A SHARP SOBOLEV TRACE INEQUALITY INVOLVING THE MEAN CURVATURE ON RIEMANNIAN MANIFOLDS

TIANLING JIN AND JINGANG XIONG

ABSTRACT. Let (M, g) be a smooth compact n -dimensional Riemannian manifold with smooth boundary ∂M for $n \geq 5$. We prove a trace inequality, that is

$$\|u\|_{L^q(\partial M)}^2 \leq S \left(\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 ds_g \right) + A \|u\|_{L^r(\partial M)}^2$$

for all $u \in H^1(M)$, where $S = \frac{2}{n-2} \omega_n^{-1/(n-1)}$ with ω_n the volume of the unit sphere in $\mathbb{R}^n$, $q = \frac{2(n-1)}{n-2}$, $r = \frac{2(n-1)}{n}$, h_g is the mean curvature of ∂M , dv_g is the volume form of (M, g) , ds_g is the induced volume form on ∂M , and A is a positive constant depending only on (M, g) . This inequality is sharp in the sense that S cannot be replaced by any smaller constant, h in general cannot be replaced by any smooth function which is smaller than h at some point on ∂M , and r cannot be replaced by any smaller number.

1. INTRODUCTION

Recall that the sharp Sobolev trace inequality in the upper half space asserts

$$(1.1) \quad \left(\int_{\partial \mathbb{R}_+^n} |u|^{\frac{2(n-1)}{n-2}} dx' \right)^{\frac{n-2}{n-1}} \leq S \int_{\mathbb{R}_+^n} |\nabla u|^2 dx \quad \forall u \in C_c^1(\mathbb{R}_+^n \cup \partial \mathbb{R}_+^n),$$

where $S = \frac{2}{n-2} \omega_n^{-1/(n-1)}$, $x = (x', x_n)$, and ω_n is the volume of the unit sphere in $\mathbb{R}^n$, $n \geq 3$. The best constant and the extremal functions of (1.1) were found by Escobar [14] and Beckner [5] independently. Indeed, as pointed out in [5], (1.1) follows from the sharp fractional Sobolev inequality, because

$$(1.2) \quad \left(\int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} dx' \right)^{\frac{n-2}{n-1}} \leq S \|v\|_{\dot{H}^{1/2}(\mathbb{R}^{n-1})}^2$$

and

$$\|v\|_{\dot{H}^{1/2}(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}_+^n} |\nabla u|^2 dx,$$

where u is the harmonic extension of v in the upper half space, and the equality of (1.2) holds if and only if $v(x')$ takes the form

$$v(x') = c \left(\frac{\lambda}{1 + \lambda^2 |x' - x'_0|^2} \right)^{\frac{n-2}{2}}$$

for some $c \in \mathbb{R}$, $\lambda > 0$, and $x'_0 \in \mathbb{R}^{n-1}$, which is proved by Lieb [27].

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For any bounded smooth domain $\Omega \subset \mathbb{R}^n$, the sharp trace inequality

$$(1.3) \quad \|u\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Omega)}^2 \leq S \|\nabla u\|_{L^2(\Omega)}^2 + A(\Omega) \|u\|_{L^2(\partial\Omega)}^2 \quad \forall u \in H^1(\Omega),$$

where $A(\Omega) > 0$ depends only on Ω , was established by Adimurthi-Yadava [1] for $n \geq 5$ and by Li-Zhu [25] on Riemannian manifolds in all dimensions. The sharp Sobolev inequality

$$(1.4) \quad \|u\|_{L^{\frac{2n}{n-2}}(\Omega)}^2 \leq 2^{\frac{2}{n}} S_n \|\nabla u\|_{L^2(\Omega)}^2 + A(\Omega) \|u\|_{L^2(\partial\Omega)}^2 \quad \forall u \in H^1(\Omega),$$

where S_n is the best Sobolev constant, was initially proved by Brezis-Lieb [6] when Ω is the unit ball, and later by Adimurthi-Yadava [1] for $n \geq 5$ and by Li-Zhu [26] on Riemannian manifolds in all dimensions. We also obtained similar sharp weighted trace inequalities in [21] and some remainder terms in sharp fractional Sobolev inequalities in [20].

In this and a forthcoming paper, we would like to examine the boundary L^2 term in (1.3) and (1.4) further. The main result of the present paper is as follows.

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 5$. There exists a positive constant $A(\Omega)$ depending only on Ω such that for all $u \in H^1(\Omega)$,*

$$(1.5) \quad \left(\int_{\partial\Omega} |u|^q ds \right)^{2/q} \leq S \left(\int_{\Omega} |\nabla u|^2 dx + \frac{n-2}{2} \int_{\partial\Omega} h u^2 ds \right) + A(\Omega) \|u\|_{L^r(\partial\Omega)}^2,$$

where $q = \frac{2(n-1)}{n-2}$, $r = q' = \frac{2(n-1)}{n}$, and $h(x)$ is the mean curvature of $\partial\Omega$ at x .

The inequality (1.5) also holds on Riemannian manifolds with boundaries.

Theorem 1.2. *Let (M, g) be a smooth compact n -dimensional Riemannian manifold with smooth boundary ∂M for $n \geq 5$. Then there exists a positive constant $A(M, g)$ depending only on M, g such that for all $u \in H^1(M)$,*

$$(1.6) \quad \left(\int_{\partial M} |u|^q ds_g \right)^{2/q} \leq S \left(\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 ds_g \right) + A \|u\|_{L^r(\partial M)}^2,$$

where h_g is the mean curvature of ∂M with respect to the metric g , dv_g is the volume form of (M, g) , and ds_g is the induced volume form on ∂M .

The inequality (1.6) is sharp in the following sense. It would fail if S is replaced by any smaller constant. In general, h cannot be replaced by any smooth function which is smaller than h at some point on ∂M , and r cannot be replaced by any smaller number.

The best constant S has been demonstrated to be crucial in the study of the Yamabe problem on manifolds with boundaries; see, e.g., Escobar [13]. The effect of mean curvatures in sharp Hardy-Sobolev inequalities with a singularity on the boundary has been studied by Ghoussoub-Robert [15]. In [21], we proved a sharp weighted Sobolev trace inequality on Riemannian manifolds which would fail if the mean curvature is positive somewhere. The effect of scalar curvatures for sharp Sobolev inequalities on compact manifolds has been studied by Li-Ricciardi [23], Hebey [18], and references therein.

The sharp Sobolev inequality (1.6) is in the same spirit of a conjecture posed by Aubin [2], which has been confirmed through the work of Hebey-Vaugon [19], Aubin-Li [3], and Druet [10, 11]; see also Druet-Hebey [12] and Hebey [17]. The

procedure to prove those types of inequalities is by contradiction. A key point is to derive the asymptotical behavior of extremal functions near their energy concentration points. However, to establish (1.6) requires more precise estimates, in particular the error estimates between the extremal functions and some properly chosen bubbles.

2. PRELIMINARIES

We prove inequality (1.6) by contradiction. Suppose that for every large $\alpha > 1$, there exists $\tilde{u} \in H^1(M)$ such that

$$\left(\int_{\partial M} |\tilde{u}|^q ds_g \right)^{2/q} > S \left(\int_M |\nabla_g \tilde{u}|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g \tilde{u}^2 \right) + \alpha \|\tilde{u}\|_{L^r(\partial M, g)}^2.$$

Define

$$I_\alpha(u) = \frac{\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 + \alpha \|u\|_{L^r(\partial M, g)}^2}{\|u\|_{L^q(\partial M)}^2}, \quad \forall u \in H^1(M) \setminus H_0^1(M).$$

It follows from the contradiction hypothesis that for all large α ,

$$(2.1) \quad \ell_\alpha := \inf_{H^1(M) \setminus H_0^1(M)} I_\alpha < \frac{1}{S}.$$

Even though the functional I_α involves a critical Sobolev exponent, the above strict inequality implies the existence of a minimizer, namely,

Proposition 2.1. *For every $\alpha > 0$, ℓ_α is achieved by a nonnegative $u_\alpha \in H^1(M) \setminus H_0^1(M)$ with*

$$\int_{\partial M} u_\alpha^q ds_g = 1.$$

Consequently, $u_\alpha \in C^\infty(M) \cap C^{1,r-1}(\overline{M})$ satisfies the Euler-Lagrange equation

$$(2.2) \quad \begin{cases} -\Delta_g u_\alpha = 0 & \text{on } M, \\ \frac{\partial_g u_\alpha}{\partial \nu} + \frac{n-2}{2} h_g u_\alpha + \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} u_\alpha^{r-1} = \ell_\alpha u_\alpha^{q-1} & \text{on } \partial M, \end{cases}$$

where $\partial_g/\partial \nu$ denotes the differentiation in the direction of the unit outer normal of ∂M with respect to g .

Proof. The existence of a minimizer follows from the standard subcritical exponent approximating method and Moser's iteration argument; see, e.g., Proposition 2.1 of [13]. A calculus of variations argument shows that u_α is a weak solution of the Euler-Lagrange equation (2.2). Finally, $u_\alpha \in C^\infty(M) \cap C^{1,r-1}(\overline{M})$ follows from the regularity theory in [8]. $\square$

Lemma 2.1. *For every $\varepsilon > 0$, there exists a constant $B(\varepsilon) > 0$ depending on ε such that*

$$\left(\int_{\partial M} |u|^q ds_g \right)^{2/q} \leq (S + \varepsilon) \int_M |\nabla_g u|^2 dv_g + B(\varepsilon) \|u\|_{L^r(\partial M, g)}^2,$$

for all $u \in H^1(M)$.

Proof. By compactness, we have that for every $\varepsilon > 0$ there exists a positive constant $\tilde{B}(\varepsilon)$ such that

$$(2.3) \quad \int_{\partial M} u^2 ds_g \leq \varepsilon \int_M |\nabla_g u|^2 dv_g + \tilde{B}(\varepsilon) \|u\|_{L^r(\partial M, g)}^2.$$

Hence, the proposition follows from the sharp trace inequality in Theorem 0.1 in [25]. $\square$

Note that

$$\begin{aligned} I_\alpha(u_\alpha) &= \int_M |\nabla_g u_\alpha|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u_\alpha^2 + \alpha \|u_\alpha\|_{L^r(\partial M, g)}^2 \\ &\geq (1 - \varepsilon \max_{\partial M} |h_g|) \int_M |\nabla_g u_\alpha|^2 dv_g + (\alpha - B(\varepsilon)) \|u_\alpha\|_{L^r(\partial M, g)}^2. \end{aligned}$$

This implies

$$\int_M |\nabla_g u_\alpha|^2 dv_g \leq \frac{2}{S}$$

and

$$\|u_\alpha\|_{L^r(\partial M, g)} \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty.$$

It follows that $u_\alpha \rightarrow \bar{u}$ in $H^1(M)$ for some $\bar{u} \in H_0^1(M)$. In fact, $\bar{u} = 0$, *a.e.* on M . This is because u_α is harmonic on M and thus it satisfies (see Proposition 2.1 in [16])

$$\|u_\alpha\|_{L^{\frac{nr}{n-r}}(M, g)} \leq C(M, g, n) \|u_\alpha\|_{L^r(\partial M, g)},$$

and $u_\alpha \rightarrow \bar{u}$ in $L^{\frac{nr}{n-r}}(M)$.

We claim that, as $\alpha \rightarrow \infty$,

$$(2.4) \quad \ell_\alpha \rightarrow \frac{1}{S}$$

and

$$(2.5) \quad \alpha \|u_\alpha\|_{L^r(\partial M, g)}^2 \rightarrow 0.$$

Indeed, by Lemma 2.1 and (2.3) for every $\varepsilon > 0$

$$\begin{aligned} 1 &\leq (S + \varepsilon) \int_M |\nabla_g u_\alpha|^2 dv_g + B(\varepsilon) \|u_\alpha\|_{L^r(\partial M, g)}^2 \\ &\leq (S + \varepsilon) \ell_\alpha + (2B(\varepsilon) - \alpha S) \|u_\alpha\|_{L^r(\partial M, g)}^2. \end{aligned}$$

Thus

$$\frac{1}{S + \varepsilon} \leq \ell_\alpha < \frac{1}{S}, \quad \frac{1}{2} \alpha S \|u_\alpha\|_{L^r(\partial M, g)}^2 \leq (S + \varepsilon) \frac{1}{S} - 1 = \frac{\varepsilon}{S},$$

if $\alpha S > 4B(\varepsilon)$. Hence, the claim follows.

By the maximum principle, there exists a point $x_\alpha \in \partial M$ such that

$$u_\alpha(x_\alpha) = \max_{\bar{M}} u_\alpha > 0.$$

In view of (2.5) and

$$1 = \int_{\partial M} u_\alpha^q \leq u_\alpha(x_\alpha)^{q-r} \int_{\partial M} u_\alpha^r,$$

we have $u_\alpha(x_\alpha) \rightarrow \infty$. Set $\mu_\alpha := u_\alpha(x_\alpha)^{-2/(n-2)}$. Then $\mu_\alpha \rightarrow 0$ as $\alpha \rightarrow \infty$.

Let $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ be the *Fermi coordinate* (see, e.g., [13]) at x_α , where $(x_1, \dots, x_{n-1})$ are normal coordinates on ∂M at x_α and $\gamma(x_n)$ is the

geodesic leaving from $(x_1, \cdot, x_{n-1})$ in the orthogonal direction to ∂M and parameterized by arc length. In this coordinate system,

$$\sum_{1 \leq i, j \leq n} g_{ij}(x) dx_i dx_j = dx_n^2 + \sum_{1 \leq i, j \leq n-1} g_{ij}(x) dx_i dx_j.$$

Moreover, g^{ij} has the following Taylor expansion near ∂M :

Lemma 2.2 (Lemma 3.2 in [13]). *For $\{x_k\}_{k=1, \dots, n}$ small,*

$$(2.6a) \quad g^{ij}(x) = \delta^{ij} + 2h^{ij}(x', 0)x_n + O(|x|^2),$$

$$(2.6b) \quad g^{ij}\Gamma_{ij}^k = O(|x|),$$

where $i, j = 1, \dots, n-1$ and h_{ij} is the second fundamental form of ∂M .

To proceed, we introduce some notation.

- For a domain $D \subset \mathbb{R}^n$ with boundary ∂D , we denote $\partial' D$ as the interior of $\overline{D} \cap \partial \mathbb{R}_+^n$ in $\mathbb{R}^{n-1} = \partial \mathbb{R}_+^n$ and $\partial'' D = \partial D \setminus \partial' D$.
- For $\bar{x} \in \mathbb{R}^n$, $B_r(\bar{x}) := \{x \in \mathbb{R}^n : |x - \bar{x}| < r\}$ and $B_r^+(\bar{x}) := B_r(\bar{x}) \cap \mathbb{R}_+^n$, where $|\cdot|$ is the Euclidean distance. We will not keep writing the center $\bar{x}$ if $\bar{x} = 0$.

For suitably small $\delta_0 > 0$ (independent of α), we define v_α in a neighborhood of $x_\alpha = 0$ by

$$v_\alpha(y) = \mu_\alpha^{(n-2)/2} u_\alpha(\mu_\alpha y), \quad x \in B_{\delta_0/\mu_\alpha}^+,$$

in the Fermi coordinate mentioned above. Then v_α satisfies

$$(2.7) \quad \begin{cases} \Delta_{g_\alpha} v_\alpha = 0, & \text{in } B_{\delta_0/\mu_\alpha}^+, \\ \frac{\partial_{g_\alpha} v_\alpha}{\partial \nu} + \frac{n-2}{2} h_\alpha v_\alpha + \epsilon_\alpha v_\alpha^{r-1} = \ell_\alpha v_\alpha^{q-1}, & \text{on } \partial' B_{\delta_0/\mu_\alpha}^+, \\ v_\alpha(0) = 1, \quad 0 \leq v_\alpha \leq 1, \end{cases}$$

where $g_\alpha(x) = g_{ij}(\mu_\alpha x) dx_i dx_j$, h_α is the mean curvature of $\partial' B_{\delta_0/\mu_\alpha}$ with respect to the metric g_α , and

$$(2.8) \quad \epsilon_\alpha := \alpha \mu_\alpha^{n-1-\frac{n-2}{2}r} \|u_\alpha\|_{L^r(\partial M)}^{2-r}.$$

Note that

$$(2.9) \quad \epsilon_\alpha = \frac{\alpha \|u_\alpha\|_{L^r(\partial M)}^2}{(\max_{\overline{M}} u_\alpha)^{q-r} \int_{\partial M} u_\alpha^r ds_g} \leq \alpha \|u_\alpha\|_{L^r(\partial M)}^2 \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty,$$

because $1 = \int_{\partial M} u_\alpha^q ds_g \leq (\max_{\overline{M}} u_\alpha)^{q-r} \int_{\partial M} u_\alpha^r ds_g$. By the standard elliptic equations theory, for all $R > 1$,

$$(2.10) \quad \|v_\alpha\|_{C^{1,r-1}(\overline{B_R^+})} \leq C(R) \quad \text{for all sufficiently large } \alpha.$$

Therefore, $v_\alpha \rightarrow U$ in $C_{loc}^1(\overline{\mathbb{R}_+^n})$ for some $U \in C_{loc}^{1,r-1}(\overline{\mathbb{R}_+^n}) \cap C^\infty(\mathbb{R}_+^n)$ which satisfies

$$(2.11) \quad \begin{cases} -\Delta U = 0, & \text{in } \mathbb{R}_+^n, \\ -\partial_{y_n} U = \frac{1}{S} U^{q-1}, & \text{on } \partial \mathbb{R}_+^n, \\ U(0) = 1, \quad 0 \leq U \leq 1. \end{cases}$$

By the Liouville type theorem in [24],

$$(2.12) \quad U(y', y_n) = \left(\frac{(n-2)^2 S^2}{|y'|^2 + (y_n + (n-2)S)^2} \right)^{\frac{n-2}{2}},$$

where $y' = (y_1, \dots, y_{n-1})$. Denote $\lambda_0 = (n-2)S$ and

$$\begin{aligned} U_{x_0, \lambda}(x', x_n) &:= \lambda^{-\frac{n-2}{2}} U((x' - x'_0)/\lambda, x_n/\lambda) \\ &= \lambda_0^{\frac{n-2}{2}} \left(\frac{\lambda \lambda_0}{|x' - x'_0|^2 + (x_n + \lambda \lambda_0)^2} \right)^{\frac{n-2}{2}}, \end{aligned}$$

where $x_0 \in \partial \mathbb{R}^n$ and $\lambda > 0$. For brevity, we write U_λ as $U_{0, \lambda}$. Hence, $U_1 = U$.

Proposition 2.2. *For every $\delta > 0$,*

$$\lim_{\alpha \rightarrow \infty} \left(\int_{B_\delta^+} |\nabla_g(u_\alpha - U_{\mu_\alpha})|^2 dv_g + \int_{\partial' B_\delta^+} |u_\alpha - U_{\mu_\alpha}|^q ds_g \right) = 0.$$

Proof. We only prove that

$$\lim_{\alpha \rightarrow \infty} \int_{B_\delta^+} |\nabla_g(u_\alpha - U_{\mu_\alpha})|^2 dv_g = 0,$$

and the other can be proved similarly.

For every given $\varepsilon > 0$, one can find $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$,

$$\int_M |\nabla_g u_\alpha|^2 dv_g \leq \frac{1}{S} + \varepsilon,$$

because $\lim_{\alpha \rightarrow \infty} \int_M |\nabla_g u_\alpha|^2 dv_g = \frac{1}{S}$. Since $\int_{\mathbb{R}_+^n} |\nabla U|^2 = \frac{1}{S}$, we can choose $R > 0$ such that

$$\int_{\mathbb{R}_+^n \setminus B_R^+} |\nabla U|^2 \leq \varepsilon.$$

Note that

$$\int_{B_{\delta/\mu_\alpha}^+} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} = \int_{B_\delta^+} |\nabla_g u_\alpha|^2 dv_g \leq \frac{1}{S} + \varepsilon.$$

Hence

$$\begin{aligned} & \int_{B_\delta^+} |\nabla_g(u_\alpha - U_{\mu_\alpha})|^2 dv_g \\ &= \int_{B_{\delta/\mu_\alpha}^+} |\nabla_{g_\alpha}(v_\alpha - U)|^2 dv_{g_\alpha} \\ &= \int_{B_R^+} |\nabla_{g_\alpha}(v_\alpha - U)|^2 dv_{g_\alpha} + \int_{B_{\delta/\mu_\alpha}^+ \setminus \overline{B}_R^+} |\nabla_{g_\alpha}(v_\alpha - U)|^2 dv_{g_\alpha} \\ &\leq 2\varepsilon + 2 \int_{B_{\delta/\mu_\alpha}^+ \setminus \overline{B}_R^+} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} + 2 \int_{B_{\delta/\mu_\alpha}^+ \setminus \overline{B}_R^+} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \\ &\leq 10\varepsilon, \end{aligned}$$

where we used $\|v_\alpha - U\|_{C^1(\overline{B}_R^+)} \leq \varepsilon$ for large α and

$$\begin{aligned} & \int_{B_{\delta/\mu_\alpha}^+ \setminus \overline{B}_R^+} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} + \int_{B_{\delta/\mu_\alpha}^+ \setminus \overline{B}_R^+} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \\ &\leq \frac{1}{S} + \varepsilon - \int_{B_R^+} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} + \frac{1}{S} + \varepsilon - \int_{B_R^+} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \\ &\leq 4\varepsilon. \end{aligned}$$

□

Proposition 2.3. *For all large α ,*

$$u_\alpha(x) \leq C\mu_\alpha^{\frac{n-2}{2}} \text{dist}_g(x, x_\alpha)^{2-n} \quad \text{for all } x \in \overline{M},$$

where $C > 0$ depends only on M, g .

Let

$$b_\alpha := \begin{cases} \min\{\frac{n-2}{2}h_g + \alpha\left(\frac{\|u_\alpha\|_{L^r}}{u_\alpha}\right)^{2-r}, 1\}, & \text{if } u_\alpha \neq 0, \\ 1, & \text{if } u_\alpha = 0. \end{cases}$$

To prove Proposition 2.3, we need the following lemma about *Neumann functions*.

Lemma 2.3. *There exists a unique weak solution G_α of*

$$(2.13) \quad \begin{cases} -\Delta_g G_\alpha = 0 & \text{on } M, \\ \frac{\partial_g G_\alpha}{\partial \nu} + b_\alpha G_\alpha = \delta_{x_\alpha} & \text{on } \partial M, \end{cases}$$

satisfying $G_\alpha \in W^{1,p}(M)$ for $1 \leq p < \frac{n}{n-1}$, where δ_{x_α} is the delta measure centered at x_α . Moreover, $G_\alpha \in C_{loc}^1(\overline{M} \setminus \{x_\alpha\})$ and

$$(2.14) \quad C^{-1} \text{dist}_g(x, x_\alpha)^{2-n} \leq G_\alpha(x) \leq C \text{dist}_g(x, x_\alpha)^{2-n} \quad \text{for all } x \in \overline{M},$$

where $C > 0$ depends only on M, g .

Proof. We claim that

$$(2.15) \quad \lim_{\alpha \rightarrow \infty} \text{vol}_g\{b_\alpha < \frac{1}{2}\} = 0.$$

Indeed, for every measurable set $E \subset \subset \partial M \cap \{u_\alpha > 0\}$, we have

$$\text{vol}_g(E) = \int_E \text{d}s_g = \int_E u_\alpha^{r/2} u_\alpha^{-r/2} \text{d}s_g \leq \|u_\alpha\|_{L^r(E)}^{r/2} \|u_\alpha^{-1}\|_{L^r(E)}^{r/2}.$$

It follows that

$$\begin{aligned} \|(\|u_\alpha\|_{L^r(\partial M)} u_\alpha^{-1})^{2-r}\|_{L^{r/(2-r)}(E)} &= \|u_\alpha\|_{L^r(\partial M)}^{2-r} \|u_\alpha^{-1}\|_{L^r(E)}^{2-r} \\ &\geq (\text{vol}_g(E))^{2(2-r)/r}. \end{aligned}$$

Note that

$$\alpha(\|u_\alpha\|_{L^r(\partial M)} u_\alpha^{-1})^{2-r} < \frac{1}{2}(1 + (n-2)|h_g|),$$

if $b_\alpha < 1/2$. Since $\{b_\alpha < 1/2\} \subset \subset \partial M \cap \{u_\alpha > 0\}$,

$$(\text{vol}_g\{b_\alpha < \frac{1}{2}\})^{2(2-r)/r} \leq \frac{C}{\alpha}.$$

We verified the claim.

Notice that b_α is uniformly bounded and Lipschitz. Thus,

$$\begin{aligned}
 & \int_M |\nabla_g u|^2 dv_g + \int_{\partial M} b_\alpha u^2 ds_g \\
 & \geq \int_M |\nabla_g u|^2 dv_g + \frac{1}{2} \int_{\partial M} u^2 ds_g - \int_{\partial M} (b_\alpha - \frac{1}{2})^- u^2 ds_g \\
 & \geq \int_M |\nabla_g u|^2 dv_g + \frac{1}{2} \int_{\partial M} u^2 ds_g - \|(b_\alpha - \frac{1}{2})^-\|_{L^{n-1}(\partial M)} \|u\|_{L^q(\partial M)}^2 \\
 & \geq \int_M |\nabla_g u|^2 dv_g + \frac{1}{2} \int_{\partial M} u^2 ds_g \\
 & \quad - C \|(b_\alpha - \frac{1}{2})^-\|_{L^{n-1}(\partial M)} \left(\int_M |\nabla_g u|^2 dv_g + \int_{\partial M} u^2 ds_g \right) \\
 & \geq \frac{1}{4} \int_M |\nabla_g u|^2 dv_g + \frac{1}{4} \int_{\partial M} u^2 ds_g,
 \end{aligned}$$

where we have used Theorem 0.1 in [25] and (2.15). Thus, it is coercive. Then it follows from standard elliptic theory (see, e.g., [7, 22]) that there exists a unique solution of (2.13) in $W^{1,p}(M)$ for $1 \leq p < \frac{n}{n-1}$ satisfying (2.14). Note that the uniqueness is actually not needed in the rest of the paper. $\square$

Proof of Proposition 2.3. Set $\varphi_\alpha = \mu_\alpha^{(n-2)/2} G_\alpha$, $w_\alpha = u_\alpha / \varphi_\alpha$, and $\hat{g} = \varphi_\alpha^{\frac{4}{n-2}} g$. By the conformal invariance (see, e.g., (1.8) in [13]), it is direct to verify that for α large, w_α satisfies

$$\begin{cases} \Delta_{\hat{g}} w_\alpha = 0 & \text{on } M, \\ \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} = \ell_\alpha w_\alpha^{q-1} - w_\alpha \varphi_\alpha^{-\frac{n}{n-2}} \left(\frac{\partial_g \varphi_\alpha}{\partial \nu} + \frac{n-2}{2} h_g \varphi_\alpha + \alpha \|u_\alpha\|_{L^r}^{2-r} u_\alpha^{r-2} \varphi_\alpha \right) & \text{on } \partial M \setminus \{x_\alpha\}. \end{cases}$$

By our choice of G_α , we have

$$\begin{cases} \Delta_{\hat{g}} w_\alpha = 0 & \text{on } M, \\ \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} \leq \ell_\alpha w_\alpha^{q-1} & \text{on } \partial M \setminus \{x_\alpha\}. \end{cases}$$

Then the Moser iterations procedure on pp. 465-471 of [25] implies that

$$\|w_\alpha\|_{L^\infty(M \setminus B_{\mu_\alpha}(x_\alpha))} \leq C.$$

Recall that $v_\alpha \rightarrow U$ in C_{loc}^2 , from which we also have $\|w_\alpha\|_{L^\infty(B_{\mu_\alpha}(x_\alpha))} \leq C$. Proposition 2.3 follows immediately. $\square$

Corollary 2.1. *For any small $\delta > 0$, there exists a constant $C > 0$ depending only on M, g, δ such that*

$$\int_{M \setminus B_{\delta/2}(x_\alpha)} |\nabla_g u_\alpha|^2 \leq C \mu_\alpha^{n-2},$$

where $B_{\delta/2}(x_\alpha)$ is centered at x_α with radius $\delta/2$. Consequently, we can select $\delta_\alpha \in [\delta/2, \delta]$ such that

$$\int_{\partial B_{\delta_\alpha}(x_\alpha)} |\nabla_g u_\alpha|^2 \leq C \mu_\alpha^{n-2}.$$

Proof. Let η be a cut-off function satisfying $\text{Supp}(\eta) \subset M \setminus B_{\delta/2}(x_\alpha)$ and $\eta \equiv 1$ in $M \setminus B_\delta(x_\alpha)$. Multiplying the Euler-Lagrange equation (2.2) by $\eta^2 u_\alpha$ and integrating by parts, we have

$$\int_M \nabla_g u_\alpha \nabla_g (\eta^2 u_\alpha) dv_g \leq -\frac{n-2}{2} \int_{\partial M} h \eta^2 u_\alpha^2 ds_g + \ell_\alpha \int_{\partial M} \eta^2 u_\alpha^q ds_g.$$

It follows that

$$\int_M \eta^2 |\nabla_g u_\alpha|^2 dv_g \leq C \int_{\partial M} (\eta^2 u_\alpha^2 + \eta^2 u_\alpha^q) ds_g + C \int_M |\nabla_g \eta|^2 u_\alpha^2 dv_g.$$

Therefore, this corollary follows immediately from Proposition 2.3. $\square$

3. ENERGY ESTIMATES AND PROOF OF THEOREM 1.2

For some small δ_0 to be determined in Lemma 3.4, let $\psi_\alpha \in C^\infty(\overline{M})$ satisfy $\psi_\alpha(x_\alpha) = 1$, $\frac{1}{2} \leq \psi_\alpha \leq 2$, $\|\psi_\alpha\|_{C^2(\overline{M})} \leq C$, and

$$\begin{cases} \Delta_g \psi_\alpha = 0, & \text{in } B_{2\delta_0}^+, \\ \frac{\partial_g \psi_\alpha}{\partial \nu} + \frac{n-2}{2} h_g \psi_\alpha = 0, & \text{on } \partial' B_{2\delta_0}^+. \end{cases}$$

As in the previous section, here we used the Fermi coordinate with respect to metric g centered at x_α . Set $\hat{g} = \psi_\alpha^{4/(n-2)} g$. It is easy to see that $h_{\hat{g}} = 0$ on $\partial' B_{\delta_0}^+$. Hence, u_α/ψ_α satisfies

$$(3.1) \quad \begin{cases} \Delta_{\hat{g}} \frac{u_\alpha}{\psi_\alpha} = 0, & \text{in } B_{2\delta_0}^+, \\ \frac{\partial_{\hat{g}} \frac{u_\alpha}{\psi_\alpha}}{\partial \nu} + \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \psi_\alpha^{1-q} u_\alpha^{r-1} = \ell_\alpha \left(\frac{u_\alpha}{\psi_\alpha}\right)^{q-1}, & \text{on } \partial' B_{2\delta_0}^+. \end{cases}$$

It follows that the maximum of u_α/ψ_α on $\overline{M}$ is achieved at some point on ∂M which is denoted as $\hat{x}_\alpha$. In view of the fact $u_\alpha(x_\alpha) \rightarrow \infty$ and Proposition 2.3, we have $|\hat{x}_\alpha - x_\alpha| \rightarrow 0$ and thus $\frac{u_\alpha(\hat{x}_\alpha)}{\psi_\alpha(\hat{x}_\alpha)u_\alpha(x_\alpha)} \rightarrow 1$ as $\alpha \rightarrow \infty$. From now on, we use the Fermi coordinate with respect to the metric $\hat{g}$ centered at x_α . Since $\psi_\alpha(x_\alpha) = 1$, by a simple blow-up argument and the same proof of Proposition 2.2 we can establish

$$(3.2) \quad \lim_{\alpha \rightarrow \infty} \left(\int_{B_\delta^+} |\nabla_{\hat{g}} \left(\frac{u_\alpha}{\psi_\alpha} - U_{\mu_\alpha} \right)|^2 dv_{\hat{g}} + \int_{\partial' B_\delta^+} \left| \frac{u_\alpha}{\psi_\alpha} - U_{\mu_\alpha} \right|^q ds_{\hat{g}} \right) = 0.$$

As in Corollary 2.1, we can select $\delta_\alpha \in [\delta_0/2, \delta_0]$ such that

$$(3.3) \quad \int_{\partial'' B_{\delta_\alpha}} |\nabla_{\hat{g}} \left(\frac{u_\alpha}{\psi_\alpha} \right)|^2 dv_{\hat{g}} \leq C \mu_\alpha^{n-2},$$

where $C > 0$ is independent of α . Let us focus on the upper half ball $B_{\delta_\alpha}^+(0)$, which is equipped with the Riemannian metric $\hat{g}$.

Let $h_{\xi, \lambda}(z)$, $z \in \overline{B_{\delta_\alpha}^+}(0)$, be the classical solution of

$$\begin{cases} \Delta_{\hat{g}} h_{\xi, \lambda} = 0 & \text{in } B_{\delta_\alpha}^+, \\ h_{\xi, \lambda} = U_{\xi, \lambda} & \text{on } \partial'' B_{\delta_\alpha}^+, \\ \frac{\partial_{\hat{g}} h_{\xi, \lambda}}{\partial \nu} = 0 & \text{on } \partial' B_{\delta_\alpha}^+, \end{cases}$$

with parameters $\xi \in \partial' B_{\delta_\alpha \mu_\alpha/2}^+$ and $\lambda > 0$, and let $\chi_\alpha(z)$ be the solution of

$$\begin{cases} \Delta_{\hat{g}} \chi_\alpha = 0 & \text{in } B_{\delta_\alpha}^+, \\ \chi_\alpha = \frac{u_\alpha}{\psi_\alpha} & \text{on } \partial'' B_{\delta_\alpha}^+, \\ \frac{\partial_{\hat{g}} \chi_\alpha}{\partial \nu} = 0 & \text{on } \partial' B_{\delta_\alpha}^+. \end{cases}$$

Then $u_\alpha/\psi_\alpha - \chi_\alpha \in H_{0,L}(B_{\delta_\alpha}^+)$ and $U_{\xi,\lambda} - h_{\xi,\lambda} \in H_{0,L}(B_{\delta_\alpha}^+)$ are the projections of u_α/ψ_α and $U_{\xi,\lambda}$ on $H_{0,L}(B_{\delta_\alpha}^+)$, respectively. Here

$$H_{0,L}(B_{\delta_\alpha}^+) := \{u \in H^1(B_{\delta_\alpha}^+) : u = 0 \text{ on } \partial'' B_{\delta_\alpha}^+ \text{ in trace sense}\}$$

is a Hilbert space with the inner product $\langle u, v \rangle_{\hat{g}} := \int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} u \nabla_{\hat{g}} v \, dv_{\hat{g}}$. Denote $\|u\| = \sqrt{\langle u, u \rangle_{\hat{g}}}$, which is a norm for $u \in H_{0,L}(B_{\delta_\alpha}^+)$.

Set

$$\sigma_{\xi,\lambda} = U_{\xi,\lambda} - h_{\xi,\lambda},$$

which satisfies $\sigma_{\xi,\lambda} \leq U_{\xi,\lambda}$ and

$$\begin{cases} \Delta_{\hat{g}} \sigma_{\xi,\lambda} = \Delta_{\hat{g}} U_{\xi,\lambda}, & \text{in } B_{\delta_\alpha}^+, \\ \sigma_{\xi,\lambda} = 0, & \text{on } \partial'' B_{\delta_\alpha}^+, \\ \frac{\partial_{\hat{g}} \sigma_{\xi,\lambda}}{\partial \nu} = \frac{\partial_{\hat{g}} U_{\xi,\lambda}}{\partial \nu} & \text{on } \partial' B_{\delta_\alpha}^+. \end{cases}$$

Let $(t_\alpha, \xi_\alpha, \lambda_\alpha) \in [\frac{1}{2}, \frac{3}{2}] \times \overline{\partial' B_{\delta_\alpha \mu_\alpha/2}^+} \times [\frac{\mu_\alpha}{2}, \frac{3\mu_\alpha}{2}]$ be such that

$$\begin{aligned} & \left\| \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - t_\alpha \sigma_{\xi_\alpha, \lambda_\alpha} \right\| \\ &= \min \left\{ \left\| \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - t \sigma_{\xi, \lambda} \right\| : (t, \xi, \lambda) \in \left[\frac{1}{2}, \frac{3}{2} \right] \times \overline{\partial' B_{\delta_\alpha \mu_\alpha/2}^+} \times \left[\frac{\mu_\alpha}{2}, \frac{3\mu_\alpha}{2} \right] \right\}, \end{aligned}$$

and

$$w_\alpha = \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - t_\alpha \sigma_{\xi_\alpha, \lambda_\alpha}.$$

Define

$$W_\alpha = \{w \in H_{0,L}(B_{\delta_\alpha}^+) : \langle \sigma_{\xi_\alpha, \lambda_\alpha}, w \rangle_{\hat{g}} = 0 \text{ and } \langle u, w \rangle_{\hat{g}} = 0 \text{ for all } u \in E_\alpha\},$$

where $E_\alpha \subset H_{0,L}(B_{\delta_\alpha}^+)$ is the tangent space at $\sigma_{\xi_\alpha, \lambda_\alpha}$ of the n -dimensional surface $\{\sigma_{\xi,\lambda} : \xi \in \partial' B_{\mu_\alpha \delta_\alpha}, \lambda > 0\} \subset H_{0,L}(B_{\delta_\alpha}^+)$. More explicitly,

$$E_\alpha = \text{span} \left\{ \frac{\partial \sigma_{\xi, \lambda_\alpha}}{\partial \xi} \Big|_{\xi=\xi_\alpha}, \frac{\partial \sigma_{\xi_\alpha, \lambda}}{\partial \lambda} \Big|_{\lambda=\lambda_\alpha} \right\}.$$

For brevity, we denote $h_\alpha = h_{\xi_\alpha, \lambda_\alpha}$ and $\sigma_\alpha = \sigma_{\xi_\alpha, \lambda_\alpha}$.

Lemma 3.1. *We have*

- 1) $\|\nabla_{\hat{g}} h_\alpha\|_{L^2(B_{\delta_\alpha})} + \|h_\alpha\|_{L^\infty(B_{\delta_\alpha})} \leq C \mu_\alpha^{(n-2)/2},$
- 2) $\|\nabla_{\hat{g}} \chi_\alpha\|_{L^2(B_{\delta_\alpha})} + \|\chi_\alpha\|_{L^\infty(B_{\delta_\alpha})} \leq C \mu_\alpha^{(n-2)/2},$

for some positive constant C independent of α , and

- 3) $\|w_\alpha\| \rightarrow 0,$
- 4) $t_\alpha \rightarrow 1,$
- 5) $\mu_\alpha^{-1} |\xi_\alpha| \rightarrow 0,$
- 6) $\mu_\alpha^{-1} \lambda_\alpha \rightarrow 1,$

as $\alpha \rightarrow \infty$. Furthermore, $w_\alpha \in W_\alpha$.

Proof. Let $\eta \in H^1(B_{\delta_\alpha})$ be an extension of $U_{\xi_\alpha, \lambda_\alpha}|_{\partial'' B_{\delta_\alpha}^+}$ such that

$$(3.4) \quad \|\eta\|_{H^1(B_{\delta_\alpha})}^2 \leq C \left(\int_{\partial'' B_{\delta_\alpha}^+} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 + U_{\xi_\alpha, \lambda_\alpha}^2 \right),$$

where $C > 0$ is independent of α . Multiplying the equation of h_α by $h_\alpha - \eta \in H_{0,L}(B_{\delta_\alpha})$ and integrating by parts we have

$$0 = \int_{B_{\delta_\alpha}} \nabla_{\hat{g}} h_\alpha \nabla_{\hat{g}} (h_\alpha - \eta) dv_{\hat{g}} \geq \frac{1}{2} (\|\nabla_{\hat{g}} h_\alpha\|_{L^2}^2 - \|\nabla_{\hat{g}} \eta\|_{L^2}^2).$$

Thus we obtained the L^2 estimate for $\nabla_{\hat{g}} h_\alpha$. The L^∞ estimates for h_α follow easily from the maximum principle. Hence, we verified 1). Similarly, we can verify 2) by taking into account Proposition 2.3 and (3.3).

By the definition of t_α and σ_α ,

$$\begin{aligned} \|t_\alpha \sigma_\alpha - \sigma_{0, \mu_\alpha}\| &\leq \left\| \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - t_\alpha \sigma_\alpha \right\| + \left\| \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - \sigma_{0, \mu_\alpha} \right\| \\ &\leq 2 \left\| \frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - \sigma_{0, \mu_\alpha} \right\| \\ &\leq 2 \left\| \frac{u_\alpha}{\psi_\alpha} - U_{\mu_\alpha} \right\| + C \mu_\alpha^{(n-2)/2} \rightarrow 0 \end{aligned}$$

as $\alpha \rightarrow \infty$, where we used 1), 2), and (3.2). It follows that $\|w_\alpha\| \rightarrow 0$, i.e., 3) holds, and

$$\|t_\alpha U_{\xi_\alpha, \lambda_\alpha} - U_{\mu_\alpha}\| \leq \|t_\alpha \sigma_\alpha - \sigma_{0, \mu_\alpha}\| + \|t_\alpha h_{\xi_\alpha, \lambda_\alpha}\| + \|h_{0, \mu_\alpha}\| \rightarrow 0$$

as $\alpha \rightarrow \infty$. A simple calculation yields 4), 5), and 6). Once we have 4), 5), and 6), the minimum of the norm is attained in the interior of $[\frac{1}{2}, \frac{3}{2}] \times \partial' B_{\delta_\alpha \mu_\alpha / 2}^+ \times [\frac{\mu_\alpha}{2}, \frac{3\mu_\alpha}{2}]$. Hence, a variational argument gives $w_\alpha \in W_\alpha$. $\square$

Proposition 3.1. *Assume as above that we have*

$$\begin{aligned} \|w_\alpha\| + |t_\alpha^{q-2} \ell_\alpha - S^{-1}| &\leq C \left\{ \mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2*'}(B_{\mu_\alpha^{-1}}^+)} \right. \\ &\quad \left. + \epsilon_\alpha \|U_1^{r-1}\|_{L^r(\partial' B_{\mu_\alpha^{-1}}^+)} + \mu_\alpha^{n-2} \|U_1^{q-2}\|_{L^r(\partial' B_{\mu_\alpha^{-1}}^+)} \right\}, \end{aligned}$$

where ϵ_α is given in (2.8), $2^* = \frac{2n}{n-2}$ and $2^{*'} = \frac{2n}{n+2}$.

We postpone the proof of Proposition 3.1 to the end of this section, and use it to prove Theorem 1.2 first.

Let

$$Y(g, u_\alpha) = \frac{\int_M |\nabla_g u_\alpha|^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u_\alpha^2 ds_g}{\left(\int_{\partial M} u_\alpha^q ds_g \right)^{2/q}}.$$

It follows from Proposition 2.3 that

$$Y(g, u_\alpha) = \frac{\int_{B_{\delta_\alpha}^+} |\nabla_g u_\alpha|^2 dv_g + \frac{n-2}{2} \int_{\partial' B_{\delta_\alpha}^+} h_g u_\alpha^2 ds_g}{\left(\int_{\partial' B_{\delta_\alpha}^+} u_\alpha^q ds_g \right)^{2/q}} + O(\mu_\alpha^{n-2}).$$

Since

$$\begin{cases} \Delta_g u_\alpha - c(n) R_g u_\alpha &= \varphi_\alpha^{\frac{n+2}{n-2}} (\Delta_{\hat{g}} \frac{u_\alpha}{\varphi_\alpha} - c(n) R_{\hat{g}} \frac{u_\alpha}{\varphi_\alpha}) & \text{on } M, \\ \frac{\partial_g u_\alpha}{\partial \nu} + \frac{n-2}{2} h_g u_\alpha &= \varphi_\alpha^{\frac{n}{n-2}} \left(\frac{\partial_{\hat{g}} u_\alpha}{\partial \nu} + \frac{n-2}{2} h_{\hat{g}} \frac{u_\alpha}{\varphi_\alpha} \right) & \text{on } \partial M, \end{cases}$$

and $h_{\hat{g}} = 0$ on $\partial' B_{\delta_\alpha}^+$, we have

$$Y(g, u_\alpha) = \frac{\int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}}(\frac{u_\alpha}{\varphi_\alpha})|^2 dv_{\hat{g}}}{\left(\int_{\partial' B_{\delta_\alpha}^+} (\frac{u_\alpha}{\varphi_\alpha})^q ds_{\hat{g}}\right)^{2/q}} + O(\mu_\alpha^{n-2}).$$

In view of $u_\alpha/\varphi_\alpha = t_\alpha U_{\xi_\alpha, \lambda_\alpha} - t_\alpha h_\alpha + \chi_\alpha + w_\alpha$,

$$\int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} \chi_\alpha \nabla_{\hat{g}} w_\alpha dv_{\hat{g}} = \int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} h_\alpha \nabla_{\hat{g}} w_\alpha dv_{\hat{g}} = 0,$$

and the estimates in Lemma 3.1, we have

$$(3.5) \quad Y(g, u_\alpha) = F(w_\alpha) + O(\mu_\alpha^{n-2}),$$

where

$$F(w) := \frac{\int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}}(t_\alpha U_{\xi_\alpha, \lambda_\alpha} + w)|^2 dv_{\hat{g}}}{\left(\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha} + w|^q ds_{\hat{g}}\right)^{2/q}}.$$

By a direct computation,

$$\begin{aligned} F'(0)w_\alpha = & \frac{2}{\left(\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^q ds_{\hat{g}}\right)^{2/q}} \left\{ \int_{B_{\delta_\alpha}^+} t_\alpha \nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} \nabla_{\hat{g}} w_\alpha dv_{\hat{g}} \right. \\ & \left. - \frac{\int_{B_{\delta_\alpha}^+} |t_\alpha \nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}}}{\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^q ds_{\hat{g}}} \int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^{q-1} w_\alpha ds_{\hat{g}} \right\}. \end{aligned}$$

Recall that $\sigma_\alpha = U_{\xi_\alpha, \lambda_\alpha} - h_\alpha$ and $\int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} \sigma_\alpha \nabla_{\hat{g}} w_\alpha = \int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} h_\alpha \nabla_{\hat{g}} w_\alpha = 0$. It follows that $\int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} \nabla_{\hat{g}} w_\alpha = 0$. Hence (recall that we are in the Fermi coordinates of $\hat{g}$),

$$\begin{aligned} |F'(0)w_\alpha| & \leq C \left| \int_{\partial' B_{\delta_\alpha}^+} |U_{\xi_\alpha, \lambda_\alpha}|^{q-1} w_\alpha ds_{\hat{g}} \right| = CS \left| \int_{\partial' B_{\delta_\alpha}^+} \frac{\partial_{\hat{g}}}{\partial \nu} U_{\xi_\alpha, \lambda_\alpha} w_\alpha ds_{\hat{g}} \right| \\ & = CS \left| \int_{B_{\delta_\alpha}^+} \Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} w_\alpha dv_{\hat{g}} \right| \\ & \leq C \|\Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}\|_{L^{2^*'}(B_{\delta_\alpha}^+)} \|w_\alpha\|, \end{aligned}$$

where $2^* = \frac{2n}{n-2}$ and $2^{*'} = \frac{2n}{n+2}$. By (3.9) we have

$$|F'(0)w_\alpha| \leq C \mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^*'}(B_{\mu_\alpha^{-1}}^+)} \|w_\alpha\|.$$

Similarly,

$$\begin{aligned} \langle F''(0)w_\alpha, w_\alpha \rangle = & \frac{2}{\left(\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^q ds_{\hat{g}}\right)^{2/q}} \left\{ \int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} w_\alpha|^2 dv_{\hat{g}} \right. \\ & - (q-1) \frac{\int_{B_{\delta_\alpha}^+} |t_\alpha \nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}}}{\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^q ds_{\hat{g}}} \int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^{q-2} w_\alpha^2 ds_{\hat{g}} \Big\} \\ & + O\left(\left(\int_{\partial' B_{\delta_\alpha}^+} |U_{\xi_\alpha, \lambda_\alpha}|^{q-1} w_\alpha ds_{\hat{g}}\right)^2\right). \end{aligned}$$

By Lemma 3.6, we have

$$\begin{aligned} \int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} w_\alpha|^2 dv_{\hat{g}} - \frac{(q-1) \int_{B_{\delta_\alpha}^+} |t_\alpha \nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}}}{\int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^q ds_{\hat{g}}} \int_{\partial' B_{\delta_\alpha}^+} |t_\alpha U_{\xi_\alpha, \lambda_\alpha}|^{q-2} w_\alpha^2 ds_{\hat{g}} \\ \geq \frac{c_1}{2} \|w_\alpha\|^2, \end{aligned}$$

for large α . It follows that

$$\langle F''(0)w_\alpha, w_\alpha \rangle \geq \frac{c_1}{2} \|w_\alpha\|^2 + O(\mu_\alpha^2) \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha^{-1}}^+)}^2 \|w_\alpha\|^2.$$

Noticing that $\mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha^{-1}}^+)}^2 \rightarrow 0$ as $\alpha \rightarrow \infty$, we have

$$\begin{aligned} F(w_\alpha) &= F(0) + F'(0)w_\alpha + \frac{1}{2} \langle F''(0)w_\alpha, w_\alpha \rangle + o(\|w_\alpha\|^2) \\ &\geq F(0) + O\left(\mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha^{-1}}^+)} \|w_\alpha\|\right). \end{aligned}$$

By (3.5), we conclude that

$$(3.6) \quad Y(g, u_\alpha) \geq \frac{\int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}}}{\left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q ds_{\hat{g}}\right)^{2/q}} + O\left(\mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha^{-1}}^+)} \|w_\alpha\| + \mu_\alpha^{n-2}\right).$$

Lemma 3.2. *We have*

$$\frac{\int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}}}{\left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q ds_{\hat{g}}\right)^{2/q}} = \begin{cases} \frac{1}{S} + O(\mu_\alpha^2), & n \geq 5, \\ \frac{1}{S} + O(\mu_\alpha^2 \log \mu_\alpha^{-1}), & n = 4. \end{cases}$$

Proof. Let $\pi = \hat{h}_{ij} dz^i dz^j$, $1 \leq i, j \leq n-1$, be the second fundamental form of $\partial' B_{\delta_\alpha}^+$ with respect to the metric $\hat{g}_{ij}$. Since $\sqrt{\det \hat{g}_{ij}} = 1 + O(|z|^2)$ on $\partial' B_{\delta_\alpha}^+$, we have

$$\begin{aligned} \left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q ds_{\hat{g}}\right)^{2/q} &= \left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q (1 + O(|z'|^2)) dz\right)^{2/q} \\ &= \left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q dz\right)^{2/q} + O\left(\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^q |z'|^2 dz\right) \\ &= \left(\int_{\partial \mathbb{R}_+^n} U_{\xi_\alpha, \lambda_\alpha}^q dz\right)^{2/q} + O(\lambda_\alpha^{n-2}) \\ &\quad + O(\lambda_\alpha^2 \int_{\partial' B_{\lambda_\alpha^{-1}}^+} (1 + |z'|^2)^{1-n} |z'|^2 dz) \\ &= 1 + O(\lambda_\alpha^{n-2}) + O(\lambda_\alpha^2 \int_{\partial' B_{\lambda_\alpha^{-1}}^+} (1 + |z'|^2)^{1-n} |z'|^2 dz), \end{aligned}$$

where we used $\lambda_\alpha^{-1} |\xi_\alpha| \rightarrow 0$ as $\alpha \rightarrow \infty$ in the last equality.

In addition, by Lemma 2.2, we have

$$\begin{aligned} \int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}|^2 dv_{\hat{g}} &= \int_{B_{\delta_\alpha}^+} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 dz + 2\hat{h}^{ij}(0) \int_{B_{\delta_\alpha}^+} \partial_i U_{\xi_\alpha, \lambda_\alpha} \partial_j U_{\xi_\alpha, \lambda_\alpha} z_n dz \\ &\quad + O\left(\int_{B_{\delta_\alpha}^+} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 |z|^2 dz\right). \end{aligned}$$

It is easy to see that

$$\int_{B_{\delta_\alpha}^+} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 dz = \int_{\mathbb{R}_+^n} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 dz + O(\lambda_\alpha^{n-2}) = \frac{1}{S} + O(\lambda_\alpha^{n-2})$$

and

$$\begin{aligned} &\int_{B_{\delta_\alpha}^+} |\nabla U_{\xi_\alpha, \lambda_\alpha}|^2 |z|^2 dz \\ &= C(n) \lambda_\alpha^2 \int_{B_{\lambda_\alpha^{-1}}^+} (|z'|^2 + (z_n + 1)^2)^{1-n} |z + \lambda_\alpha^{-1} \xi_\alpha|^2 dz + O(\lambda_\alpha^{n-2}). \end{aligned}$$

By symmetry, we have

$$\begin{aligned} &\sum_{i,j=1}^{n-1} \hat{h}^{ij}(0) \int_{B_{\delta_\alpha}^+} \partial_i U_{\xi_\alpha, \lambda_\alpha} \partial_j U_{\xi_\alpha, \lambda_\alpha} z_n dz \\ &= \sum_{i,j=1}^{n-1} \hat{h}^{ij}(0) \int_{B_{\delta_\alpha/2}^+(\xi_\alpha)} \partial_i U_{\xi_\alpha, \lambda_\alpha} \partial_j U_{\xi_\alpha, \lambda_\alpha} z_n dz + O(\mu_\alpha^{n-2}) \\ &= \sum_{i=1}^{n-1} \hat{h}^{ii}(0) \int_{B_{\delta_\alpha/2}^+(\xi_\alpha)} |\partial_1 U_{\xi_\alpha, \lambda_\alpha}|^2 z_n dz + O(\mu_\alpha^{n-2}) = O(\mu_\alpha^{n-2}), \end{aligned}$$

where we used $\sum_{i=1}^{n-1} \hat{h}^{ii}(0) = 0$ since the mean curvature of $\hat{g}$ is vanishing at x_α . The lemma follows immediately from Lemma 3.1. $\square$

Proof of Theorem 1.2. Notice that

$$\frac{1}{S} > I_\alpha(u_\alpha) = Y(g, u_\alpha) + \alpha \|u_\alpha\|_{L^r(\partial M)}^2.$$

By (3.6) and Lemma 3.2, we have

$$\alpha \|u_\alpha\|_{L^r(\partial M)}^2 = O(\mu_\alpha^2) + O\left(\mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha}^+)} \|w_\alpha\| + \mu_\alpha^{n-2}\right).$$

By Proposition 3.1, we find

$$\begin{aligned} (3.7) \quad \alpha \|u_\alpha\|_{L^r(\partial M)}^2 &\leq C \left\{ \mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha}^+)} \left(\mu_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha}^+)} \right. \right. \\ &\quad \left. \left. + \epsilon_\alpha \|U_1^{r-1}\|_{L^r(\partial' B_{\mu_\alpha}^+)} + \mu_\alpha^{n-2} \|U_1^{q-2}\|_{L^r(\partial' B_{\mu_\alpha}^+)} \right) + \mu_\alpha^2 \right\}. \end{aligned}$$

Due to $n \geq 5$, we have

$$\begin{aligned} \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\mu_\alpha^-}^+)} &\leq C, \\ \mu_\alpha \|U_1^{r-1}\|_{L^r(\partial' B_{\mu_\alpha^-}^+)} &\leq C\mu_\alpha(1 + \mu_\alpha^{\frac{n^2-8n+8}{2n}}) = o(1), \\ \|U_1^{q-2}\|_{L^r(\partial' B_{\mu_\alpha^-}^+)} &\leq C(1 + \mu_\alpha^{(4-n)/2}). \end{aligned}$$

From (2.9), i.e., $\epsilon_\alpha \leq \alpha \|u_\alpha\|_{L^r(\partial M)}^2$, it follows that

$$\alpha \|u_\alpha\|_{L^r(\partial M)}^2 \leq C\mu_\alpha^2.$$

On the other hand,

$$\|u_\alpha\|_{L^r(\partial M)} \geq \|u_\alpha\|_{L^r(B_{\mu_\alpha^+}(x_\alpha) \cap \partial M)} \geq C^{-1}\mu_\alpha \|v_\alpha\|_{L^r(\partial' B_1^+)} \geq C^{-1}\mu_\alpha,$$

where v_α is as in (2.7). Hence

$$\alpha \leq C.$$

This is a contradiction. $\square$

3.1. Proof of Proposition 3.1. The rest of the paper is devoted to the proof of Proposition 3.1. We start with the equation that w_α satisfies.

Lemma 3.3. w_α satisfies

$$(3.8) \quad \begin{cases} -\Delta_{\hat{g}} w_\alpha = t_\alpha \Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}, & \text{in } B_{\delta_\alpha}^+, \\ \frac{\partial_{\hat{g}} w}{\partial \nu} - \ell_\alpha(q-1)|\Theta_\alpha|^{q-3}\Theta_\alpha w_\alpha + b'|\Theta_\alpha|^{q-3}w_\alpha^2 + b''|w_\alpha|^{q-1} = f_\alpha, & \text{on } \partial' B_{\delta_\alpha}^+, \end{cases}$$

where

$$\Theta_\alpha = t_\alpha \sigma_\alpha + \chi_\alpha,$$

$$f_\alpha = t_\alpha(\ell_\alpha t_\alpha^{q-2} - \frac{1}{S})U_{\xi_\alpha, \lambda_\alpha}^{q-1} - \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \psi_\alpha^{1-q} u_\alpha^{r-1} + O(\mu_\alpha^{(n-2)/2})U_{\xi_\alpha, \lambda_\alpha}^{q-2},$$

and b', b'' are bounded functions with $b' \equiv 0$ if $n \geq 4$.

Proof. The proof follows from straightforward computations. First of all, by the definition of w_α and by equation (3.1) of u_α/ψ_α , we have

$$-\Delta_{\hat{g}} w_\alpha = -\Delta_{\hat{g}} \left(\frac{u_\alpha}{\psi_\alpha} - \chi_\alpha - t_\alpha \sigma_\alpha \right) = t_\alpha \Delta_{\hat{g}} \sigma_\alpha = t_\alpha \Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} \quad \text{in } B_{\delta_\alpha}^+,$$

and

$$\begin{aligned} \frac{\partial_{\hat{g}} w_\alpha}{\partial \nu} &= \ell_\alpha(\Theta_\alpha + w_\alpha)^{q-1} - \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \psi_\alpha^{1-q} u_\alpha^{r-1} - t_\alpha \frac{\partial_{\hat{g}} \sigma_\alpha}{\partial \nu} \\ &= \ell_\alpha(\Theta_\alpha + w_\alpha)^{q-1} - \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \psi_\alpha^{1-q} u_\alpha^{r-1} - t_\alpha \frac{\partial_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}}{\partial \nu} \\ &= \ell_\alpha(\Theta_\alpha + w_\alpha)^{q-1} - \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \psi_\alpha^{1-q} u_\alpha^{r-1} - \frac{t_\alpha}{S} U_{\xi_\alpha, \lambda_\alpha}^{q-1}, \end{aligned}$$

where we used that $\frac{\partial_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}}{\partial \nu} = \frac{\partial U_{\xi_\alpha, \lambda_\alpha}}{\partial x_n} = U_{\xi_\alpha, \lambda_\alpha}^{q-1}$ in Fermi coordinate systems. Note that we have the elementary expansion

$$(k+l)^{q-1} = |k|^{q-2}k + (q-1)|k|^{q-3}kl + b'(k, l)|k|^{q-3}l^2 + b''(k, l)|l|^{q-1},$$

for all k, l such that $k + l > 0$, where b', b'' are bounded and $b' \equiv 0$ if $n \geq 4$. For $k = \Theta_\alpha$, $l = w_\alpha$, we obtain

$$\begin{aligned} (\Theta_\alpha + w_\alpha)^{q-1} &= |\Theta_\alpha|^{q-2} \Theta_\alpha + (q-1) |\Theta_\alpha|^{q-3} \Theta_\alpha w_\alpha \\ &\quad + b' |\Theta_\alpha|^{q-3} w_\alpha^2 + b'' |w_\alpha|^{q-1}, \end{aligned}$$

and Lemma 3.3 follows from

$$\begin{aligned} |\Theta_\alpha|^{q-2} \Theta_\alpha &= |t_\alpha U_{\xi_\alpha, \lambda_\alpha} - t_\alpha h_\alpha + \chi_\alpha|^{q-2} (t_\alpha U_{\xi_\alpha, \lambda_\alpha} - t_\alpha h_\alpha + \chi_\alpha) \\ &= (t_\alpha U_{\xi_\alpha, \lambda_\alpha})^{q-1} - (t_\alpha U_{\xi_\alpha, \lambda_\alpha})^{q-2} (t_\alpha h_\alpha - \chi_\alpha) \\ &\quad + |t_\alpha U_{\xi_\alpha, \lambda_\alpha} - \theta(t_\alpha h_\alpha + \chi_\alpha)|^{q-3} (t_\alpha h_\alpha + \chi_\alpha) (t_\alpha U_{\xi_\alpha, \lambda_\alpha} - t_\alpha h_\alpha + \chi_\alpha) \\ &= (t_\alpha U_{\xi_\alpha, \lambda_\alpha})^{q-1} + O(\mu_\alpha^{(n-2)/2} U_{\xi_\alpha, \lambda_\alpha}^{q-2}), \end{aligned}$$

where $\theta \in (0, 1)$ and we have used 1) and 2) from Lemma 3.1. $\square$

Define

$$Q_\alpha(\varphi, \phi) = \int_{B_{\delta_\alpha}^+} \nabla_{\hat{g}} \varphi \nabla_{\hat{g}} \phi \, dv_{\hat{g}} - \ell_\alpha (q-1) \int_{\partial' B_{\delta_\alpha}^+} |\Theta_\alpha|^{q-3} \Theta_\alpha \varphi \phi \, ds_{\hat{g}},$$

for all $\varphi, \phi \in H_{0,L}(B_{\delta_\alpha}^+)$.

Lemma 3.4. *There exist $0 < \delta_0 \ll 1$, $\alpha_0 \geq 1$ and $c_0 > 0$ independent of α such that*

$$Q_\alpha(w, w) \geq c_0 \int_{B_{\delta_\alpha}^+} |\nabla_{\hat{g}} w|^2 \, dv_{\hat{g}}, \quad \forall w \in W_\alpha, \quad \alpha \geq \alpha_0.$$

Proof. The proof follows from Lemma 3.1 and Lemma 3.6. $\square$

Proof of Proposition 3.1. Multiplying both sides of (3.8) by w_α , we arrive at

$$Q_\alpha(w_\alpha, w_\alpha) + o(\|w_\alpha\|^2) = t_\alpha \int_{B_{\delta_\alpha}^+} w_\alpha \Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} \, dv_{\hat{g}} + \int_{\partial' B_{\delta_\alpha}^+} w_\alpha f_\alpha \, ds_{\hat{g}},$$

where we used

$$\int_{\partial' B_{\delta_\alpha}^+} b'' |w_\alpha|^q \, ds_{\hat{g}} \leq C \|\nabla_{\hat{g}} w_\alpha\|_{L^2(B_{\delta_\alpha}^+)}^q = C \|w_\alpha\|^2 \|w_\alpha\|^{q-2}$$

and

$$\int_{\partial' B_{\delta_\alpha}^+} b' |\Theta_\alpha|^{q-3} |w_\alpha|^3 \leq C \|w_\alpha\|^3$$

if $n = 3$ and used also Lemma 3.1. By Hölder's inequality, Sobolev's inequality, and Lemma 3.4, we have

$$\begin{aligned} \|w_\alpha\| &\leq C \left\{ \|\Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}\|_{L^{2^*}(B_{\delta_\alpha}^+)} + \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \|u_\alpha^{r-1}\|_{L^{q'}(\partial' B_{\delta_\alpha}^+)} \right. \\ &\quad \left. + \mu_\alpha^{\frac{n-2}{2}} \|U_{\xi_\alpha, \lambda_\alpha}^{q-2}\|_{L^{q'}(\partial' B_{\delta_\alpha}^+)} \right\} \end{aligned}$$

with $q' = r = q/(q-1)$. Since $\Delta U_{\xi_\alpha, \lambda_\alpha} = 0$, a direct computation in the Fermi coordinate system yields

$$\begin{aligned}
 & \|\Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha}\|_{L^{2^{*'}}(B_{\delta_\alpha}^+)} \\
 &= \|\Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} - \Delta U_{\xi_\alpha, \lambda_\alpha}\|_{L^{2^{*'}}(B_{\delta_\alpha}^+)} \\
 &\leq \|(\hat{g}^{ij} - \delta^{ij}) \frac{\partial^2 U_{\xi_\alpha, \lambda_\alpha}}{\partial z_i \partial z_j} - \hat{g}^{ij} \hat{\Gamma}_{ij}^k \frac{\partial u}{\partial z_k}\|_{L^{2^{*'}}(B_{\delta_\alpha}^+)} \\
 &\leq C \left(\int_{B_{\delta_\alpha}^+} (|z| \lambda_\alpha^{(n-2)/2} (|z' - \xi'_\alpha|^2 + (z_n + \lambda_0 \lambda_\alpha)^2)^{-n/2})^{2^{*'}} dz \right)^{1/2^{*'}} \\
 (3.9) \quad &+ C \left(\int_{B_{\delta_\alpha}^+} (\lambda_\alpha^{(n-2)/2} (|z' - \xi'_\alpha|^2 + (z_n + \lambda_0 \lambda_\alpha)^2)^{-(n-1)/2})^{2^{*'}} dz \right)^{1/2^{*'}} \\
 &\leq C \lambda_\alpha \left(\int_{B_{\lambda_\alpha^{-1}}^+} (|y + \lambda_\alpha^{-1} \xi_\alpha| (|y'|^2 + (y_n + \lambda_0)^2)^{-n/2})^{2^{*'}} dy \right)^{1/2^{*'}} \\
 &+ C \lambda_\alpha \left(\int_{B_{\lambda_\alpha^{-1}}^+} (|y' - \lambda_\alpha^{-1} \xi_\alpha|^2 + (y_n + \lambda_0)^2)^{-(n-1)/2})^{2^{*'}} dy \right)^{1/2^{*'}} \\
 &\leq C \lambda_\alpha \|U_1^{\frac{n-1}{n-2}}\|_{L^{2^{*'}}(B_{\lambda_\alpha^{-1}}^+)},
 \end{aligned}$$

where $2^* = \frac{2n}{n-2}$, $2^{*'} = \frac{2n}{n+2}$ and we used the variables transformation $y = \lambda_\alpha^{-1}(z - \xi_\alpha)$ and the fact $\lambda_\alpha^{-1}|\xi_\alpha| \rightarrow 0$ as $\alpha \rightarrow \infty$. By Proposition 2.3, we have $u_\alpha \leq C U_{\mu_\alpha}$. Hence

$$\|u_\alpha^{r-1}\|_{L^{q'}(\partial' B_{\delta_\alpha}^+)} \leq C \|U_{\mu_\alpha}^{r-1}\|_{L^{q'}(\partial' B_{\delta_\alpha}^+)} \leq \mu_\alpha^{n-1-\frac{n-2}{2}r} \|U_1^{r-1}\|_{L^{q'}(\partial' B_{\mu_\alpha^{-1}}^+)}.$$

Finally, it is clear that

$$\|U_{\xi_\alpha, \lambda_\alpha}^{q-2}\|_{L^{q'}(\partial' B_{\delta_\alpha}^+)} \leq C \mu_\alpha^{\frac{n-2}{2}} \|U_1^{q-2}\|_{L^{q'}(\partial' B_{\mu_\alpha^{-1}}^+)}.$$

Therefore, we obtained the estimate for $\|w_\alpha\|$.

Multiplying (3.8) by σ_α and integrating over $B_{\delta_\alpha}^+$, we find that

$$\begin{aligned}
 & |t_\alpha(\ell_\alpha t_\alpha^{q-2} - \frac{1}{S})| \int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^{q-1} \sigma_\alpha ds_{\hat{g}} \\
 &\leq C \left(\int_{B_{\delta_\alpha}^+} |\Delta_{\hat{g}} U_{\xi_\alpha, \lambda_\alpha} \sigma_\alpha| dv_{\hat{g}} + \int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^{q-1} |w_\alpha| ds_{\hat{g}} \right. \\
 &\quad \left. + \alpha \|u_\alpha\|_{L^r(\partial M)}^{2-r} \int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^r ds_{\hat{g}} + \mu_\alpha^{(n-2)/2} \int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^{q-1} ds_{\hat{g}} \right),
 \end{aligned}$$

where we used $|\sigma_\alpha| + |\Theta_\alpha| + |w_\alpha| + u_\alpha \leq C U_{\xi_\alpha, \lambda_\alpha}$. Since $\int_{\partial' B_{\delta_\alpha}^+} U_{\xi_\alpha, \lambda_\alpha}^{q-1} \sigma_\alpha ds_{\hat{g}} \geq C^{-1} > 0$, the estimate of $|\ell_\alpha t_\alpha^{q-2} - \frac{1}{S}|$ follows by Hölder's inequality and the estimate for $\|w_\alpha\|$. $\square$

3.2. Proof of Lemma 3.4. Let $\mathcal{D}^{1,2}(\mathbb{R}_+^n)$ be the closure of $C_c^\infty(\mathbb{R}_+^n \cup \partial\mathbb{R}_+^n)$ under the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^n)} = \left(\int_{\mathbb{R}_+^n} |\nabla u|^2 dy \right)^{1/2}.$$

In fact, $\mathcal{D}^{1,2}(\mathbb{R}_+^n)$ is a Hilbert space with the inner product

$$\langle \varphi, \phi \rangle_1 := \int_{\mathbb{R}_+^n} \nabla \varphi \nabla \phi dy$$

for any $\varphi, \phi \in \mathcal{D}^{1,2}(\mathbb{R}_+^n)$. Define the functional

$$Q_1(\varphi, \phi) := \int_{\mathbb{R}_+^n} \nabla \varphi \nabla \phi dy - \frac{q-1}{S} \int_{\partial\mathbb{R}_+^n} U_1^{q-2} \varphi \phi dy'$$

for all $\varphi, \phi \in E_0$, where $y = (y', 0)$ and

$$E_1 = \left\{ w \in \mathcal{D}^{1,2}(\mathbb{R}_+^n) : \begin{aligned} &\langle \frac{\partial U_{\xi,1}}{\partial \xi_i} \Big|_{\xi=0}, w \rangle_1 = \langle \frac{\partial U_{0,\lambda}}{\partial \lambda} \Big|_{\lambda=1}, w \rangle_1 = \langle U_{0,1}, w \rangle_1 = 0, \\ &i = 1, 2, \dots, n-1 \end{aligned} \right\}.$$

Then we have

Lemma 3.5. *There exists a constant $c_1 > 0$ depending only on n such that*

$$Q_1(w, w) \geq c_1 \|w\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^n)}^2,$$

for all $w \in E_1$.

Lemma 3.5 follows from an analysis of the eigenvalues of

$$\begin{cases} \Delta v = 0, & \text{in } B_1, \\ \frac{\partial v}{\partial \nu} = \lambda v, & \text{on } \partial B_1. \end{cases}$$

The details can be found in, e.g., [9].

Lemma 3.6. *For $R > 0$ and $x \in \partial\mathbb{R}^n$ with $|x| \leq R/10$, let h_{ij} be a Riemannian metric on $B_R^+(x)$, $k > 0$ and $\Theta \in L^q(\partial' B_R)$. Denote Q_2 as the continuous bilinear form on $H_{0,L}(B_R^+(x)) \times H_{0,L}(B_R^+(x))$*

$$Q_2(\varphi, \phi) = \int_{B_R^+(x)} \nabla_h \varphi \nabla_h \phi dv_h - k \int_{\partial' B_R^+(x)} |\Theta|^{q-3} \Theta \varphi \phi, ds_h.$$

There exists a small positive ε_0 depending only on n such that if

$$\|\Theta - U_1\|_{L^q(\partial' B_R^+(x))} + |k - \frac{q-1}{S}| + \|h_{ij} - \delta_{ij}\|_{L^\infty(B_R^+(x))} \leq \varepsilon_0,$$

then

$$Q_2(w, w) \geq \frac{c_1}{2} \int_{B_R^+(x)} |\nabla_h w|^2 dv_h,$$

for all $w \in E_2$, where

$$E_2 = \left\{ w \in H_{0,L}(B_R^+(x)) : \begin{aligned} &\langle U_{0,1}, w \rangle_1 \leq \varepsilon_0 \|w\|_h, \langle \frac{\partial U_{0,\lambda}}{\partial \lambda} \Big|_{\lambda=1}, w \rangle_1 \leq \varepsilon_0 \|w\|_h \\ &\langle \frac{\partial U_{\xi,1}}{\partial \xi_i} \Big|_{\xi=0}, w \rangle_1 \leq \varepsilon_0 \|w\|_h, i = 1, 2, \dots, n-1 \end{aligned} \right\}.$$

Proof. From the assumptions, there exist $\delta_1, \dots, \delta_{n+1}$ satisfying $|\delta_j| = O(\varepsilon_0 \|w\|_h)$ for every j such that

$$\tilde{w} = w - \sum_{j=1}^{n-1} \delta_j \frac{\partial U_{\xi,1}}{\partial \xi_i} \Big|_{\xi=0} - \delta_n \frac{\partial U_{0,\lambda}}{\partial \lambda} \Big|_{\lambda=1} - \delta_{n+1} U_{0,1}$$

belongs to E_1 . It follows from Lemma 3.5 that

$$Q_1(\tilde{w}, \tilde{w}) \geq c_1 \|\tilde{w}\|_{\mathcal{D}^{1,2}(\mathbb{R}_+^n)}.$$

The lemma follows by choosing sufficiently small ε_0 and making use of the Sobolev trace inequality. $\square$

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVENUE, CHICAGO, ILLINOIS 60637

E-mail address: tj@math.uchicago.edu

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA

E-mail address: jxiong@math.pku.edu.cn