BLOW-UP PROFILE FOR THE COMPLEX-VALUED SEMILINEAR WAVE EQUATION

ASMA AZAIEZ

ABSTRACT. In this paper, we consider a blow-up solution for the complex-valued semilinear wave equation with power non-linearity in one space dimension. We first characterize all the solutions of the associated stationary problem as a two-parameter family. Then, we use a dynamical system formulation to show that the solution in self-similar variables approaches some particular stationary one in the energy norm, in the non-characteristic case. This gives the blow-up profile for the original equation in the non-characteristic case. Our analysis is not just a simple adaptation of the already handled real case. In particular, there is one more neutral-direction in our problem, which we control thanks to a modulation technique.

1. INTRODUCTION

1.1. The problem and known results. We consider the following complex-valued one-dimensional semilinear wave equation:

$$(1) \quad \begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1} u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $u(t) : x \in \mathbb{R} \rightarrow u(x, t) \in \mathbb{C}$, $p > 1$, $u_0 \in H_{loc,u}^1$ and $u_1 \in L_{loc,u}^2$, with

$$\|v\|_{L_{loc,u}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a| < 1} |v(x)|^2 dx \text{ and } \|v\|_{H_{loc,u}^1}^2 = \|v\|_{L_{loc,u}^2}^2 + \|\nabla v\|_{L_{loc,u}^2}^2.$$

The Cauchy problem for equation (1) in the space $H_{loc,u}^1 \times L_{loc,u}^2$ follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$. See for instance Ginibre, Soffer and Velo [15], Ginibre and Velo [16], and Lindblad and Sogge [29] (for the local in time wellposedness in $H^1 \times L^2$). The existence of blow-up solutions for equation (1) is a consequence of the finite speed of propagation and ODE techniques (see for example Levine [28] and Antonini and Merle [3]). More blow-up results can be found in Caffarelli and Friedman [4], Alinhac [1] and [2], Kichenassamy and Littman [23], [22], and Shatah and Struwe [42].

The real case (in one space dimension) has been understood completely in a series of papers by Merle and Zaag [33], [34], [37] and [38] and in Côte and Zaag [5] (see also the note [35]). Some of those results have been extended to higher

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dimensions for conformal or subconformal p :

$$(2) \quad 1 < p \leq p_c \equiv 1 + \frac{4}{N-1},$$

under radial symmetry outside the origin in [36]. For non-radial solutions, we would like to mention [30] and [31] where the blow-up rate was obtained. We also mention the recent contribution of [39] and [40] where the blow-up behavior is given, together with some stability results.

Considering the behavior of radial solutions at the origin, Donniger and Schörkhuber [8] were able to prove the stability of the space-independent solution (i.e. the solution of the associated ODE $u'' = u^p$) with respect to perturbation in the initial data. Willing to be as exhaustive as possible in our bibliography about the blow-up question for equation (1), we would like to mention some blow-up results in the superconformal, Sobolev critical and supercritical ranges for equation (1).

When

$$N \geq 2 \text{ and } p_c < p < p_s \equiv \frac{N+2}{N-2},$$

Killip, Stoval and Visan found in [24] an upper bound on the blow-up rate. That bound is larger than the solution of the associated ODE $u'' = u^p$, and is therefore thought to be non-optimal. In [17] Hamza and Zaag gives a different proof of the results of [24], improving some of their estimates.

When

$$N \geq 3 \text{ and } p = p_s,$$

equation (1) has attracted much interest. Many authors addressed the question of obtaining sufficient conditions for scattering and blow-up through energy estimates, in relation to the ground state (see Kenig and Merle [20] and Duyckaerts and Merle [13]). Furthermore, dynamics around the soliton were studied: see Krieger and Schlag [25] and Krieger, Nakanishi and Schlag [26] and [27]. There are also some remarkable classification theorems by Duyckaerts, Kenig and Merle [9], [11], [10] and [12]. Concerning the blow-up behavior, we would like to mention that Donniger, Huang, Krieger and Schlag prove in [6] the existence of so-called “exotic” blow-up solutions when $N = 3$, whose blow-up rate oscillates between several pure-power laws.

When

$$N \geq 3 \text{ and } p > p_s,$$

much less is known. We would just like to mention that the stability result of Donniger and Schörkhuber proved in [7] for the superconformal range does hold in the Sobolev supercritical range too, at least when $N = 3$. There is also a remarkable result by Kenig and Merle [21] on the dynamics of solutions with some compactness property.

In this work, our aim is to study the profile of blow-up solutions in the complex case of equation (1). In particular, relying on the work of Merle and Zaag in [33], we give a trapping result near the set of non-zero stationary solutions in self-similar variables. This study is far from being trivial since the complex structure introduces an additional zero eigenfunction in the linearized equation around the expected profile, and also because of the coupling between the real and the imaginary parts.

If u is a blow-up solution of (1), we define (see for example Alinhac [1]) a continuous curve Γ as the graph of a function $x \mapsto T(x)$ such that the domain of definition

of u (or the maximal influence domain of u) is

$$D_u = \{(x, t) | t < T(x)\}.$$

From the finite speed of propagation, T is a 1-Lipschitz function. The time $\tilde{T} = \inf_{x \in \mathbb{R}} T(x)$ and the graph Γ are called (respectively) the blow-up time and the blow-up graph of u .

Let us introduce the following non-degeneracy condition for Γ . If we introduce for all $x \in \mathbb{R}$, $t \leq T(x)$ and $\delta > 0$, the cone

$$\mathcal{C}_{x,t,\delta} = \{(\xi, \tau) \neq (x, t) | 0 \leq \tau \leq t - \delta|\xi - x|\},$$

then our non-degeneracy condition is the following: x_0 is a non-characteristic point if

$$(3) \quad \exists \delta = \delta(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0, T(x_0), \delta_0}.$$

If condition (3) is not true, then we call x_0 a characteristic point. Already when u is real-valued, we know from [37] and [5] that there exist blow-up solutions with characteristic points.

Given some $x_0 \in \mathbb{R}$, we introduce the following self-similar change of variables:

$$(4) \quad w_{x_0}(y, s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T(x_0) - t}, \quad s = -\log(T(x_0) - t).$$

This change of variables transforms the backward light cone with vertex $(x_0, T(x_0))$ into the infinite cylinder $(y, s) \in (-1, 1) \times [-\log T(x_0), +\infty)$. The function w_{x_0} (we write w for simplicity) satisfies the following equation for all $|y| < 1$ and $s \geq -\log T(x_0)$:

$$(5) \quad \partial_s^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_y w,$$

where

$$(6) \quad \mathcal{L}w = \frac{1}{\rho} \partial_y (\rho(1-y^2) \partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}.$$

This equation will be studied in the space

$$(7) \quad \mathcal{H} = \{q \in H_{loc}^1 \times L_{loc}^2((-1, 1), \mathbb{C}) \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 (|q_1|^2 + |q_1'|^2(1-y^2) + |q_2|^2) \rho \, dy < +\infty\},$$

which is the energy space for w . Note that $\mathcal{H} = \mathcal{H}_0 \times L_\rho^2$ where

$$(8) \quad \mathcal{H}_0 = \{r \in H_{loc}^1((-1, 1), \mathbb{C}) \mid \|r\|_{\mathcal{H}_0}^2 \equiv \int_{-1}^1 (|r'|^2(1-y^2) + |r|^2) \rho \, dy < +\infty\}.$$

Let us define

$$(9) \quad E(w, \partial_s w) = \int_{-1}^1 \left(\frac{1}{2} |\partial_s w|^2 + \frac{1}{2} |\partial_y w|^2 (1-y^2) + \frac{p+1}{(p-1)^2} |w|^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho \, dy.$$

By the argument of Antonini and Merle [3], which works straightforwardly in the complex case, we see that E is a Lyapunov functional for equation (5). Similarly, some arguments of the real case can be adapted with no problems to the complex-case, others cannot. As a matter of fact, the derivation of the blow-up rate works as in the real case whereas the convergence to the profile needs intricate estimates,

and this is the goal of our paper. Let us first briefly state the result for the blow-up rate, then focus on the convergence question.

1.2. Blow-up rate. Only in this subsection, the space dimension will be extended to any $N \geq 1$. We assume in addition that p is conformal or subconformal (see (2)). We recall that for the real case of equation (1), Merle and Zaag determined in [30] and [31] the blow-up rate for (1) in the region $\{(x, t) \mid t < \tilde{T}\}$ in a first step. Then in [32], they extended their result to the whole domain of definition $\{(x, t) \mid t < T(x)\}$. In the following, we give the growth estimate near the blow-up surface for solutions of equation (1).

Proposition 1 (Growth estimate near the blow-up surface for solutions of equation (1)). *If u is a solution of (1) with blow-up surface $\Gamma : \{x \rightarrow T(x)\}$, and if $x_0 \in \mathbb{R}^N$ is non-characteristic (in the sense of (3)), then:*

(i) **(Uniform bounds on w)** For all $s \geq -\log \frac{T(x_0)}{4}$:

$$0 < \epsilon_0(N, p) \leq \|w_{x_0}(s)\|_{H^1(B)} + \|\partial_s w_{x_0}(s)\|_{L^2(B)} \leq K.$$

(ii) **(Uniform bounds on u)** For all $t \in [\frac{3}{4}T(x_0), T(x_0))$:

$$0 < \epsilon_0(N, p) \leq (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{N/2}} \\ + (T(x_0) - t)^{\frac{2}{p-1}+1} \left(\frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{N/2}} \right) \leq K,$$

where the constant K depends only on N, p , and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$ and the initial data in $H_{loc, u}^1 \times L_{loc, u}^2$.

Proof. The idea of the proof is the same as in [32]. For the sake of completeness, we give a sketch of the proof in Appendix A. $\square$

With the bounds in Proposition 1, we ask the question of compactness of the solution and the question of convergence of w to a stationary solution of (5).

1.3. Blow-up profile. From now on, we assume again that

$$N = 1.$$

This subsection is the heart of our work. Indeed, unlike for the blow-up rate, it is not a simple adaptation of the real case. It involves many new ideas of ours. The first step towards the determination of the blow-up profile is to characterize all stationary solutions in $\mathcal{H}_0$ of equation (5).

Proposition 2 (Characterization of all stationary solution of equation (5) in $\mathcal{H}_0$).

(i) Consider $w \in \mathcal{H}_0$ a stationary solution of (5). Then, either $w \equiv 0$ or there exist $\delta \in (-1, 1)$ and $\theta \in \mathbb{R}$ such that $w(y) = e^{i\theta} \kappa(\delta, y)$ where

$$(10) \quad \forall (\delta, y) \in (-1, 1)^2, \quad \kappa(\delta, y) = \kappa_0 \frac{(1 - \delta^2)^{\frac{1}{p-1}}}{(1 + \delta y)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}.$$

(ii) It holds that

$$(11) \quad E(0, 0) = 0 \text{ and } \forall \delta \in (-1, 1), \forall \theta \in \mathbb{R}, E(e^{i\theta} \kappa(\delta, \cdot), 0) = E(\kappa_0, 0) > 0,$$

where E is given by (9).

Remark. Note that the proof of this proposition is very different from the real case. Indeed, in the real case, the result follows from a transformation of the hyperbolic plane, which gives nothing in the complex case. We succeed in proving the result relying on ODE techniques for complex-valued equation.

Remark. Unlike the real case where the set of stationary solutions is made of three connected components: $\{0\}$, $\{+\kappa(\delta, y)\}$ and $\{-\kappa(\delta, y)\}$, we have only two connected components: $\{0\}$ and $\{e^{i\theta}\kappa(\delta, y) \mid \theta \in \mathbb{R}, |\delta| < 1\}$. This is one of the novelties of our approach. Indeed, here we need a modulation technique to control the parameter θ which may take any real value, unlike the real case, where it was equal to $k\pi$ only.

The second step is the same as in the real case, and involves no novelty on our behalf. It uses the Lyapunov functional to show that when x_0 is non-characteristic, then w_{x_0} approaches the set of non-zero stationary solutions. This is the result:

Proposition 3 (Approaching the set of non-zero stationary solutions near a non-characteristic point). *Consider u a solution of (1) with blow-up curve $\Gamma : \{x \rightarrow T(x)\}$. If $x_0 \in \mathbb{R}$ is non-characteristic, then:*

- (A.i) $\inf_{\{\theta \in \mathbb{R}, |\delta| < 1\}} \|w_{x_0}(\cdot, s) - e^{i\theta}\kappa(\delta, \cdot)\|_{H^1(-1,1)} + \|\partial_s w_{x_0}\|_{L^2(-1,1)} \rightarrow 0$ as $s \rightarrow \infty$.
 (A.ii) $E(w_{x_0}(s), \partial_s w_{x_0}(s)) \rightarrow E(\kappa_0, 0)$ as $s \rightarrow \infty$.

Proof. The idea of the proof is the same as in [33]. For the sake of completeness, we give a sketch of the proof in Appendix A. $\square$

From this result, we wonder whether θ and δ have limits as $s \rightarrow \infty$; in other words, whether $w_{x_0}(\cdot, s)$ converges to some $e^{i\theta_\infty(x_0)}\kappa(\delta_\infty(x_0))$ for some $\theta_\infty(x_0) \in \mathbb{R}$ and $|\delta_\infty(x_0)| < 1$. The answer is in fact positive, as one sees in Theorem 5 below that the following trapping result of solutions of equation (5) near non-zero stationary solutions is a major tool towards this result.

In the following, we consider $w \in C([s^*, \infty), \mathcal{H})$ and show that if $w(s^*)$ is close enough to some non-zero stationary solution and satisfies an energy barrier, then $w(s)$ converges to a neighboring stationary solution as $s \rightarrow \infty$.

Theorem 4 (Trapping near the set of non-zero stationary solutions of (5)). *There exist positive ϵ_0 , μ_0 and C_0 such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (5) such that*

$$(12) \quad \forall s \geq s^*, E(w(s), \partial_s w(s)) \geq E(\kappa_0, 0),$$

and

$$(13) \quad \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - e^{i\theta^*} \begin{pmatrix} \kappa(\delta^*, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^*$$

for some $\delta^* \in (-1, 1)$, $\theta^* \in \mathbb{R}$ and $\epsilon^* \in (0, \epsilon_0]$, then there exists $\delta_\infty \in (-1, 1)$ and $\theta_\infty \in \mathbb{R}$ such that

$$|\operatorname{arctanh} \delta_\infty - \operatorname{arctanh} \delta^*| + |\theta_\infty - \theta^*| \leq C_0 \epsilon^*$$

and for all $s \geq s^*$:

$$(14) \quad \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - e^{i\theta_\infty} \begin{pmatrix} \kappa(\delta_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \epsilon^* e^{-\mu_0(s-s^*)}.$$

Remark. Condition (12) is crucial for the conclusion. Indeed, if (13) is satisfied but not (12), we may have a different conclusion, as with the explicit solution

$w(y, s) = \kappa_0 \frac{(1-\delta^2)^{\frac{1}{p-1}}}{(1+\mu e^s + \delta y)^{\frac{2}{p-1}}}$ which may be made arbitrarily close to $\kappa(\delta, y)$ and satisfies convergences to 0 as $s \rightarrow +\infty$.

As we said earlier, the third step towards the derivation of the blow-up profile simply uses Proposition 3 and Theorem 4 to get the following:

Theorem 5 (Blow-up profile near a non-characteristic point). *If u is a solution of (1) with blow-up curve $\Gamma : \{x \rightarrow T(x)\}$ and $x_0 \in \mathbb{R}$ is non-characteristic (in the sense as in (3)), then there exist $\delta_\infty(x_0) \in (-1, 1)$, $\theta_\infty(x_0) \in \mathbb{R}$ and $s^*(x_0) \geq -\log T(x_0)$ such that for all $s \geq s^*(x_0)$, (14) holds with $\epsilon^* = \epsilon_0$, where C_0 and ϵ_0 are given in Theorem 4. Moreover,*

$$\|w_{x_0}(s) - e^{i\theta_\infty(x_0)}\kappa(\delta_\infty(x_0))\|_{H^1(-1,1)} + \|\partial_s w_{x_0}(s)\|_{L^2(-1,1)} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

Remark. From the Sobolev embedding, we know that the convergence takes place also in L^∞ , in the sense that

$$\|w_{x_0}(s) - e^{i\theta_\infty(x_0)}\kappa(\delta_\infty(x_0))\|_{L^\infty(-1,1)} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

Remark. As we mentioned above one of our difficulties comes from the invariance of the solution under complex rotation, which induces an additional zero-mode in the linearization of equation (5) around $\kappa(\delta, y)$. In order to overcome that difficulty, we use a modulation technique (see Proposition 4.1 below). Let us mention that the extension from the real to complex case has been successfully performed by Filippas and Merle [14] in the case of the semilinear heat equation with Sobolev subcritical non-linearity:

$$(15) \quad \partial_t u = \Delta u + |u|^{p-1}u, \text{ with } p > 1 \text{ and } (N-2)p < N+2.$$

We mention that the modulation technique was already crucial in [14] to control the additional zero-mode coming from the invariance of equation (15) under complex rotation. Note however that the adaptation from the real to the complex case for the wave equation is far more difficult, since we have additional problems, coming from the fact that we have to handle non-self-adjoint operators.

This paper is organized as follows:

In Section 2, we characterize the set of stationary solutions, proving Proposition 2.

In Section 3, we study the properties of the linearized operator of equation (5) around a non-zero stationary solution.

Finally, Section 4 is devoted to the proof of Proposition 3, Theorem 4 and Theorem 5.

2. CHARACTERIZATION OF THE STATIONARY SOLUTIONS IN SELF-SIMILAR VARIABLES

In this section, we prove Proposition 2 which characterizes all $\mathcal{H}_0$ solutions of

$$(16) \quad \frac{1}{\rho}(\rho(1-y^2)w')' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0,$$

the stationary version of (5). As our solution is a complex-valued one, we will use in addition to the techniques of Section 2.3 in [33], other techniques as the determination of the phase and some projections. Note that since 0 and $\kappa_0 e^{i\theta}$

are trivial solutions to equation (5) for any $\theta \in \mathbb{R}$, we see from Lemma C.5 that $\mathcal{T}_\delta(e^{i\theta}\kappa_0) = e^{i\theta}\kappa(\delta, y)$ is also a stationary solution to (5). Let us introduce the set

$$(17) \quad S \equiv \{0, e^{i\theta}\kappa(\delta, \cdot), |\delta| < 1, \theta \in \mathbb{R}\}$$

and prove that there are no more solutions of (16) in $\mathcal{H}_0$ outside the set S .

Proof. We first prove (ii), since its proof is short.

(ii) Since we clearly have from the definition (9) that $E(0, 0) = 0$, we will compute $E(e^{i\theta}\kappa(\delta, \cdot), 0)$. From (9) and the proof of the real case (page 59 in [33]), we see that

$$E(e^{i\theta}\kappa(\delta, \cdot), 0) = E(\kappa(\delta, \cdot), 0) = E(\kappa_0, 0) > 0.$$

Thus, (11) follows.

(i) Consider $w \in \mathcal{H}_0$ a non-zero solution of (16). Let us prove that there are some $\delta \in (-1, 1)$ and $\theta \in \mathbb{R}$ such that $w = e^{i\theta}\kappa(\delta, \cdot)$. For this purpose, define

$$(18) \quad \xi = \frac{1}{2} \log \left(\frac{1+y}{1-y} \right) \text{ (that is } y = \tanh \xi \text{) and } v(\xi) = w(y)(1-y^2)^{\frac{1}{p-1}}.$$

As in the real case, we see from straightforward calculations that $v \not\equiv 0$ is an $H^1(\mathbb{R})$ solution to

$$(19) \quad \partial_\xi^2 v + |v|^{p-1}v - \frac{4}{(p-1)^2}v = 0, \quad \forall \xi \in \mathbb{R}.$$

Our aim is to prove the existence of $\theta_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}$ such that $v(\xi) = e^{i\theta_0}\check{k}(\xi + \xi_0)$ where

$$\check{k}(\xi) = \frac{\kappa_0}{\cosh^{\frac{2}{p-1}}(\xi)}.$$

Since $v \in H^1(\mathbb{R}) \subset C^{\frac{1}{2}}(\mathbb{R})$, we see that v is a strong C^2 solution of equation (19). Since $v \not\equiv 0$, there exists $\xi_0 \in \mathbb{R}$ such that $v(\xi_0) \neq 0$. By invariance of (19) under translation, we may suppose that $\xi_0 = 0$. Let

$$(20) \quad G^* = \{\xi \in \mathbb{R} \mid \check{w}(\xi) \neq 0\},$$

a non-empty open set by continuity. Note that G^* contains some non-empty interval I containing 0. We also introduce $R(\xi) = |v(\xi)|$, θ and $\check{\theta}$ two determinations of the phase given by

$$(21) \quad \theta(\xi) = \arctan \left(\frac{\operatorname{Im} v(\xi)}{\operatorname{Re} v(\xi)} \right) \text{ and } \check{\theta}(\xi) = \operatorname{arccotan} \left(\frac{\operatorname{Re} v(\xi)}{\operatorname{Im} v(\xi)} \right),$$

and $h : G^* \rightarrow \mathbb{R}$ given by

$$(22) \quad \forall \xi \in G^*, \quad h(\xi) = \begin{cases} \theta'(\xi), & \text{if } \operatorname{Re} v(\xi) \neq 0, \\ \check{\theta}'(\xi), & \text{if } \operatorname{Im} v(\xi) \neq 0. \end{cases}$$

We claim that h is well defined and that $h \in C^1(G^*)$. Indeed, let $\xi_0 \in \mathbb{R}$ such that $v(\xi_0) \neq 0$. Necessarily, either its real or its imaginary part is non-zero. If for instance $\operatorname{Re} v(\xi_0) \neq 0$, by continuity

$$\exists \delta_0 > 0, \quad \forall |\xi - \xi_0| < \delta_0, \quad \operatorname{Re} \check{w}(\xi) \neq 0,$$

so θ is well defined in $(\xi_0 - \delta_0, \xi_0 + \delta_0)$, and h is well defined and C^1 in $(\xi_0 - \delta_0, \xi_0 + \delta_0)$. Now, if $\operatorname{Im} v(\xi_0) \neq 0$, by the same way, we prove that h given by $h(\xi) = \check{\theta}'(\xi)$ is well defined and C^1 in a small interval $(\xi_0 - \delta_0, \xi_0 + \delta_0)$.

This definition is non-ambiguous. Indeed, if ever both θ and $\check{\theta}$ are defined on the same interval (a, b) with $a < b$, then there exists $k \in \mathbb{Z}$ such that

$$\forall \xi \in (a, b), \theta(\xi) = \check{\theta}(\xi) + 2k\pi.$$

Differentiating this, we get

$$\forall \xi \in (a, b), \theta'(\xi) = \check{\theta}'(\xi).$$

Thus, (22) defines $h(\xi)$ with no ambiguity.

Take $\xi \in G^*$. Using one of the angle determinations in (21) and projecting equation (19), we see that

$$(23) \quad \forall \xi \in G^*, \begin{cases} R''(\xi) - R(\xi)h^2(\xi) - c_0R(\xi) + R^p(\xi) = 0, & c_0 = \frac{4}{(p-1)^2}, \\ 2R'(\xi)h(\xi) + R(\xi)h'(\xi) = 0. \end{cases}$$

Integrating the second equation on the interval $I \subset G^*$, we see that for all $\xi \in I$, $h(\xi) = \frac{h(0)R^2(0)}{R^2(\xi)}$. Plugging this in the first equation, we get

$$(24) \quad \forall \xi \in I, R''(\xi) - \frac{\mu}{R^3(\xi)} - c_0R(\xi) + R^p(\xi) = 0, \text{ where } \mu = h^2(0)R^4(0).$$

Now let

$$(25) \quad G = \left\{ \xi \in G^*, \forall \xi' \in I_\xi, h(\xi') = \frac{h(0)R^2(0)}{R^2(\xi')} \right\},$$

where $I_\xi = [0, \xi]$ if $\xi \geq 0$ or $I_\xi = (\xi, 0]$ if $\xi \leq 0$. Note that $I \subset G$. Now, we give the following:

Lemma 2.1. *There exists $\epsilon_0 > 0$ such that*

$$\forall \xi \in G, \forall \xi' \in I_\xi, 0 < \epsilon_0 \leq |v(\xi')| \leq \frac{1}{\epsilon_0}.$$

Proof. Take $\xi \in G$. By definition (25) of G , we see that equation (24) is satisfied for all $\xi' \in I_\xi$. Multiplying $R''(\xi) - \frac{\mu}{R^3(\xi)} - c_0R(\xi) + R^p(\xi) = 0$ by R' and integrating between 0 and ξ , we get:

$$\forall \xi \in I_\xi, \mathcal{E}(\xi') = \mathcal{E}(0), \text{ where } \mathcal{E}(\xi') = \frac{1}{2}R'^2(\xi') + \frac{\mu}{2R^2(\xi')} - \frac{c_0}{2}R^2(\xi') + \frac{R^{p+1}(\xi')}{p+1},$$

or equivalently,

$$\forall \xi' \in I_\xi, F(R(\xi')) = \frac{1}{2}R'(\xi')^2 \geq 0, \text{ where } F(r) = -\frac{\mu}{2r^2} + \frac{c_0}{2}r^2 - \frac{r^{p+1}}{p+1} + \mathcal{E}(0).$$

Since $F(r) \rightarrow -\infty$ as $r \rightarrow 0$ or $r \rightarrow \infty$, there exists $\epsilon_0 = \epsilon_0(\mu, E(0)) > 0$ such that $\epsilon_0 \leq R(\xi') \leq \frac{1}{\epsilon_0}$, which yields the conclusion of Lemma 2.1. $\square$

We claim the following:

Lemma 2.2. *It holds that $G = \mathbb{R}$.*

Proof. Note first that by construction, G is a non-empty interval (note that $0 \in I \subset G$ where I is defined right after (20)). We have only to prove that $\sup G = +\infty$, since the fact that $\inf G = -\infty$ can be deduced by replacing $v(\xi)$ by $v(-\xi)$.

By contradiction, suppose that $\sup G = a < +\infty$. By continuity, we have

$$(26) \quad \forall \xi \in [0, a], h(\xi) = \frac{h(0)R(0)^2}{R(\xi)^2},$$

on the one hand. On the other hand, by Lemma 2.1, for all $\xi' \in [0, a)$, $0 < \epsilon_0 \leq |v(\xi')| \leq \frac{1}{\epsilon_0}$. Using (26) $v(a) \neq 0$, and $a \in G^*$. Using (26), we see that $a \in G$. By continuity, we can write for all $\xi \in (a - \delta, a + \delta)$, where $\delta > 0$ is small enough,

$$\begin{cases} R''(\xi) - R(\xi)\theta'(\xi)^2 - c_0 R(\xi) + R(\xi)^p = 0, & c_0 = \frac{4}{(p-1)^2}, \\ 2R'(\xi)h(\xi) + R(\xi)h'(\xi) = 0. \end{cases}$$

From the second equation and (26) applied with $\xi = a$, we see that $h(\xi) = \frac{h(a)(R(a))^2}{(R(\xi))^2} = \frac{h(0)(R(0))^2}{(R(\xi))^2}$. Therefore, it follows that $(a, a + \delta) \in E$, which contradicts the fact that $a = \sup G$. $\square$

Note from Lemma 2.2 that (23) and (24) hold for all $\xi \in \mathbb{R}$. We claim that $h(0) = 0$. Indeed, if not, then by (24), we have $\mu \neq 0$, and since $G = \mathbb{R}$, we see from Lemma 2.1 that for all $\xi \in \mathbb{R}$, $|v(\xi)| \geq \epsilon_0$. Therefore $v \notin L^2(\mathbb{R})$, which contradicts the fact that $v \in H^1(\mathbb{R})$. Thus, $h(0) = 0$, and $\mu = 0$. By uniqueness of solutions to the second equation of (23), we see that $h(\xi) = 0$ for all $\xi \in \mathbb{R}$, so $\theta(\xi) = \theta(0)$ and $v'(0) = \lambda e^{i\theta(0)}$ ($\lambda \in \mathbb{R}$). Thus

$$\begin{cases} v(0) = R(0)e^{i\theta(0)}, \\ v'(0) = \lambda e^{i\theta(0)}. \end{cases}$$

Let $W \in H^1(\mathbb{R})$ be a real solution of

$$(27) \quad \begin{cases} W'' - c_0 W + |W|^{p-1} W = 0, \\ W(0) = R(0), \\ W'(0) = \lambda. \end{cases}$$

By uniqueness of the Cauchy problem of equation (19), we have for all $\xi \in \mathbb{R}$, $v(\xi) = W(\xi)e^{i\theta(0)}$, and as $v \in H^1(\mathbb{R})$, W is also in $H^1(\mathbb{R})$. Since it is well known that the real solutions of (27) in $H^1(\mathbb{R})$ are

$$(28) \quad \text{either } W \equiv 0 \text{ or } W(\xi) = \pm \tilde{k}(\xi + \xi_0) \text{ for some } \xi_0 \in \mathbb{R},$$

for the reader's convenience, we recall the proof in Appendix B. It follows that $v(\xi) = e^{i\theta_0} \tilde{k}(\xi + \xi_0)$ for some $\xi_0 \in \mathbb{R}$, because $W \not\equiv 0$ and $W > 0$. Thus, for $d = \tanh \xi_0 \in (-1, 1)$ and $y = \tanh \xi$, we get

$$\begin{aligned} v(\xi) &= e^{i\theta_0} \kappa_0 [1 - \tanh(\xi + \xi_0)^2]^{\frac{1}{p-1}} = e^{i\theta_0} \kappa_0 \left[1 - \left(\frac{\tanh \xi + \tanh \xi_0}{1 + \tanh \xi \tanh \xi_0} \right)^2 \right]^{\frac{1}{p-1}} \\ &= e^{i\theta_0} \kappa_0 \left[1 - \left(\frac{y + d}{1 + dy} \right)^2 \right]^{\frac{1}{p-1}} = e^{i\theta_0} \kappa_0 \left[\frac{(1 - d^2)(1 - y^2)^2}{(1 + dy)^2} \right]^{\frac{1}{p-1}} \\ &= e^{i\theta_0} \kappa(\delta, y) (1 - y^2)^{\frac{1}{p-1}}. \end{aligned}$$

By (18), we see that $w(y) = e^{i\theta_0} \kappa(\delta, y)$. This concludes the proof of Proposition 2. $\square$

3. THE LINEARIZED OPERATOR AROUND A NON-ZERO STATIONARY SOLUTION

In this section, we study the properties of the linearized operator of equation (5) around the stationary solution $\kappa(\delta, y)$ (10). We recall that in [33] the authors have treated the real case by introducing $q = (q_1, q_2) \in \mathbb{R} \times \mathbb{R}$ and linearizing around $\kappa(\delta, y)$. It turns out that the real part of our linearized operator for complex-valued solution is identical to the real part of the linearized operator in the real case treated

in [33]. As a matter of fact, we will rely on [33] for the real part and have to invent new methods for the imaginary part.

For any complex number z , we use in the following the notation

$$\check{z} = \operatorname{Re}(z) \text{ and } \tilde{z} = \operatorname{Im}(z).$$

If we introduce $q = (q_1, q_2) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{C} \times \mathbb{C}$ for all $s \in [s_0, \infty)$, for a given $s_0 \in \mathbb{R}$, by

$$\begin{pmatrix} w(y, s) \\ \partial_s w(y, s) \end{pmatrix} = \begin{pmatrix} \kappa(\delta, y) \\ 0 \end{pmatrix} + \begin{pmatrix} q_1(y, s) \\ q_2(y, s) \end{pmatrix},$$

then we see from equation (5) that q satisfies the following equation for all $s \geq s_0$:

$$(29) \quad \frac{\partial}{\partial s} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = L_\delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f_\delta(q_1) \end{pmatrix},$$

where

$$\begin{aligned} L_\delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} \mathcal{L}q_1 + \check{\psi}(\delta, y)\check{q}_1 + i\tilde{\psi}(\delta, y)\tilde{q}_1 - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2 \\ \check{\psi}(\delta, y) = p\kappa(\delta, y)^{p-1} - \frac{2(p+1)}{(p-1)^2}, \\ \tilde{\psi}(\delta, y) = \kappa(\delta, y)^{p-1} - \frac{2(p+1)}{(p-1)^2}, \\ f_\delta(q_1) &= \check{f}_\delta(\check{q}_1, \check{q}_1) + i\tilde{f}_\delta(\tilde{q}_1, \tilde{q}_1), \end{pmatrix}, \end{aligned}$$

$$\text{where } \check{f}_\delta(\check{q}_1, \check{q}_1) = |\kappa(\delta, y) + q_1|^{p-1}(\kappa(\delta, y) + \check{q}_1) - \kappa(\delta, y)^p - p\kappa^{p-1}(\delta, y)\check{q}_1,$$

$$(30) \quad \tilde{f}_\delta(\tilde{q}_1, \tilde{q}_1) = |\kappa(\delta, y) + q_1|^{p-1}\tilde{q}_1 - \kappa^{p-1}(\delta, y)\tilde{q}_1.$$

From (29), dissociating the real and imaginary parts, we get for all $s \geq s_0$:

$$(31) \quad \frac{\partial}{\partial s} \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} = \check{L}_\delta \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \check{f}_\delta \end{pmatrix},$$

where

$$(32) \quad \check{L}_\delta \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}\check{q}_1 + \check{\psi}(\delta, y)\check{q}_1 - \frac{p+3}{p-1}\check{q}_2 - 2y\partial_y \check{q}_2 \\ \end{pmatrix},$$

and

$$(33) \quad \frac{\partial}{\partial s} \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \tilde{L}_\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{f}_\delta \end{pmatrix},$$

where

$$(34) \quad \tilde{L}_\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}\tilde{q}_1 + \tilde{\psi}(\delta, y)\tilde{q}_1 - \frac{p+3}{p-1}\tilde{q}_2 - 2y\partial_y \tilde{q}_2 \\ \end{pmatrix}.$$

Remark. From (29) we see that for $q = \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} + i \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}$, we have

$$L_\delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \check{L}_\delta \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} + i\tilde{L}_\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}.$$

Note that the operator $\check{L}_\delta$ (32) already appears in the real case studied in [33]. For that reason, we recall from that paper the properties of $\check{L}_\delta$, and focus here on the

properties of $\tilde{L}_\delta$, which is one of the novelties of our work. Note from (7) that we have

$$\|q\|_{\mathcal{H}} = [\phi(q, q)]^{\frac{1}{2}} < +\infty,$$

where the hermitian inner product ϕ is defined by

$$\phi(q, r) = \phi\left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right) = \int_{-1}^1 (q_1 \bar{r}_1 + q_1' \bar{r}_1' (1 - y^2) + q_2 \bar{r}_2) \rho(y) dy.$$

Using integration by parts and the definition of $\mathcal{L}$ (6), we have the following:

$$(35) \quad \phi(q, r) = \int_{-1}^1 (q_1 (-\mathcal{L} \bar{r}_1 + \bar{r}_1) + q_2 \bar{r}_2) \rho(y) dy.$$

We note that $q \in \mathcal{H}$ if and only if $\check{q} \in \mathcal{H}$ and $\tilde{q} \in \mathcal{H}$, and

$$\|q\|_{\mathcal{H}}^2 = \|\check{q}\|_{\mathcal{H}}^2 + \|\tilde{q}\|_{\mathcal{H}}^2.$$

This section is organized as follows:

-We first recall some spectral properties of $\tilde{L}_\delta$ which was proved by Merle and Zaag in [33].

-Then, we focus on the study of $\tilde{L}_\delta$, precisely, we compute $\tilde{L}_\delta^*$ the conjugate operator of $\tilde{L}_\delta$ and we give a zero direction for it.

-Using the projection on the eigenspace of $\tilde{L}_\delta$, we introduce a function which will capture the dispersive character of equation (29), and give some dispersive estimates in order to prove Theorem 4.

3.1. Spectral properties of $\tilde{L}_\delta$. From Section 4 in [33], we know that $\tilde{L}_\delta$ has two non-negative eigenvalues $\lambda = 1$ and $\lambda = 0$ with eigenfunctions

$$(36) \quad \check{F}_1(\delta, y) = (1 - \delta^2)^{\frac{p}{p-1}} \begin{pmatrix} (1 + \delta y)^{-\frac{p+1}{p-1}} \\ (1 + \delta y)^{-\frac{p+1}{p-1}} \end{pmatrix} \text{ and } \check{F}_0(\delta, y) = (1 - \delta^2)^{\frac{1}{p-1}} \begin{pmatrix} \frac{y + \delta}{(1 + \delta y)^{\frac{p+1}{p-1}}} \\ 0 \end{pmatrix}.$$

Note that for some $C_0 > 0$ and any $\lambda \in \{0, 1\}$, we have

$$(37) \quad \forall |\delta| < 1, \quad \frac{1}{C_0} \leq \|\check{F}_\lambda^\delta\|_{\mathcal{H}} \leq C_0 \quad \text{and} \quad \|\partial_\delta \check{F}_\lambda^\delta\|_{\mathcal{H}} \leq \frac{C_0}{1 - \delta^2}.$$

We also know that $\tilde{L}_\delta^*$ the conjugate operator of $\tilde{L}_\delta$ with respect to ϕ is given by

$$\tilde{L}_\delta^* \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \check{R}_\delta(r_2) \\ -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr_2' - \frac{8}{(p-1)} \frac{r_2}{(1-y^2)} \end{pmatrix}$$

for any $(r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2$, where $r = \check{R}_\delta(r_2)$ is the unique solution of

$$-\mathcal{L}r + r = \mathcal{L}r_2 + \check{\psi}(\delta, y)r_2.$$

Here, the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ defined in (6) is the set of all $r \in L_\rho^2$ such that $\mathcal{L}r \in L_\rho^2$. Furthermore, $\tilde{L}_\delta^*$ has two non-negative eigenvalues $\lambda = 0$ and $\lambda = 1$ with eigenfunctions $\check{W}_\lambda$ such that

$$\check{W}_{1,2}(\delta, y) = \check{c}_1 \frac{(1 - y^2)(1 - \delta)^{\frac{1}{p-1}}}{(1 + \delta y)^{\frac{p+1}{p-1}}}, \quad \check{W}_{0,2}(\delta, y) = \check{c}_0 \frac{(y + \delta)(1 - \delta)^{\frac{1}{p-1}}}{(1 + \delta y)^{\frac{p+1}{p-1}}},$$

with¹

$$\frac{1}{\check{c}_\lambda} = 2\left(\frac{2}{p-1} + \lambda\right) \int_{-1}^1 \left(\frac{y^2}{1-y^2}\right)^{1-\lambda} \rho(y) dy,$$

and $\check{W}_{\lambda,1}(\delta, \cdot)$ is the unique solution of the equation

$$-\mathcal{L}r + r = \left(\lambda - \frac{p+3}{p-1}\right) r_2 - 2yr'_2 + \frac{8}{p-1} \frac{r_2}{1-y^2}$$

with $r_2 = \check{W}_{\lambda,2}(\delta, \cdot)$. Note that we have the following relations for $\lambda = 0$ or $\lambda = 1$:

$$\phi(\check{W}_\lambda(\delta, \cdot), \check{F}_\lambda(\delta, \cdot)) = 1 \text{ and } \phi(\check{W}_\lambda(\delta, \cdot), \check{F}_{1-\lambda}(\delta, \cdot)) = 0.$$

Let us introduce for $\lambda \in \{0, 1\}$ the projectors $\tilde{\pi}_\lambda^\delta(r)$, and $\tilde{\pi}_-^\delta(r)$ for any $r \in \mathcal{H}$ by

$$(38) \quad \tilde{\pi}_\lambda^\delta(r) = \phi(\check{W}_\lambda(\delta, \cdot), r),$$

$$(39) \quad r = \tilde{\pi}_0^\delta(r) \check{F}_0(\delta, \cdot) + \tilde{\pi}_1^\delta(r) \check{F}_1(\delta, \cdot) + \tilde{\pi}_-^\delta(r),$$

and the space

$$\mathcal{H}_-^\delta \equiv \{r \in \mathcal{H} \mid \tilde{\pi}_1^\delta(r) = \tilde{\pi}_0^\delta(r) = 0\}.$$

Introducing the bilinear form

$$(40) \quad \check{\varphi}_\delta(q, r) = \int_{-1}^1 (-\check{\psi}(\delta, \cdot) q_1 r_1 + q'_1 r'_1 (1-y^2) + q_2 r_2) \rho dy,$$

where $\check{\psi}(\delta, y)$ is defined in (30), we recall from Proposition 4.7 on page 90 in [33] that there exists $C_0 > 0$ such that for all $|\delta| < 1$, for all $r \in \mathcal{H}_-^\delta$,

$$(41) \quad \frac{1}{C_0} \|r\|_{\mathcal{H}}^2 \leq \check{\varphi}_\delta(r, r) \leq C_0 \|r\|_{\mathcal{H}}^2.$$

In the following sections, we follow the method of [33] to study the spectral properties of $\tilde{L}_\delta$.

3.2. A zero direction of $\tilde{L}_\delta$. Let us show that $\lambda = 0$ is an eigenvalue for $\tilde{L}_\delta$. We claim the following:

Lemma 3.1 (Zero direction of $\tilde{L}_\delta$).

(i) For all $|\delta| < 1$, $\lambda = 0$ is an eigenvalue of the linear operator $\tilde{L}_\delta$ and its corresponding eigenfunction is

$$(42) \quad \tilde{F}_0(\delta, y) = \begin{pmatrix} \kappa(\delta, y) \\ 0 \end{pmatrix}.$$

(ii) Moreover, it holds for some $C_0 > 0$ that

$$(43) \quad \forall |\delta| < 1, \quad \frac{1}{C_0} \leq \|\tilde{F}_0(\delta, \cdot)\|_{\mathcal{H}} \leq C_0 \quad \text{and} \quad \|\partial_\delta \tilde{F}_0(\delta, \cdot)\|_{\mathcal{H}} \leq \frac{C_0}{1-\delta^2}.$$

Remark. There is a more geometrical way to see that $\lambda = 0$ is an eigenvalue for $\tilde{L}_\delta$ and $\tilde{L}_\delta$ (in other words, a double eigenvalue for L_δ given in (30)): simply note that equation (5) has a two-parameter family of stationary solutions

$$K(\delta, \theta, y) = (e^{i\theta} \kappa(\delta, y), 0)$$

¹In section 4 of [33], we had non-explicit normalizing constants $\check{c}_\lambda = \check{c}_\lambda(\delta)$. In Lemma 2.4 in [39], the authors compute the explicit dependence of $\check{c}_\lambda(\delta)$.

hence, $\partial_\delta K(\delta, 0, y) = (\partial_\delta \kappa(\delta, y), 0)$ and $\partial_\theta K(\delta, 0, y) = (i\kappa(\delta, y), 0)$ are eigenfunctions of the linearized operator of equation (5) around $K(\delta, 0, y) = (\kappa(\delta, y), 0)$, which is precisely the operator L_δ . Splitting L_δ into real and imaginary parts shows that $(\partial_\delta \kappa(\delta, y), 0)$ and $(\kappa(\delta, y), 0)$ are eigenfunctions of $\tilde{L}_\delta$ and $\tilde{L}_\delta$, respectively. A simple calculation shows indeed that $(\partial_\delta \kappa(\delta, y), 0)$ is proportional to $\tilde{F}_0(\delta, y)$ given in (36). The fact that $\lambda = 1$ is an eigenvalue of L_δ follows from similar ideas: noting that

$$\bar{K}(\delta, \mu, y, s) = \left(\frac{\kappa_0(1 - \delta^2)^{\frac{1}{p-1}}}{(1 + \mu e^s + \delta y)^{\frac{2}{p-1}}}, -\frac{2\mu e^s}{p-1} \frac{\kappa_0(1 - \delta^2)^{\frac{1}{p-1}}}{(1 + \mu e^s + \delta y)^{\frac{p+1}{p-1}}} \right)$$

is an explicit solution of equation (5) with $\bar{K}(\delta, 0, y, s) = (\kappa(\delta, y), 0)$, when $\mu = 0$, differentiating with respect to the new parameter μ , we obtain an eigenfunction for L_δ with $\lambda = 1$.

Proof. (i) As $\kappa(\delta, y)$ is a stationary solution of (5), it satisfies (16), hence

$$\mathcal{L}\kappa(\delta, y) - \frac{2(p+1)}{(p-1)^2} \kappa(\delta, y) + \kappa(\delta, y)^p = 0.$$

By definition (34) of $\tilde{L}_\delta$, we see that $\tilde{L}_\delta \begin{pmatrix} \kappa(\delta, y) \\ 0 \end{pmatrix} = 0$.

(ii) Noting that $\kappa(\delta, y) = \mathcal{T}_\delta(\kappa_0)$ where the transformation $\mathcal{T}_\delta$ is defined in (176), applying Lemma C.6 and using (10), we get the first bound. In order to prove the second one, we recall the following integral calculation rules from [33]:

Claim 3.2. Consider for some $\alpha > -1$ and $\beta \in \mathbb{R}$ the following integral:

$$I(\delta) = \int_{-1}^1 \frac{(1 - y^2)^\alpha}{(1 + \delta y)^\beta} dy.$$

- (i) if $\alpha + 1 - \beta > 0$, then $\frac{1}{K} \leq I(\delta) \leq K$;
- (ii) if $\alpha + 1 - \beta = 0$, then $\frac{1}{K} \leq \frac{I(\delta)}{|\log(1 - \delta^2)|} \leq K$;
- (iii) if $\alpha + 1 - \beta < 0$, then $\frac{1}{K} \leq I(\delta)(1 - \delta^2)^{-(\alpha+1)+\beta} \leq K$.

Proof. See page 84 of [33]. □

Using the definition of $\tilde{F}_0$ (42), the fact that

$$(44) \quad \forall(\delta, y) \in (-1, 1)^2, |y + \delta| + |1 - \delta^2| + (1 - y^2) \leq C(1 + \delta y),$$

and straightforward computations we see that

$$|\partial_\delta \tilde{F}_0(\delta, y)| \leq C \frac{(1 - \delta^2)^{\frac{2-p}{p-1}}}{(1 + \delta y)^{\frac{2}{p-1}}}, \quad |\partial_{\delta, y}^2 \tilde{F}_0(\delta, y)| \leq C \frac{(1 - \delta^2)^{\frac{2-p}{p-1}}}{(1 + \delta y)^{\frac{p+1}{p-1}}}.$$

Using this and Claim 3.2, we see that (43) holds for $\tilde{F}_0$. □

3.3. The conjugate operator $\tilde{L}_\delta^*$. In this step of the work of [33], the authors have computed $\tilde{L}_\delta^*$ by simple calculations using the definition of the conjugate, namely that $\phi(\tilde{L}_\delta(q), r) = \phi(q, \tilde{L}_\delta^*(r))$ and the fact that $\mathcal{L}$ is self-adjoint. In the same way, we introduce, in the following, the conjugate operator of $\tilde{L}_\delta$ with respect to ϕ .

Lemma 3.3 (The conjugate operator of $\tilde{L}_\delta$ with respect to ϕ). *For all $|\delta| < 1$, the operator $\tilde{L}_\delta^*$ conjugate of $\tilde{L}_\delta$ with respect to ϕ is given by*

$$(45) \quad \tilde{L}_\delta^* \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \tilde{R}_\delta(r_2) \\ -\mathcal{L}r_1 + r_1 + \frac{p+3}{p-1}r_2 + 2yr'_2 - \frac{8}{(p-1)}\frac{r_2}{(1-y^2)} \end{pmatrix}$$

for any $(r_1, r_2) \in (\mathcal{D}(\mathcal{L}))^2$, where $g = \tilde{R}_\delta(r_2)$ is the unique solution of

$$(46) \quad -\mathcal{L}g + g = \mathcal{L}r_2 + \tilde{\psi}(\delta, y)r_2.$$

Proof. The proof is the same as the proof of Lemma 4.1 on page 81 in [33]. $\square$

In the following, we give an eigenfunction of $\tilde{L}_\delta^*$ associated to the eigenvalue $\lambda = 0$.

Lemma 3.4 (Eigenfunction of $\tilde{L}_\delta^*$ for the eigenvalue $\lambda = 0$).

(i) (Existence) *For all $|\delta| < 1$, there exists $\tilde{W}_0 \in \mathcal{H}$ continuous in terms of d such that $\tilde{L}_\delta^*(\tilde{W}_0) = 0$ where*

$$(47) \quad \tilde{W}_{0,2}(\delta, y) = \tilde{c}_0 \kappa(\delta, y) \text{ and } \frac{1}{\tilde{c}_0} = \frac{4\kappa_0^2}{p-1} \int_{-1}^1 \frac{\rho(y)}{1-y^2} dy$$

and $\tilde{W}_{0,1}$ is the unique solution of the equation

$$(48) \quad -\mathcal{L}g + g = -\frac{p+3}{p-1}r_2 - 2yr'_2 + \frac{8}{p-1}\frac{r_2}{1-y^2}$$

with $r_2 = \tilde{W}_{0,2}$. Moreover, we have

$$(49) \quad \phi(\tilde{W}_0, \tilde{F}_0) = 1.$$

(ii) (Normalization) *There exists $C_0 > 0$ such that for $|\delta| < 1$,*

$$(50) \quad \|\tilde{W}_0(\delta, \cdot)\|_{\mathcal{H}} \leq C_0 \text{ and } \|\partial_\delta \tilde{W}_0(\delta, \cdot)\|_{\mathcal{H}} \leq \frac{C_0}{1-\delta^2}.$$

Before proving this lemma, let us recall the result from [33].

Claim 3.5. *For any $r_2 \in \mathcal{H}_0$, the equation (48) has a unique solution $g \in \mathcal{H}_0$ (8) such that*

$$\|g\|_{\mathcal{H}_0} \leq C\|r_2\|_{\mathcal{H}_0}.$$

Proof. See Claim 4.5 on page 86 in [33]. $\square$

Proof of Lemma 3.4. (i) From the definition of $\tilde{L}_\delta^*$ (see (45)), $\tilde{W}_0(\delta, \cdot) = (\tilde{W}_{0,1}(\delta, \cdot), \tilde{W}_{0,2}(\delta, \cdot))$ is an eigenfunction for the eigenvalue $\lambda = 0$ if and only if

$$(51) \quad \begin{cases} \tilde{R}_\delta(\tilde{W}_{0,2}(\delta, \cdot)) = 0, \\ -\mathcal{L}\tilde{W}_{0,1}(\delta, \cdot) + \tilde{W}_{0,1}(\delta, \cdot) + \frac{p+3}{p-1}\tilde{W}_{0,2}(\delta, \cdot) + 2y\partial_y \tilde{W}_{0,2}(\delta, \cdot) - \frac{8}{(p-1)}\frac{\tilde{W}_{0,2}(\delta, \cdot)}{(1-y^2)} = 0. \end{cases}$$

Note that $\tilde{R}_\delta(r_2)$ is the unique solution of (46). Therefore, if $\tilde{W}_0$ is a solution of (51)-(52), then we have

$$\mathcal{L}\tilde{W}_{0,2}(\delta, \cdot) + \tilde{\psi}(\delta, \cdot)\tilde{W}_{0,2}(\delta, \cdot) = 0.$$

Note also that, since $\kappa(\delta, y)$ is a stationary solution of equation (5), it follows that

$$(53) \quad \mathcal{L}\kappa(\delta, y) + \tilde{\psi}\kappa(\delta, y) = 0.$$

This suggests that we take $\tilde{W}_{0,2}(\delta, y) = \tilde{c}_0(\delta)\kappa(\delta, y)$ with $\tilde{c}_0(\delta) \neq 0$ and $\tilde{W}_{0,1}$ the unique solution of (52) (note that $\kappa(\delta, \cdot) \in \mathcal{H}_0$ by definition (10) and use Claim 3.5 for the existence and uniqueness of $\tilde{W}_{0,1}(\delta, \cdot)$). In this step, we will try to normalize $\tilde{W}_0$. From the definition of ϕ (35), Lemma 3.1 and (48), we write

$$\begin{aligned} \phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) &= \int_{-1}^1 ((-\mathcal{L}\tilde{W}_{0,1}(\delta, y) + \tilde{W}_{0,1}(\delta, y))\kappa(\delta, y) \\ &\quad + \tilde{W}_{0,1}(\delta, y)\kappa(\delta, y))\rho(y)dy \\ &= \int_{-1}^1 \left(-\frac{p+3}{p-1}\tilde{W}_{0,2}(\delta, y) - 2y\tilde{W}'_{0,2}(\delta, y) \right. \\ &\quad \left. + \frac{8}{p-1}\frac{\tilde{W}_{0,2}(\delta, y)}{(1-y^2)}\right)\kappa(\delta, y)\rho(y)dy. \end{aligned}$$

Note in particular that $\tilde{W}_{0,2}(\delta, y) = \tilde{c}_0(\delta)\kappa(\delta, y)$, so

$$\begin{aligned} \phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) &= \tilde{c}_0(\delta) \left[\int_{-1}^1 \left(-\frac{p+3}{p-1} + \frac{8}{(p-1)(1-y^2)}\right)\kappa^2(\delta, y)\rho(y)dy + \int_{-1}^1 \kappa^2(\delta, y)(y\rho(y))'dy \right] \\ &= \tilde{c}_0(\delta) \int_{-1}^1 \left(-\frac{p+3}{p-1} + \frac{8}{(p-1)(1-y^2)} + 1 - \frac{4y^2}{(p-1)(1-y^2)}\right)\kappa^2(\delta, y)\rho(y)dy \\ &= \tilde{c}_0(\delta)\frac{4}{p-1}\kappa_0^2(1-\delta^2)^{\frac{2}{p-1}} \int_{-1}^1 \frac{1}{(1+\delta y)^{\frac{4}{p-1}}} \frac{\rho(y)}{1-y^2} dy. \end{aligned}$$

Performing the change of variable $Y = \frac{y+\delta}{1+\delta y}$, we get

$$\phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) = \tilde{c}_0(\delta)\frac{4}{p-1}\kappa_0^2 \int_{-1}^1 (1-Y^2)^{\frac{3-p}{p-1}} dY.$$

Therefore, in order to get $\phi(\tilde{W}_0, \tilde{F}_0) = 1$, it is enough to fix $\tilde{c}_0(\delta)$ as a positive constant independent from δ as stated in (47).

(ii) (*Normalization*) Since $\tilde{W}_{0,1}$ and $\partial_\delta \tilde{W}_{0,1}$ are solutions to equation (48), respectively with $r_2 = \tilde{W}_{0,2}$ and $r_2 = \partial_\delta \tilde{W}_{0,2}$, we see from Claim 3.5 that for all $|\delta| < 1$,

$$(54) \quad \|\tilde{W}_0\|_{\mathcal{H}} \leq C_0 \|\tilde{W}_{0,2}\|_{\mathcal{H}_0} \quad \text{and} \quad \|\partial_\delta \tilde{W}_0\|_{\mathcal{H}} \leq C_0 \|\partial_\delta \tilde{W}_{0,2}\|_{\mathcal{H}_0}.$$

Using (44) together with the definition of $\tilde{W}_{0,2}$ and straightforward computations, we see that for all $|\delta| < 1$ and $|y| < 1$,

$$\begin{aligned} |\tilde{W}_{0,2}(\delta, y)| &\leq C \frac{(1-\delta^2)^{\frac{1}{p-1}}}{(1+\delta y)^{\frac{2}{p-1}}}, \quad |\partial_y \tilde{W}_{0,2}(\delta, y)| \leq C \frac{(1-\delta^2)^{\frac{1}{p-1}}}{(1+\delta y)^{\frac{p+1}{p-1}}}, \\ |\partial_\delta \tilde{W}_{0,2}(\delta, y)| &\leq C \frac{(1-\delta^2)^{\frac{2-p}{p-1}}}{(1+\delta y)^{\frac{2}{p-1}}}, \quad |\partial_{\delta,y}^2 \tilde{W}_{0,2}(\delta, y)| \leq C \frac{(1-\delta^2)^{\frac{2-p}{p-1}}}{(1+\delta y)^{\frac{p+1}{p-1}}}. \end{aligned}$$

Since we have by this, by Claim 3.2 and by the definition of the norm in $\mathcal{H}_0$, $\|\tilde{W}_{0,2}\|_{\mathcal{H}_0} + (1-\delta^2)\|\partial_\delta \tilde{W}_{0,2}\|_{\mathcal{H}_0} \leq C_0$, we see that (50) follows by (54). This concludes the proof of Lemma 3.4. $\square$

3.4. Expansion of q with respect to the eigenspaces of $\tilde{L}_\delta$. In the following, we expand any $r \in \mathcal{H}$ with respect to the eigenspaces of $\tilde{L}_\delta$ partially computed in Lemma 3.1. We claim the following:

Definition 3.6 (Expansion of r with respect to the eigenspaces of $\tilde{L}_\delta$). Consider $r \in \mathcal{H}$ and introduce

$$(55) \quad \tilde{\pi}_0^\delta(r) = \phi(\tilde{W}_0(\delta, \cdot), r),$$

where $\tilde{W}_0(\delta, \cdot)$ is the eigenfunction of $\tilde{L}_\delta^*$ computed in Lemma 3.4, and $\tilde{\pi}_-^\delta(r)$ is defined by

$$(56) \quad r = \tilde{\pi}_0^\delta(r)\tilde{F}_0(\delta, \cdot) + \tilde{\pi}_-^\delta(r).$$

Applying the operator $\tilde{\pi}_0^\delta$ to (56), we write

$$\tilde{\pi}_0^\delta(r) = \tilde{\pi}_0^\delta(r)\tilde{\pi}_0^\delta(\tilde{F}_0(\delta, \cdot)) + \tilde{\pi}_0^\delta(\tilde{\pi}_-^\delta(r)).$$

By (49),

$$\tilde{\pi}_0^\delta(\tilde{F}_0(\delta, \cdot)) = \phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) = 1;$$

therefore,

$$\phi(\tilde{W}_0(\delta, \cdot), \tilde{\pi}_-^\delta(r)) = \tilde{\pi}_0^\delta(\tilde{\pi}_-^\delta(r)) = 0.$$

Thus, we have

$$(57) \quad \tilde{\pi}_-^\delta(r) \in \tilde{\mathcal{H}}_-^\delta \equiv \{r \in \mathcal{H} \mid \tilde{\pi}_0^\delta(r) = 0\}.$$

Remark. Note that if $r \in \tilde{\mathcal{H}}_-^\delta$, then $\tilde{\pi}_-^\delta(r) = r$ (just use (56) and (57)) and $\tilde{L}_\delta r \in \tilde{\mathcal{H}}_-^\delta$. Indeed, using the definition of $\tilde{\pi}_0^\delta$ (55), the definition of $\tilde{L}_\delta^*$ and Lemma 3.4, we write $\tilde{\pi}_0^\delta(\tilde{L}_\delta r) = \phi(\tilde{W}_0(\delta, \cdot), \tilde{L}_\delta r) = \phi(\tilde{L}_\delta^* \tilde{W}_0(\delta, \cdot), r) = 0$. Moreover $\tilde{\pi}_-^\delta(\tilde{F}_0(\delta, \cdot)) = 0$ (just use (56) with $r = \tilde{F}_0(\delta, \cdot)$ and (49)).

Remark. Note that $\tilde{\pi}_0^\delta(r)$ is the projection of r on the eigenfunction of $\tilde{L}_\delta$ associated to $\lambda = 0$, and $\tilde{\pi}_-^\delta(r)$ is the negative part of r .

3.5. Equivalent norms on $\mathcal{H}$ and $\tilde{\mathcal{H}}_-^\delta$ adapted to the dispersive structure. We introduce

$$(58) \quad \begin{aligned} \tilde{\varphi}_\delta(q, r) &= \int_{-1}^1 (-\tilde{\psi}(\delta, \cdot)q_1r_1 + q_1'r_1'(1-y^2) + q_2r_2)\rho dy \\ &= \int_{-1}^1 (-q_1(\mathcal{L}r_1 + \tilde{\psi}(\delta, \cdot)r_1) + q_2r_2)\rho dy. \end{aligned}$$

Proposition 3.7 (Equivalence in $\tilde{\mathcal{H}}_-^\delta$ of the $\mathcal{H}$ norm and the $\tilde{\varphi}_\delta$ norm). *There exists $C_0 > 0$ such that for all $|\delta| < 1$, the following hold:*

(i) **(Equivalence of norms in $\tilde{\mathcal{H}}_-^\delta$)** For all $r \in \tilde{\mathcal{H}}_-^\delta$,

$$\frac{1}{C_0} \|r\|_{\mathcal{H}}^2 \leq \tilde{\varphi}_\delta(r, r) \leq C_0 \|r\|_{\mathcal{H}}^2.$$

(ii) **(Equivalence of norms in $\mathcal{H}$)** For all $r \in \mathcal{H}$,

$$\frac{1}{C_0} \|r\|_{\mathcal{H}} \leq \left(|\tilde{\pi}_0^\delta(r)| + \sqrt{\tilde{\varphi}_\delta(r_-, r_-)} \right) \leq C_0 \|r\|_{\mathcal{H}}, \text{ where } r_- = \tilde{\pi}_-^\delta(r).$$

We introduce for all $\epsilon > 0$

(59)

$$\begin{aligned}\tilde{\varphi}_{\delta,\epsilon}(q, r) &= \int_{-1}^1 q_1 \left(-(1-\epsilon)\mathcal{L}r_1 + \left(-(1-\epsilon)\tilde{\psi}(\delta, y) - \epsilon \frac{2p(p+1)}{(p-1)^2} \frac{(1-\delta^2)}{(1+\delta y)^2} \right) r_1 \right) \rho dy \\ &\quad + (1-\epsilon) \int_{-1}^1 q_2 r_2 \rho dy.\end{aligned}$$

To prove this proposition, we use the following:

Lemma 3.8 (Reduction of the proof of Proposition 3.7). *There exists $\epsilon_0 \in (0, 1)$ such that for all $|\delta| < 1$ and $r \in \tilde{\mathcal{H}}_-^\delta$, $\varphi_{\delta,\epsilon_0}(r, r) \geq 0$.*

Lemma 3.8 implies Proposition 3.7. As we have $|\tilde{\psi}(\delta, y)| \leq \frac{C}{1-y^2}$, we proceed exactly like in [33] page 91. $\square$

Proof of Lemma 3.8. We proceed in three parts:

- In Part 1, we find a hyperplane of $\mathcal{H}$ where $\tilde{\varphi}_{\delta,\epsilon}$ is non-negative.
- In Part 2, we find a straight line in $\mathcal{H}$, where $\tilde{\varphi}_{\delta,\epsilon}$ is negative and which is “orthogonal” to $\tilde{\mathcal{H}}_-^\delta$ with respect to $\tilde{\varphi}_{\delta,\epsilon}$.
- In Part 3, we proceed by contradiction and prove that $\tilde{\varphi}_{\delta,\epsilon}$ is non-negative on $\tilde{\mathcal{H}}_-^\delta$.

Part 1 : $\tilde{\varphi}_{\delta,\epsilon}$ is non-negative on a hyperplane. We claim the following:

Lemma 3.9. *There exists $\epsilon_1 > 0$ such that for all $|\delta| < 1$ and $\epsilon \in (0, \epsilon_1]$, $\tilde{\varphi}_{\delta,\epsilon}$ is non-negative on the hyperplane*

$$(60) \quad E_1 = \left\{ q \in \mathcal{H} \mid \int_{-1}^1 \mathcal{T}_{-\delta}(q_1) \rho(y) dy = 0 \right\},$$

where $\mathcal{T}_{-\delta}$ is defined in (176).

Proof. Define from (175) $\epsilon_1 = \min(1, \frac{\gamma_1}{\gamma_1 - \frac{2p(p+1)}{(p-1)^2}}) > 0$ and fix $\epsilon \in (0, \epsilon_1]$. We consider $u = (u_1, u_2) \in E_1$, and write from (59)

(61)

$$\begin{aligned}\tilde{\varphi}_{\delta,\epsilon}(u, u) &= \int_{-1}^1 u_1 \left(-(1-\epsilon)\mathcal{L}u_1 + \left[-(1-\epsilon)\tilde{\psi}(\delta, y) - \epsilon \frac{2p(p+1)}{(p-1)^2} \frac{(1-\delta^2)}{(1+\delta y)^2} \right] u_1 \right) \rho(y) dy \\ &\quad + (1-\epsilon) \int_{-1}^1 u_2^2 \rho(y) dy.\end{aligned}$$

If $U_1 = \mathcal{T}_{-\delta}u_1$, then $u_1 = \mathcal{T}_\delta U_1$ and we have from (176)

$$\begin{aligned}u_1(y) &= \frac{(1-\delta^2)^{\frac{1}{p-1}}}{(1+\delta y)^{\frac{2}{p-1}}} U_1(z) \quad \text{with } z = \frac{y+\delta}{1+\delta y}, \\ \mathcal{L}u_1(y) + \tilde{\psi}(\delta, y)u_1(y) &= \frac{(1-\delta^2)^{\frac{p}{p-1}}}{(1+\delta y)^{\frac{2p}{p-1}}} \mathcal{L}U_1(z), \\ \rho(y)dy &= \frac{(1+\delta y)^{\frac{2(p+1)}{p-1}}}{(1-\delta^2)^{\frac{p+1}{p-1}}} \rho(z)dz, \\ 0 &= \int U_1(z) \rho(z) dz.\end{aligned}$$

Therefore, we see from (61) and Lemma C.4 that

$$\begin{aligned}\tilde{\varphi}_{\delta,\epsilon}(u, u) &= \int_{-1}^1 U_1(z) \left(-(1-\epsilon)\mathcal{L}U_1(z) - \epsilon \frac{2p(p+1)}{(p-1)^2} U_1(z) \right) \rho(z) dz \\ &\quad + (1-\epsilon) \int_{-1}^1 u_2^2 \rho(y) dy \\ &\geq \left(-(1-\epsilon)\gamma_1 - \epsilon \frac{2p(p+1)}{(p-1)^2} \right) \int_{-1}^1 U_1^2(z) \rho(z) dz + (1-\epsilon) \int_{-1}^1 u_2^2 \rho(y) dy \geq 0\end{aligned}$$

since $\epsilon \leq \epsilon_1$ hence $-(1-\epsilon)\gamma_1 - \epsilon \frac{2p(p+1)}{(p-1)^2} \geq 0$ and $1-\epsilon \geq 0$. This concludes the proof of Lemma 3.9. $\square$

Part 2 : $\tilde{\varphi}_{\delta,\epsilon}$ is negative on a straight line orthogonal to $\tilde{\mathcal{H}}_-^\delta$. We need to find $\tilde{V}_0(\delta, \epsilon, \cdot)$ in $\mathcal{H}$ such that $\tilde{\varphi}_{\delta,\epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), r) = 0$ for all $r \in \tilde{\mathcal{H}}_-^\delta$. Since we know from the definition of $\tilde{\mathcal{H}}_-^\delta$ (57) that

$$\forall r \in \tilde{\mathcal{H}}_-^\delta, \phi(\tilde{W}_0, r) = \tilde{\pi}_0^\delta(r) = 0,$$

we proceed as on page 93 in [33] and search $\tilde{V}_0(\delta, \epsilon, \cdot)$ such that

$$(62) \quad \forall r \in \mathcal{H}, \phi(\tilde{W}_0(\delta, \cdot), r) = \tilde{\varphi}_{\delta,\epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), r).$$

Then, we will show that $\tilde{\varphi}_{\delta,\epsilon}$ is negative on the straight line spanned by $\tilde{V}_0(\delta, \epsilon, \cdot)$. Consider $\epsilon > 0$ small enough and take $|\delta| < 1$. We claim the following:

Lemma 3.10. *There exists $\epsilon_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2)$ and $|\delta| < 1$:*

(i) *There exists $\tilde{V}_0(\delta, \epsilon, \cdot) \in \mathcal{H}_0$ such that (62) holds.*

(ii) *Moreover there exists $c > 0$ such that*

$$(63) \quad \sup_{|\delta| < 1} \|\epsilon \tilde{V}_0(\delta, \epsilon, \cdot) + c \tilde{F}_0(\delta, \cdot)\|_{\mathcal{H}} \rightarrow 0 \text{ as } \epsilon \rightarrow 0^+.$$

(iii) *The bilinear form $\tilde{\varphi}_{\delta,\epsilon}$ is negative on a line of $\mathcal{H}$ spanned by $\tilde{V}_0(\delta, \epsilon, \cdot)$.*

Proof of Lemma 3.10. We proceed in three steps:

-In Step 1, we find a PDE satisfied by $\tilde{V}_{0,1}(\delta, \epsilon, \cdot)$ and transform it with the Lorentz transform in similarity variables defined in (176).

In Step 2, we solve the transformed PDE and find the asymptotic behavior of $\tilde{V}_{0,1}(\delta, \epsilon, \cdot)$ as $\epsilon \rightarrow 0^+$, uniformly in $|\delta| < 1$, which gives (i) and (ii).

In Step 3, we use that asymptotic behavior to show that $\tilde{\varphi}_{\delta,\epsilon}$ is negative on a straight line spanned by $\tilde{V}_0(\delta, \epsilon, \cdot)$, which gives (iii).

Step 1 (Reduction to the solution of some PDE). From the definition of $\tilde{\varphi}_{\delta,\epsilon}$ (59) and ϕ (35), we see that in order to satisfy (62), it is enough to take

$$(64) \quad \tilde{V}_{0,2}(\delta, \epsilon, \cdot) = \tilde{W}_{0,2}(\delta, \cdot)/(1-\epsilon)$$

and to prove the existence of a $\tilde{V}_{0,1}(\delta, \epsilon, \cdot)$ solution to

$$\begin{aligned}-(1-\epsilon)\mathcal{L}\tilde{V}_{0,1}(\delta, \epsilon, \cdot) &+ \left(-(1-\epsilon)\tilde{\psi}(\delta, \cdot) - \epsilon \frac{2p(p+1)}{(p-1)^2} \frac{(1-\delta^2)}{(1+\delta y)^2} \right) \tilde{V}_{0,1}(\delta, \epsilon, \cdot) \\ (65) \quad &= -\mathcal{L}\tilde{W}_{0,1}(\delta, \cdot) + \tilde{W}_{0,1}(\delta, \cdot).\end{aligned}$$

Claim 3.11 (Reduction to an explicitly solvable PDE). *Consider $\tilde{V}_{0,1}(\delta, \epsilon, \cdot)$ and introduce $\tilde{v}_{0,1}(\delta, \epsilon, \cdot)$ defined by*

$$(66) \quad \tilde{v}_{0,1}(\delta, \epsilon, \cdot) = \mathcal{T}_{-\delta} \tilde{V}_{0,1}(\delta, \epsilon, \cdot),$$

where $\mathcal{T}_{\delta}$ is defined in (176). Then,

(i) $\tilde{V}_{0,1}(\delta, \epsilon, \cdot)$ is a solution to (65) if and only if $\tilde{v}_{0,1}(\delta, \epsilon, \cdot)$ is a solution to the equation

$$(67) \quad \begin{aligned} (1 - \epsilon) \mathcal{L} \tilde{v}_{0,1}(\delta, \epsilon, z) + \epsilon \frac{2p(p+1)}{(p-1)^2} \tilde{v}_{0,1}(\delta, \epsilon, z) &= f_0^{\delta}(z) \\ &\equiv \frac{1 - \delta^2}{(1 - \delta z)^2} \mathcal{T}_{-\delta} \left(\mathcal{L} \tilde{W}_{0,1}(\delta, y) - \tilde{W}_{0,1}(\delta, y) \right). \end{aligned}$$

(ii) The linear form $h \mapsto \int_{-1}^1 f_0^{\delta} h \rho$ defined for all $h \in \mathcal{H}_0$ is continuous and for some $C_0 > 0$, we have

$$\forall \delta \in (-1, 1), \|f_0^{\delta}\|_{\mathcal{H}'_0} \leq C_0 \|\tilde{W}_0(\delta, \cdot)\|_{\mathcal{H}} \leq C_0^2.$$

Proof. (i) Using (176) we see that

$$\begin{aligned} \tilde{V}_{0,1}(\delta, \epsilon, y) &= \frac{(1 - \delta^2)^{\frac{1}{p-1}}}{(1 + \delta y)^{\frac{2}{p-1}}} \tilde{v}_{0,1}(\delta, \epsilon, z) \quad \text{with } z = \frac{y + \delta}{1 + \delta y}, \\ \mathcal{L} \tilde{V}_{0,1}(\delta, \epsilon, y) + \tilde{\psi}(\delta, y) \tilde{V}_{0,1}(\delta, \epsilon, y) &= \frac{(1 - \delta^2)^{\frac{p}{p-1}}}{(1 + \delta y)^{\frac{2p}{p-1}}} \mathcal{L} \tilde{v}_{0,1}(\delta, \epsilon, z). \end{aligned}$$

Since $\frac{(1 - \delta z)^2}{1 - \delta^2} = \frac{1 - \delta^2}{(1 + \delta y)^2}$, we see that equation (65) and (67) are equivalent.

(ii) The proof of (ii) is the same as the proof of Claim 4.11 on page 94 in [33]. $\square$

Step 2 (Solution of equation (67) and asymptotic behavior as $\epsilon \rightarrow 0^+$). We prove (i) and (ii) of Lemma 3.10 here. Let us first recall the following result from [33].

Claim 3.12 (Solution of equation (67)). *Consider*

$$f(y) = \sum_{n=0}^{\infty} \tilde{f}_n h_n(y) \in \mathcal{H}'_0,$$

where h_n are the eigenfunctions of $\mathcal{L}$ defined in Proposition C.3. Then, for any $\epsilon \in (0, \frac{1}{2})$, the equation

$$(1 - \epsilon) \mathcal{L} v + \epsilon \frac{2p(p+1)}{(p-1)^2} v = f$$

has a unique solution in $\mathcal{H}_0$ given by

$$v = \sum_{n=0}^{\infty} \frac{\tilde{f}_n}{\gamma_n + \left(\frac{2(p+1)}{(p-1)^2} - \gamma_n \right) \epsilon} h_n,$$

where $\gamma_n \leq 0$ are the eigenvalues of $\mathcal{L}$ introduced in Proposition C.3.

Now, we use this claim to prove (i) and (ii).

Proof of (i) of Lemma 3.10. Using (ii) in Claim 3.11, we see that $f_0^{\delta} \in \mathcal{H}'_0$. Therefore, Claim 3.12 applies, and we have a unique solution $\tilde{v}_{0,2}(\delta, \epsilon, \cdot) \in \mathcal{H}_0$ to equation (67). Using (i) of Lemma 3.4 and Lemma C.6 below we get a solution $\tilde{V}_{0,1}(\delta, \epsilon, \cdot) \in \mathcal{H}_0$ to equation (65). $\square$

Proof of (ii) of Lemma 3.10. Note that the spectral properties of $\mathcal{L}$ are given in Proposition C.3 below. Since $h_0 = c_0$ by Proposition C.3, we see from Claim 3.12 and (ii) in Claim 3.11 that for ϵ small enough,

$$(68) \quad \|\tilde{v}_{0,1}(\delta, \epsilon, \cdot) - \frac{\tilde{f}_0}{\frac{2(p+1)}{(p-1)^2}\epsilon} c_0\|_{\mathcal{H}_0} \leq C \|f_0^\delta\|_{\mathcal{H}'_0} \leq C,$$

where from (ii) in Lemma C.6 and the fact that $c_0 = \mathcal{T}_{-\delta}(c_0 \frac{\kappa(\delta, y)}{\kappa_0})$ (see (176)), we have

$$\begin{aligned} \tilde{f}_0^\delta &= c_0 \int_{-1}^1 f(z) \rho(z) dz = \frac{c_0}{\kappa_0} \int_{-1}^1 \left(\mathcal{L} \tilde{W}_{0,1}(\delta, y) - \tilde{W}_{0,1}(\delta, y) \right) \kappa(\delta, y) \rho(y) dy \\ &= -\frac{c_0}{\kappa_0} \phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) = -\frac{c_0}{\kappa_0} \end{aligned}$$

(use also the expression (35) of ϕ together with (42) and (49)). As $\tilde{V}_{0,2}(\delta, \epsilon, \cdot)$ is explicitly given by (64) and (47), we see that (63) follows from (68), (66), the fact that $\mathcal{T}_\delta(\kappa_0) = \kappa(\delta, y)$ and the expression of $\tilde{F}_0$ (42). $\square$

Step 3. Sign of $\tilde{\varphi}_{\delta, \epsilon}$ on the line spanned by $\tilde{V}_0(\delta, \epsilon, \cdot)$.

Proof of (iii) of Lemma 3.10. We will prove now that $\tilde{\varphi}_{\delta, \epsilon}$ is negative on the straight line spanned by $\tilde{V}_0(\delta, \epsilon, \cdot)$. From (62), (63) and (49), we see that

$$\begin{aligned} \tilde{\varphi}_{\delta, \epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), \tilde{V}_0(\delta, \epsilon, \cdot)) &= \phi(\tilde{W}_0(\delta, \cdot), \tilde{V}_0(\delta, \epsilon, \cdot)) \\ &\sim -\frac{c}{\epsilon} \phi(\tilde{W}_0(\delta, \cdot), \tilde{F}_0(\delta, \cdot)) = -\frac{c}{\epsilon}, \text{ as } \epsilon \rightarrow 0, \end{aligned}$$

uniformly in $|\delta| < 1$. So $\tilde{\varphi}_{\delta, \epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), \tilde{V}_0(\delta, \epsilon, \cdot)) < 0$. This concludes the proof of Lemma 3.10. $\square$

Part 3 : End of the proof of Lemma 3.8. From Lemmas 3.9 and 3.10, we define $\epsilon_0 = \min(\epsilon_1, \epsilon_2) \in (0, 1)$. We will now prove by contradiction that $\tilde{\varphi}_{\delta, \epsilon_0}$ is positive on $\tilde{\mathcal{H}}_-^\delta$ for all $|\delta| < 1$.

We note that from (57) and (62), for all $|\delta| < 1$ and $\epsilon \in (0, \epsilon_0]$, the definition of $\tilde{\mathcal{H}}_-^\delta$ (57) is as follows:

$$(69) \quad \tilde{\mathcal{H}}_-^\delta = \{r \in \mathcal{H} \mid \tilde{\varphi}_{\delta, \epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), r) = 0\}.$$

Consider $|\delta| < 1$. By contradiction, assume that $\tilde{\varphi}_{\delta, \epsilon}$ is negative so

$$(70) \quad \text{there is a non-zero } r \in \tilde{\mathcal{H}}_-^\delta \text{ such that } \tilde{\varphi}_{\delta, \epsilon}(r, r) < 0.$$

We mention that r is not collinear $\tilde{V}_0(\delta, \epsilon, \cdot)$. Indeed, if $r = \alpha \tilde{V}_0(\delta, \epsilon, \cdot)$ with $\alpha \in \mathbb{R}^*$, then we would have

$$\tilde{\varphi}_{\delta, \epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), r) = \alpha \tilde{\varphi}_{\delta, \epsilon}(\tilde{V}_0(\delta, \epsilon, \cdot), \tilde{V}_0(\delta, \epsilon, \cdot)) \neq 0,$$

by (iii) in Lemma 3.10, which contradicts (69). Thus, the vector subspace

$$E_2 = \text{span}(\tilde{V}_0(\delta, \epsilon, \cdot), r)$$

is of dimension 2. Therefore, as the subspace E_1 (60) is of codimension 1, there exists a non-zero $u \in E_1 \cap E_2$.

On the one hand, since $u \in E_1$, we have from Lemma 3.9 that

$$(71) \quad \tilde{\varphi}_{\delta, \epsilon}(u, u) \geq 0.$$

On the other hand, since $\tilde{\varphi}_{\delta,\epsilon}$ is negative on E_2 by (iii) of Lemma 3.10, we must have from (69) and (70),

$$\tilde{\varphi}_{\delta,\epsilon}(u, u) < 0.$$

This contradicts (71). So, $\tilde{\varphi}_{\delta,\epsilon}$ is non-negative on $\tilde{\mathcal{H}}_-^\delta$. This concludes the proof of Lemma 3.8 and Proposition 3.7. $\square$

4. TRAPPING NEAR THE SET OF STATIONARY SOLUTIONS

In this part of the work in the real case in [33], the authors have assumed that (13) holds for some $s^* \in \mathbb{R}$ and $d^* \in (-1, 1)$ and use modulation theory to introduce a parameter $d(s)$ adapted to the linearized equation and derive from the energy barrier the smallness of the unstable direction with respect to the stable. Then they use this to show that $(w(s), \partial_s w(s))$ to some $\kappa(\delta_\infty, \cdot)$ as $s \rightarrow \infty$ in the norm of $\mathcal{H}$.

This section is devoted to the proofs of Proposition 3, Theorem 4 and Theorem 5. Let us first give the proof of Proposition 3 and then derive Theorem 5 from Theorem 4. Later we prove Theorem 4.

4.1. Convergence to a stationary solution. We give the proofs of Proposition 3 and Theorem 5 here.

Proof of Proposition 3. From Proposition C.1 and Proposition 1, one can see that the proof given in the real case in Section 3.1 in [33] holds here with no change. Indeed, all the estimates remain valid in the complex case, in particular, the Sobolev embedding and the Duhamel formulation of the wave equation (1). $\square$

Proof of Theorem 5 assuming Theorem 4. Consider $w = w_{x_0}$ where x_0 is non-characteristic. The conclusion will follow from the application of Theorem 4 to w_{x_0} . In order to conclude, we have to check conditions (12) and (13). From the monotonicity of functional E (See Proposition C.1 below) and (A.ii) of Proposition 3, we see that

$$\forall s \geq -\log(T(x_0)), E(w(s), \partial_s w(s)) \geq E(\kappa_0, 0)$$

and (12) follows. Consider ϵ^* defined in Theorem 4. From (A.i) of Proposition 3, we have the existence of $s^* \geq -\log T(x_0)$ such that

$$\inf_{\{|\delta| < 1, \theta \in \mathbb{R}\}} \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - e^{i\theta} \begin{pmatrix} \kappa(\delta, \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2} \leq \frac{\epsilon^*}{2}.$$

Therefore, there exists $|\delta^*| < 1$ and $\theta^* \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - e^{i\theta^*} \begin{pmatrix} \kappa(\delta^*, \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2} \leq \epsilon^*.$$

Since $0 \leq \rho(y) \leq 1$, it follows that

$$\inf_{\{|\delta| < 1, \theta \in \mathbb{R}\}} \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - e^{i\theta} \begin{pmatrix} \kappa(\delta, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^*$$

and (13) follows. Applying Theorem 4, we get the conclusion of Theorem 5. $\square$

4.2. A modulation technique. We introduce two parameters $\delta(s)$ and $\theta(s)$ and we use a modulation technique to claim the following:

Proposition 4.1 (Modulation of w with respect to $e^{i\theta}\kappa(\delta, \cdot)$). *There exists $\epsilon_1 > 0$ and $K_1 > 0$ such that if $(w, \partial_s w) \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution to equation (5) which satisfies (13) for some $|\delta^*| < 1, \theta^* \in \mathbb{R}$ and $\epsilon^* < \epsilon_1$, then the following is true:*

(i) **(Choice of the modulation parameter)** *There exist $\delta(s) \in C^1([s^*, \infty), (-1, 1))$ and $\theta(s) \in C^1([s^*, \infty), \mathbb{R})$ such that for all $s \in [s^*, \infty)$,*

$$(72) \quad \tilde{\pi}_0^{\delta(s)}(\check{q}(s)) = \tilde{\pi}_0^{\delta(s)}(\tilde{q}(s)) = 0$$

where $\tilde{\pi}_0^{\delta}$ and $\tilde{\pi}_0^{\delta}$ are defined in (38), (55) and $q = (q_1, q_2)$ is defined for all $s \in [s_0, \infty)$ by

$$(73) \quad \begin{pmatrix} w(y, s) \\ \partial_s w(y, s) \end{pmatrix} = e^{i\theta(s)} \left[\begin{pmatrix} \kappa(\delta(s), y) \\ 0 \end{pmatrix} + \begin{pmatrix} q_1(y, s) \\ q_2(y, s) \end{pmatrix} \right].$$

Moreover,

$$|\theta(s^*) - \theta^*| + \left| \log \left(\frac{1 + \delta(s^*)}{1 - \delta(s^*)} \right) - \log \left(\frac{1 + \delta^*}{1 - \delta^*} \right) \right| + \|q(s^*)\|_{\mathcal{H}} \leq K_1 \epsilon^*.$$

(ii) **(Equation on q)** *For all $s \in [s^*, \infty)$,*

$$(74) \quad \frac{\partial}{\partial s} \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} = \tilde{L}_{\delta(s)} \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \check{f}_{\delta(s)}(q_1) \end{pmatrix} - \delta'(s) \begin{pmatrix} \partial_\delta \kappa(\delta, y) \\ 0 \end{pmatrix} + \theta'(s) \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix},$$

$$(75) \quad \frac{\partial}{\partial s} \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \tilde{L}_{\delta(s)} \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{f}_{\delta(s)}(q_1) \end{pmatrix} - \theta'(s) \begin{pmatrix} \kappa(\delta, y) + \check{q}_1 \\ \check{q}_2 \end{pmatrix},$$

where $\tilde{L}_{\delta(s)}, \tilde{L}_{\delta(s)}, \check{f}_{\delta(s)}$ and $\tilde{f}_{\delta(s)}$ are defined in (30), (32) and (34).

Proof. (i) From (38) and (56), we see that the condition (72) becomes

$$\Phi((w(s), \partial_s w(s)), \delta(s), \theta(s)) = 0,$$

where $\Phi \in C(\mathcal{H} \times (-1, 1) \times \mathbb{R}, \mathbb{R} \times \mathbb{R})$ is defined by

$$(76) \quad \Phi(v, \delta, \theta) = \begin{pmatrix} \check{\Phi}(v, \delta, \theta) \\ \tilde{\Phi}(v, \delta, \theta) \end{pmatrix} = \begin{pmatrix} \phi(\operatorname{Re}(e^{-i\theta}v - (\kappa(\delta, \cdot), 0)), \check{W}_0) \\ \phi(\operatorname{Im}(e^{-i\theta}v), \tilde{W}_0) \end{pmatrix}.$$

We recall the following inequality which has been proved on page 102 in [33]:

$$(77) \quad \forall \delta_1, \delta_2 \in (-1, 1), \|\kappa(\delta_1, \cdot) - \kappa(\delta_2, \cdot)\|_{\mathcal{H}_0} \leq C_0 |\lambda_1 - \lambda_2|, \text{ where } \lambda_i = \log \left(\frac{1 + \delta_i}{1 - \delta_i} \right).$$

We apply the implicit function theorem to Φ near the point $(e^{i\theta^*}(\kappa(\delta^*, \cdot), 0), \delta^*, \theta^*)$. Three facts have to be checked:

1-First, note that

$$\Phi(e^{i\theta^*}(\kappa(\delta^*, \cdot), 0), \delta^*, \theta^*) = 0.$$

2-Then, we compute from (76), for all $u \in \mathcal{H}$,

$$D_v \check{\Phi}(v, \delta, \theta)(u) = \phi(\operatorname{Re}(e^{-i\theta}u), \check{W}_0),$$

$$D_v \tilde{\Phi}(v, \delta, \theta)(u) = \phi(\operatorname{Im}(e^{-i\theta}u), \tilde{W}_0),$$

so we have

$$(78) \quad \|D_v \check{\Phi}(v, \delta, \theta)\| \leq C_0 \text{ and } \|D_v \tilde{\Phi}(v, \delta, \theta)\| \leq C_0.$$

3-Let $J(\check{\Phi}, \check{\Phi})$ be the jacobian matrix of Φ , and D its determinant so

$$J = \begin{pmatrix} \partial_{\delta}\check{\Phi} & \partial_{\theta}\check{\Phi} \\ \partial_{\delta}\check{\Phi} & \partial_{\theta}\check{\Phi} \end{pmatrix},$$

where

$$\begin{aligned} \partial_{\delta}\check{\Phi} &= \phi((\partial_{\delta}\kappa(\delta, \cdot), 0), \check{W}_0) + \phi(\operatorname{Re}(e^{-i\theta}v - (\kappa(\delta, \cdot), 0)), \partial_{\delta}\check{W}_0) \\ &= \frac{2\kappa_0}{(p-1)(1-\delta^2)} + \phi(\operatorname{Re}(e^{-i\theta}v - (\kappa(\delta, \cdot), 0)), \partial_{\delta}\check{W}_0), \\ \partial_{\theta}\check{\Phi} &= \phi(\operatorname{Im}(e^{-i\theta}v), \check{W}_0), \\ \partial_{\delta}\check{\Phi} &= \phi(\operatorname{Im}(e^{-i\theta}v), \partial_{\delta}\check{W}_0), \\ \partial_{\theta}\check{\Phi} &= \phi(-\kappa(\delta, \cdot), \check{W}_0) + \phi(-\operatorname{Re}(e^{-i\theta}v) + \kappa(\delta, \cdot), \check{W}_0) \\ &= -1 + \phi(-\operatorname{Re}(e^{-i\theta}v) + \kappa(\delta, \cdot), \check{W}_0), \end{aligned}$$

referring to Lemma 4.4 in [33] for the first equation and the orthogonality relation (49) for the last one. Using the Cauchy-Schwarz inequality, the continuity of ϕ in $\mathcal{H}$, the bound (50), Lemma 4.4 in [33], and (77), we see that if

$$(79) \quad |\theta - \theta^*| + \left| \log \left(\frac{1+\delta}{1-d} \right) - \log \left(\frac{1+\delta^*}{1-\delta^*} \right) \right| + \|v - e^{i\theta^*}(\kappa(\delta^*, \cdot), 0)\|_{\mathcal{H}} \leq \epsilon_1$$

for some $\epsilon_1 > 0$ small enough independent of δ^* , then we have

$$(80) \quad \left| \partial_{\delta}\check{\Phi} - \frac{2\kappa_0}{(p-1)(1-\delta^2)} \right| \leq \frac{C}{1-\delta^2} \left(\|\kappa(\delta^*, \cdot) - \kappa(\delta, \cdot)\|_{\mathcal{H}_0} + \|\operatorname{Re}(e^{-i\theta^*}v - (\kappa(\delta^*, \cdot), 0))\|_{\mathcal{H}} + \|\operatorname{Re}(v(e^{-i\theta} - e^{-i\theta^*}))\|_{\mathcal{H}} \right) \leq \frac{C\epsilon_1}{1-\delta^2},$$

$$(81) \quad |\partial_{\delta}\check{\Phi}| \leq \frac{C}{1-\delta^2} \|\operatorname{Im}(e^{-i\theta}v)\|_{\mathcal{H}} \leq \frac{C\epsilon_1}{1-\delta^2},$$

$$(82) \quad |\partial_{\theta}\check{\Phi}| \leq C \|\operatorname{Im}(e^{-i\theta}v)\|_{\mathcal{H}} \leq C\epsilon_1,$$

$$(83) \quad |\partial_{\theta}\check{\Phi} + 1| = |\phi(-\operatorname{Re}(e^{-i\theta}v) + \kappa(\delta, \cdot), \check{W}_0)| \leq C \left(\|\operatorname{Re}(e^{-i\theta^*}v) + \kappa(\delta^*, \cdot)\|_{\mathcal{H}} + \|\operatorname{Re}(v(e^{-i\theta} - e^{-i\theta^*}))\|_{\mathcal{H}} + \|\kappa(\delta, \cdot) - \kappa(\delta^*, \cdot)\|_{\mathcal{H}_0} \right) \leq C\epsilon_1.$$

Collecting (80)-(83), we see that for ϵ_1 small enough, we have

$$(84) \quad \left| D + \frac{2\kappa_0}{(p-1)(1-\delta^2)} \right| \leq \frac{C\epsilon_1}{1-\delta^2},$$

so we have the non-degeneracy of $\check{\Phi}$ near the point $(e^{i\theta^*}(\kappa(\delta^*, \cdot), 0), \delta^*, \theta^*)$. Applying the implicit function theorem, we see from (78) and (84) that there exist $\epsilon_2, \epsilon_3 > 0$, C^1 applications $(f, g) : \mathcal{H} \rightarrow (-1, 1) \times \mathbb{R}$ such that for all $v \in \mathcal{H}$ satisfying $\|e^{i\theta^*}(\kappa(\delta^*, \cdot), 0) - v\|_{\mathcal{H}} \leq \epsilon_2$ and for all $(\delta, \theta) \in (-1, 1) \times \mathbb{R}$ satisfying $|\delta - \delta^*| + |\theta - \theta^*| \leq \epsilon_3$ we have

$$(85) \quad \Phi(v, \delta, \theta) = 0 \Leftrightarrow (\delta, \theta) = (f(v), g(v)).$$

Take $\epsilon_0 = \frac{\epsilon_2}{2}$ and consider $\epsilon^* \leq \epsilon_0$. From (13) and the continuity of $(w, \partial_s w)$, we see that for some $\sigma^* > s^*$, we have:

$$\forall s \in [s^*, \sigma^*], \|(w(s), \partial_s w(s)) - e^{i\theta^*}(\kappa(\delta^*, \cdot), 0)\|_{\mathcal{H}} \leq 2\epsilon^* \leq \epsilon_2.$$

Therefore, from (76) and (85), we see that requiring (72) is equivalent to having $d = f(w(s))$ and $\theta = g(w(s))$. Since f and g are C^1 , we get the conclusion with C^1 functions $\delta(s)$ and $\theta(s)$ such that (72) holds for all $s \in [s^*, \sigma^*]$.

Now, let's prove that $\sigma^* = +\infty$. By contradiction, suppose $\sigma^* < +\infty$. We apply the implicit function theorem to Φ at the point $(v_n, \delta_n, \theta_n) \equiv ((w(s_n), \partial_s w(s_n)), \delta(s_n), \theta(s_n))$ where $s_n = \sigma^* - \frac{1}{n}$, and the uniform continuity of $(w(s), \partial_s w(s))$ from $[\sigma^* - \eta_0, \sigma^* + \eta_0]$ to $\mathcal{H}$ for some $\eta_0 > 0$. In fact, from (85), $\Phi((w(s_n), \partial_s w(s_n)), \delta(s_n), \theta(s_n)) = 0$. Moreover, (78) and (84) are uniformly satisfied, so as above we see that we can define $\delta(s)$ for all $s \in [s_n, s_n + \epsilon_0]$ for some $\epsilon_0 > 0$ independent of n . Therefore, for n large enough, $\delta(s)$ exists beyond σ^* , which is a contradiction. Thus, $\sigma^* = +\infty$.

(ii) This is a direct consequence of the equation (5) satisfied by w put in vectorial form:

$$\begin{aligned} \partial_s w &= v, \\ \partial_s v &= \mathcal{L}w - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w - \frac{p+3}{p-1}v - 2y\partial_y v \end{aligned}$$

and the fact that $(\kappa(\delta, \cdot), 0)$ satisfies

$$\mathcal{L}\kappa(\delta, \cdot) - \frac{2(p+1)}{(p-1)^2}\kappa(\delta, \cdot) + |\kappa(\delta, \cdot)|^{p-1}\kappa(\delta, \cdot) = 0$$

as a stationary solution. We have from (73)

$$\begin{aligned} \partial_s q_1 &= q_2 - i\theta'(\kappa(\delta(s), y) + q_1) - \delta' \partial_s \kappa, \\ \partial_s q_2 &= \mathcal{L}\kappa(\delta(s), y) + \mathcal{L}q_1 - \frac{2(p+1)}{(p-1)^2}(\kappa(\delta(s), y) + q_1) \\ &\quad + |\kappa(\delta(s), y) + q_1|^{p-1}(\kappa(\delta(s), y) + q_1) - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2 - i\theta' q_2. \end{aligned}$$

Dissociating the real and the imaginary part of these equations, we get (74) and (75). $\square$

4.3. Projection on the eigenspaces of the operator L_δ . Given $s \geq s^*$ and following the previous section, we make in this section the following a priori estimate

$$(86) \quad \|q(s)\|_{\mathcal{H}} \leq \epsilon$$

for some $\epsilon > 0$. From (72), we will expand $\check{q}$ and $\tilde{q}$ respectively according to the spectrum of the linear operators $\check{L}_\delta$ and $\tilde{L}_\delta$ as in (39) and (56):

$$(87) \quad \check{q}(y, s) = \check{\alpha}_1 \check{F}_1(\delta, y) + \check{q}_-(y, s),$$

$$(88) \quad \tilde{q}(y, s) = \tilde{q}_-(y, s),$$

where

$$(89) \quad \check{\alpha}_1 = \tilde{\pi}_1^{\delta(s)}(\check{q}), \quad \check{\alpha}_0 = \tilde{\pi}_0^{\delta(s)}(\check{q}) = 0, \quad \check{\alpha}_-(s) = \sqrt{\tilde{\varphi}_\delta(\check{q}_-, \check{q}_-)},$$

$$(90) \quad \tilde{\alpha}_0 = \tilde{\pi}_0^{\delta(s)}(\tilde{q}) = 0, \quad \tilde{\alpha}_-(s) = \sqrt{\tilde{\varphi}_\delta(\tilde{q}_-, \tilde{q}_-)},$$

and

$$\begin{aligned}\check{q}_- &= \begin{pmatrix} \check{q}_{-,1} \\ \check{q}_{-,2} \end{pmatrix} = \tilde{\pi}_-^\delta(\check{q}) = \tilde{\pi}_-^\delta \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix}, \\ \tilde{q}_- &= \begin{pmatrix} \tilde{q}_{-,1} \\ \tilde{q}_{-,2} \end{pmatrix} = \tilde{\pi}_-^\delta(\tilde{q}) = \tilde{\pi}_-^\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}.\end{aligned}$$

From (87), (88), (41) and Proposition 3.7, we see that for all $s \geq s_0$,

$$\begin{aligned}\frac{1}{C_0}\check{\alpha}_-(s) &\leq \|\check{q}_-(s)\|_{\mathcal{H}} \leq C_0\check{\alpha}_-(s), \\ (91) \quad \frac{1}{C_0}(|\check{\alpha}_1(s)| + \check{\alpha}_-(s)) &\leq \|\check{q}(s)\|_{\mathcal{H}} \leq C_0(|\check{\alpha}_1(s)| + \check{\alpha}_-(s)), \\ \frac{1}{C_0}\tilde{\alpha}_-(s) &\leq \|\tilde{q}(s)\|_{\mathcal{H}} \leq C_0\tilde{\alpha}_-(s),\end{aligned}$$

for some $C_0 > 0$. Let us introduce

$$(92) \quad R_-(s) = - \int_{-1}^1 \mathcal{F}_\delta(q_1)\rho dy,$$

where

$$\begin{aligned}(93) \quad \mathcal{F}_{\delta(s)}(q_1(y, s)) &= \frac{|\kappa(\delta, \cdot) + q_1|^{p+1}}{p+1} - \frac{\kappa(\delta, \cdot)^{p+1}}{p+1} \\ &\quad - \kappa(\delta, \cdot)^p \check{q}_1 - \frac{p}{2}\kappa(\delta, \cdot)^{p-1}\check{q}_1^2 - \frac{\kappa(\delta, \cdot)^{p-1}}{2}\check{q}_1^2.\end{aligned}$$

In the following proposition, we derive from (74) and (75) differential inequalities satisfied by $\check{\alpha}_1(s)$, $\check{\alpha}_-(s)$, $\tilde{\alpha}_-(s)$, $\theta(s)$ and $\delta(s)$.

Proposition 4.2. *There exists C_0 and $\epsilon_2 > 0$ such that if w is a solution to equation (5) satisfying (72) and (86) at some time s for some $\epsilon \leq \epsilon_2$, where q is defined in (73), then:*

(i) **(Control of the modulation parameter)**

$$(94) \quad |\theta'| + \frac{|\delta'|}{1-\delta^2} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2).$$

(ii) **(Projection of equation (74) on the different eigenspaces of $\check{L}_\delta$ and $\tilde{L}_\delta$)**

$$(95) \quad |\check{\alpha}'_1 - \check{\alpha}_1| \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2),$$

$$(96) \quad \left(R_- + \frac{1}{2}(\check{\alpha}_-^2 + \tilde{\alpha}_-^2)\right)' \leq -\frac{4}{p-1} \int_{-1}^1 (\check{q}_{-,2}^2 + \tilde{q}_{-,2}^2) \frac{\rho}{1-y^2} dy + C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}},$$

for $R_-(s)$, as defined in (92), satisfying

$$(97) \quad |R_-(s)| \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{1+\check{p}}{2}}, \text{ where } \check{p} = \min(p, 2) > 1.$$

(iii) **(An additional relation)**

$$(98) \quad \frac{d}{ds} \int_{-1}^1 \check{q}_1 \check{q}_2 \rho \leq -\frac{4}{5} \check{\alpha}_-^2 + C_0 \int_{-1}^1 \check{q}_{-,2}^2 \frac{\rho}{1-y^2} + C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2),$$

$$(99) \quad \frac{d}{ds} \int_{-1}^1 \tilde{q}_1 \tilde{q}_2 \rho \leq -\frac{4}{5} \tilde{\alpha}_-^2 + C_0 \int_{-1}^1 \tilde{q}_{-,2}^2 \frac{\rho}{1-y^2} + C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2).$$

(iv) **(Energy barrier)** If moreover (12) holds, then

$$(100) \quad \check{\alpha}_1(s) \leq C_0 \check{\alpha}_-(s) + C_1 \tilde{\alpha}_-(s).$$

Remark. Estimate (96) shows a kind of Lyapunov functional for system (74)-(75). Indeed, if we imagine for a second that δ and θ do not depend on s (in other words, if we forget the modulation technique), then proving (96) reduces to finding a Lyapunov functional for system (74)-(75), which follows, as for equation (5), by multiplication by the conjugate of the time derivative and then by integration over $(-1, 1)$.

Because of the modulation, we need to be more careful and use (i) to show that $|\delta'|$ and $|\theta'|$ are quadratic in $\|q\|_{\mathcal{H}}$.

Remark. The estimates concerning $\theta'(s)$ and R_- are among the novelties of our paper, since they directly involve the complex structure. The other estimates are parallel to those of the real case treated in [33].

Remark. The behavior of the solution will be derived in Section 4.4 below, thanks to the differential inequalities stated in Proposition 4.2 above. One issue will be to show that the unstable direction $\check{\alpha}_1$, which satisfies (95) never dominates the other components. This fact is true from (100), which is a direct consequence of the energy barrier hypothesis $E(w(s), \partial_s w(s)) \geq E(\kappa_0, 0)$ given in (12). Let us stress the fact that such a hypothesis is natural, since Theorem 4 will be applied with $w = w_{x_0}$ where x_0 is non-characteristic, and thanks to Proposition 3, we know that (100) holds.

Let us give the following estimate:

Claim 4.3. For all $y \in (-1, 1)$, $q_1 = \check{q}_1 + i\tilde{q}_1$

$$(101) \quad |f_{\delta(s)}(q_1(y, s))| \leq C_0 mM (k(\delta(s), y)^{p-2} |q_1(y, s)|^2, |q_1(y, s)|^p),$$

$$(102) \quad |\mathcal{F}_{\delta(s)}(q_1(y, s))| \leq C_0 mM (k(\delta(s), y)^{p-2} |q_1(y, s)|^3, |q_1(y, s)|^{p+1}),$$

where $f_{\delta}(q_1)$ and $\mathcal{F}_{\delta}(q_1)$ are introduced in (30) and (93), $mM = \min$ if $1 < p < 2$ and $mM = \max$ if $p \geq 2$.

Proof of Claim 4.3. Introducing $\xi = \check{\xi} + i\tilde{\xi} = q_1/\kappa(\delta(s), y)$ and considering the cases $|\xi| \leq 1$ and $|\xi| \geq 1$, for $f_{\delta}(q_1) = \check{f}_{\delta}(q_1) + i\tilde{f}_{\delta}(q_1)$ given in (30), we get the conclusion. $\square$

Proof of Proposition 4.2. We proceed in four parts in order to prove Proposition 4.2:

-In Part 1, we project equations (74) and (75) respectively with the projectors $\tilde{\pi}_{\lambda}$ ($\lambda \in \{0, 1\}$) (38) and $\tilde{\pi}_0$ (55) and we derive the smallness condition on δ' and θ' , together with (95).

-In Part 2, we first give some preliminary estimations; then from the derivatives of $\check{\alpha}_-^2$ and $\tilde{\alpha}_-^2$ given by the quadratic form $\check{\phi}_{\delta}$ and $\tilde{\phi}_{\delta}$, we get (96) and (97).

-In Part 3, writing equations satisfied by $\check{q}$ (74), $\tilde{q}$ (75) and using (72) we prove (98) and (99).

-In Part 4, we prove (iv).

Part 1 : Projection of equations (74) and (75). Projecting equation (74) with the projector $\tilde{\pi}_{\lambda}^{\delta}$ (38) for $\lambda = 0$ and $\lambda = 1$, we write

$$\tilde{\pi}_{\lambda}^{\delta}(\partial_s \check{q}) = \tilde{\pi}_{\lambda}^{\delta}(\check{L}_{\delta(s)} \check{q}) + \tilde{\pi}_{\lambda}^{\delta} \begin{pmatrix} 0 \\ \check{f}_{\delta(s)}(q_1) \end{pmatrix} - \delta'(s) \tilde{\pi}_{\lambda}^{\delta} \begin{pmatrix} \partial_{\delta} \kappa(\delta, y) \\ 0 \end{pmatrix} + \theta'(s) \tilde{\pi}_{\lambda}^{\delta} \begin{pmatrix} \check{q}_1 \\ \check{q}_2 \end{pmatrix}.$$

Proceeding exactly as on page 105 in [33] with (86) and (91), and using the fact that

$$|\pi_\lambda^\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}| = |\phi(\tilde{W}_\lambda(\delta, \cdot), \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix})| \leq \|\tilde{W}_\lambda(\delta, \cdot)\|_{\mathcal{H}} \|\tilde{q}\|_{\mathcal{H}} \leq C\tilde{\alpha}_-,$$

we get

$$(103) \quad \frac{2\kappa_0}{(p-1)(1-\delta^2)} |\delta'| \leq \frac{C_0}{1-\delta^2} |\delta'| (|\tilde{\alpha}_1| + \tilde{\alpha}_-) + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2) + C_0|\theta'| \tilde{\alpha}_-,$$

$$(104) \quad |\tilde{\alpha}'_1(s) - \tilde{\alpha}_1(s)| \leq \frac{C_0}{1-\delta^2} |\delta'| (|\tilde{\alpha}_1| + \tilde{\alpha}_-) + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2) + C_0|\theta'| \tilde{\alpha}_-.$$

Now, projecting equation (75) with the projector $\tilde{\pi}_0^\delta$ (55), we get

$$(105) \quad \tilde{\pi}_0^\delta(\partial_s \tilde{q}) = \tilde{\pi}_0^\delta(\tilde{L}_{\delta(s)} \tilde{q}) + \tilde{\pi}_0^\delta \begin{pmatrix} 0 \\ \tilde{f}_{\delta(s)}(q_1) \end{pmatrix} - \theta'(s) \tilde{\pi}_0^\delta \begin{pmatrix} \kappa(\delta, y) + \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}.$$

Since $\tilde{\alpha}_0(s) = \tilde{\pi}_0^\delta(\tilde{q}) = \phi(\tilde{W}_0(\delta, \cdot), \tilde{q}) = 0$ by (90) and the definition of $\tilde{\pi}_0^\delta$ (55), we write

$$0 = \tilde{\alpha}'_0(s) = \tilde{\pi}_0^\delta(\partial_s \tilde{q}) + \delta'(s) \tilde{\phi}(\partial_s \tilde{W}_0(\delta, \cdot), \tilde{q}).$$

Using (50) and (91), we get

$$(106) \quad |\tilde{\pi}_0^\delta(\partial_s \tilde{q})| \leq \frac{C_0}{1-\delta^2} |\delta'| \tilde{\alpha}_-.$$

Using (i) of Lemma 3.4, the definition of $\tilde{\pi}_0^\delta$ (55), we write

$$(107) \quad \tilde{\pi}_0^\delta(\tilde{L}_\delta(\tilde{q})) = \tilde{\phi}(\tilde{W}_0(\delta, \cdot), \tilde{L}_\delta(\tilde{q})) = \tilde{\phi}(\tilde{L}_\delta^*(\tilde{W}_0(\delta, \cdot)), \tilde{q}) = 0.$$

From the definitions of $\tilde{\pi}_0^\delta$ (55) and ϕ (35), together with Claim 4.3, we see that

$$(108) \quad \begin{aligned} \left| \tilde{\pi}_0^\delta \begin{pmatrix} 0 \\ \tilde{f}_{\delta(s)}(q_1) \end{pmatrix} \right| &\leq C \int_{-1}^1 \kappa(\delta, y) |\tilde{f}_\delta(q_1)| \rho(y) dy \\ &\leq C_0 \int_{-1}^1 \kappa(\delta, y)^{p-1} |q_1(y, s)|^2 \rho dy + C_0 \delta_{\{p \geq 2\}} \int_{-1}^1 \kappa(\delta, y) |q_1(y, s)|^p \rho(y) dy \\ &\leq C_0 \|q_1\|_{L^{p+1}}^2 \|\kappa(\delta, \cdot)\|_{L^{p+1}}^{p-1} + C_0 \delta_{\{p \geq 2\}} \|q_1\|_{L^{p+1}}^p \|\kappa(\delta, \cdot)\|_{L^{p+1}}, \end{aligned}$$

where $\delta_{\{p \geq 2\}}$ is 0 if $1 < p < 2$ and 1 otherwise. Therefore, using (108), Lemma C.2, (86) and (91), we get

$$(109) \quad \left| \tilde{\pi}_0^\delta \begin{pmatrix} 0 \\ \tilde{f}_{\delta(s)}(q_1) \end{pmatrix} \right| \leq C_0(\tilde{\alpha}_1(s)^2 + \tilde{\alpha}_-(s)^2 + \tilde{\alpha}_-(s)^2).$$

Since $\tilde{\pi}_0^\delta \begin{pmatrix} \kappa(\delta, y) \\ 0 \end{pmatrix} = 1$ from Lemmas 3.1 and 3.4, and using (50) and (91), we write

$$(110) \quad \left| \tilde{\pi}_0^\delta \begin{pmatrix} \kappa(\delta, y) + \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} - 1 \right| = \left| \tilde{\pi}_0^\delta \begin{pmatrix} \kappa(\delta, y) + \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} - \tilde{\pi}_0^\delta \begin{pmatrix} \kappa(\delta, y) \\ 0 \end{pmatrix} \right| \leq C_0(|\tilde{\alpha}_1(s)| + \tilde{\alpha}_-(s)).$$

Using (106), (107), (109) and (110), to bound the terms of equation (105) we get:

$$(111) \quad |\theta'| \leq \frac{C_0}{1-\delta^2} |\delta'| \tilde{\alpha}_- + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2) + C_0|\theta'| (|\tilde{\alpha}_1| + \tilde{\alpha}_-).$$

Using (86) and (91), we see that

$$|\theta'| + \frac{2\kappa_0}{p-1} \frac{|\delta'|}{1-\delta^2} \leq C_0\epsilon \frac{|\delta'|}{1-\delta^2} + C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2) + C_0\epsilon|\theta'|,$$

hence

$$(1 - C_0\epsilon)|\theta'| + \left(\frac{2\kappa_0}{p-1} - C_0\epsilon\right) \frac{|\delta'|}{1-\delta^2} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2).$$

Taking ϵ small enough, we get (94). Then using (94) to bound the term of the right hand side of (104) we get (95).

Part 2 : A kind of Lyapunov functional for system (74)-(75). We need to put together information from $\check{q}_-$ and $\tilde{q}_-$ in order to conclude. Handling each one alone doesn't allow us to control the terms $\int_{-1}^1 \check{q}_2 \check{f}_\delta(q_1) \rho dy$ and $\int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy$ which appear in the differential inequalities satisfied by $\check{\alpha}_-$ and $\tilde{\alpha}_-$ (use (115) and (121) below). We claim that (96) follows from the following lemmas:

Lemma 4.4 (Preliminary estimates for $\check{q}_-$). *There exists $\epsilon_3 > 0$ such that if $\epsilon \leq \epsilon_3$ in the hypotheses of Proposition 4.2, then*

$$\left\| \partial_s \check{q}_- - \check{L}_\delta(\check{q}_-) - \check{\pi}_-^\delta \begin{pmatrix} 0 \\ \check{f}_\delta(q_1) \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}},$$

$$\left| \check{\varphi}_\delta \left(\check{q}_-, \check{\pi}_-^\delta \begin{pmatrix} 0 \\ \check{f}_\delta(q_1) \end{pmatrix} \right) - \int_{-1}^1 \check{q}_2 \check{f}_\delta(q_1) \rho dy \right| \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}}.$$

Proof. Since the equation (74) satisfied by $\check{q}$ is the same as in the real case treated in [33], except for the last term $\theta'(\check{q}_1, \check{q}_2)$, we refer the reader to Claim 5.4 on page 106 in [33], and focus only on the last term. Using (91) and (94), we see that

$$\|\theta'(\check{q}_1, \check{q}_2)\|_{\mathcal{H}} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}},$$

which is precisely the error in the right hand sides of the inequalities in Lemma 4.4. Since our equation (5) satisfies (86) and (91), we see that the proof of Merle and Zaag on page 108 in [33] can be adapted in our case. $\square$

Lemma 4.5 (Preliminary estimates for $\tilde{q}_-$). *There exists $\epsilon_4 > 0$ such that if $\epsilon \leq \epsilon_4$ in the hypotheses of Proposition 4.2, then*

$$(112) \quad \left\| \partial_s \tilde{q}_- - \tilde{L}_\delta(\tilde{q}_-) - \tilde{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}},$$

$$(113) \quad \left| \tilde{\varphi}_\delta \left(\tilde{q}_-, \tilde{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right) - \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy \right| \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}},$$

$$(114) \quad \left| \int_{-1}^1 \check{q}_2 \check{f}_\delta(q_1) \rho dy + \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy - \frac{d}{ds} \int_{-1}^1 \mathcal{F}_{\delta(s)} \rho dy \right| \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^2.$$

Remark. Note that (114) is one of the new features of our paper. Indeed, it directly involves the complex structure.

Let us derive (96) and (97) from Lemmas 4.4 and 4.5. Then we prove Lemma 4.5.

Proof of (96) and (97). Using the definition of $\check{\alpha}_-$ (89), we proceed as on page 107 in [33] and we apply the bound (94) on $|\delta'|$, so we get

$$|\check{\alpha}'_- \check{\alpha}_- - \check{\varphi}_\delta(\check{q}_-, \partial_s \check{q}_-)| \leq C_0 |\delta'| \frac{\check{\alpha}_-^2}{1-\delta^2} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \tilde{\alpha}_-^2)^2.$$

Using Lemma 4.4, we write

$$\begin{aligned} & \left| \check{\alpha}'_- \check{\alpha}_- - \check{\varphi}_\delta(\check{q}_-, \check{L}_\delta \check{q}_-) - \int_{-1}^1 \check{q}_2 \check{f}_\delta(q_1) \rho dy \right| \\ & \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} + \left| \check{\varphi}_\delta \left(\check{q}_-, \partial_s \check{q}_- - \check{L}_\delta(\check{q}_-) - \check{\pi}_-^\delta \begin{pmatrix} 0 \\ \check{f}_\delta(q_1) \end{pmatrix} \right) \right| \\ & \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} + \|\check{q}_-\|_{\mathcal{H}}(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}}. \end{aligned}$$

Since we easily get from the definition (32) of $\check{L}_\delta$ that

$$\check{\varphi}_\delta(\check{q}, \check{L}_\delta(\check{q})) = -\frac{4}{p-1} \int_{-1}^1 \check{q}_{-,2}^2 \frac{\rho}{1-y^2} dy,$$

we conclude that

$$(115) \quad \left| \check{\alpha}'_- \check{\alpha}_- - \int_{-1}^1 \check{q}_2 \check{f}_\delta(q_1) \rho dy \right| \leq -\frac{4}{p-1} \int_{-1}^1 \check{q}_{-,2}^2 \frac{\rho}{1-y^2} dy + C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}}.$$

Arguing similarly for $\tilde{\alpha}_-$ (89), we see that

$$\tilde{\alpha}_-^2(s) = \tilde{\varphi}_\delta(\tilde{q}(s), \tilde{q}(s)).$$

Using the definition (58) of $\tilde{\varphi}_\delta$, we have by differentiation

$$(116) \quad \tilde{\alpha}'_- \tilde{\alpha}_- = \tilde{\varphi}_\delta(\tilde{q}, \partial_s \tilde{q}) - \frac{1}{2} \delta'(s) \int_{-1}^1 \partial_s \tilde{\psi}(\delta, y) \tilde{q}_1^2 \rho.$$

Using the Hölder inequality, the Hardy-Sobolev estimate of Lemma C.2 and (91), we write

$$(117) \quad \left| \int_{-1}^1 \partial_s \tilde{\psi}(\delta, y) \tilde{q}_1^2 \rho \right| \leq \|\partial_s \tilde{\psi}(\delta, y)\|_{L_\rho^{\frac{p+1}{p-1}}} \|\tilde{q}_1\|_{L_\rho^{p+1}}^2 \leq C_0 \|\partial_s \tilde{\psi}(\delta, y)\|_{L_\rho^{\frac{p+1}{p-1}}} \tilde{\alpha}_-(s)^2.$$

Since $|\partial_s \tilde{\psi}(\delta, y)| \leq C/(1+\delta y)^2$ for all $(\delta, y) \in (-1, 1)^2$ from the expression of $\tilde{\psi}$ in (30), using Claim 3.2, we see that $\|\partial_s \tilde{\psi}(\delta, y)\|_{L_\rho^{\frac{p+1}{p-1}}} \leq C/(1-\delta^2)$. Therefore, using (116), (117), and the bound (94) on $|\delta'(s)|$, we get

$$(118) \quad |\tilde{\alpha}'_- \tilde{\alpha}_- - \tilde{\varphi}_\delta(\tilde{q}, \partial_s \tilde{q})| \leq C_0 |\delta'| \frac{\tilde{\alpha}_-^2}{1-\delta^2} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^2.$$

From (118), the continuity of $\tilde{\varphi}_\delta$, and Lemma 4.5, we write

$$\begin{aligned} & \left| \tilde{\alpha}'_- \tilde{\alpha}_- - \tilde{\varphi}_\delta(\tilde{q}, \tilde{L}_\delta \tilde{q}) - \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy \right| \\ & \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} + \left| \tilde{\varphi}_\delta \left(\tilde{q}, \partial_s \tilde{q} - \tilde{L}_\delta(\tilde{q}) - \check{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right) \right| \\ (119) \quad & \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} + \|\tilde{q}\|_{\mathcal{H}}(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}} \leq C_0(\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2)^{\frac{3}{2}}. \end{aligned}$$

Besides, using the expressions of $\tilde{L}_\delta$ (34) and $\tilde{\varphi}_\delta$ (58), we have

$$\begin{aligned}
 \tilde{\varphi}_\delta(\tilde{q}, \tilde{L}_\delta(\tilde{q})) &= \tilde{\varphi}_\delta \left(\left(\mathcal{L}\tilde{q}_1 + \tilde{\psi}(\delta, y)\tilde{q}_1 - \frac{\tilde{q}_2}{p-1}\tilde{q}_2 - 2y\partial_y\tilde{q}_2 \right), \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} \right) \\
 &= - \int_{-1}^1 \tilde{q}_2(\mathcal{L}\tilde{q}_1 + \tilde{\psi}(\delta, y)\tilde{q}_1)\rho dy \\
 &\quad + \int_{-1}^1 (\mathcal{L}\tilde{q}_1 + \tilde{\psi}(\delta, y)\tilde{q}_1 - \frac{p+3}{p-1}\tilde{q}_2 - 2y\tilde{q}_2')\tilde{q}_2\rho dy \\
 &= -\frac{p+3}{p-1} \int_{-1}^1 \tilde{q}_2^2\rho dy - \int_{-1}^1 y(\tilde{q}_2^2)'\rho dy = -\frac{p+3}{p-1} \int_{-1}^1 \tilde{q}_2^2 + \int_{-1}^1 \tilde{q}_2^2(\rho - y\rho') dy \\
 (120) \quad &= -\frac{4}{p-1} \left[\int_{-1}^1 \tilde{q}_{-,2}^2\rho dy + \int_{-1}^1 \tilde{q}_{-,2}^2 \frac{y^2\rho}{1-y^2} dy \right] = -\frac{4}{p-1} \int_{-1}^1 \tilde{q}_{-,2}^2 \frac{\rho}{1-y^2} dy.
 \end{aligned}$$

Using (119) and (120), we see that

$$(121) \quad \left| \tilde{\alpha}'_- \tilde{\alpha}_- - \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy \right| \leq -\frac{4}{p-1} \int_{-1}^1 \tilde{q}_{-,2}^2 \frac{\rho}{1-y^2} dy + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}}.$$

Therefore, using (115) with (121), we write

$$\begin{aligned}
 (122) \quad &\left| \frac{1}{2}(\tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)' - \left[\int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) + \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy \right] \right| \\
 &\leq -\frac{4}{p-1} \int_{-1}^1 (\tilde{q}_{-,2}^2 + \tilde{q}_{-,2}^2) \frac{\rho}{1-y^2} dy + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}}.
 \end{aligned}$$

Better yet, by (114) we see that estimate (96) holds with R_- given by (92). Using Claim 4.3, Lemma C.2 and condition (86) (considering first the case $p \geq 2$ and then the case $1 < p < 2$), we see that (97) holds. It remains to prove Lemma 4.5 in order to conclude the proof of (98) and (100).

Proof of Lemma 4.5.

• *Proof of (112):* We first project equation (75) using the negative projector $\tilde{\pi}_-^\delta$ introduced in Definition 3.6:

$$(123) \quad \tilde{\pi}_-^\delta(\partial_s \tilde{q}) = \tilde{\pi}_-^\delta(\tilde{L}_{\delta(s)} \tilde{q}) + \tilde{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} - \theta'(s) \tilde{\pi}_-^\delta \begin{pmatrix} \kappa(\delta, \cdot) + \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}.$$

We write the expansion (56) with $\partial_s \tilde{q}$

$$(124) \quad \partial_s \tilde{q} = \tilde{\pi}_0^\delta(\partial_s \tilde{q}) \tilde{F}_0(\delta, \cdot) + \tilde{\pi}_-^\delta(\partial_s \tilde{q}).$$

Using (72) and (124), we see that

$$(125) \quad \tilde{\pi}_-^\delta(\partial_s \tilde{q}) - \partial_s \tilde{q} = 0.$$

From the remark after Definition 3.6, we see that $\tilde{L}_\delta(\tilde{q}) \in \mathcal{H}_-$ (as $\tilde{q} \in \mathcal{H}_-$) and

$$(126) \quad \tilde{\pi}_-^\delta(\tilde{L}_\delta(\tilde{q})) = \tilde{L}_\delta(\tilde{q}).$$

Using (56) with $(\kappa(\delta, y), 0)$, (49) and (55), we get

$$\tilde{\pi}_-^\delta \begin{pmatrix} \kappa(\delta, \cdot) \\ 0 \end{pmatrix} = 0;$$

therefore, using (56) with $\tilde{q}$, we write

$$(127) \quad \left\| \tilde{\pi}_-^\delta \begin{pmatrix} \kappa(\delta, y) + \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} \right\|_{\mathcal{H}} = \left\| \tilde{\pi}_-^\delta \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} \right\|_{\mathcal{H}} \leq \|\tilde{q}\|_{\mathcal{H}} + |\tilde{\pi}_0^\delta(\tilde{q})| \|\tilde{F}_0(\delta, \cdot)\|_{\mathcal{H}} \\ \leq C \|\tilde{q}\|_{\mathcal{H}} \leq C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{1}{2}},$$

where we used (55), (50), (43) and (91) to get the last line. Using (123), (125), (126), (127) and (94) we get (112).

•*Proof of (113):* Note from (56) that

$$\begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} = \tilde{B}_0(s) \tilde{F}_0(\delta, \cdot) + \tilde{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix},$$

where $\tilde{B}_0(s) = \tilde{\pi}_0^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix}$. So from the definition (58), the bilinearity of $\tilde{\varphi}_\delta$ and the bound on the norm of $\tilde{F}_0$, we have

$$\begin{aligned} & \left| \tilde{\varphi}_\delta \left(\tilde{q}, \tilde{\pi}_-^\delta \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right) - \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy \right| \\ &= \left| \tilde{\varphi}_\delta \left(\tilde{q}, \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right) - \tilde{B}_0(s) \tilde{F}_0(\delta, y) - \tilde{\varphi}_\delta \left(\tilde{q}, \begin{pmatrix} 0 \\ \tilde{f}_\delta(q_1) \end{pmatrix} \right) \right| \\ &= \left| \tilde{\varphi}_\delta \left(\tilde{q}, \tilde{B}_0(s) \tilde{F}_0(\delta, y) \right) \right| \leq C |\tilde{B}_0(s)| \|\tilde{q}\|_{\mathcal{H}} \leq C(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}}. \end{aligned}$$

From (108) and (91), (113) follows.

[• *Proof of (114)*] We see from the expression of $\mathcal{F}_{\delta(s)}(q_1)$ (93) that

$$(128) \quad \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) + \int_{-1}^1 \tilde{q}_2 \tilde{f}_\delta(q_1) \rho dy = \int_{-1}^1 \partial_s \mathcal{F}_{\delta(s)}(q_1) \rho dy \\ + \int_{-1}^1 \frac{p-1}{2} \kappa^{p-2}(\delta) \partial_\delta \kappa(\delta) \delta'(s) (p\tilde{q}_1^2 + \tilde{q}_1^2) \rho dy \\ + \theta'(s) \int_{-1}^1 \left[(\kappa(\delta, y) + \tilde{q}_1) \tilde{f}_\delta(q_1) - \tilde{q}_1 \tilde{f}_\delta(q_1) \right] \rho dy.$$

Since we have $\|\partial_\delta \kappa(\delta, \cdot) \kappa(\delta, \cdot)^{p-2}\|_{L_\rho^{\frac{p+1}{p-1}}} \leq C_0/(1-\delta^2)$, from the expression of $\partial_\delta \kappa(\delta, y)$,

$$\partial_\delta \kappa(\delta, y) = -\frac{2\kappa_0}{p-1} \frac{(y+\delta)(1-\delta^2)^{\frac{2-p}{p-1}}}{(1+\delta y)^{-\frac{p+1}{p-1}}},$$

the definition of (10) $\kappa(\delta, y)$ and Claim 3.2, we use the Hölder inequality and the Hardy-Sobolev inequality of Lemma C.2 to derive that

$$\left| \int_{-1}^1 \partial_\delta \kappa(\delta, y) \kappa(\delta, y)^{p-2} (p\tilde{q}_1^2 + \tilde{q}_1^2) \rho dy \right| \leq \frac{C_0}{1-\delta^2} \|q(s)\|_{\mathcal{H}}^2,$$

so, by (91) and (94), we get

$$(129) \quad \left| (p-1)\delta'(s) \int_{-1}^1 \kappa^{p-1}(\delta, y) \partial_\delta \kappa(\delta, y) (p\tilde{q}_1^2 + \tilde{q}_1^2) \rho dy \right| \leq C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^2.$$

Since $\forall |\delta| < 1$, $\|\kappa(\delta, \cdot)\|_{\mathcal{H}_0} \leq C_0$ by definition (10), using Claim 4.3, Lemma C.2 and (91) we see that

$$\begin{aligned}
 & \left| \int_{-1}^1 \tilde{q}_1 \tilde{f}_\delta(q_1) \rho(y) dy \right| + \left| \int_{-1}^1 \tilde{q}_1 \check{f}_\delta(q_1) \rho(y) dy \right| \\
 & \leq C_0 \left(\delta_{\{p \geq 2\}} \int_{-1}^1 \kappa(\delta, y)^{p-2} |q_1(y, s)|^3 \rho dy + \int_{-1}^1 |q_1(y, s)|^{p+1} \rho dy \right) \\
 & \leq C_0 \left(\delta_{\{p \geq 2\}} \|\kappa(\delta, \cdot)\|_{L^{p+1}_\rho}^{p-2} \|q_1\|_{L^{p+1}_\rho}^{p+1} + \|q_1\|_{L^{p+1}_\rho}^{p+1} \right) \\
 & \leq C_0 \|q_1(s)\|_{\mathcal{H}}^{\tilde{p}+1}, \text{ where } \tilde{p} = \min(p, 2) > 1 \\
 (130) \quad & \leq C_0 (\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2).
 \end{aligned}$$

Using (128), (129), (108), (130) and (94) we see that estimate (114) holds. This concludes the proof of Lemma 4.5. $\square$

Part 3 : An additional relation. (iii) First, note from (91) that

$$(131) \quad \int_{-1}^1 \tilde{q}_1^2 \rho + \int_{-1}^1 (\partial_y \tilde{q}_1)^2 (1 - y^2) \rho + \int_{-1}^1 \tilde{q}_2^2 \rho \leq C_0 (\check{\alpha}_1^2 + \check{\alpha}_-^2),$$

$$(132) \quad \int_{-1}^1 \tilde{q}_1^2 \rho + \int_{-1}^1 (\partial_y \tilde{q}_1)^2 (1 - y^2) \rho + \int_{-1}^1 \tilde{q}_2^2 \rho \leq C_0 \check{\alpha}_-^2,$$

and, using equation (74) and the definition of $\check{L}_\delta$ (32), we write

$$\begin{aligned}
 & \frac{d}{ds} \int_{-1}^1 \tilde{q}_1 \tilde{q}_2 \rho = \int_{-1}^1 (\theta'(s) \tilde{q}_1 \tilde{q}_2 + \tilde{q}_2^2 - \delta'(s) \tilde{q}_2 \partial_\delta \kappa(\delta, \cdot)) \rho \\
 (133) \quad & + \int_{-1}^1 \tilde{q}_1 (\mathcal{L} \tilde{q}_1 + \tilde{\psi}(\delta, \cdot) \tilde{q}_1 - \frac{p+3}{p-1} \tilde{q}_2 - 2y \partial_y \tilde{q}_2 + \check{f}_\delta(q_1) + \theta'(s) \tilde{q}_2) \rho.
 \end{aligned}$$

Almost all of the terms in the right hand side of (133) have been studied in [33], except for the two terms with $\theta'(s)$.

Using the Cauchy-Schwartz inequality, (86), (91) and (94), we see that

$$(134) \quad \left| \int_{-1}^1 \theta'(s) \tilde{q}_1 \tilde{q}_2 \rho \right| \leq |\theta'(s)| \left(\int_{-1}^1 \tilde{q}_1^2 \rho \right)^{\frac{1}{2}} \left(\int_{-1}^1 \tilde{q}_2^2 \rho \right)^{\frac{1}{2}} \leq \frac{1}{100} (\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2),$$

and

$$(135) \quad \left| \int_{-1}^1 \theta'(s) \tilde{q}_1 \tilde{q}_2 \rho \right| \leq |\theta'(s)| \left(\int_{-1}^1 \tilde{q}_1^2 \rho \right)^{\frac{1}{2}} \left(\int_{-1}^1 \tilde{q}_2^2 \rho \right)^{\frac{1}{2}} \leq \frac{1}{100} (\check{\alpha}_1^2 + \check{\alpha}_-^2 + \check{\alpha}_-^2),$$

for ϵ small enough. Using the proof of Proposition 5.2 on page 103 in [33] to control the other terms we get (98).

In the same way, in order to prove (99), we will bound all the terms on the right hand side of the following:

$$\begin{aligned}
 & \frac{d}{ds} \int_{-1}^1 \tilde{q}_1 \tilde{q}_2 \rho = \int_{-1}^1 (-\theta'(s) \tilde{q}_2 (\kappa(\delta, \cdot) + \tilde{q}_1) + \tilde{q}_2^2) \rho \\
 (136) \quad & + \int_{-1}^1 \tilde{q}_1 (\mathcal{L} \tilde{q}_1 + \tilde{\psi}(\delta, \cdot) \tilde{q}_1 - \frac{p+3}{p-1} \tilde{q}_2 - 2y \partial_y \tilde{q}_2 + \check{f}_\delta(q_1) - \theta'(s) \tilde{q}_2) \rho.
 \end{aligned}$$

We use the Cauchy-Schwartz inequality, (94), (131), (132), (86) and bound $\kappa(\delta, y)$ to write for ϵ small enough,

$$\begin{aligned} \left| \int_{-1}^1 \theta'(s) \tilde{q}_2(\kappa(\delta, y) + \tilde{q}_1) \rho \right| &\leq |\theta'(s)| \left(\|\kappa(\delta, \cdot)\|_{L_\rho^2} + \left(\int_{-1}^1 \tilde{q}_1^2 \rho \right)^{\frac{1}{2}} \right) \left(\int_{-1}^1 \tilde{q}_2^2 \rho \right)^{\frac{1}{2}} \\ (137) \quad &\leq C(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2)^{\frac{3}{2}} \leq \frac{1}{100}(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2). \end{aligned}$$

From the definition of $\tilde{\varphi}_\delta$ (58) and the definition of $\tilde{\alpha}_-$ (90), we write

$$(138) \quad \int_{-1}^1 \tilde{q}_1(\mathcal{L}\tilde{q}_1 + \tilde{\psi}(\delta, \cdot)\tilde{q}_1)\rho = -\tilde{\varphi}_\delta \left(\begin{pmatrix} \tilde{q}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{q}_1 \\ 0 \end{pmatrix} \right) = -\tilde{\alpha}_-^2.$$

Using integration by parts, the fact that $|y\partial_y\rho(y)| \leq C\frac{\rho}{1-y^2}$, the Cauchy-Schwartz inequality, Lemma C.2, (131) and (132), we write

$$\begin{aligned} &\left| -\frac{p+3}{p-1} \int_{-1}^1 \tilde{q}_1 \tilde{q}_1 \rho - 2 \int_{-1}^1 \tilde{q}_1 y \partial_y \tilde{q}_2 \rho \right| \\ &= \left| 2 \int_{-1}^1 \tilde{q}_2 \partial_y \tilde{q}_1 y \rho + \left(2 - \frac{p+3}{p-1} \right) \int_{-1}^1 \tilde{q}_1 \tilde{q}_1 \rho + 2 \int_{-1}^1 \tilde{q}_2 \tilde{q}_1 y \partial_y \rho \right| \\ &\leq C \int_{-1}^1 \left(|\tilde{q}_2| |\partial_y \tilde{q}_1| \rho + |\tilde{q}_2| |\tilde{q}_1| \frac{\rho}{1-y^2} \right) \\ (139) \quad &\leq \left(\int_{-1}^1 \tilde{q}_2^2 \frac{\rho}{1-y^2} \right)^{1/2} \left[\int_{-1}^1 (\partial_y \tilde{q}_1)^2 (1-y^2) \rho + \int_{-1}^1 \tilde{q}_1^2 \frac{\rho}{1-y^2} \right]^{1/2} \\ &\leq C_0 \tilde{\alpha}_- (\int_{-1}^1 \tilde{q}_2^2 \frac{\rho}{1-y^2})^{1/2} \leq \frac{1}{100} \tilde{\alpha}_-^2 + C \int_{-1}^1 \tilde{q}_2^2 \frac{\rho}{1-y^2}. \end{aligned}$$

Arguing as for (130) and (134) and using (86) we write for ϵ small enough

$$(140) \quad \left| \int_{-1}^1 \tilde{q}_1 \tilde{f}_\delta(q_1) \rho(y) dy \right| \leq C_0 \|q_1(s)\|_{\mathcal{H}}^{\tilde{p}+1} \leq \frac{1}{100}(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2),$$

$$(141) \quad \left| \int_{-1}^1 \theta'(s) \tilde{q}_2 \tilde{q}_1 \rho \right| \leq C |\theta'(s)| (\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2) \leq \frac{1}{100}(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2).$$

Collecting (136)-(141), we get

$$\frac{d}{ds} \int_{-1}^1 \tilde{q}_1 \tilde{q}_2 \rho \leq -\frac{4}{5} \tilde{\alpha}_-^2 + C_0 \int_{-1}^1 \tilde{q}_2^2 \frac{\rho}{1-y^2} + C_0(\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2).$$

Part 4 : Energy barrier. (iv) Using the definition of $q(y, s)$ (73), we can make an expansion of $E(w(s), \partial_s w(s))$ (9) for $q \rightarrow 0$ in $\mathcal{H}$ and obtain from straightforward computations:

$$(142) \quad E(w(s), \partial_s w(s)) = E(\kappa_0, 0) + \frac{1}{2}(\tilde{\varphi}_\delta(\tilde{q}, \tilde{q}) + \tilde{\varphi}_\delta(\tilde{q}, \tilde{q})) - \int_{-1}^1 \mathcal{F}_\delta(q_1) \rho dy,$$

where $\tilde{\varphi}_\delta$, $\tilde{\varphi}_\delta$ and $\mathcal{F}_\delta(q_1)$ are defined in (40), (58) and (93).

Since we have (92), (97), (86) and (91):

$$(143) \quad \left| \int_{-1}^1 \mathcal{F}_\delta(q_1) \rho dy \right| \leq C \|q(s)\|_{\mathcal{H}}^{\tilde{p}+1} \leq C \epsilon^{\tilde{p}-1} (\tilde{\alpha}_1^2 + \tilde{\alpha}_-^2 + \tilde{\alpha}_-^2),$$

we note that for some $C_1 > 0$

$$(144) \quad \tilde{\varphi}_\delta(\tilde{q}, \tilde{q}) \leq C_0 \tilde{\alpha}_1^2 - C_1 \tilde{\alpha}_-^2,$$

which was proved in [33] on page 113. From (12), (142), (144) and (143), we see that taking ϵ small enough so that $C\epsilon^{\bar{p}-1} < \frac{C_1}{4}$, we get

$$0 \leq E(w(s), \partial_s w(s)) - E(\kappa_0, 0) \leq \left(\frac{C_0}{2} + \frac{C_1}{4} \right) \check{\alpha}_-^2 - \frac{C_1}{4} \check{\alpha}_1^2 + \left(\frac{1}{2} + \frac{C_1}{4} \right) \check{\alpha}_-^2,$$

which yields (100). $\square$

4.4. Exponential decay of the different components. Our aim is to show that $\|q(s)\|_{\mathcal{H}} \rightarrow 0$ and that both θ and δ converge as $s \rightarrow \infty$. An important issue will be to show that the unstable mode $\check{\alpha}_1$, which satisfies equation (91), never dominates. This is true thanks to item (iv) in Proposition 4.2 (see the third remark following that Proposition for more details). Let us first introduce a more adapted notation and rewrite Proposition 4.2.

If we introduce

$$(145) \quad \lambda(s) = \frac{1}{2} \log \left(\frac{1 + \delta(s)}{1 - \delta(s)} \right), a(s) = \check{\alpha}_1(s)^2 \text{ and } b(s) = \check{\alpha}_-(s)^2 + \tilde{\alpha}_-(s)^2 + R_-(s)$$

(note that $\delta(s) = \tanh(\lambda(s))$), then we see from (97) and (91) that if (86) holds, then $|b - (\check{\alpha}_-(s)^2 + \tilde{\alpha}_-(s)^2)| \leq C_0 \epsilon^{\bar{p}-1} (\check{\alpha}_1(s)^2 + \check{\alpha}_-(s)^2 + \tilde{\alpha}_-(s)^2)$. Hence

$$(146) \quad \frac{99}{100} \check{\alpha}_-(s)^2 + \frac{99}{100} \tilde{\alpha}_-(s)^2 - \frac{1}{100} a \leq b \leq \frac{101}{100} \check{\alpha}_-(s)^2 + \frac{101}{100} \tilde{\alpha}_-(s)^2 + \frac{1}{100} a$$

for ϵ small enough. Therefore, using Proposition 4.2, estimate (86), (91) and the fact that $\lambda'(s) = \frac{\delta'(s)}{1-\delta(s)^2}$, we derive the following:

Claim 4.6 (Relations between a , b , λ , θ , $\int_{-1}^1 \check{q}_1 \check{q}_2 \rho$ and $\int_{-1}^1 \tilde{q}_1 \tilde{q}_2 \rho$). *There exist positive ϵ_4 , K_4 and K_5 such that if w is a solution to equation (5) such that (72) and (86) hold at some time s for some $\epsilon \leq \epsilon_4$, where q is defined in (73), then using the notation (145), we have:*

(i) **(Size of the solution)**

$$(147) \quad \frac{1}{K_4} (a(s) + b(s)) \leq \|q(s)\|_{\mathcal{H}}^2 \leq K_4 (a(s) + b(s)) \leq K_4^2 \epsilon^2,$$

$$(148) \quad |\theta'(s)| + |\lambda'(s)| \leq K_4 (a(s) + b(s)) \leq K_4^2 \|q(s)\|_{\mathcal{H}}^2,$$

$$(149) \quad \left| \int_{-1}^1 \check{q}_1 \check{q}_1 \rho \right| \leq K_4 (a(s) + b(s)),$$

$$(150) \quad \left| \int_{-1}^1 \tilde{q}_1 \tilde{q}_1 \rho \right| \leq K_4 b(s),$$

and (146) holds.

(ii) **(Equations)**

$$(151) \quad \frac{3}{2} a - K_4 \epsilon b \leq a' \leq \frac{5}{2} a - K_4 \epsilon b,$$

$$(152) \quad b' \leq -\frac{8}{p-1} \int_{-1}^1 (\check{q}_{-,2}^2 + \tilde{q}_{-,2}^2) \frac{\rho}{1-y^2} dy + K_4 \epsilon (a+b),$$

$$\frac{d}{ds} \int_{-1}^1 (\check{q}_1 \check{q}_2 + \tilde{q}_1 \tilde{q}_2) \rho \leq -\frac{3}{5} b + K_4 \int_{-1}^1 (\check{q}_{-,2}^2 + \tilde{q}_{-,2}^2) \frac{\rho}{1-y^2} + K_4 a.$$

(iii) **(Energy barrier)** If (12) holds, then

$$a(s) \leq K_5 b(s).$$

Proof of Theorem 4. Consider $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ a solution of equation (5) such that (12) and (13) hold for some $\delta^* \in (-1, 1)$, $\theta^* \in [0, 2\pi)$ and $\epsilon^* \in (0, \epsilon_0]$. Consider then

$$(153) \quad \epsilon = 2K_0K_1\epsilon^*$$

where K_1 is given in Proposition 4.1 and K_0 will be fixed later. If

$$\epsilon^* \leq \epsilon_1 \text{ and } \epsilon \leq \epsilon_4,$$

then we see that Proposition 4.1, Claim 4.6 and (146) apply respectively with ϵ^* and ϵ . In particular, there is a maximal solution $\delta(s) \in C^1([s^*, \infty), (-1, 1))$ such that (72) holds for all $s \in [s^*, \infty)$ where $q(y, s)$ is defined in (73) and

$$(154) \quad |\theta - \theta^*| + |\lambda(s^*) - \lambda^*| + \|q(s^*)\|_{\mathcal{H}} \leq K_1\epsilon^* \text{ with } \lambda^* = \log \left(\frac{1 + \delta^*}{1 - \delta^*} \right).$$

If in addition we have

$$(155) \quad K_0 \geq 1, \text{ hence } \epsilon \geq 2K_1\epsilon^*,$$

then, we can give two definitions:

- We define first from (154) and (155) $s_1^* \in (s^*, \infty)$ such that for all $s \in [s^*, s_1^*]$,

$$(156) \quad \|q(s)\|_{\mathcal{H}} < \epsilon$$

and if $s_1^* < \infty$, then $\|q(s_1^*)\|_{\mathcal{H}} = \epsilon$.

-Then, we define $s_2^* \in [s^*, s_1^*]$ as the first $s \in [s^*, s_1^*]$ such that

$$(157) \quad a(s) \geq \frac{b(s)}{5K_4}$$

where K_4 is introduced in Claim 4.6, or $s_2^* = s_1^*$ if (157) is never satisfied on $[s^*, s_1^*]$. We claim the following:

Claim 4.7. *There exist positive ϵ_6 , μ_6 , K_6 and $f \in C^1([s^*, s_2^*])$ such that if $\epsilon \leq \epsilon_6$, then for all $s \in [s^*, s_2^*]$:*

(i)

$$\frac{1}{2}f(s) \leq b(s) \leq 2f(s) \text{ and } f'(s) \leq -2\mu_6 f(s),$$

(ii)

$$\|q(s)\|_{\mathcal{H}} \leq K_6\|q(s^*)\|_{\mathcal{H}}e^{-\mu_6(s-s^*)} \leq K_6K_1\epsilon^*e^{-\mu_6(s-s^*)}.$$

Proof. The proof of Claim 5.6 on page 115 in [33] remains valid where $f(s)$ is given by

$$f(s) = b(s) + \eta_6 \int_{-1}^1 (\tilde{q}_1 \tilde{q}_2 + \tilde{q}_1 \tilde{q}_2) \rho,$$

where $\eta_6 > 0$ is fixed small independent of ϵ . □

Claim 4.8. (i) *There exists $\epsilon_7 > 0$ such that for all $\sigma > 0$, there exists $K_7(\sigma) > 0$ such that if $\epsilon \leq \epsilon_7$, then*

$$\forall s \in [s_2^*, \min(s_2^* + \sigma, s_1^*)], \|q(s)\|_{\mathcal{H}} \leq K_7\|q(s^*)\|_{\mathcal{H}}e^{-\mu_6(s-s^*)} \leq K_7K_1\epsilon^*e^{-\mu_6(s-s^*)}$$

where μ_6 has been introduced in Claim 4.7.

(ii) *There exists $\epsilon_8 > 0$ such that if $\epsilon \leq \epsilon_8$, then*

$$(158) \quad \forall s \in (s_2^*, s_1^*], b(s) \leq a(s) \left(5K_4e^{-\frac{(s-s_2^*)}{2}} + \frac{1}{4K_5} \right)$$

where K_4 and K_5 have been introduced in Claim 4.6.

Proof. The proof is the same as the proof of Claim 5.7 on page 117 in [33]. $\square$

Now, in order to conclude the proof of Theorem 4, we fix $\sigma_0 > 0$ such that

$$5K_4^{-\frac{\sigma_0}{2}} + \frac{1}{4K_5} \leq \frac{1}{2K_5},$$

where K_4 and K_5 are introduced in Claim 4.6. Then, we impose the condition

$$(159) \quad \epsilon = 2K_0K_1\epsilon^*, \text{ where } K_0 = \max(2, K_6, K_7(\sigma_0)),$$

and the constants are defined in Proposition 4.1 and Claims 4.7 and 4.8. Then, we fix

$$\epsilon_0 = \min \left(1, \epsilon_1, \frac{\epsilon_i}{2K_0K_1} \text{ for } i \in \{4, 6, 7, 8\} \right)$$

and the constants are defined in Proposition 4.1 and Claims 4.6, 4.7 and 4.8. Now, if $\epsilon^* \leq \epsilon_0$, then Claim 4.6, Claim 4.7 and Claim 4.8 apply. We claim that for all $s \in [s^*, s_1^*]$,

$$(160) \quad \|q(s)\|_{\mathcal{H}} \leq K_0 \|q(s^*)\|_{\mathcal{H}} e^{-\mu_6(s-s^*)} \leq K_0K_1\epsilon^* e^{-\mu_6(s-s^*)} = \frac{\epsilon}{2} e^{-\mu_6(s-s^*)}.$$

Indeed, if $s \in [s^*, \min(s_2^* + \sigma_0, s_1^*)]$, then, this comes from (ii) of Claim 4.7 or (i) of Claim 4.8 and the definition of K_0 (159).

Now, if $s_2^* + \sigma_0 < s_1^*$ and $s \in [s_2^* + \sigma_0, s_1^*]$, then we have from (158) and the definition of σ_0 , $b(s) \leq \frac{a(s)}{2K_5}$ on the one hand. On the other hand, from (iii) in Claim 4.6, we have $a(s) \leq K_5b(s)$, hence $a(s) = b(s) = 0$ and from (147), $q(y, s) \equiv 0$, hence (160) is satisfied trivially.

In particular, we have for all $s \in [s^*, s_1^*]$, $\|q\|_{\mathcal{H}} \leq \frac{\epsilon}{2}$, hence, by definition of s_1^* , this means that $s_1^* = \infty$.

From (i) of Claim 4.8 and (148), we have

$$(161) \quad \forall s \geq s^*, \|q(s)\|_{\mathcal{H}} \leq \frac{\epsilon}{2} e^{-\mu_6(s-s^*)} \text{ and } |\theta'(s)| + |\lambda'(s)| \leq K_4^2 \frac{\epsilon^2}{4} e^{-2\mu_6(s-s^*)}.$$

Hence, there is $\theta_\infty, \lambda_\infty \in \mathbb{R}$ such that $\theta(s) \rightarrow \theta_\infty, \lambda(s) \rightarrow \lambda_\infty$ as $s \rightarrow \infty$ and

$$(162) \quad \forall s \geq s^*, |\lambda_\infty - \lambda(s)| \leq C_1 \epsilon^{*2} e^{-2\mu_6(s-s^*)} = C_2 \epsilon^2 e^{-2\mu_6(s-s^*)},$$

$$(163) \quad \forall s \geq s^*, |\theta_\infty - \theta(s)| \leq C_1 \epsilon^{*2} e^{-2\mu_6(s-s^*)} = C_2 \epsilon^2 e^{-2\mu_6(s-s^*)},$$

for some positive C_1 and C_2 . Taking $s = s^*$ here, and using (154) and (153), we see that $|\lambda_\infty - \lambda^*| + |\theta_\infty - \theta^*| \leq C_0 \epsilon^*$. If $\delta_\infty = \tanh \lambda_\infty$, then we see that $|\delta_\infty - \delta^*| \leq C_3(1 - \delta^{*2})\epsilon^*$.

Using the definition of q (73), (79), (161), (162) and (163) we write

$$\begin{aligned} & \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - e^{i\theta_\infty} \begin{pmatrix} \kappa(\delta_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \\ & \leq \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - e^{i\theta(s)} \begin{pmatrix} \kappa(\delta(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} + \|e^{i\theta(s)}(\kappa(\delta(s), \cdot) - \kappa(\delta_\infty, \cdot))\|_{\mathcal{H}_0} \\ & + \|\kappa(\delta_\infty, \cdot)\|_{\mathcal{H}_0} |e^{i\theta(s)} - e^{i\theta_\infty}| \\ & \leq \|q(s)\|_{\mathcal{H}} + C|\lambda_\infty - \lambda(s)| + C|\theta_\infty - \theta(s)| \leq C_4 \epsilon^* e^{-\mu_6(s-s^*)}. \end{aligned}$$

This concludes the proof of Theorem 4.

APPENDIX A. ENERGY ESTIMATES IN SIMILARITY VARIABLES

For the sake of completeness, we give in this section sketches of the proofs of Propositions 1 and 3 proved in the real case in [32] and [33], and which extend to the complex case straightforwardly.

Sketch of the proof of Proposition 1. Consider $x_0 \in \mathbb{R}$. If $T(x_0) = \min_{x \in \mathbb{R}} T(x)$, the proof is given in [30]. If not, then we have to use a geometrical covering argument in addition to the ideas of [30]. In order to keep this sketch to a reasonable length, we don't mention this covering argument and refer the reader to [32]. Thus, we only focus on the real case where

$$T_0 \equiv T(x_0) = \min_{x \in \mathbb{R}} T(x).$$

• *The lower bound:*

Note first that the lower bound follows from the finite speed of propagation and scaling. Indeed, if $(w, \partial_s w)$ is small in $H^1 \times L^2$ at some time $s = s_0$, then using back the similarity variables transformation (4), we see that initial data for $(u, \partial_t u)$ is small on the basis of the light cone, which means that the solution cannot blow-up at time T . See Remark after Theorem 1 on page 1149 in [30].

• *The upper bound:*

The proof is performed in similarity variables and relies on two arguments:

- The fact that the functional $E(w)$ defined in (9) is a Lyapunov functional for equation (5) which satisfies

$$\frac{d}{ds} E(w(s)) = -\frac{4}{p-1} \int_{-1}^1 |\partial_s w|^2 \frac{\rho}{1-y^2} dy.$$

- A blow-up criterion from Antonini and Merle [3] stating that a solution w of the equation cannot be defined for all $s \in [s_0, +\infty)$ if $E(w(s_0)) < 0$.

From these two facts, we see that for any $\bar{x} \in \mathbb{R}$, $w_{\bar{x}}$ satisfies

$$(164) \quad \forall s \geq -\log T_0, \begin{cases} 0 \leq E(w_{\bar{x}}(s)) \leq C_0, \\ \int_{-\log T_0}^{+\infty} \int_{|y|<1} |\partial_s w|^2 \frac{\rho}{1-y^2} \leq C_0. \end{cases}$$

With this identity, the proof is done in three steps:

Step 1. Multiplying (5) by $\bar{w}\rho$ and integrating for $x \in (-1, 1)$, we obtain a new identity. Combining that identity with (164) we end up proving that

$$(165) \quad \forall \bar{x} \in \mathbb{R}, \forall s \leq -\log T_0 + 1, \int_s^{s+1} \int_{|y|<\frac{1}{2}} (|\partial_s w_{\bar{x}}|^2 + |\partial_y w_{\bar{x}}|^2 (1-|y|^2) + |w_{\bar{x}}|^{p+1}) dy \leq C_0.$$

Note that it is important to get (164) and (165) for any $\bar{x} \in \mathbb{R}$ and not just for $\bar{x} = x_0$.

Step 2. Using interpolation, Sobolev embeddings and a covering argument we end up with the fact that

$$(166) \quad \forall \bar{x} \in \mathbb{R}, \forall s \geq -\log T_0 + 1, \int_{|y|<1} |w_{\bar{x}}|^{p+1} dy \leq C_0.$$

Step 3. Given $s \geq -\log T_0 + 1$, we need to work at $\tilde{x} = \tilde{x}(s)$ such that

$$(167) \quad \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 dy \geq \frac{1}{2} \sup_{x \in \mathbb{R}} \int_{|y|<1} |\nabla w_{x(s)}(y, s)|^2 dy.$$

Thanks to a covering technique, we see that at such an $\tilde{x}(s)$, we have the equivalence of pure and weighted L^2 norm of the gradient, in the sense that

$$\begin{aligned} \frac{1}{C_0} \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 dy &\leq \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 \rho(1-y^2) dy \\ &\leq C_0 \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 dy. \end{aligned}$$

This equivalence of norms is really crucial to finish the proof. Then we need the following and the Gagliardo-Nirenberg inequality (see Proposition 3.2 on page 1158 in [30]):

$$(168) \quad \int_{|y|<1} |w|^{P+1} dy \leq C \left(\int_{|y|<1} (\partial_y w)^2 dy \right)^\beta \quad \text{for some } \beta < 1.$$

Indeed, thanks to interpolation estimates in Sobolev spaces and the Gagliardo-Nirenberg inequality (168), we use the energy boundedness (164) to derive that

$$\forall s \geq -\log T_0 + 1, \quad \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 dy \leq K(C_0)$$

for some $K > 0$. Using (167), we see that

$$\forall s \geq -\log T_0 + 1, \forall \tilde{x} \in \mathbb{R}, \quad \int_{|y|<1} |\nabla w_{\tilde{x}(s)}(y, s)|^2 dy \leq 2K(C_0).$$

Using the definition of the functional $E(w)$ (9) and a covering argument together with (166), we conclude the proof of the upper bound of Proposition 1. For details, see [30] and [32].

Sketch of the proof of Proposition 3. The idea is simple: equation (5) has a Lyapunov functional $E(w)$ defined in (9) which satisfies

$$(169) \quad \forall s \geq -\log T(x_0), \quad \frac{d}{ds} E(w_{x_0}(s)) = -\frac{4}{p-1} \int_{-1}^1 (\partial_s w_{x_0})^2 \frac{\rho}{1-y^2} dy.$$

From (164), it follows that

$$(170) \quad \int_{-\log T(x_0)}^{+\infty} \int_{|y|<1} (\partial_s w)^2 \frac{\rho}{1-y^2} dy ds \leq E(w_{x_0}(-\log T(x_0))) \leq C_0.$$

From this identity, we see that for any $\epsilon > 0$

$$\int_s^{s+1} \int_{|y|<1-\epsilon} (\partial_s w)^2 \frac{\rho}{1-y^2} dy ds \rightarrow 0 \quad \text{as } s \rightarrow +\infty.$$

This means that $\partial_s w \rightarrow 0$, in a certain sense, which means that w would approach the set of stationary solutions. From the lower bound in Proposition 1, w cannot approach the zero solutions. Since the set of non-zero solutions of (5) is given by $\{0, e^{i\theta} \kappa(\delta, \cdot), |\delta| < 1, \theta \in \mathbb{R}\}$, we get the conclusion. For the actual proof and for details, see Theorem 2 on page 47 in [33].

APPENDIX B. EXPLICIT SOLUTION OF THE ELLIPTIC EQUATION IN ONE SPACE DIMENSION IN H^1

Here we solve equation (27), deriving the well-known KDV solutions. Our aim is to prove (28). Multiplying equation (27) by W' and integrating in space we have

$$(171) \quad \frac{1}{2}W'^2 - \frac{c}{2}W^2 + \frac{|W|^{p+1}}{p+1} = K.$$

As $W \in H^1$, all the terms of the left hand side of (171) are integrable, so $K = 0$. We claim that

$$(172) \quad \exists \xi_0 \in \mathbb{R}, W'(\xi_0) = 0,$$

otherwise, if

$$(173) \quad \forall \xi \in \mathbb{R}, W'(\xi) \neq 0,$$

W would be monotonic, with limits (in $\bar{\mathbb{R}}$) at $\pm$ infinity. Since $W \in L^2$, those limits have to be zero, leading to $W \equiv 0$, contradicting (173). Thus (172) holds, and from (171), we see that either

$$(174) \quad W(\xi_0) = 0 \text{ or } W(\xi_0) = \pm \kappa_0,$$

given in (10). Since we already know two solutions satisfying (172) and (174), namely,

$$W \equiv 0 \text{ or } W(\xi) = \check{k}(\xi - \xi_0),$$

this concludes the proof of (28).

APPENDIX C. BASIC PROPERTIES AND SOME RESULTS

In the following, we recall some results which we have used in this work. We first give the boundedness for E .

Proposition C.1 (Boundedness of the Lyapunov functional for equation (5)). *Consider $w(y, s)$ a solution to (5) defined for all $(y, s) \in (-1, 1) \times [-\log T, +\infty)$ such that*

$$(w, \partial_s w)(-\log T) \in H^1 \times L^2(-1, 1).$$

For all $s \geq -\log T$, we have

$$0 \leq E(w(s), \partial_s w(s)) \leq E(w(-\log T), \partial_s w(-\log T)),$$

and

$$\int_{-\log T}^{\infty} \int_{-1}^1 (\partial_s w(y, s))^2 \frac{\rho(y)}{1-y^2} dy ds \leq \frac{p-1}{4} E(w(-\log T), \partial_s w(-\log T)).$$

Proof. See Antonini and Merle [3].

These following properties have been cited and proved in Section 2 in [33]. We first give Hardy-Sobolev identities in the space $\mathcal{H}_0$ (8).

Lemma C.2 (A Hardy-Sobolev type identity). *For all $h \in \mathcal{H}_0$, it holds that*

$$\left(\int_{-1}^1 h(y)^2 \frac{\rho(y)}{1-y^2} dy \right)^{1/2} + \|h\|_{L^{p+1}} + \|h(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} \leq C \|h\|_{\mathcal{H}_0}.$$

The operator $\mathcal{L}$ introduced in equation (5) has the following properties:

Proposition C.3 (Properties of the operator $\mathcal{L}$ (6)). *The operator $\mathcal{L}$ is self-adjoint in L_ρ^2 . For each $n \in \mathbb{N}$, there exists a polynomial h_n of degree n such that*

$$(175) \quad \mathcal{L}h_n = \gamma_n h_n \text{ where } \gamma_n = -n \left(n + \frac{p+3}{p-1} \right).$$

The family $\{h_n | n \in \mathbb{N}\}$ is orthogonal and spans the whole space L_ρ^2 . When $n = 0$ and $n = 1$, the eigenfunctions are $h_0 = c_0$ and $h_1 = c_1 y$ for some positive c_0 and c_1 , and

$$\mathcal{L}c_0 = 0, \mathcal{L}c_1 y = -\frac{2(p+1)}{p-1} c_1 y.$$

We also claim the following:

Lemma C.4. *Consider $u \in L_\rho^2$ such that $\mathcal{L}u \in L_\rho^2$ and*

$$\int_{-1}^1 u(y) \rho(y) dy = 0.$$

Then, $\int_{-1}^1 u \mathcal{L}u \rho dy \leq \gamma_1 \int u^2 \rho dy$, where $\gamma_1 = -2\frac{p+1}{p-1}$.

Using the Lorentz transform we get a one-dimensional group which keeps invariant equation (5):

Lemma C.5 (The Lorentz transform in similarity variables). *Consider $w(y, s)$ a solution of equation (5) defined for all $|y| < 1$ and $s \in (s_0, s_1)$ for some s_0 and s_1 in $\mathbb{R}$, and introduce for any $\delta \in (-1, 1)$, the function $W \equiv \mathcal{T}_\delta(w)$ defined by*

$$(176) \quad W(Y, S) = \frac{(1 - \delta^2)^{\frac{1}{p-1}}}{(1 + \delta Y)^{\frac{2}{p-1}}} w(y, s), \text{ where } y = \frac{Y + \delta}{1 + \delta Y} \text{ and } s = S - \log \frac{1 + \delta Y}{\sqrt{1 - \delta^2}}.$$

Then $W(Y, S) = \mathcal{T}_\delta(w)$ is also a solution of (5) defined

$$\text{for all } |Y| < 1 \text{ and } S \in \left(s_0 + \frac{1}{2} \log \frac{1 + |\delta|}{1 - |\delta|}, s_1 - \frac{1}{2} \log \frac{1 + |\delta|}{1 - |\delta|} \right).$$

In the following, we recall from [33] some properties of the transformation $\mathcal{T}_\delta$:

Lemma C.6 (Continuity of $\mathcal{T}_\delta$). *There exists $C_0 > 0$ such that for all $\delta \in (-1, 1)$ and $v \in \mathcal{H}_0$, we have*

(i) (Continuity of $\mathcal{T}_\delta$ in $\mathcal{H}_0$)

$$\frac{1}{C_0} \|v\|_{\mathcal{H}_0} \leq \|\mathcal{T}_\delta(v)\|_{\mathcal{H}_0} \leq C_0 \|v\|_{\mathcal{H}_0}.$$

(ii) For any V_1 and V_2 in L_ρ^2 , we have

$$\int_{-1}^1 V_1(y) V_2(y) \rho(y) dy = \int_{-1}^1 \frac{1 - \delta^2}{(1 - \delta z)^2} v_1(z) v_2(z) \rho(z) dz$$

where $v_i = \mathcal{T}_{-\delta} V_i$, $i \in \{1, 2\}$.

Proof. (i) See Lemma 2.8 on page 57 in [33].

(ii) This is a direct consequence of the change of variable (176) and the definition (6) of $\rho(y)$.

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INSTITUT GALILÉE, LABORATOIRE ANALYSE GÉOMETRIE ET APPLICATIONS, CNRS-UMR 7539,
UNIVERSITÉ PARIS 13, 99 AVENUE J.B. CLÉMENT 93430, VILLETANEUSE, FRANCE
E-mail address: `azaiez@math.univ-paris13.fr`