{aligned}\mathcal{L}u_t &= \frac{2}{W}\left(\frac{u_{;t}^i u^j}{W} - \frac{u^i}{W}\frac{u^j}{W}W_t\right)u_{i;j} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k \\ &= \frac{2}{W}\left((W_i - \frac{\gamma_i}{2W})u_{;t}^i - (W_i - \frac{\gamma_i}{2W})\frac{u^i}{W}\frac{u^k}{W}u_{t;k}\right) \\ &\quad - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k \\ &= \frac{2}{W}(W_i - \frac{\gamma_i}{2W})(\sigma^{ik} - \frac{u^i}{W}\frac{u^k}{W})u_{t;k} - \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k + \frac{1}{W}\mathcal{H}u^k(u_t)_k.\end{aligned}$$

Hence it follows that

$$(3.3) \quad \mathcal{L}u_t - \frac{2}{W}g^{ik}(W_i - \frac{\gamma_i}{2W})(u_t)_k + \frac{1}{W^4}\langle\nabla\gamma, \nabla u\rangle u^k(u_t)_k - \frac{1}{W}\mathcal{H}u^k(u_t)_k = 0.$$

Thus fix $T^* \in [0, T)$ and let (x_0, t_0) be a point in $\bar{\Omega} \times [0, T^*]$ such that

$$u_t(x_0, t_0) = \max_{\bar{\Omega} \times [0, T^*]} |u_t|.$$

Hence we choose a coordinate system adapted to the boundary Γ in such a way that $\frac{\partial}{\partial x^n} = \nu$ at x_0 . Then, at the point (x_0, t_0) we have

$$u_{i;t} = u_{t;i} = 0$$

for $1 \leq i < n$ which implies that

$$W_t = \frac{1}{W}u^n u_{n;t} = -\phi(x_0)u_{n;t},$$

where we used (2.19) and (3.2). On the other hand, (2.19) implies that

$$(3.4) \quad u_{t;n} = u_{n;t} = -(\phi W)_t = -\phi(x_0)W_t$$

at (x_0, t_0) . We conclude that

$$(1 - \phi^2(x_0))u_{n;t} = 0.$$

However since $|\phi| < 1$, it follows that $u_{t;n} = 0$ which contradicts the parabolic Hopf Lemma [5].

From this contradiction we conclude that $t_0 = 0$. Since T^* is arbitrary, the conclusion follows. $\square$

4. BOUNDARY GRADIENT ESTIMATES

Now we will prove a gradient bound for a solution of (2.17)-(2.19) by applying a modification of Korevaar's technique [4] which formerly appeared in [2].

From now on, we consider a nonnegative extension $d : \bar{\Omega} \rightarrow \mathbb{R}$ of the distance function $\text{dist}_P(\cdot, \Gamma)$ satisfying $|\nabla d| \leq 1$ in $\bar{\Omega}$. In the same way, we consider a C^∞ extension of the boundary data ϕ to the domain $\bar{\Omega}$ which we also denote by ϕ . Then we define

$$(4.1) \quad \eta = e^{Ku}h,$$

where

$$(4.2) \quad h = 1 + \alpha d - \phi \langle \nabla d, N \rangle,$$

where K and α are positive numbers to be fixed later.

Proposition 2. *For $\alpha > 0$ sufficiently large independent of K and t , if for some $t \geq 0$ fixed, $\eta W(\cdot, t)$ attains a local maximum value at a point $x_0 \in \partial\Omega$, then $W(x_0, t) \leq K$.*

Proof. Let $t \geq 0$ be such that

$$\max_{\bar{\Omega}} \eta W(t, \cdot) = \eta W(t, x_0)$$

for a point $x_0 \in \Gamma$. Hence we choose a coordinate system adapted to Γ such that $\frac{\partial}{\partial x^n} = \nu$ at x_0 and

$$(4.3) \quad u_1(x_0) \geq 0 \quad \text{and} \quad u_i(x_0) = 0, \quad \text{for } 2 \leq i \leq n-1.$$

We have at x_0

$$(4.4) \quad 0 = (\eta W)_1 = \eta_1 W + \eta W_1 = e^{Ku} (W K u_1 (1 - \phi^2) - 2W \phi \phi_1 + W_1 (1 - \phi^2))$$

from which follows that

$$(4.5) \quad W_1 = -K u_1 W + \frac{2\phi\phi_1}{(1-\phi^2)} W.$$

On the other hand, at x_0 we have

$$\begin{aligned} \eta_n &= e^{Ku} (K u_n (1 - \phi^2) + \alpha - \phi \phi_n - \phi (\langle \nabla_{\nabla d} N, \nabla d \rangle + \langle N, \nabla_{\nabla d} \nabla d \rangle)) \\ &= e^{Ku} (K u_n (1 - \phi^2) + \alpha - \phi \phi_n - \phi (\langle \partial_n \frac{1}{W} (\gamma Y - \nabla u), \partial_n \rangle \\ &\quad + \langle \frac{1}{W} \nabla_{\partial_n} (\gamma Y - \nabla u), \partial_n \rangle)) \\ &= e^{Ku} (K u_n (1 - \phi^2) + \alpha - \phi \phi_n - \frac{1}{W^2} \phi u_n W_n + \frac{1}{W} \phi u_{n;n}). \end{aligned}$$

Since $(\eta W)_n \leq 0$ at x_0 it holds that

$$\begin{aligned} 0 &\geq W K u_n (1 - \phi^2) + \alpha W - W \phi \phi_n - \frac{1}{W} \phi u_n W_n + \phi u_{n;n} + (1 - \phi^2) W_n \\ &= W K u_n (1 - \phi^2) + \alpha W + W_n + \phi u_{n;n} + u_n \phi_n \\ &= W K u_n (1 - \phi^2) + \alpha W + W_n + \phi u_{n;n} - W \phi \phi_n. \end{aligned}$$

On the other hand,

$$(4.6) \quad W_n = \frac{\gamma_n}{2W} + \frac{1}{W}(u_1 u_{1;n} + u_n u_{n;n}) = \frac{\gamma_n}{2W} - \frac{1}{W}\phi u_1 W_1 - \phi_1 u_1 - \phi u_{n;n},$$

which implies that

$$\begin{aligned} W_n &= \frac{\gamma_n}{2W} - \frac{1}{W}\phi u_1 \left(\frac{2\phi\phi_1 W}{1-\phi^2} - K u_1 W \right) - \phi_1 u_1 - \phi u_{n;n} \\ &= \frac{\gamma_n}{2W} - \frac{1+\phi^2}{1-\phi^2} u_1 \phi_1 + K \phi u_1^2 - \phi u_{n;n}. \end{aligned}$$

Therefore since

$$u_1^2 = |\nabla u|^2 - u_n^2 = W^2 - \gamma - \phi^2 W^2 = W^2(1 - \phi^2) - \gamma,$$

we conclude that

$$\begin{aligned} 0 &\geq \alpha + \frac{\gamma_n}{2W^2} - \frac{1+\phi^2}{1-\phi^2} \frac{u_1}{W} \phi_1 + \frac{K \phi u_1^2}{W} - \phi \phi_n + K u_n(1 - \phi^2) \\ &= \alpha + \frac{\gamma_n}{2W^2} + \frac{1+\phi^2}{1-\phi^2} N_1 \phi_1 + K \phi \left(W(1 - \phi^2) - \frac{\gamma}{W} \right) - \phi \phi_n - K \phi W(1 - \phi^2) \\ &= \alpha + \frac{\gamma_n}{2W^2} + \frac{1+\phi^2}{1-\phi^2} N_1 \phi_1 - \frac{K \phi \gamma}{W} - \phi \phi_n \\ &\geq \alpha + C - \frac{K \gamma}{W} \end{aligned}$$

for a given constant C depending solely on γ and ϕ . It follows that $W(x_0, t) \leq K$ if α is chosen large enough and independent of K and t . $\square$

5. INTERIOR GRADIENT ESTIMATES

In this section we deduce a global gradient bound using the techniques in [1] and [2]. However the more general context of warped product gives rise to a long list of additional terms which require careful tracking along the calculations.

In the sequel, we consider the parabolic linear operator given by

$$(5.1) \quad Lv = g^{ij} v_{i;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \gamma^i v_i - v_t,$$

where $v \in C^\infty(\Omega \times [0, T])$.

Proposition 3. *For fixed $T^* < T$ there exists $K > 0$ sufficiently large so that if*

$$\eta W(x_0, t_0) = \max_{\bar{\Omega} \times [0, T^*]} \eta W$$

for some $(x_0, t_0) \in \bar{\Omega} \times [0, T^]$, then $W(x_0, t_0) \leq C$ for some constant C .*

Proof. We can assume $x_0 \in \Omega$ and $t_0 > 0$. At a point (x_0, t_0) where ηW attains maximum value we have

$$(5.2) \quad \eta_i W + \eta W_i = 0$$

and

$$(5.3) \quad \frac{1}{\eta} L\eta + \frac{1}{W} \left(LW - \frac{2}{W} g^{ij} W_i W_j \right) \leq 0.$$

We conclude that

$$\begin{aligned}\frac{1}{\eta}L\eta &= KLu + \frac{1}{h}Lh + K^2g^{ij}u_iu_j + 2Kg^{ij}u_i\frac{h_j}{h} \\ &= K\mathcal{H}W + \frac{1}{h}Lh + K^2\frac{\gamma|\nabla u|^2}{W^2} + 2Kg^{ij}u_i\frac{h_j}{h}.\end{aligned}$$

Now we have

$$g^{ij}u_ih_j = \frac{\gamma}{W^2}u^jh_j = -\frac{\gamma}{W}(\alpha\langle N, \nabla d \rangle - \langle N, \nabla \phi \rangle \theta - \phi \langle N, \nabla \theta \rangle).$$

However

$$\langle N, \nabla \theta \rangle = W\langle AY^T, \nabla^\Sigma d \rangle + \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle - \kappa \frac{|\nabla u|^2}{W^2}.$$

Therefore

$$\begin{aligned}g^{ij}u_ih_j &= -\alpha\frac{\gamma}{W}\langle N, \nabla d \rangle + \frac{\gamma}{W}\langle N, \nabla \phi \rangle \theta + \gamma\phi\langle AY^T, \nabla^\Sigma d \rangle \\ &\quad + \frac{\gamma}{W}\phi\langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle - \gamma\phi\kappa\frac{|\nabla u|^2}{W^3}.\end{aligned}$$

Thus the expression for Lh in the Appendix allows us to conclude that

$$\begin{aligned}\frac{1}{\eta}L\eta &= K\mathcal{H}W + K^2\frac{\gamma|\nabla u|^2}{W^2} \\ &\quad + \frac{2K}{h}\left(-\alpha\frac{\gamma}{W}\langle N, \nabla d \rangle + \frac{\gamma}{W}\langle N, \nabla \phi \rangle \theta + \gamma\phi\langle AY^T, \nabla^\Sigma d \rangle \right. \\ &\quad \left. + \frac{\gamma}{W}\phi\langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle - \gamma\phi\kappa\frac{|\nabla u|^2}{W^3}\right) \\ &\quad + \frac{1}{h}|A|^2\phi\theta + n\frac{1}{h}\phi HW\langle AY^T, \nabla^\Sigma d \rangle + \frac{1}{h}\left(\kappa\gamma - \frac{1}{2}\langle \nabla d, \nabla \gamma \rangle\right)\phi\langle AY^T, Y^T \rangle \\ &\quad + \frac{2}{h}\langle A\nabla^\Sigma d, \nabla^\Sigma \phi \rangle + \frac{2}{h}\langle A, \nabla^2 d \rangle_\Sigma \phi - \frac{1}{hW^2}\phi\langle A\nabla^\Sigma d, X_*\nabla \gamma \rangle \\ &\quad + \frac{1}{h}(nH - \mathcal{H})\left(\langle N, \nabla \phi \rangle \theta - \alpha\theta - \frac{1}{2W^2}\langle \nabla \gamma, \nabla d \rangle \phi + \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle \phi\right) - n\frac{1}{h}\kappa H\phi \\ &\quad - n\frac{\alpha}{h}Hd + \frac{1}{h}\left(2\langle N, \nabla \phi \rangle - \alpha\right)\langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle + \frac{2}{h}\langle \nabla_{\frac{\nabla u}{W}} \nabla d, \nabla \phi \rangle \\ &\quad - \frac{1}{h}\phi\langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle + \frac{1}{h}\phi\langle \nabla^\Sigma \mathcal{H}, \bar{\nabla} d \rangle + n\frac{1}{h}\langle \nabla H_d, N \rangle \phi - \frac{1}{h}\phi\nabla^3 d\left(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W}\right) \\ &\quad + \frac{1}{h}\text{Ric}(\nabla d, \frac{\nabla u}{W})\phi + \frac{\gamma}{hW^2}\langle N, \nabla \kappa \rangle \phi - \frac{1}{h}\left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\alpha\langle \nabla d, \nabla \gamma \rangle \\ &\quad + \frac{1}{h}\left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\langle \nabla \phi, \nabla \gamma \rangle \theta \\ &\quad - \kappa\frac{1}{h}\phi\langle N, \frac{\nabla \gamma}{2\gamma} \rangle + \frac{2}{h}\kappa\frac{\gamma}{W^2}\langle \nabla \phi, N \rangle - \frac{1}{h}(\Delta\phi - \langle \nabla_{\frac{\nabla u}{W}} \nabla \phi, \frac{\nabla u}{W} \rangle)\theta.\end{aligned}$$

On the other hand Lemma 3 yields

$$\begin{aligned} \frac{1}{W} \left(LW - \frac{2}{W} g^{ij} W_i W_j \right) &= |A|^2 + nHW^2 \langle AY^T, Y^T \rangle - nHW^2 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle \\ &\quad - 3 \frac{1}{W} \gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle + g^{ij} \frac{\gamma_{i;j}}{2\gamma} - \frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 + \gamma \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle \\ &\quad - \langle \nabla^\Sigma \mathcal{H}, N \rangle - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W^2} - \frac{W_t}{W}. \end{aligned}$$

Now we use the fact that x_0 is a critical point to ηW . We have

$$e^{Ku} (Ku_i h + h_i) W = -e^{Ku} h W_i,$$

which implies that

$$-KW^2 h N_i N^i + W h_i N^i = -h W_i N^i$$

and then

$$-Kh|\nabla u|^2 + W h_i N^i = -h W_i N^i.$$

However

$$\begin{aligned} W_i N^i &= \frac{\gamma_i}{2\gamma} N^i W + N_i N^i W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W^2 \langle AY^T, N^i X_* \frac{\partial}{\partial x^i} \rangle \\ &= \frac{1}{2\gamma} \langle \nabla \gamma, N \rangle W + |\nabla u|^2 W \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - W^3 \langle AY^T, Y^T \rangle \end{aligned}$$

and

$$\begin{aligned} h_i N^i &= \alpha \theta - \langle \nabla \phi, N \rangle \theta + \phi a_i^j N^i d_j - \phi (d_{i;j} N^i N^j - \kappa \sigma_{ij}) N^i N^j \\ &= \alpha \theta - \langle \nabla \phi, N \rangle \theta - \phi W \langle AY^T, \nabla^\Sigma d \rangle - \phi \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle + \phi \kappa \frac{|\nabla u|^2}{W^2}. \end{aligned}$$

We then conclude that

$$\begin{aligned} &-K \frac{|\nabla u|^2}{W} + \frac{\alpha \theta}{h} - \frac{1}{h} \langle \nabla \phi, N \rangle \theta - \frac{\phi}{h} W \langle AY^T, \nabla^\Sigma d \rangle - \frac{\phi}{h} \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle + \frac{\phi}{h} \kappa \frac{|\nabla u|^2}{W^2} \\ &= -\frac{1}{2\gamma} \langle \nabla \gamma, N \rangle - |\nabla u|^2 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W^2 \langle AY^T, Y^T \rangle. \end{aligned}$$

Moreover

$$\begin{aligned} -\frac{W_t}{W} &= \frac{\eta_t}{\eta} = Ku_t + \frac{h_t}{h} = WK(nH - \mathcal{H}) + \frac{h_t}{h} \\ &= nH KW - KW \mathcal{H} - \frac{1}{h} (nH - \mathcal{H}) \left(\langle \nabla \phi, N \rangle \theta - \alpha \theta - \frac{\phi}{2W^2} \langle \nabla \gamma, \nabla d \rangle \right) \\ &\quad + \frac{\phi}{h} \langle \frac{\nabla u}{W}, \nabla_{\frac{\nabla u}{W}} \nabla d \rangle. \end{aligned}$$

Then we have

$$\begin{aligned}
& \frac{1}{W} \left(LW - \frac{2}{W} g^{ij} W_i W_j \right) \\
&= |A|^2 + KnH \frac{\gamma}{W} + \frac{\alpha\theta}{h} nH - \frac{1}{h} nH \langle \nabla \phi, N \rangle \theta - \frac{\phi}{h} nHW \langle AY^T, \nabla^\Sigma d \rangle \\
&\quad - \frac{1}{h} (nH - \mathcal{H}) \left(\langle \nabla \phi, N \rangle \theta - \alpha\theta - \frac{\phi}{2W^2} \langle \nabla \gamma, \nabla d \rangle \right) - \frac{\phi}{h} nH \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle \\
&\quad + \frac{\phi}{h} nH \kappa \frac{|\nabla u|^2}{W^2} - 3 \frac{1}{W} \gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle + g^{ij} \frac{\gamma_{i;j}}{2\gamma} - \frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 \\
&\quad + \gamma \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - \langle \nabla^\Sigma \mathcal{H}, N \rangle - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W^2} - KW\mathcal{H} + \frac{\phi}{h} \langle \frac{\nabla u}{W}, \nabla_{\frac{\nabla u}{W}} \nabla d \rangle.
\end{aligned}$$

We conclude that

$$\frac{1}{\eta} L\eta + \frac{1}{W} \left(LW - \frac{2}{W} g^{ij} W_i W_j \right) = K^2 \frac{\gamma |\nabla u|^2}{W^2} + \mathcal{A} + \mathcal{B},$$

where

$$\begin{aligned}
\mathcal{A} &= \left(1 + \frac{\phi\theta}{h} \right) |A|^2 + \frac{2K}{h} \gamma \phi \langle AY^T, \nabla^\Sigma d \rangle + \frac{\phi}{h} \left(\kappa\gamma - \frac{1}{2} \langle \nabla d, \nabla \gamma \rangle \right) \langle AY^T, Y^T \rangle \\
&\quad + \frac{2}{h} \langle A \nabla^\Sigma d, \nabla^\Sigma \phi \rangle + \frac{2}{h} \langle A, \nabla^2 d \rangle_\Sigma \phi - \frac{1}{hW^2} \phi \langle A \nabla^\Sigma d, X_* \nabla \gamma \rangle \\
&\quad + KnH \frac{\gamma}{W} + \frac{\alpha\theta}{h} nH - \frac{1}{h} nH \langle \nabla \phi, N \rangle \theta - \frac{\phi}{h} nH \kappa \frac{\gamma}{W^2} - 3 \frac{1}{W} \gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= \frac{2K}{h} \left(-\alpha \frac{\gamma}{W} \langle N, \nabla d \rangle + \frac{\gamma}{W} \langle N, \nabla \phi \rangle \theta + \frac{\gamma}{W} \phi \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle - \gamma \phi \kappa \frac{|\nabla u|^2}{W^3} \right) \\
&\quad - \mathcal{H} \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle \phi - n \frac{\alpha}{h} H_d + \frac{1}{h} (2 \langle N, \nabla \phi \rangle - \alpha) \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle \\
&\quad + \frac{2}{h} \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \nabla \phi \rangle - \frac{1}{h} \phi \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle + \frac{1}{h} \phi \langle \nabla^\Sigma \mathcal{H}, \bar{\nabla} d \rangle + n \frac{1}{h} \langle \nabla H_d, N \rangle \phi \\
&\quad - \frac{1}{h} \phi \nabla^3 d \left(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W} \right) + \frac{1}{h} \text{Ric} \left(\nabla d, \frac{\nabla u}{W} \right) \phi + \frac{\gamma}{hW^2} \langle N, \nabla \kappa \rangle \phi \\
&\quad - \frac{1}{h} \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \alpha \langle \nabla d, \nabla \gamma \rangle + \frac{1}{h} \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \langle \nabla \phi, \nabla \gamma \rangle \theta - \kappa \frac{1}{h} \phi \langle N, \frac{\nabla \gamma}{2\gamma} \rangle \\
&\quad + \frac{2}{h} \kappa \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle - \frac{1}{h} (\Delta \phi - \langle \nabla_{\frac{\nabla u}{W}} \nabla \phi, \frac{\nabla u}{W} \rangle) \theta + g^{ij} \frac{\gamma_{i;j}}{2\gamma} - \frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} \\
&\quad - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 + \gamma \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - \langle \nabla^\Sigma \mathcal{H}, N \rangle - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W^2} + \frac{\phi}{h} \langle \frac{\nabla u}{W}, \nabla_{\frac{\nabla u}{W}} \nabla d \rangle.
\end{aligned}$$

However, using some standard inequalities, we obtain

$$\begin{aligned}
\mathcal{A} &\geq \left(1 + \frac{\phi\theta}{h} \right) |A|^2 - \left(\frac{2K\gamma}{h\sqrt{\gamma}} + \frac{\kappa}{h} + \frac{1}{2h\gamma} |\nabla \gamma| + \frac{2}{h} |\nabla \phi| + \frac{2}{h} |\nabla^2 d|_\Sigma \right. \\
&\quad \left. + \frac{1}{hW^2} |X_* \nabla \gamma| + \frac{K\gamma\sqrt{n}}{W} + \frac{\alpha\theta\sqrt{n}}{h} + \frac{\theta\sqrt{n}}{h} |\nabla \phi| + \frac{\gamma\sqrt{n}\kappa}{hW^2} + \frac{3\gamma}{\sqrt{\gamma}W} |X_* \frac{\nabla \gamma}{2\gamma}| \right) |A|.
\end{aligned}$$

Using that $W^2 \geq \gamma$ and choosing α sufficiently large and depending only on n, γ, ϕ and κ , we have

$$\begin{aligned} \mathcal{A} &\geq \frac{1}{2}|A|^2 - \left(\epsilon + 2\sqrt{\gamma}\frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W}|X_*\frac{\nabla\gamma}{2\gamma}| \right)|A| \\ &\geq -\left(\epsilon + 2\sqrt{\gamma}\frac{K}{h} + \frac{K\gamma\sqrt{n}}{W} + \frac{3\sqrt{\gamma}}{W}|X_*\frac{\nabla\gamma}{2\gamma}| \right)^2. \end{aligned}$$

Moreover

$$\mathcal{B} \geq -C \left(1 + \frac{\alpha}{h} + \frac{\alpha}{hW^2} + \frac{1}{h} + \frac{1}{W^2} + \frac{1}{hW^2} + K\frac{\alpha}{h} + \frac{K}{h} \right),$$

where C is a constant depending on $n, \gamma, \phi, d, \kappa$ and $\mathcal{H}$.

Hence we obtain

$$\begin{aligned} \frac{1}{\eta}L\eta + \frac{1}{W} \left(\mathcal{L}W - \frac{2}{W}g^{ij}W_iW_j \right) &\geq K^2\frac{\gamma|\nabla u|^2}{W^2} - C(\epsilon) - \frac{K}{W}C(\epsilon, \gamma, n) - \frac{K^2}{W^2}C(\gamma, n) \\ &- \frac{1}{W}C(\gamma, \epsilon) - \frac{K}{W^2}C(\gamma, n) - \frac{K^2}{h^2}C(\gamma) - \frac{K^2}{hW}C(\gamma, n) - \frac{1}{W^2}C(\gamma) - \frac{K}{hW}C(\gamma) \\ &- K\frac{\alpha}{h}C - \frac{K}{h}C(\epsilon, \gamma) - C - \frac{\alpha}{h}C - \frac{\alpha}{hW^2}C - \frac{1}{h}C - \frac{1}{W^2}C - \frac{1}{hW^2}C. \end{aligned}$$

Then

$$\begin{aligned} -K^2\frac{\gamma|\nabla u|^2}{W^2} &\geq -C \left(\frac{K^2}{W^2} + \frac{K}{W^2} + \frac{1}{W} + \frac{K}{hW} + \frac{K^2}{h^2} + \frac{\alpha}{hW^2} + \frac{1}{W^2} + \frac{1}{hW^2} \right. \\ &\quad \left. + \frac{K}{W} + \frac{1}{W} + K\frac{\alpha}{h} + \frac{K}{h} + \frac{\alpha}{h} + \frac{1}{h} + 1 \right). \end{aligned}$$

It follows that

$$\begin{aligned} \left(K^2\gamma - \left(\frac{K^2}{h^2} + K\frac{\alpha}{h} + \frac{K}{h} + \frac{1+\alpha}{h} + 1 \right) C \right) W^2 &\leq \left(K^2 + K + \frac{1+\alpha}{h} + 1 \right) C \\ &+ (K + \frac{K}{h} + 1) CW. \end{aligned}$$

Now suppose that $W(x_0, t_0) \geq 1$. Otherwise we are done. In this case we have $W \leq W^2$ and absorbing the terms with W into the one with W^2 transforms the inequality above into

$$\begin{aligned} \left(K^2\gamma - \frac{K^2}{h^2}C - \frac{K}{h}C - KC - C - \frac{1}{h}(\alpha + 1)(K + 1)C \right) W^2 \\ \leq \left(K^2 + K + 1 + \frac{1}{h}(\alpha + 1) \right) C. \end{aligned}$$

If $d_0 = d(x_0)$, then choosing $\alpha \geq 1/(C(d_0)d_0 - 1)$ for some constant $C(d_0) > 1/d_0$, we obtain $(1 + \alpha)/h \leq C(d_0)$ which implies that

$$\left(K^2\gamma - \frac{K^2}{h^2}C - \frac{K}{h}C - KC(d_0) - C(d_0) \right) W^2 \leq (K^2 + K + C(d_0))C.$$

Then for $\alpha > \frac{1}{d_0} \max\{1, \sqrt{2C/\gamma}\}$ we have

$$\left(K^2\frac{\gamma}{2} - KC(d_0) - C(d_0) \right) W^2 \leq (K^2 + K + C(d_0))C.$$

It follows that for $K > \frac{C(d_0) + \sqrt{C(d_0)^2 + 2\gamma C(d_0)}}{\gamma}$ we have $K^2 \frac{\gamma}{2} - KC(d_0) - C(d_0) > 0$ and

$$(5.4) \quad W^2 \leq \frac{C(K^2 + K + C(d_0))}{K^2 \frac{\gamma}{2} - KC(d_0) - C(d_0)}.$$

This finishes the proof of the proposition. $\square$

Theorem 2. *There exists a unique solution $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{I}$ to the problem (1.2)-(1.4).*

Proof. Propositions 1, 2 and 3 yield the global gradient bound

$$(5.5) \quad W(x, t) \leq W(x_0, t_0) \frac{\eta(x, t)}{\eta(x_0, t_0)} \leq C_1 e^{C_2 MT^*}$$

for $(x, t) \in \bar{\Omega} \times [0, T^*]$, where C_1 and C_2 are positive constants and

$$M = \max_{\bar{\Omega} \times [0, T^*]} |u - u_0|.$$

It results that (2.17) is uniformly parabolic and then the standard theory of quasi-linear parabolic PDEs may be applied to assure the existence of a unique smooth solution to (2.17)-(2.19). $\square$

6. ASYMPTOTIC BEHAVIOR

Suppose from now on that $\mathcal{H} = 0$ and $\phi = 0$. In the particular case when the evolving functions have the form $u(x, t) = v(x) + Ct$, $(x, t) \in \bar{\Omega} \times [0, T]$, the initial value problem (2.17)-(2.19) becomes

$$(6.1) \quad \operatorname{div} \frac{\nabla v}{W} - \gamma \langle \bar{\nabla}_Y Y, \frac{\nabla v}{W} \rangle = \frac{C}{W} \quad \text{in } \Omega,$$

$$(6.2) \quad \langle \nu, N \rangle = 0 \quad \text{on } \partial\Omega.$$

Conversely, notice that if $v(x)$ is a solution of (6.1)-(6.2), then $u = v + Ct$ is a solution of (2.17) which is translating along the flow lines of Y with speed C .

Now observe that

$$\operatorname{div} \frac{\nabla v}{W} - \gamma \langle \bar{\nabla}_Y Y, \frac{\nabla v}{W} \rangle = \operatorname{div} \frac{\nabla v}{W} + \gamma \langle \bar{\nabla}_{\frac{\nabla v}{W}} Y, Y \rangle = \operatorname{div} \frac{\nabla v}{W} + \gamma \langle \bar{\nabla}_Y \frac{\nabla v}{W}, Y \rangle = \operatorname{div}_M \frac{\nabla v}{W}.$$

Therefore it follows from the divergence theorem that

$$(6.3) \quad \int_{\vartheta([0, s] \times \bar{\Omega})} \frac{C}{W} + \mathcal{H} = - \int_{\vartheta([0, s] \times \Gamma)} \left\langle \frac{\nabla v}{W}, \nu \right\rangle = \int_{\vartheta([0, s] \times \Gamma)} \langle N, \nu \rangle = \int_{\vartheta([0, s] \times \Gamma)} \phi.$$

Since the integrands do not depend on s we have

$$(6.4) \quad \int_{\Omega} C \frac{1}{\sqrt{\gamma} W} = \int_{\Gamma} \frac{1}{\sqrt{\gamma}} \phi,$$

from which results that

$$(6.5) \quad C = 0.$$

We then obtain the following height estimate.

Proposition 4. *Given a solution $u(x, t)$ of (2.17) there exists a constant M such that*

$$(6.6) \quad |u(x, t)| \leq M$$

for $(x, t) \in \bar{\Omega} \times [0, +\infty)$.

Proof. We observe that since C is necessarily zero, $v = \text{cte}$ is a solution to (6.1). In particular the constant functions $v_1 = \inf_{\Omega} u_0$ and $v_2 = \sup_{\Omega} u_0$ are solutions of (6.1) with $v_1 \leq u_0 \leq v_2$. Hence the parabolic maximum principle implies that

$$v_1 \leq u(\cdot, t) \leq v_2$$

for $t \in [0, T]$ from which we obtain (6.6). □

Now, proceeding as in [3], we prove the following convergence result.

Theorem 3. *Suppose that $\mathcal{H} = 0$ and $\phi = 0$. Then $\lim_{t \rightarrow \infty} u_t = 0$. In particular the mean curvature flow converges to a slice of the form $\vartheta(\{s\} \times \bar{\Omega})$ for some $s \in \mathbb{I}$.*

Proof. It is immediate that $v = s$ is a trivial solution to (6.1) with (necessarily) $C = 0$. We also have

$$\frac{d}{dt} \int_{\Omega} W = \int_{\Omega} \frac{u^i u_{i;t}}{W} = - \int_{\Omega} \frac{u_t^2}{W} - \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle - \int_{\Omega} \frac{|\nabla u|^2}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle.$$

Therefore

$$(6.7) \quad - \int_{\Omega} \frac{u_t^2}{W} = \frac{d}{dt} \left(\int_{\Omega} W \right) + \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle + \int_{\Omega} \frac{|\nabla u|^2}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle.$$

It follows that

$$\begin{aligned} \int_0^T \int_{\Omega} \frac{u_t^2}{W} &= - \int_{\Omega} W(x, T) + \int_{\Omega} W(x, 0) \\ &+ \int_0^T \int_{\Omega} \frac{1}{2W^3} \langle \nabla u, \nabla \gamma \rangle + \int_0^T \int_{\Omega} \frac{|\nabla u|^2}{2\gamma W^2} \langle \nabla u, \nabla \gamma \rangle \leq \tilde{C} \end{aligned}$$

for some positive constant $\tilde{C}$. It also follows that $\lim_{t \rightarrow \infty} \frac{u_t^2}{W} = 0$. Since W is bounded, then $\lim_{t \rightarrow \infty} u_t = 0$. This finishes the proof of the theorem. □

7. APPENDIX

In what follows, II and A denote respectively the second fundamental form and the Weingarten map of Σ_t . Their components are given by

$$(7.1) \quad a_{ij} = II(X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j}) := \langle AX_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j} \rangle.$$

Some lemmata will be needed in the sequel. Their content could also be of independent interest for other applications.

Lemma 1. *Denote $\theta = \langle \nabla d, N \rangle$. The differentials of the functions θ and h have components given by*

$$(7.2) \quad \theta_i = -a_i^j d_j + (d_{i;j} - \kappa \sigma_{ij}) N^j$$

and

$$(7.3) \quad h_i = (\alpha \delta_i^j + \phi a_i^j) d_j - (\phi (d_{i;j} - \kappa \sigma_{ij}) + \phi_i d_j) N^j,$$

respectively, where $\kappa = \langle \gamma \bar{\nabla}_Y Y, \nabla d \rangle$.

Proof. We have

$$\begin{aligned}
 \frac{\partial \theta}{\partial x^i} &= X_* \frac{\partial}{\partial x^i} \langle N, \bar{\nabla} d \rangle = \langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} N, \bar{\nabla} d \rangle + \langle N, \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} \bar{\nabla} d \rangle \\
 &= -\langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla} d \rangle + \langle N, \bar{\nabla}_{\frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial x^0}} \bar{\nabla} d \rangle \\
 &= -\langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla} d \rangle + \frac{\gamma}{W} \langle \frac{\partial}{\partial x^0}, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \bar{\nabla} d \rangle - \langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \bar{\nabla} d \rangle \\
 &\quad + u_i \frac{\gamma}{W} \langle \frac{\partial}{\partial x^0}, \bar{\nabla}_{\frac{\partial}{\partial x^0}} \bar{\nabla} d \rangle - u_i \langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\partial}{\partial x^0}} \bar{\nabla} d \rangle.
 \end{aligned}$$

Since P is totally geodesic we have

$$\langle \frac{\partial}{\partial x^0}, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \bar{\nabla} d \rangle = \langle \frac{\partial}{\partial x^0}, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \nabla d \rangle = 0.$$

Moreover we compute

$$\langle \frac{\partial}{\partial x^0}, \bar{\nabla}_{\frac{\partial}{\partial x^0}} \bar{\nabla} d \rangle = |Y|^2 \langle \frac{Y}{|Y|}, \bar{\nabla}_{\frac{Y}{|Y|}} \bar{\nabla} d \rangle = |Y|^2 \kappa = \frac{1}{\gamma} \kappa$$

and

$$\langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\partial}{\partial x^0}} \bar{\nabla} d \rangle = \langle \frac{\nabla u}{W}, \bar{\nabla}_{\nabla d} \frac{\partial}{\partial x^0} \rangle + \langle \frac{\nabla u}{W}, [\frac{\partial}{\partial x^0}, \bar{\nabla} d] \rangle = 0,$$

where we used the fact that $[\frac{\partial}{\partial x^0}, \bar{\nabla} d] = 0$ and that P is totally geodesic.

Thus we conclude that

$$\frac{\partial \theta}{\partial x^i} = -\langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla} d \rangle - \langle \frac{\nabla u}{W}, \nabla_{\frac{\partial}{\partial x^i}} \nabla d \rangle + \kappa \frac{u_i}{W}.$$

However

$$\langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla} d \rangle = a_i^j \langle X_* \frac{\partial}{\partial x^j}, \bar{\nabla} d \rangle = a_i^j \langle \frac{\partial}{\partial x^j} + u_j Y, \bar{\nabla} d \rangle = a_i^j d_j = g^{jk} a_{ik} d_j.$$

Therefore we write

$$(7.4) \quad \theta_i = -g^{jk} a_{ik} d_j + (d_{i,j} - \kappa \sigma_{ij}) N^j.$$

This finishes the proof of the proposition. $\square$

We denote the components of the tensor $X^* II$ in P by

$$(7.5) \quad b_{ij} = X^* II(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) := \langle AX_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j} \rangle.$$

Notice that the covariant derivatives of $X^* II$ and II are related by

$$\begin{aligned}
 \nabla_k b_{ij} &= \langle (\nabla_{X_* \frac{\partial}{\partial x^k}}^\Sigma A) X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j} \rangle + \langle AX_* \frac{\partial}{\partial x^j}, \bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} \rangle \\
 &\quad + \langle AX_* \frac{\partial}{\partial x^i}, \bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^j} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \rangle.
 \end{aligned}$$

However since $X_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + u_i Y$ we compute

$$\begin{aligned}
 \bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} &= \bar{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} + u_{i,k} Y + u_i \bar{\nabla}_{\frac{\partial}{\partial x^k}} Y \\
 &\quad + u_k \bar{\nabla}_Y \frac{\partial}{\partial x^i} + u_i u_k \bar{\nabla}_Y Y - \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \langle \nabla u, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} \rangle Y.
 \end{aligned}$$

Therefore

$$\bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = u_{i;k} Y + u_i \bar{\nabla}_{\frac{\partial}{\partial x^k}} Y + u_k \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y + u_i u_k \bar{\nabla}_Y Y.$$

Hence using (2.6), (2.10) and (2.11) we obtain

$$\begin{aligned} \bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} &= (W a_{ik} + u_i u_k u^l \frac{\gamma_l}{2\gamma^2}) Y + \frac{1}{2} u_i u_k \frac{\nabla \gamma}{\gamma^2} \\ &= W a_{ik} Y + \frac{1}{2\gamma^2} u_i u_k (\langle \nabla u, \nabla \gamma \rangle Y + \nabla \gamma) = W a_{ik} Y + \frac{1}{2\gamma^2} u_i u_k X_* \nabla \gamma. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \langle A X_* \frac{\partial}{\partial x^j}, \bar{\nabla}_{X_* \frac{\partial}{\partial x^k}} X_* \frac{\partial}{\partial x^i} - X_* \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} \rangle &= \langle A X_* \frac{\partial}{\partial x^j}, \frac{u_i u_k}{2\gamma^2} X_* \nabla \gamma + W a_{ik} Y \rangle \\ &= \frac{1}{\gamma} W a_{ik} a_j^l u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l. \end{aligned}$$

We conclude that

$$\begin{aligned} \nabla_k b_{ij} &= \langle (\nabla_{X_* \frac{\partial}{\partial x^k}}^\Sigma A) X_* \frac{\partial}{\partial x^i}, X_* \frac{\partial}{\partial x^j} \rangle + \frac{1}{\gamma} W a_{ik} a_j^l u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l \\ &\quad + \frac{1}{\gamma} W a_{jk} a_i^l u_l + \frac{u_j u_k}{2\gamma^2} a_{il} \gamma^l, \end{aligned}$$

that is,

$$(7.6) \quad \nabla_k b_{ij} = \nabla_k^\Sigma a_{ij} + \frac{1}{\gamma} W a_{ik} a_j^l u_l + \frac{1}{\gamma} W a_{jk} a_i^l u_l + \frac{u_i u_k}{2\gamma^2} a_{jl} \gamma^l + \frac{u_j u_k}{2\gamma^2} a_{il} \gamma^l.$$

Now we use (7.6) for computing the Hessian of the function θ .

Lemma 2. *The trace of the Hessian of θ in Ω calculated with respect to the metric in Σ is given by*

$$\begin{aligned} g^{ik} \theta_{i;k} &= -|A|^2 \theta - 2 \langle \nabla^2 d, X^* II \rangle_\Sigma - n \langle \nabla^\Sigma H, \nabla^\Sigma d \rangle - n H W \langle AY^T, \nabla^\Sigma d \rangle \\ &\quad - \text{Ric}(\nabla d, \frac{\nabla u}{W}) - \text{tr}_\Sigma \nabla_{\frac{\nabla u}{W}} \nabla^2 d - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_* \frac{\nabla \gamma}{2\gamma} \rangle + \frac{1}{2} \langle AY^T, Y^T \rangle \langle \nabla d, \nabla \gamma \rangle \\ &\quad - \frac{1}{2W^2} \nabla^2 d \langle \frac{\nabla u}{W}, \nabla \gamma \rangle - \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle + \kappa (nH - \gamma \langle AY^T, Y^T \rangle) - \kappa \frac{1}{2W^2} \langle N, \nabla \gamma \rangle. \end{aligned}$$

Proof. Notice that we may write (7.4) as

$$(7.7) \quad \theta_i = -g^{jl} b_{il} d_j + (d_{i;j} - \kappa \sigma_{ij}) N^j.$$

Hence we have

$$\begin{aligned} n g^{ik} \theta_{i;k} &= -g^{ik} (g^{jl} b_{il} d_j)_{;k} + g^{ik} (d_{i;jk} - \kappa_k \sigma_{ij}) N^j + g^{ik} (d_{i;j} - \kappa \sigma_{ij}) N_{;k}^j \\ &= -g^{ik} (g^{jl} b_{il} d_j)_{;k} + g^{ik} (d_{i;kj} + R_{jk i}^l d_l - \kappa_k \sigma_{ij}) N^j - g^{ik} (d_{i;j} - \kappa \sigma_{ij}) (a_k^j - N_k \frac{\gamma^j}{2\gamma}). \end{aligned}$$

However

$$\begin{aligned} g^{ik} (g^{jl} b_{il} d_j)_{;k} &= g^{jl} g^{ik} b_{il;k} d_j + g^{ik} g_{;k}^{jl} b_{il} d_j + g^{ik} g^{jl} b_{il} d_{j;k} \\ &= g^{jl} g^{ik} (\nabla_k^\Sigma a_{il} + \frac{1}{\gamma} W a_{ik} a_l^m u_m + \frac{1}{\gamma} W a_{lk} a_i^m u_m + u_i u_k a_{lm} \frac{\gamma^m}{2\gamma^2} + u_l u_k a_{im} \frac{\gamma^m}{2\gamma^2}) d_j \\ &\quad + g^{ik} g_{;k}^{jl} b_{il} d_j + g^{ik} g^{jl} b_{il} d_{j;k}. \end{aligned}$$

Hence using Codazzi's equation we obtain

$$\begin{aligned} g^{ik}(g^{jl}b_{il}d_j)_{;k} &= g^{jl}(nH_l + n\frac{1}{\gamma}WHa_l^m u_m + \frac{1}{\gamma}Wa_l^i a_i^m u_m + \frac{|\nabla u|^2}{W^2}a_{lm}\frac{\gamma^m}{2\gamma} \\ &\quad + u_l u_k a_m^k \frac{\gamma^m}{2\gamma^2})d_j + g^{jl}g^{ik}\langle \bar{R}(X_*\frac{\partial}{\partial x^i}, X_*\frac{\partial}{\partial x^k})N, X_*\frac{\partial}{\partial x^l} \rangle d_j + g^{ik}g_{;k}^{jl}b_{il}d_j \\ &\quad + g^{ik}g^{jl}b_{il}d_{j;k}. \end{aligned}$$

Using that $g^{jl}u_l = \frac{\gamma}{W^2}u^j$ we conclude that

$$\begin{aligned} g^{ik}(g^{jl}b_{il}d_j)_{;k} &= ng^{jl}H_l d_j - n\frac{1}{\gamma}W^2Hg^{jl}a_l^m N_m d_j - \frac{1}{\gamma}W^2g^{jl}a_l^i a_i^m N_m d_j \\ &\quad + \frac{|\nabla u|^2}{W^2}a_m^j \frac{\gamma^m}{2\gamma}d_j + N^j N_k a_m^k \frac{\gamma^m}{2\gamma}d_j + g^{ik}g_{;k}^{jl}b_{il}d_j + g^{ik}g^{jl}b_{il}d_{j;k}. \end{aligned}$$

However we have

$$g_{;k}^{jl} = (\sigma^{jl} - N^j N^l)_{;k} = -N_{;k}^j N^l - N^j N_{;k}^l = (a_k^j - N_k \frac{\gamma^j}{2\gamma})N^l + N^j(a_k^l - N_k \frac{\gamma^l}{2\gamma})$$

and

$$\begin{aligned} \bar{\nabla}_{\frac{\partial}{\partial x^k}} N &= \bar{\nabla}_{X_*\frac{\partial}{\partial x^k}} N - \bar{\nabla}_{u_k Y} N = -AX_*\frac{\partial}{\partial x^k} - u_k \bar{\nabla}_Y (\frac{\gamma}{W}Y - \frac{\nabla u}{W}) \\ &= -AX_*\frac{\partial}{\partial x^k} - \frac{u_k}{2W}(\frac{\nabla \gamma}{\gamma} + \langle \nabla u, \frac{\nabla \gamma}{\gamma} \rangle Y), \end{aligned}$$

from which follows that

$$\begin{aligned} g^{ik}(g^{jl}b_{il}d_j)_{;k} &= ng^{jl}H_l d_j - n\frac{1}{\gamma}W^2Ha_l^m N_m g^{jl}d_j - \frac{1}{\gamma}W^2a_l^i a_i^m N_m g^{jl}d_j \\ &\quad + \frac{|\nabla u|^2}{W^2}a_m^j \frac{\gamma^m}{2\gamma}d_j + N^j N_k a_m^k \frac{\gamma^m}{2\gamma}d_j + a_l^k a_k^j N^l d_j \\ &\quad - a_l^k N_k N^l \frac{\gamma^j}{2\gamma}d_j + a_l^k a_k^l N^j d_j - a_l^k N_k \frac{\gamma^j}{2\gamma}N^j d_j + g^{jl}a_l^k d_{j;k}. \end{aligned}$$

Therefore

$$\begin{aligned} g^{ik}\theta_{i;k} &= -ng^{jl}H_l d_j + n\frac{1}{\gamma}W^2Ha_l^m N_m g^{jl}d_j + \frac{1}{\gamma}W^2a_l^i a_i^m N_m g^{jl}d_j \\ &\quad - \frac{|\nabla u|^2}{W^2}a_m^j \frac{\gamma^m}{2\gamma}d_j - a_l^k a_k^j N^l d_j + a_l^k N_k N^l \langle \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle - a_l^k a_k^l \theta - g^{jl}a_l^k d_{j;k} \\ &\quad + g^{ik}(d_{i;kj} + R_{jki}^l d_l - \kappa_k \sigma_{ij})N^j - g^{ik}(d_{i;j} - \kappa \sigma_{ij})(a_k^j - N_k \frac{\gamma^j}{2\gamma}). \end{aligned}$$

Now using the fact that $g^{ij}u_j = \frac{\gamma}{W^2}u^i$ and therefore $g^{ij}N_j = \frac{\gamma}{W^2}N^i$ we obtain

$$\begin{aligned} a_i^m N_m &= g^{km}a_{ik}N_m = \frac{\gamma}{W^2}a_{ik}N^k = \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N^k X_*\frac{\partial}{\partial x^k} \rangle \\ &= \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N^k \frac{\partial}{\partial x^k} + \langle N^k \frac{\partial}{\partial x^k}, \nabla u \rangle Y \rangle \\ &= \frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, N - \frac{\gamma}{W}Y + \langle N, \nabla u \rangle Y \rangle \\ &= -\frac{\gamma}{W^2}\langle AX_*\frac{\partial}{\partial x^i}, Y \rangle (\frac{\gamma}{W} + \frac{|\nabla u|^2}{W}) = -\frac{\gamma}{W}\langle AX_*\frac{\partial}{\partial x^i}, Y \rangle = -\frac{\gamma}{W}\langle AY^T, X_*\frac{\partial}{\partial x^i} \rangle. \end{aligned}$$

Therefore

$$a_l^m N_m g^{jl} d_j = -\frac{\gamma}{W} \langle AY^T, g^{jl} d_j X_* \frac{\partial}{\partial x^l} \rangle = -\frac{\gamma}{W} \langle AY^T, \nabla^\Sigma d \rangle.$$

Moreover notice that

$$a_l^k N^l = g^{km} a_{ml} N^l = -g^{km} W \langle AY^T, X_* \frac{\partial}{\partial x^m} \rangle$$

and

$$a_{ik} N^k = -W \langle AY^T, X_* \frac{\partial}{\partial x^i} \rangle.$$

Similarly we have

$$a_k^j d_j = g^{jm} d_j \langle AX_* \frac{\partial}{\partial x^k}, X_* \frac{\partial}{\partial x^m} \rangle = \langle AX_* \frac{\partial}{\partial x^k}, \nabla^\Sigma d \rangle = \langle A \nabla^\Sigma d, X_* \frac{\partial}{\partial x^k} \rangle.$$

Replacing the above we obtain

$$\begin{aligned} g^{ik} \theta_{i;k} &= -n \langle \nabla^\Sigma H, \nabla^\Sigma d \rangle - n H W \langle AY^T, \nabla^\Sigma d \rangle - W \langle AY^T, A \nabla^\Sigma d \rangle \\ &\quad - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_* \frac{\nabla \gamma}{2\gamma} \rangle \\ &\quad + W \langle AY^T, A \nabla^\Sigma d \rangle + \gamma \langle AY^T, Y^T \rangle \langle \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle - |A|^2 \theta - g^{jl} a_l^k d_{j;k} \\ &\quad + g^{ik} (d_{i;kj} + R_{jki}^l d_l - \kappa_k \sigma_{ij}) N^j - g^{ik} (d_{i;j} - \kappa \sigma_{ij}) (a_k^j - N_k \frac{\gamma^j}{2\gamma}). \end{aligned}$$

Therefore

$$\begin{aligned} g^{ik} \theta_{i;k} &= -n \langle \nabla^\Sigma H, \nabla^\Sigma d \rangle - n H W \langle AY^T, \nabla^\Sigma d \rangle - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_* \frac{\nabla \gamma}{2\gamma} \rangle \\ &\quad + \gamma \langle AY^T, Y^T \rangle \langle \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle - |A|^2 \theta - g^{jl} a_l^k d_{j;k} \\ &\quad + g^{ik} (d_{i;kj} + R_{jki}^l d_l - \kappa_k \sigma_{ij}) N^j - g^{ik} (d_{i;j} - \kappa \sigma_{ij}) (a_k^j - N_k \frac{\gamma^j}{2\gamma}). \end{aligned}$$

However

$$g^{ik} \sigma_{ij} = g^{ik} (g_{ij} - \frac{u_i u_j}{\gamma}) = \delta_j^k - \frac{1}{W^2} u^k u_j = \delta_j^k - N^k N_j.$$

Hence we have

$$\begin{aligned} g^{ik} \theta_{i;k} &= -n \langle \nabla^\Sigma H, \nabla^\Sigma d \rangle - n H W \langle AY^T, \nabla^\Sigma d \rangle - \frac{|\nabla u|^2}{W^2} \langle A \nabla^\Sigma d, X_* \frac{\nabla \gamma}{2\gamma} \rangle \\ &\quad + \frac{1}{2} \langle AY^T, Y^T \rangle \langle \nabla d, \nabla \gamma \rangle - |A|^2 \theta - 2g^{ik} g^{jl} d_{i;j} a_{kl} + \frac{1}{2W^2} d_{i;j} N^i \gamma^j \\ &\quad + g^{ik} d_{i;kj} N^j - \text{Ric}(\nabla d, \frac{\nabla u}{W}) - \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle + \kappa (nH - \gamma \langle AY^T, Y^T \rangle) \\ &\quad - \kappa \frac{1}{2W^2} \langle N, \nabla \gamma \rangle. \end{aligned}$$

This finishes the proof of the lemma. $\square$

Using Lemma 2 we will obtain an expression for Lh . Notice that

$$h_{i;k} = \alpha d_{i;k} - \phi_i \theta_k - \phi_k \theta_i - \phi_{i;k} \theta - \phi \theta_{i;k}.$$

Moreover it holds that

$$\begin{aligned} 2g^{ik}\phi_i\theta_k &= 2g^{ik}\phi_i\langle A\nabla^\Sigma d, X_*\frac{\partial}{\partial x^k}\rangle - 2g^{ik}d_{k;l}\phi_iN^l + 2\kappa g^{ik}\sigma_{kl}\phi_iN^l \\ &= 2\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle - 2g^{ik}d_{k;l}\phi_iN^l + 2\kappa\frac{\gamma}{W^2}\langle \nabla\phi, N\rangle. \end{aligned}$$

We conclude that

$$\begin{aligned} g^{ik}h_{i;k} &= \alpha g^{ik}d_{i;k} + 2\langle A\nabla^\Sigma d, \nabla^\Sigma\phi\rangle - 2g^{ik}d_{k;l}\phi_iN^l + 2\kappa\frac{\gamma}{W^2}\langle \nabla\phi, N\rangle - g^{ik}\phi_{i;k}\theta \\ &\quad + n\phi\langle \nabla^\Sigma H, \nabla^\Sigma d\rangle + n\phi HW\langle AY^T, \nabla^\Sigma d\rangle + \frac{|\nabla u|^2}{W^2}\phi\langle A\nabla^\Sigma d, X_*\frac{\nabla\gamma}{2\gamma}\rangle \\ &\quad - \frac{1}{2}\phi\langle AY^T, Y^T\rangle\langle \nabla d, \nabla\gamma\rangle + |A|^2\phi\theta + 2g^{ik}g^{jl}d_{i;j}a_{kl}\phi - \frac{1}{2W^2}\phi d_{i;j}N^i\gamma^j \\ &\quad - g^{ik}d_{i;kj}N^j\phi + \text{Ric}(\nabla d, \frac{\nabla u}{W})\phi + \frac{\gamma}{W^2}\langle N, \nabla\kappa\rangle\phi - \kappa(nH - \gamma\langle AY^T, Y^T\rangle)\phi \\ &\quad + \kappa\frac{1}{2W^2}\langle N, \nabla\gamma\rangle\phi. \end{aligned}$$

Now we compute the derivatives with respect to t . We have

$$\begin{aligned} \theta_t &= X_*\frac{\partial}{\partial t}\langle N, \bar{\nabla}d\rangle = \langle \bar{\nabla}_{X_*\frac{\partial}{\partial t}}N, \bar{\nabla}d\rangle + \langle N, \bar{\nabla}_{X_*\frac{\partial}{\partial t}}\bar{\nabla}d\rangle \\ &= -\langle \nabla^\Sigma(nH - \mathcal{H}), \bar{\nabla}d\rangle + (nH - \mathcal{H})\langle N, \bar{\nabla}_N\bar{\nabla}d\rangle. \end{aligned}$$

However

$$\langle N, \bar{\nabla}_N\bar{\nabla}d\rangle = -\frac{1}{2W^2}\langle \bar{\nabla}\gamma, \bar{\nabla}d\rangle + \langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\nabla u}{W}}\bar{\nabla}d\rangle.$$

Hence we have

$$\theta_t = -\langle \nabla^\Sigma(nH - \mathcal{H}), \bar{\nabla}d\rangle + (nH - \mathcal{H})\left(-\frac{1}{2W^2}\langle \nabla\gamma, \nabla d\rangle + \langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\nabla u}{W}}\bar{\nabla}d\rangle\right).$$

Moreover we have

$$(7.8) \quad d_t = \langle X_*\frac{\partial}{\partial t}, \bar{\nabla}d\rangle = (nH - \mathcal{H})\langle N, \bar{\nabla}d\rangle = (nH - \mathcal{H})\theta.$$

Therefore

$$\begin{aligned} h_t &= \alpha(nH - \mathcal{H})\theta - (nH - \mathcal{H})\langle N, \bar{\nabla}\phi\rangle\theta + \phi\langle \nabla^\Sigma(nH - \mathcal{H}), \bar{\nabla}d\rangle \\ &\quad - \phi(nH - \mathcal{H})\left(-\frac{1}{2W^2}\langle \nabla\gamma, \nabla d\rangle + \langle \frac{\nabla u}{W}, \bar{\nabla}_{\frac{\nabla u}{W}}\bar{\nabla}d\rangle\right). \end{aligned}$$

We also compute

$$\langle \nabla\gamma, \nabla h\rangle = \alpha\langle \nabla d, \nabla\gamma\rangle + \phi\langle A\nabla^\Sigma d, X_*\nabla\gamma\rangle - \langle \nabla\phi, \nabla\gamma\rangle\theta - \phi d_{i;j}\gamma^iN^j + \kappa\phi\langle N, \nabla\gamma\rangle.$$

Now we obtain

$$g^{ik}d_{i;k} = \Delta d - \langle \nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle = -nH_d - \langle \nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle$$

and

$$\begin{aligned} g^{ik}d_{k;l}\phi_iN^l &= d_{k;l}\phi^kN^l - d_{k;l}N^kN^lN^i\phi_i \\ &= -\langle \nabla_{\frac{\nabla u}{W}}\nabla d, \nabla\phi\rangle - \langle \nabla_{\frac{\nabla u}{W}}\nabla d, \frac{\nabla u}{W}\rangle\langle N, \nabla\phi\rangle. \end{aligned}$$

Moreover we have

$$g^{ij}\phi_{i;j} = \Delta\phi - \langle \nabla_{\frac{\nabla u}{W}}\nabla\phi, \frac{\nabla u}{W}\rangle$$

and

$$\begin{aligned} g^{ik} d_{i;kj} N^j &= (\sigma^{ik} d_{i;k})_{;j} N^j - d_{i;kj} N^i N^k N^j \\ &= -n(H_d)_j N^j + \nabla^3 d \left(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W} \right). \end{aligned}$$

Therefore grouping and rearranging these expressions we obtain

$$\begin{aligned} Lh &= |A|^2 \phi \theta + n \phi H W \langle AY^T, \nabla^\Sigma d \rangle + \left(\kappa \gamma - \frac{1}{2} \langle \nabla d, \nabla \gamma \rangle \right) \phi \langle AY^T, Y^T \rangle \\ &\quad + 2 \langle A \nabla^\Sigma d, \nabla^\Sigma \phi \rangle + 2 \langle A, \nabla^2 d \rangle_\Sigma \phi - \frac{1}{W^2} \phi \langle A \nabla^\Sigma d, X_* \nabla \gamma \rangle \\ &\quad (nH - \mathcal{H}) \left(\langle N, \nabla \phi \rangle \theta - \alpha \theta - \frac{1}{2W^2} \langle \nabla \gamma, \nabla d \rangle \phi + \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle \phi \right) - n \kappa H \phi \\ &\quad - n \alpha H_d + \left(2 \langle N, \nabla \phi \rangle - \alpha \right) \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla u}{W} \rangle + 2 \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \nabla \phi \rangle \\ &\quad - \phi \langle \nabla_{\frac{\nabla u}{W}} \nabla d, \frac{\nabla \gamma}{2\gamma} \rangle + \phi \langle \nabla^\Sigma \mathcal{H}, \bar{\nabla} d \rangle + n \langle \nabla H_d, N \rangle \phi \\ &\quad - \phi \nabla^3 d \left(\frac{\nabla u}{W}, \frac{\nabla u}{W}, \frac{\nabla u}{W} \right) + \text{Ric}(\nabla d, \frac{\nabla u}{W}) \phi \\ &\quad + \frac{\gamma}{W^2} \langle N, \nabla \kappa \rangle \phi - \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \alpha \langle \nabla d, \nabla \gamma \rangle + \left(\frac{1}{2\gamma} + \frac{1}{2W^2} \right) \langle \nabla \phi, \nabla \gamma \rangle \theta \\ &\quad - \kappa \phi \langle N, \frac{\nabla \gamma}{2\gamma} \rangle + 2 \kappa \frac{\gamma}{W^2} \langle \nabla \phi, N \rangle - \left(\Delta \phi - \langle \nabla_{\frac{\nabla u}{W}} \nabla \phi, \frac{\nabla u}{W} \rangle \right) \theta. \end{aligned}$$

Lemma 3. *We have*

$$\begin{aligned} LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + n H W^3 \langle AY^T, Y^T \rangle - n H W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - 3 \gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle \\ &\quad + g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - W \langle \nabla^\Sigma \mathcal{H}, N \rangle \\ &\quad - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W} - W_t. \end{aligned}$$

Proof. Notice that

$$\begin{aligned} W_i &= -W^2 \left(\langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} Y, N \rangle + \langle Y, \bar{\nabla}_{X_* \frac{\partial}{\partial x^i}} N \rangle \right) \\ &= -W^2 \left(\langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} Y, N \rangle + u_i \langle \bar{\nabla}_Y Y, N \rangle - \langle Y, A X_* \frac{\partial}{\partial x^i} \rangle \right) \\ &= -W^2 \left(-\frac{\gamma_i}{2\gamma} \langle Y, N \rangle + u_i \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \langle Y, A X_* \frac{\partial}{\partial x^i} \rangle \right). \end{aligned}$$

Therefore

$$W_i = \frac{\gamma_i}{2\gamma} W + N_i W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + W^2 \langle AY^T, X_* \frac{\partial}{\partial x^i} \rangle.$$

However

$$\langle AY^T, X_* \frac{\partial}{\partial x^i} \rangle = g^{kl} \langle Y, X_* \frac{\partial}{\partial x^k} \rangle \langle X_* \frac{\partial}{\partial x^l}, A X_* \frac{\partial}{\partial x^i} \rangle = g^{kl} \langle Y, u_k Y \rangle b_{il} = \frac{1}{W^2} u^l b_{il}.$$

Hence it follows that

$$W_i = \frac{\gamma_i}{2\gamma} W + N_i W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - W N^l b_{il}.$$

Hence we obtain

$$\begin{aligned} \frac{1}{W} g^{ij} W_i W_j &= \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W + W \langle \nabla \gamma, N \rangle \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + \langle AY^T, \frac{\nabla^\Sigma \gamma}{\gamma} \rangle W^2 \\ &\quad + \gamma |\nabla u|^2 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle^2 W - \langle AY^T, Y^T \rangle \langle \frac{\nabla \gamma}{\gamma}, N \rangle W^3 + \langle AY^T, AY^T \rangle W^3. \end{aligned}$$

Now we compute

$$\begin{aligned} W_{i;j} &= \left(\frac{\gamma_{i;j}}{2\gamma} - \frac{\gamma_i \gamma_j}{2\gamma^2} \right) W + \frac{\gamma_i}{2\gamma} W_j + N_{i;j} W^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + 3N_i W^2 W_j \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle \\ &\quad + N_i W^3 \left(\langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^j}} \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - \langle \frac{\nabla \gamma}{2\gamma^2}, AX_* \frac{\partial}{\partial x^j} \rangle \right) - W_j N^l b_{il} - W N_{i;j}^l b_{il} - W N^l b_{il;j}. \end{aligned}$$

However we have

$$g^{ij} \frac{\gamma_i}{2\gamma} W_j = \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W + \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W + W^2 \langle AY^T, \frac{\nabla^\Sigma \gamma}{2\gamma} \rangle$$

and

$$\begin{aligned} g^{ij} N_{i;j} &= g^{ij} \sigma_{ik} N_{i;j}^k = -(\delta_k^j - N^j N_k)(a_j^k - N_j \frac{\gamma^k}{2\gamma}) \\ &= -nH + \frac{\gamma}{W^2} \langle N, \frac{\nabla \gamma}{2\gamma} \rangle + \gamma \langle AY^T, Y^T \rangle. \end{aligned}$$

Moreover we compute

$$g^{ij} N_i W_j = \frac{\gamma}{W} \langle N, \frac{\nabla \gamma}{2\gamma} \rangle + \frac{\gamma |\nabla u|^2}{W} \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle - \gamma W \langle AY^T, Y^T \rangle$$

and

$$\begin{aligned} g^{ij} N_i W^3 &\left(\langle \bar{\nabla}_{X_* \frac{\partial}{\partial x^j}} \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - \langle \frac{\bar{\nabla} \gamma}{2\gamma^2}, AX_* \frac{\partial}{\partial x^j} \rangle \right) \\ &= \gamma W \left(\langle \bar{\nabla}_{N-WY} \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - \langle A \frac{\nabla^\Sigma \gamma}{2\gamma^2}, -WY \rangle \right) \\ &= \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle + W \frac{|\nabla \gamma|^2}{4\gamma^2} + \gamma W^2 \langle A \frac{\nabla^\Sigma \gamma}{2\gamma^2}, Y^T \rangle. \end{aligned}$$

We also have

$$\begin{aligned} 2W g^{ij} W_j \langle AY^T, X_* \frac{\partial}{\partial x^i} \rangle &= 2W^2 \langle AY^T, \frac{\nabla^\Sigma \gamma}{2\gamma} \rangle - W^3 \langle \frac{\nabla \gamma}{\gamma}, N \rangle \langle AY^T, Y^T \rangle \\ &\quad + 2W^3 \langle AY^T, AY^T \rangle. \end{aligned}$$

Now we compute

$$\begin{aligned} g^{ij} W N^l b_{il;j} &= W N^l g^{ij} \nabla_j^\Sigma a_{il} + \frac{1}{\gamma} W^2 g^{ij} a_{ij} a_l^m N^l u_m + \frac{1}{\gamma} W^2 g^{ij} a_{lj} N^l a_i^m u_m \\ &\quad + W g^{ij} \frac{u_i u_j}{2\gamma^2} a_{lm} N^l \gamma^m + W g^{ij} N^l \frac{u_l u_j}{2\gamma^2} a_{im} \gamma^m. \end{aligned}$$

Hence we have

$$\begin{aligned}
 g^{ij}WN^lb_{il;j} &= WN^l(n\nabla_l^\Sigma H + g^{ij}\langle \bar{R}(X_*\frac{\partial}{\partial x^i}, X_*\frac{\partial}{\partial x^j})N, X_*\frac{\partial}{\partial x^l}\rangle) \\
 &\quad + nHW^2\langle AY^T, N^kX_*\frac{\partial}{\partial x^k}\rangle \\
 &\quad + W^2g^{ij}\langle AY^T, X_*\frac{\partial}{\partial x^j}\rangle(-W\langle AY^T, X_*\frac{\partial}{\partial x^i}\rangle) \\
 &\quad - |\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{2\gamma}\rangle + \frac{|\nabla u|^2}{2\gamma W}(-W\langle AY^T, X_*\frac{\partial}{\partial x^m}\rangle)\gamma^m.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 g^{ij}WN^lb_{il;j} &= nWN^l\nabla_l^\Sigma H - nHW^3\langle AY^T, Y^T\rangle - W^3\langle AY^T, AY^T\rangle \\
 &\quad - |\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{\gamma}\rangle.
 \end{aligned}$$

Moreover

$$g^{ij}W_jN^lb_{il} = -W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle + W^3\langle \frac{\nabla\gamma}{2\gamma}, N\rangle\langle AY^T, Y^T\rangle - W^3\langle AY^T, AY^T\rangle$$

and

$$Wg^{ij}N_{;j}^lb_{il} = -Wg^{ij}(a_j^l - N_j\frac{\gamma^l}{2\gamma})a_{il} = -|A|^2W - \frac{1}{2}\langle AY^T, X_*\nabla\gamma\rangle.$$

We conclude that

$$\begin{aligned}
 g^{ij}W_{i;j} &= |A|^2W + 2W^3\langle AY^T, AY^T\rangle + (nH - 3\langle \frac{\nabla\gamma}{2\gamma}, N\rangle)W^3\langle AY^T, Y^T\rangle \\
 &\quad + 3W^2\langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma}\rangle + |\nabla u|^2\langle AY^T, X_*\frac{\nabla\gamma}{\gamma}\rangle + \frac{1}{2}\langle AY^T, X_*\nabla\gamma\rangle \\
 &\quad + g^{ij}\frac{\gamma_{i;j}}{2\gamma}W - \frac{|\nabla^\Sigma\gamma|^2}{4\gamma^2}W + \frac{|\nabla\gamma|^2}{4\gamma^2}W + (5W + 3\frac{W}{\gamma}|\nabla u|^2)\langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 \\
 &\quad - nHW^3\langle \frac{\nabla\gamma}{2\gamma^2}, N\rangle + \gamma W\langle \bar{\nabla}_N\frac{\bar{\nabla}\gamma}{2\gamma^2}, N\rangle - nWN^l\nabla_l^\Sigma H.
 \end{aligned}$$

Now

$$\langle \nabla\gamma, \nabla W\rangle = \frac{|\nabla\gamma|^2}{2\gamma}W + \frac{1}{2\gamma^2}\langle \nabla\gamma, N\rangle^2W^3 + W^2\langle AY^T, X_*\nabla\gamma\rangle.$$

Hence

$$\begin{aligned}
LW - \frac{2}{W}g^{ij}W_iW_j &= |A|^2W + (nH + \langle \frac{\nabla\gamma}{2\gamma}, N \rangle)W^3 \langle AY^T, Y^T \rangle \\
&\quad - W^2 \langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma} \rangle + |\nabla u|^2 \langle AY^T, X_* \frac{\nabla\gamma}{\gamma} \rangle + \frac{1}{2} \langle AY^T, X_* \nabla\gamma \rangle \\
&\quad - (\frac{1}{2\gamma} + \frac{1}{2W^2})W^2 \langle AY^T, X_* \nabla\gamma \rangle \\
&\quad + g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4} \frac{|\nabla^\Sigma\gamma|^2}{4\gamma^2} W + \frac{|\nabla\gamma|^2}{4\gamma^2} W + (5W + 3\frac{W}{\gamma} |\nabla u|^2) \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 \\
&\quad - nHW^3 \langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla}\gamma}{2\gamma^2}, N \rangle - nWN^l \nabla_l^\Sigma H \\
&\quad - (\frac{1}{2\gamma} + \frac{1}{2W^2}) (\frac{|\nabla\gamma|^2}{2\gamma} W + \frac{1}{2\gamma^2} \langle \nabla\gamma, N \rangle^2 W^3) - \frac{1}{\gamma^2} \langle \nabla\gamma, N \rangle^2 W \\
&\quad - 2\gamma |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle^2 W - W_t.
\end{aligned}$$

However

$$3\frac{W}{\gamma} |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 - 2\gamma |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma^2}, N \rangle^2 W = \frac{W}{\gamma} |\nabla u|^2 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2,$$

$$5W \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 - \frac{1}{\gamma^2} \langle \nabla\gamma, N \rangle^2 W = \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W$$

and

$$\begin{aligned}
&-W^2 \langle AY^T, \frac{\nabla^\Sigma\gamma}{2\gamma} \rangle + |\nabla u|^2 \langle AY^T, X_* \frac{\nabla\gamma}{\gamma} \rangle + \frac{1}{2} \langle AY^T, X_* \nabla\gamma \rangle \\
&\quad - (\frac{1}{2\gamma} + \frac{1}{2W^2})W^2 \langle AY^T, X_* \nabla\gamma \rangle \\
&\quad = -3\gamma \langle AY^T, X_* \frac{\nabla\gamma}{2\gamma} \rangle - W^3 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle \langle AY^T, Y^T \rangle.
\end{aligned}$$

Moreover we compute

$$\begin{aligned}
&(\frac{1}{2\gamma} + \frac{1}{2W^2}) (\frac{|\nabla\gamma|^2}{2\gamma} W + \frac{1}{2\gamma^2} \langle \nabla\gamma, N \rangle^2 W^3) \\
&\quad = \frac{|\nabla\gamma|^2}{4\gamma^2} W + \frac{1}{\gamma} W^3 \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 + \frac{|\nabla\gamma|^2}{4\gamma} \frac{1}{W} + \langle \frac{\nabla\gamma}{2\gamma}, N \rangle^2 W
\end{aligned}$$

and

$$-nWN^l \nabla_l^\Sigma H = -nW \langle \nabla^\Sigma H, N \rangle = -W \langle \nabla^\Sigma \mathcal{H}, N \rangle.$$

We conclude that

$$\begin{aligned} LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - 3\gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle \\ &\quad + g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4} \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W + \frac{|\nabla u|^2}{\gamma} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W \\ &\quad - nHW^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - W \langle \nabla^\Sigma \mathcal{H}, N \rangle \\ &\quad - \frac{1}{\gamma} W^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W} - W_t. \end{aligned}$$

However

$$\frac{|\nabla u|^2}{\gamma} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W - \frac{1}{\gamma} W^3 \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 = - \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W$$

and

$$\begin{aligned} -\frac{3}{4} \frac{|\nabla^\Sigma \gamma|^2}{4\gamma^2} W - \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W &= -\frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} W + \frac{3}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W - \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W \\ &= -\frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W. \end{aligned}$$

Hence we obtain

$$\begin{aligned} LW - \frac{2}{W} g^{ij} W_i W_j &= |A|^2 W + nHW^3 \langle AY^T, Y^T \rangle - nHW^3 \langle \frac{\nabla \gamma}{2\gamma^2}, N \rangle \\ &\quad - 3\gamma \langle AY^T, X_* \frac{\nabla \gamma}{2\gamma} \rangle + g^{ij} \frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4} \frac{|\nabla \gamma|^2}{4\gamma^2} W - \frac{1}{4} \langle \frac{\nabla \gamma}{2\gamma}, N \rangle^2 W \\ &\quad + \gamma W \langle \bar{\nabla}_N \frac{\bar{\nabla} \gamma}{2\gamma^2}, N \rangle - W \langle \nabla^\Sigma \mathcal{H}, N \rangle - \frac{|\nabla \gamma|^2}{4\gamma} \frac{1}{W} - W_t. \end{aligned}$$

This finishes the proof of the lemma. $\square$

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