THE GLASSEY CONJECTURE ON ASYMPTOTICALLY FLAT MANIFOLDS

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ABSTRACT. We verify the 3-dimensional Glassey conjecture on asymptotically flat manifolds $(\mathbb{R}^{1+3}, g)$, where the metric g is a certain small space-time perturbation of the flat metric, as well as the nontrapping asymptotically Euclidean manifolds. Moreover, for radial asymptotically flat manifolds $(\mathbb{R}^{1+n}, g)$ with $n \geq 3$, we verify the Glassey conjecture in the radial case. High dimensional wave equations with higher regularity are also discussed. The main idea is to exploit local energy and KSS estimates with variable coefficients, together with the weighted Sobolev estimates including trace estimates.

1. INTRODUCTION

The purpose of this article is to study the longtime existence of solutions for the Cauchy problem of the nonlinear wave equations of the type

$$\square u = c_1 |u_t|^p + c_2 |\nabla_x u|^p$$

on certain asymptotically flat manifolds, with small initial data.

In the 1980s, Glassey in [6] made a conjecture that the critical exponent p for the problem, posed on the Minkowski space-time, to admit global solutions with small, smooth initial data with compact support is

$$p_c = 1 + \frac{2}{n-1}$$

(see also Schaeffer [21] and Rammaha [20]), where n is the spatial dimension. The conjecture was verified for dimension $n = 2, 3$ for general data (Hidano and Tsutaya [8] and Tzvetkov [29] independently, as well as the radial case in Sideris [22] for $n = 3$). For higher dimension $n \geq 4$, when $c_2 = 0$ and $c_1 > 0$, the blow up results (together with an explicit upper bound of the lifespan) for $p \leq p_c$ were obtained in Zhou [32] (see also Zhou and Han [33] for problems with time-independent compact metric perturbation). Recently, for the radial data, the existence results with sharp lifespan for any $p \in (1, 1 + 2/(n-2))$ was proved in Hidano, Yokoyama and the author [10] (see also Fang and the author [5] for the critical case $n = 2$ and $p = 3$), which particularly proved the Glassey conjecture in the radial case.

In this paper, we prove analogs of the results on the Glassey conjecture of [8], [29] and [10], on certain asymptotically flat manifolds. One of the main ingredients

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in the proof is the local energy estimates with variable coefficients of Metcalfe and Sogge [16] and Hidano, Yokoyama and the author [9]. The local energy estimates first appeared in Morawetz [19] and subsequently in many works; see e.g. [11, 12, 14, 16, 23, 26, 27] and the references therein. Such estimates are known to be fairly robust and variants for metric perturbations were proved in e.g. [1–3, 9, 10, 15–17, 24] and the references therein. Another key ingredient is the weighted Sobolev estimates, including the trace estimates. Together with the local energy estimates, estimates of this kind have been proved to be very useful; see e.g. [4, 7, 12, 13, 15, 16].

Let us begin with the space-time manifolds we will work on. As usual, we use $(x^0, x^1, \dots, x^n) = (t, x)$ to denote points in $\mathbb{R}^{1+n}$, and $\partial_\alpha = \partial/\partial x^\alpha$ as partial derivatives, with the abbreviations $\partial = (\partial_0, \partial_1, \dots, \partial_n) = (\partial_t, \nabla)$. We consider the asymptotically flat Lorentzian manifolds $(\mathbb{R}^{1+n}, \mathbf{g})$ with

$$\mathbf{g} = g_{\alpha\beta}(t, x) dx^\alpha dx^\beta = \sum_{\alpha, \beta=0}^n g_{\alpha\beta}(t, x) dx^\alpha dx^\beta.$$

Here, we have used the convention that Greek indices α, β, γ range from 0 to n and Latin indices i, j, k from 1 to n . We will also use a, b, c to denote multi-indices. Moreover, the Einstein summation convention will be performed over repeated indices.

We will consider two types of manifolds. The first one is a small, time-dependent, asymptotically flat perturbation of the Minkowski metric. More precisely, with Euclidean radius $r = \sqrt{|x|^2}$, we assume $g_{\alpha\beta} \in C^\infty(\mathbb{R}^{1+n})$ and, for some fixed $\rho_1 > 0$, $\rho_2 > 1$ and $\delta \ll 1$,

$$(H1) \quad \mathbf{g} = m + g_1(t, r) + g_2(t, x),$$

where $(m_{\alpha\beta}) = \text{Diag}(1, -1, -1, \dots, -1)$ is the standard Minkowski metric, $\langle x \rangle = \sqrt{1 + |x|^2}$,

$$(H1.1) \quad |\partial_{t,x}^a g_{i,\alpha\beta}| \lesssim_a \delta \langle x \rangle^{-|a|-\rho_i}, i = 1, 2, \rho = \min(\rho_1, \rho_2 - 1), \rho_1 < \rho_2,$$

and, moreover, we assume the first perturbation g_1 is radial. By radial metric, we mean that, when writing out the metric $\mathbf{g}$, with $g_2 = 0$, in polar coordinates $(t, x) = (t, r\omega)$ with $\omega \in \mathbb{S}^{n-1}$, we have

$$m + g_1(t, r) = \tilde{g}_{00}(t, r) dt^2 + 2\tilde{g}_{01}(t, r) dt dr + \tilde{g}_{11}(t, r) dr^2 + \tilde{g}_{22}(t, r) r^2 d\omega^2.$$

In this form, the assumption (H1.1) for g_1 on asymptotically flatness is equivalent to the following requirement:

$$(H1.2) \quad |\partial_{t,x}^a (\tilde{g}_{00} - 1, \tilde{g}_{11} + 1, \tilde{g}_{22} + 1, \tilde{g}_{01})| \lesssim_a \delta \langle x \rangle^{-|a|-\rho}.$$

An example of such a metric can be

$$g_{\alpha\beta} = m_{\alpha\beta} + \delta \langle x \rangle^{-\rho} + \delta \phi(t/\langle x \rangle) \langle x \rangle^{-\rho}$$

with $\phi \in C_0^\infty(\mathbb{R})$. Here, and in what follows, we use $A \lesssim B$ to stand for $A \leq CB$ where the constant C may change from line to line. When $\delta \geq 0$ is small enough, it is clear that the metric $\mathbf{g}$ is a nontrapping perturbation. Notice that the form of the metrics mimic that in Tataru [28] and Metcalfe, Tataru and Tohaneanu [18], and g_1 could be a long range perturbation. Also, the metric models the black hole metrics occurring in mathematical relativity, near spatial infinity. For example, for large enough radius r , a Schwarzschild metric is a radial metric (H1) with $\rho_1 = 1$, and a Kerr metric is a metric (H1) with $\rho_1 = 1$ and $\rho_2 = 2$. The separation of the

radial part from the general nonradial metric is crucial in many problems; see e.g. [15, 18, 28].

We will also consider nontrapping asymptotically Euclidean manifolds. That is, we assume that

$$(H2) \quad \mathbf{g} = m + g_1(r) + g_2(x), \mathbf{g} \text{ is nontrapping,}$$

where we suppose g_1 and g_2 are of the form $g_{jk}dx^jdx^k$,

$$(H2.1) \quad |\nabla_x^a g_{i,jk}| \lesssim_a \langle x \rangle^{-|a|-\rho_i}, i = 1, 2, \rho = \min(\rho_1, \rho_2 - 1), \rho_1 < \rho_2,$$

and as in (H1.2), we assume the first perturbation g_1 is radial.

With $|\mathbf{g}| = (-1)^n \det(g_{\alpha\beta})$, the d'Alembertian operator associated with $\mathbf{g}$ is given by

$$\square_{\mathbf{g}} = \sqrt{|\mathbf{g}|}^{-1} \partial_{\alpha} g^{\alpha\beta} \sqrt{|\mathbf{g}|} \partial_{\beta},$$

where $(g^{\alpha\beta}(t, x))$ denotes the inverse matrix of $(g_{\alpha\beta}(t, x))$.

Let $n \geq 3$, $p > 1$. We consider initial value problems for the following nonlinear wave equations:

$$(1.1) \quad \begin{cases} \square_{\mathbf{g}} u = \sum_{\alpha=0}^n a_{\alpha}(u) |\partial_{\alpha} u|^p \equiv F_p(u), & x \in \mathbb{R}^n, \\ u(0, x) = f(x), \partial_t u(0, x) = g(x), \end{cases}$$

for given smooth functions a_{α} , as well as the radial problems (with $g_2 = 0$)

$$(1.2) \quad \begin{cases} \square_{\mathbf{g}} u = c_1 |\partial_t u|^p + c_2 |\nabla_x u|^p \equiv G_p(u), & x \in \mathbb{R}^n, \\ u(0, x) = f(x), \partial_t u(0, x) = g(x), \end{cases}$$

for constants c_1, c_2 . We will study the longtime existence of such problems with small enough initial data, according to certain norm.

Before stating our results, let us give some more notation. At first, the vector fields to be used will be labeled as

$$Y = (Y_1, \dots, Y_{n(n+1)/2}) = (\nabla, \Omega)$$

with rotational vector fields

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, 1 \leq i < j \leq n.$$

For fixed $T > 0$, the space-time norm $L_T^q L_x^r$ is simply $L_t^q([0, T], L_x^r(\mathbb{R}^n))$. In the case of $T = \infty$, we use $L_t^q L_x^r$ to denote $L_t^q([0, \infty), L_x^r(\mathbb{R}^n))$. As usual, we use $\|\cdot\|_{E_m}$ to denote the energy norm of order $m \geq 0$,

$$(1.3) \quad \|u\|_E = \|u\|_{E_0} = \|\partial u\|_{L_T^{\infty} L_x^2}, \|u\|_{E_m} = \sum_{|a| \leq m} \|Y^a u\|_E.$$

We will use $\|\cdot\|_{LE}$ to denote the (strong) local energy norm

$$(1.4) \quad \|u\|_{LE} = \|u\|_E + \|\partial u\|_{l_{\infty}^{-1/2}(L_t^2 L_x^2)} + \|r^{-1} u\|_{l_{\infty}^{-1/2}(L_t^2 L_x^2)},$$

where we write

$$\|u\|_{l_q^s(A)} = \|(\phi_j(x)u(t, x))\|_{l_q^s(A)},$$

for a partition of unity subordinate to the dyadic (spatial) annuli, $\sum_{j \geq 0} \phi_j^2(x) = 1$. The local energy includes the energy norm and could control the KSS norm; see Lemma 3.4. On the basis of the local energy norm, we can similarly define $\|u\|_{LE_m}$, and the dual norm $LE^* = l_1^{1/2} L_t^2 L_x^2$.

In proving the radial Glassey conjecture, we need to obtain stronger control on the local information, in contrast to the local energy norm (1.4). For this purpose, we introduce a modified local energy norm, mimicking that which occurred in Hidano and Yokoyama [11] (see also [9]),

$$(1.5) \quad \|u\|_{\widetilde{LE}} = \left\| r^{-1/2+\mu} \langle r \rangle^{-\mu'} \partial u \right\|_{L_t^2 L_x^2} + \left\| r^{-3/2+\mu} \langle r \rangle^{-\mu'} u \right\|_{L_t^2 L_x^2} + \|\partial u\|_{L_t^\infty L_x^2}$$

with $\mu \in (0, 1/2)$ and $\mu' > \mu$ to be specified. On the basis of $\widetilde{LE}$, we can similarly define

$$\|u\|_{\widetilde{LE}_m} = \sum_{|a| \leq m} \|\nabla^a u\|_E, \quad \|F\|_{\widetilde{LE}^*} = \|r^{1/2-\mu} \langle r \rangle^{\mu'} F\|_{L_t^2 L_x^2}.$$

Here, we do not use the vector field Ω , since we will only apply such norms in the radial problems.

We can now state our main results. The first result is about the problem (1.1) with general data, which verifies the 3-dimensional Glassey conjecture on asymptotically flat manifolds and nontrapping asymptotically Euclidean manifolds.

Theorem 1.1. *Consider the problem (1.1) on the manifold $(\mathbb{R}^{1+3}, \mathbf{g})$ satisfying (H1) or (H2) with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Let $p > 2$. There exist small positive constants ε_0 and δ_0 , such that the Cauchy problem with $\delta \leq \delta_0$ has a unique global solution $u \in C([0, \infty); H^3(\mathbb{R}^3)) \cap C^1([0, \infty); H^2(\mathbb{R}^3))$ when the initial data satisfy*

$$(1.6) \quad \sum_{|a| \leq 2} \|\partial Y^a u(0)\|_{L^2(\mathbb{R}^3)} = \varepsilon \leq \varepsilon_0, \quad \|u(0)\|_{L^2(\mathbb{R}^3)} < \infty.$$

Moreover, it satisfies $\|u\|_{LE_2} \lesssim \varepsilon$.

As a byproduct, provided that the nonlinearity is sufficiently smooth, we can also deal with the case $p \geq 2$ and $n \geq 3$, which generalizes the works for $p = 2$ on asymptotically Euclidean manifolds (H2) (with $g_1 = 0$) in Bony and Häfner [2] and Sogge and the author [24]. For small asymptotically flat manifolds (H1) (with $g_1 = 0$) and $p = 2$, the almost global existence and global existence for the solutions was essentially proved, even for the quasilinear problems, in Yu and the author [31]. Notice also that we have improved the condition on the regularity. For simplicity, we only consider the case of constant coefficients ($a_\alpha(u) = a_\alpha$).

Theorem 1.2. *Let $n \geq 3$, $p \geq 2$, $m = [\frac{n+2}{2}]$, $p > m$ or $p \in \mathbb{N}$ and $F_p(u)$ smooth in ∂u (e.g. $p = 2$ and $F_p(u) = |\partial u|^2$, $p = 3$ and $F_p(u) = |\partial u|^2 \partial_t u$). Consider the problem (1.1) with $a_\alpha(u) = a_\alpha$ on the manifold $(\mathbb{R}^{1+n}, \mathbf{g})$ satisfying (H1) or (H2) with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Then there exist small positive constants ε_0 and δ_0 , such that for any data*

$$(1.7) \quad \sum_{|a| \leq m} \|\partial Y^a u(0)\|_{L^2(\mathbb{R}^n)} = \varepsilon \leq \varepsilon_0, \quad \|u(0)\|_{L^2(\mathbb{R}^n)} < \infty,$$

the Cauchy problem with $\delta \leq \delta_0$ has a unique global solution $u \in C([0, T_); H^{m+1}(\mathbb{R}^n)) \cap C^1([0, T_*); H^m(\mathbb{R}^n))$ with $\|u\|_{LE_m} \lesssim \varepsilon$. Here $T_* = \infty$, except for the case $n = 3$ and $p = 2$, where $T_* = \exp(c/\varepsilon)$ for some constant $c > 0$.*

Turning to the problem (1.2) with radial data, we can prove the longtime existence of the radial solutions, in the spirit of [10]. To simplify the exposition, we will only prove a weaker version of the global existence theorem, comparing that in

[10]. It is not hard to see that our proof could be adapted to prove the same set of Theorems 1.1-1.3 (with $n \geq 3$) in [10], by noticing Lemma 3.8 and using

$$\Lambda_2 = \|\partial \nabla u(0)\|_{L^2(\mathbb{R}^n)} + \delta \|\partial u(0)\|_{L_x^2}.$$

We leave the details to the reader.

We shall use H_{rad}^m to denote the space of spherically symmetric functions in the usual Sobolev space H^m . We have the following global existence theorem for $p > p_c$ and $n \geq 3$, which can be viewed as a positive solution for the radial Glassey conjecture in the setting of asymptotically flat manifolds.

Theorem 1.3. *Let $n \geq 3$ and $1 + 2/(n-1) < p < 1 + 2/(n-2)$. Consider the problem (1.2) with initial data $(f, g) \in H_{\text{rad}}^2 \times H_{\text{rad}}^1$, posed on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H1) with $g_2 = 0$ and $\rho > 0$. Then there exist constants $\varepsilon_0, \delta_0 > 0$, such that if*

$$\|\partial u(0)\|_{H^1} = \varepsilon \leq \varepsilon_0, \quad \delta \leq \delta_0,$$

then we have a unique global solution u to (1.2) satisfying

$$u \in C([0, \infty); H_{\text{rad}}^2(\mathbb{R}^n)) \cap C^1([0, \infty); H_{\text{rad}}^1(\mathbb{R}^n)),$$

$$\|u\|_{\widetilde{LE}_1} \lesssim \varepsilon, \quad \mu = \frac{1}{2} - \frac{n-2}{4}(p-1), \quad \mu' = \frac{p-1}{4}.$$

In Theorem 1.3, the technical restriction $p < 1 + 2/(n-2)$ is partly due to the regularity, since we are assuming H^2 regularity of the data, while the critical scaling regularity for the problem is $s_c = n/2 + 1 - 1/(p-1)$, and $s_c < 2$ if and only if $p < 1 + 2/(n-2)$. We could expect to obtain more general results if we relax the regularity assumption. However, we could not expect to completely overcome this difficulty just by increasing the regularity, due to the limited regularity of the nonlinearity (noticing that $1 + 2/(n-2) < 2$ for $n > 4$ and so $|x|^{1+2/(n-2)} \notin C^2$). Because of this, current technology can only be adapted for some specific situations.

Theorem 1.4. *Let $n \geq 4$ be even numbers and $p > n/2$. Consider the problem (1.2), posed on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H1) with $g_2 = 0$ and $\rho > 0$. When $(u(0), \partial_t u(0)) \in H_{\text{rad}}^{n/2+1} \times H_{\text{rad}}^{n/2}$,*

$$\|\partial u(0)\|_{H^{n/2}} = \varepsilon \ll 1, \quad \|\partial u(0)\|_{L_x^\infty} \lesssim 1$$

and $\delta \ll 1$, we have global existence and uniqueness of the radial solutions for (1.2) in $C([0, \infty); H_{\text{rad}}^{n/2+1}) \cap C^1([0, \infty); H_{\text{rad}}^{n/2})$ with the property

$$\sum_{|a| \leq n/2} \|\partial^a u\|_{\widetilde{LE}} \lesssim \varepsilon, \quad \mu = \frac{1}{4}, \quad \mu' = \frac{(n-1)(p-1)-1}{4}.$$

In addition, for $n = 5$ and $p \in [2, 3)$, we have a similar result in $C([0, \infty); H_{\text{rad}}^3) \cap C^1([0, \infty); H_{\text{rad}}^2)$, for small $H_{\text{rad}}^3 \times H_{\text{rad}}^2$ data (without the boundedness assumption on $\partial u(0)$), with $\mu = \frac{3-p}{4}$ and $\mu' = \frac{3}{4}(p-1)$.

Remark 1. Comparing Theorems 1.3, 1.4 and 1.2 applied to radial solutions, we see that we could prove global existence of radial solutions for $p > 1 + 2/(n-1)$, except the case $p \in [5/3, 2)$ for $n = 5$ and $p \in [1 + 2/(n-2), [(n+1)/2]]$ for $n \geq 6$. As we have mentioned, it is mainly due to the limited regularity of the nonlinearity. It will be interesting to fill this gap. By exploiting Strichartz estimates, one may shrink the gap further. However, it seems that the known Strichartz estimates for $\square_{\mathfrak{g}}$ have more restriction on the assumption of the metric than (H1) (for example,

an additional requirement $|g_1(t, r)| \lesssim \delta r^\mu$ for some $\mu > 0$ may be required; see e.g. Metcalfe and Tataru [17] and the references therein).

Remark 2. One may ask if we can prove similar results as Theorem 1.3 on nontrapping asymptotically Euclidean manifolds, as in [2], [24] and [30] for other related problems. However, it seems to us that the current available local energy estimates are not enough to control all of the required local information.

This paper is organized as follows. In the next section, we give several Sobolev-type estimates, including the trace estimates. In Section 3, we give various forms of the local energy estimates and KSS estimates for $\square_g$ satisfying (H1) or (H2), together with the higher order estimates. The remaining sections are devoted to the proof of the theorems.

2. SOBOLEV-TYPE ESTIMATES

In this section, we give several Sobolev-type estimates, including the trace estimates. At first, let us recall the trace estimates (see (1.3), (1.7) in Fang and the author [4] and the references therein).

Lemma 2.1 (Trace estimates). *Let $n \geq 2$ and $1/2 < s < n/2$. Then*

$$(2.1) \quad \|r^{n/2-s}u\|_{L_r^\infty L_\omega^2} \lesssim_s \|u\|_{\dot{H}^s}, \|r^{(n-1)/2}u\|_{L_r^\infty L_\omega^2} \lesssim_s \|u\|_{H^s},$$

in particular, for radial functions,

$$(2.2) \quad \|r^{n/2-s}\langle r \rangle^{s-1/2}u\|_{L_x^\infty} \lesssim_s \|u\|_{H_{grad}^s}.$$

We will also need the following variant of the Sobolev embeddings.

Lemma 2.2. *Let $n \geq 2$. For any $m \in \mathbb{R}$ and $k \geq n/2 - n/q$ with $q \in [2, \infty)$, we have*

$$(2.3) \quad \|\langle r \rangle^{(n-1)(1/2-1/q)+m}u\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{|a| \leq k} \|\langle r \rangle^m Y^a u\|_{L^2(\mathbb{R}^n)}.$$

Moreover, we have

$$(2.4) \quad \|\langle r \rangle^{(n-1)/2+m}u\|_{L^\infty(\mathbb{R}^n)} \lesssim \sum_{|a| \leq [(n+2)/2]} \|\langle r \rangle^m Y^a u\|_{L^2(\mathbb{R}^n)},$$

where $[k]$ stands for the integer part of k .

Proof. We employ a similar proof of related estimates as in Lemma 3.1 of Lindblad, Metcalfe, Sogge, Tohaneanu and the author [15]. For $q < \infty$, by Sobolev's lemma $H^k \subset L^q$ on $\mathbb{R} \times \mathbb{S}^{n-1}$, we have for each $j \geq 0$ the uniform bounds

$$\left(\int_{j+1}^{j+2} \int_{\mathbb{S}^{n-1}} |v|^q d\omega dr \right)^{\frac{1}{q}} \lesssim \sum_{|a| \leq k} \left(\int_j^{j+3} \int_{\mathbb{S}^{n-1}} |Y^a v|^2 d\omega dr \right)^{\frac{1}{2}}.$$

Hence,

$$\|v\|_{L_x^q(r \in [j+1, j+2])} \lesssim j^{-(n-1)(1/2-1/q)} \sum_{|a| \leq k} \|Y^a v\|_{L_x^2(r \in [j, j+3])}.$$

Or more generally, for any $m \in \mathbb{R}$,

$$\|r^{m+(n-1)(1/2-1/q)}v\|_{L_x^q(r \in [j+1, j+2])} \lesssim \sum_{|a| \leq k} \|r^m Y^a v\|_{L_x^2(r \in [j, j+3])}.$$

The factor $j^{-(n-1)(1/2-1/q)}$ on the right comes from the fact that the volume element for $\mathbb{R}^n$ is $r^{n-1}drd\omega$. Recall that Sobolev's lemma on $\mathbb{R}^n$ also gives us

$$\|\langle r \rangle^{m+(n-1)(1/2-1/q)} v\|_{L_x^q(r \in [0,1])} \lesssim \sum_{|a| \leq k} \|\langle r \rangle^m Y^a v\|_{L_x^2(r \in [0,2])}.$$

The above two inequalities together imply (2.3) if we l^q -sum over $j \geq 0$ using the Minkowski integral inequality. A direct modification of the proof yields the case $q = \infty$, (2.4), which completes the proof. $\square$

In the last section, we will also need the weighted Hardy-Littlewood-Sobolev inequalities (Stein and Weiss [25]). Here, we just record a particular case which we will use.

Lemma 2.3. *Let $n \geq 2$. For any $k \in (0, n)$ and $\alpha \in (-n/2, n/2 - k)$, we have*

$$(2.5) \quad \|r^{-\alpha-k} u\|_{L^2(\mathbb{R}^n)} \lesssim \|r^{-\alpha} \sqrt{-\Delta}^k u\|_{L^2(\mathbb{R}^n)}.$$

In particular, when $k = 2l$ with $l \in \mathbb{Z}_+$, we have

$$(2.6) \quad \|r^{-\alpha-k} u\|_{L^2(\mathbb{R}^n)} \lesssim \|r^{-\alpha} (-\Delta)^l u\|_{L^2(\mathbb{R}^n)}.$$

3. LOCAL ENERGY ESTIMATES

In this section, we collect the required local energy estimates for $\square_{\mathfrak{g}}$, together with the higher order estimates.

3.1. Local energy estimates.

3.1.1. Asymptotically flat manifolds.

Lemma 3.1. *Let $n \geq 3$ and consider the linear problem $\square_{\mathfrak{g}} u = F$ on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H1) with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Then there exists a constant δ_0 , such that for any $\delta \leq \delta_0$, we have the following local energy estimates,*

$$(3.1) \quad \|u\|_{LE} \lesssim \|\partial u(0)\|_{L_x^2} + \|F\|_{LE^* + L_t^1 L_x^2}.$$

This result was essentially proved in Section 2 of [9] and [16]; see also Lemma 3.2 of [10] and [31]. Here, let us record one version of the local energy estimates, which are consequences of the classical positive commutator method (also known as Friedrichs' abc method).

Lemma 3.2. *Let $f = f(r)$ be any fixed differential function. For any solution $u \in C^\infty([0, T], C_0^\infty(\mathbb{R}^n))$ to the equation*

$$(3.2) \quad (\partial_t^2 - \Delta + h^{\alpha\beta}(t, x) \partial_\alpha \partial_\beta) u = F$$

in $S_T = [0, T] \times \mathbb{R}^n$ with $h^{\alpha\beta} = h^{\beta\alpha}$, $\sum_{0 \leq \alpha, \beta \leq n} |h^{\alpha\beta}| \leq 1/2$ and $n \geq 3$, we have

$$(3.3) \quad \int_{S_T} Q dx dt = \int_{S_T} f F \left(\partial_r + \frac{n-1}{2r} \right) u dx dt - \int_{\mathbb{R}^n} P^0(t, \cdot) dx \Big|_0^T,$$

where $P^0 = f(m+h)^{0\beta} \partial_\beta u \left(\partial_r + \frac{n-1}{2r} \right) u$,

$$(3.4) \quad Q_0 = \frac{2f - rf'}{r} \frac{|\nabla u|^2}{2} + f' \frac{|\partial_r u|^2 + |\partial_t u|^2}{2} - \frac{n-1}{4} \Delta \left(\frac{f}{r} \right) u^2$$

and

$$\begin{aligned} Q &= Q_0 + \left(\frac{f}{2} \partial_r h^{\alpha\beta} + \frac{f'}{2} h^{\alpha\beta} \right) \partial_\alpha u \partial_\beta u - f \partial_\gamma h^{\gamma\beta} \partial_r u \partial_\beta u - h^{i\beta} \frac{f}{r} \partial_i u \partial_\beta u \\ &\quad - \omega_i h^{i\beta} f' \partial_r u \partial_\beta u - \frac{n-1}{2} \left(\omega_i h^{i\beta} \left(\frac{f}{r} \right)' + \frac{f}{r} \partial_\gamma h^{\gamma\beta} \right) u \partial_\beta u \end{aligned}$$

with $\omega_i = x^i/r$, $\partial_i u = \partial_i u - \omega_i \partial_r u$, $|\nabla u|^2 = |\nabla u|^2 - |\partial_r u|^2$.

This is essentially coming from multiplying $f(r) (\partial_r + \frac{n-1}{2r}) u$ to the wave equation and a tedious calculation of integration by parts. See e.g. [16, p. 200 (5.4)]. Similar to the energy estimates, Lemma 3.2 is robust enough for us to obtain many interesting and useful estimates, particularly local energy and KSS estimates.

Typically, f is chosen to be differential functions satisfying

$$(3.5) \quad f \leq 1, 2f \geq r f'(r) \geq 0, -\Delta(f/r) \geq 0,$$

which ensure that Q_0 is positive semidefinite. In the literature, the typical choices include $f = 1$ [19], $1 - (3+r)^{-\delta}$ ($\delta > 0$) [27], $r/(R+r)$ [16, 26], $(r/(R+r))^{2\mu}$ ($\mu \in (0, 1/2)$, [9, 10]).

With the help of Lemma 3.2 with $f_j = r/(2^j + r)$ and $j \geq 0$, and the classical energy estimates, we can give the proof of Lemma 3.1.

At first, we notice that

$$(3.6) \quad \square_g = \square + (g^{\alpha\beta} - m^{\alpha\beta}) \partial_\alpha \partial_\beta + r_1 \partial.$$

Here, and in what follows, we use r_m to denote functions such that

$$|\partial_{t,x}^a r_m(t, x)| \leq C_a \delta \langle r \rangle^{-\rho-m-|a|}.$$

Then the equation $\square_g u = F$ is equivalent to

$$(3.7) \quad \square u + (g^{\alpha\beta} - m^{\alpha\beta}) \partial_\alpha \partial_\beta u = F - r_1 \partial u \equiv G.$$

With $f_j = r/(R+r)$ and $R \geq 1$, through a direct calculation, it is easy to check from (3.4) that Q_0 is comparable to

$$(3.8) \quad \frac{R}{(R+r)^2} \left[|\partial_r u|^2 + |\partial_t u|^2 + \frac{R+r}{R} |\nabla u|^2 + \left(1 + (n-3) \frac{R+r}{R} \right) \frac{|u|^2}{r(R+r)} \right].$$

By restricting the integral region to $\{r \leq R\}$, we get

$$\int_{S_T \cap \{r \leq R\}} \frac{1}{R} \left(|\partial u|^2 + \frac{|u|^2}{rR} \right) dx dt \lesssim \int_{S_T} Q_0 dx dt.$$

Together with Hardy's inequality and a cutoff argument¹ for the part $\{r < 1\}$, i.e.,

$$\|u/r\|_{L_x^2(r<1)} \leq \|\phi(r)u/r\|_{L_x^2} \lesssim \|\nabla(\phi(r)u)\|_{L_x^2} \lesssim \|u\|_{L_x^2(1<r<2)} + \|\partial u\|_{L_x^2(r<2)},$$

for some smooth cutoff function ϕ , we see that

$$\|u\|_{LE}^2 \lesssim \sup_{R \geq 1} \int_{S_T} Q_0 dx dt.$$

¹The author learned this simple but clever argument from Jason Metcalfe.

Recalling (H1.1) and (3.12), we could control the error terms $\int (Q - Q_0) dx dt$ by $\delta \|u\|_{LE}^2$ (independent of $R \geq 1$), which yields

$$\|u\|_{LE}^2 \lesssim \sup_{R \geq 1} \int_{S_T} Q dx dt ,$$

for small enough δ .

Applying Lemma 3.2 and the standard energy inequality to equation (3.7), we could now obtain

$$(3.9) \quad \|u\|_{LE} \lesssim \|\partial u(0)\|_{L_x^2} + \|G\|_{L_t^1 L_x^2 + LE^*} .$$

Since $\rho > 0$, we know that

$$\|r_1 \partial u\|_{LE^*} \lesssim \delta \|u\|_{LE} .$$

This completes the proof of (3.1), in view of (3.9) and the smallness of δ .

3.1.2. Nontrapping asymptotically Euclidean manifolds.

Lemma 3.3. *Let $n \geq 3$ and consider the linear problem $\square_{\mathfrak{g}} u = F$ on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H2) with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Then for any positive constant $\mu > 0$, we have the following local energy estimates:*

$$(3.10) \quad \|u\|_{LE} \lesssim_{\mu} \|\partial u(0)\|_{L_x^2} + \|F\|_{L_t^1 L_x^2 + l_2^{1/2+\mu} L_t^2 L_x^2} .$$

Proof. It is essentially proved in Sogge and the author [24], where the authors used a cutoff argument to prove KSS estimates based on a weaker version of the local energy estimates of Bony and Häfner [2], which states that for any $\mu > 0$,

$$(3.11) \quad \|\partial u\|_{l_2^{-1/2-\mu} L_t^2 L_x^2} + \|u\|_{l_2^{-3/2-\mu} L_t^2 L_x^2} \lesssim_{\mu} \|\partial u(0)\|_{L_x^2} + \|F\|_{L_t^1 L_x^2 + l_2^{1/2+\mu} L_t^2 L_x^2} .$$

We need only to adapt that proof and notice that we could apply Lemma 3.1 for the part near spatial infinity. We leave the details to the reader. Here, it may be interesting to point out that such an argument could yield (3.10) for any $\mu > 0$ from (3.11) with certain $\mu_0 > 0$. $\square$

Remark 3. As is clear from the proof, we do not need the radial assumption for g_1 , in Lemmas 3.1, 3.3 and 3.6. The radial assumption will be required, however, for the higher order estimates in Lemmas 3.5 and 3.7.

3.1.3. KSS estimates. To prove almost global or longtime existence, one may exploit the KSS estimates (see e.g. [12, 16]). It is known that the KSS estimates are essentially contained in the local energy estimates (for $|x| \leq 2 + T$) and energy estimates (for $|x| \geq 2 + T$), which we record as the following lemma. One may consult [16] for the proof.

Lemma 3.4. *For any $\mu_1 > 0$, $\mu_2 \in [0, 1/2)$, there are positive constants C_{μ_1} , C_{μ_2} and C , independent of $T > 0$, such that*

$$(3.12) \quad \|\partial u\|_{l_2^{-1/2-\mu_1}(L_T^2 L_x^2)} + \|r^{-1} u\|_{l_2^{-1/2-\mu_1}(L_T^2 L_x^2)} \leq C_{\mu_1} \|u\|_{LE([0, T] \times \mathbb{R}^n)} ,$$

$$(3.13) \quad \|\partial u\|_{l_2^{-1/2}(L_T^2 L_x^2)} + \|r^{-1} u\|_{l_2^{-1/2}(L_T^2 L_x^2)} \leq C(\ln(2 + T))^{1/2} \|u\|_{LE([0, T] \times \mathbb{R}^n)} ,$$

$$(3.14) \quad \|\partial u\|_{l_2^{-\mu_2}(L_T^2 L_x^2)} + \|r^{-1} u\|_{l_2^{-\mu_2}(L_T^2 L_x^2)} \leq C_{\mu_2} (2 + T)^{1/2-\mu} \|u\|_{LE([0, T] \times \mathbb{R}^n)} .$$

3.1.4. Higher order estimates.

Lemma 3.5. *Let $n \geq 3$ and consider the linear problem $\square_{\mathfrak{g}} u = F$ on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H1) or (H2), with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Then for small enough δ (in the case of (H1)), and for any positive constant $\mu > 0$, we have the following higher order local energy estimates:*

$$(3.15) \quad \|u\|_{LE_k} \lesssim \sum_{|a| \leq k} \|\partial Y^a u(0)\|_{L_x^2} + \|Y^a F\|_{L_t^1 L_x^2 + l_2^{1/2+\mu} L_t^2 L_x^2}.$$

Proof. The estimates for the asymptotically Euclidean manifolds are essentially contained in [2, 24], based on Lemma 3.3. Here, we give only the proof for the asymptotically flat manifolds (H1), in a manner similar to that of [2, 24], by induction. In this case, let us prove a stronger result

$$(3.16) \quad \|u\|_{LE_k} \lesssim \sum_{|a| \leq k} \|\partial Y^a u(0)\|_{L_x^2} + \|Y^a F\|_{L_t^1 L_x^2 + LE^*}.$$

At first, Lemma 3.1 tells us that it is true for $k = 0$. Assuming it is true for some $m \geq 0$, then since $\square_{\mathfrak{g}} u = F$ and (H1), we have

$$(3.17) \quad \square_{\mathfrak{g}} Yu = [\square_{\mathfrak{g}}, Y]u + YF = r_1 \partial^2 u + r_2 \partial u + YF = r_1 \partial \nabla u + r_2 \partial u + YF + r_1 F,$$

where we have used the facts that $\partial_t^2 u = \Delta u + r_0 \partial \nabla u + F + r_0 F$ and the radial part of the metric has no effect in the commutator with rotational vector fields Ω . Applying (3.16) to Yu with $k = m$ and noticing (3.17), we see that $\|Yu\|_{LE_m}$ can be controlled by

$$\begin{aligned} &\lesssim \sum_{|a| \leq m} (\|\partial Y^a Yu(0, \cdot)\|_{L_x^2} + \|Y^a \square_{\mathfrak{g}} Yu\|_{L_t^1 L_x^2 + LE^*}) \\ &\lesssim \sum_{|a| \leq m} (\|\partial Y^a Yu(0, \cdot)\|_{L_x^2} + \|Y^a (r_1 \partial \nabla u + r_2 \partial u + YF + r_1 F)\|_{L_t^1 L_x^2 + LE^*}) \\ &\lesssim \sum_{|a| \leq m+1} (\|\partial Y^a u(0, \cdot)\|_{L_x^2} + \|Y^a F\|_{L_t^1 L_x^2 + LE^*}) + \sum_{|a| \leq k-1} \|Y^a (r_1 \partial \nabla u + r_2 \partial u)\|_{LE^*}. \end{aligned}$$

Recall that

$$Y^a (r_1 \partial \nabla u) = r_1 \partial Y^a \nabla u + \sum_{|b| \leq |a|} r_1 \partial Y^b u,$$

$$Y^a (r_2 \partial u) = r_2 \partial Y^a u + \sum_{|b| \leq |a|} r_2 \partial Y^b u.$$

Since $\rho > 0$, it is easy to see that

$$\sum_{|a| \leq m} \|Y^a (r_1 \partial \nabla u + r_2 \partial u)\|_{LE^*} \lesssim \sum_{|a| \leq m+1} \|r_1 \partial Y^a u\|_{LE^*} \lesssim \delta \sum_{|a| \leq m+1} \|Y^a u\|_{LE},$$

which completes the proof of (3.16), for small enough δ . $\square$

3.2. Local energy estimates, version 2. In this subsection, for asymptotically flat manifolds, we will prove another version of the local energy estimates, which gives better control on the local part.

Lemma 3.6 (Local energy estimates). *Let $n \geq 3$ and consider the linear problem $\square_{\mathfrak{g}} u = F$ on the manifold $(\mathbb{R}^{1+n}, \mathfrak{g})$ satisfying (H1) with $\rho = \min(\rho_1, \rho_2 - 1) > 0$. Then for any $\mu \in (0, 1/2)$, $\mu' - \mu \in (0, \rho/2]$, and small enough $\delta > 0$,*

$$(3.18) \quad \|u\|_{\widetilde{LE}} \lesssim \|\partial u(0, \cdot)\|_{L_x^2} + \|\square_{\mathfrak{g}} u\|_{L_t^1 L_x^2 + \widetilde{LE}^*}.$$

Proof. This result has essentially been proved in Lemma 2.2 of [9], using multipliers with $f_j = r/(2^j + r)$ ($j \geq 1$) and $f_0 = (r/(1+r))^{2\mu}$ ($\mu \in (0, 1/2)$). Here, we use a single multiplier² to give an alternative proof of the required local energy estimates.

The proof proceeds as that of Lemma 3.1 and we need only to point out the differences here. For $\mu \in (0, 1/2)$, $\mu' > \mu$, we set

$$(3.19) \quad f(r) = \left(\frac{r}{3+r} \right)^{2\mu} \left(1 - (3+r)^{-2(\mu'-\mu)} \right).$$

Through a direct calculation, it is easy to check from (3.4) that Q_0 is comparable to

$$(3.20) \quad r^{2\mu-1} \langle r \rangle^{-2\mu'} \left[|\partial_r u|^2 + |\partial_t u|^2 + \langle r \rangle^{2(\mu'-\mu)} |\nabla u|^2 + (1 + (n-3) \langle r \rangle^{2(\mu'-\mu)}) r^{-2} u^2 \right],$$

which controls the integrand in the local energy norm. Applying Lemma 3.2 and the standard energy inequality to equation (3.7), we obtain

$$(3.21) \quad \|u\|_{\widetilde{LE}} \lesssim \|\partial u(0)\|_{L_x^2} + \|G\|_{L_t^1 L_x^2 + \widetilde{LE}^*}.$$

Since $2\mu' \leq \rho + 2\mu$, we have

$$\|r_1 \partial u\|_{\widetilde{LE}^*} \lesssim \delta \|u\|_{\widetilde{LE}},$$

and this completes the proof of (3.18), for small enough δ . $\square$

To prove the radial Glassey conjecture, we need to obtain higher order local energy estimates involving the vector fields.

Lemma 3.7. *Under the same assumption of Lemma 3.6, we have the higher order local energy estimates for solutions to $\square_{\mathfrak{g}} u = F$ with order $m \geq 0$:*

$$(3.22) \quad \|u\|_{\widetilde{LE}_m} \leq C \sum_{|a| \leq m} (\|\partial \nabla^a u(0, \cdot)\|_{L_x^2} + \|\nabla^a F\|_{L_t^1 L_x^2 + \widetilde{LE}^*}).$$

In addition, we have

$$(3.23) \quad \sum_{|a| \leq m} \|\partial^a u\|_{\widetilde{LE}} \leq C \sum_{|a| \leq m} (\|\partial \partial^a u(0, \cdot)\|_{L_x^2} + \|\partial^a F\|_{L_t^1 L_x^2 + \widetilde{LE}^*}).$$

The proof is basically the same as that of Lemma 3.5 and we omit the details here.

²The idea of constructing one multiplier is inspired by the discussion with Hans Lindblad when the author was working on [15].

3.2.1. *Higher order local energy estimates.* Here, we record an alternative higher order local energy estimate, which shall be sufficient for us to adapt the current proof to obtain the same set of Theorems in [10], for $n \geq 3$.

Lemma 3.8. *Under the same assumption of Lemma 3.6, we have the higher order local energy estimates for solutions to $\square_{\mathbf{g}} u = F$:*

$$(3.24) \quad \|\nabla u\|_{\widehat{LE}} \lesssim \|\partial \nabla u(0, \cdot)\|_{L_x^2} + \|\nabla F\|_{\widehat{LE}^*} + \|r_1 F\|_{\widehat{LE}^*} + \|r_2 \partial_t u\|_{\widehat{LE}^*}.$$

In particular, letting $\|u\|_{\widehat{E}_2} = \|\nabla u\|_E + \delta \|u\|_E$ and $\widehat{LE}_2, \widehat{LE}_2^*$ by similar fashion, we have

$$(3.25) \quad \|u\|_{\widehat{LE}_2} \lesssim \|\partial \nabla u(0, \cdot)\|_{L_x^2} + \delta \|\partial u(0, \cdot)\|_{L_x^2} + \|F\|_{\widehat{LE}_2^*}.$$

Proof. Since $\square_{\mathbf{g}} u = F$ and (H1), we have

$$(3.26) \quad \square_{\mathbf{g}} \nabla u = [\square_{\mathbf{g}}, \nabla] u + \nabla F = r_1 \partial^2 u + r_2 \partial u + \nabla F = r_1 \partial \nabla u + r_2 \partial u + \nabla F + r_1 F,$$

where we have used the fact that $\partial_t^2 u = \Delta u + r_0 \partial \nabla u + F + r_0 F$. Notice that

$$\|r_1 \partial \nabla u + r_2 \nabla u\|_{\widehat{LE}^*} \leq C \delta \|\nabla u\|_{\widehat{LE}}.$$

We see that, by applying Lemma 3.6,

$$\begin{aligned} \|\nabla u\|_{\widehat{LE}} &\lesssim \|\partial \nabla u(0, \cdot)\|_{L_x^2} + \|\square_{\mathbf{g}} \nabla u\|_{\widehat{LE}^*} \\ &\lesssim \|\partial \nabla u(0, \cdot)\|_{L_x^2} + \delta \|\nabla u\|_{\widehat{LE}} + \|\nabla F\|_{\widehat{LE}^*} + \|r_1 F\|_{\widehat{LE}^*} + \|r_2 \partial_t u\|_{\widehat{LE}^*}, \end{aligned}$$

which gives us (3.24). For (3.25), since $\|r_2 \partial_t u\|_{\widehat{LE}^*} \leq C \delta \|u\|_{\widehat{LE}}$, we need only to combine (3.18) with (3.24). $\square$

4. GLASSEY CONJECTURE WITH DIMENSION 3

In this section, we will prove Theorem 1.1, mainly based on Lemmas 2.2 and 3.5, for given f and g such that (1.6) is satisfied.

As usual, we shall use iteration to give the proof. We set $u_0 \equiv 0$ and recursively define u_{k+1} to be the solution to the linear equation

$$\square_{\mathbf{g}} u_{k+1} = F_p(u_k), u(0, x) = f(x), \partial_t u(0, x) = g(x).$$

Boundedness: By the smallness condition (1.6) on the data, it follows from Lemma 3.5 that there is a universal constant C_1 so that

$$(4.1) \quad \|u_1\|_{LE_2} \leq C_1 \varepsilon, \quad \|u_{k+1}\|_{LE_2} \leq C_1 \varepsilon + C_1 \sum_{|a| \leq 2} \|Y^a F_p(u_k)\|_{L_t^1 L_x^2}.$$

We shall argue inductively to prove that

$$(4.2) \quad \|u_{k+1}\|_{LE_2} \leq 2C_1 \varepsilon.$$

By the above, it suffices to show

$$(4.3) \quad \sum_{|a| \leq 2} \|Y^a F_p(u)\|_{L_t^1 L_x^2} \leq \varepsilon,$$

for any u with $\|u\|_{LE_2} \leq 2C_1 \varepsilon \leq 1$.

Notice that there exist smooth functions b_i , $1 \leq i \leq 5$, such that

$$\begin{aligned} Y^{\leq 2} F_p(u) &= b_1(u) |\partial u|^{p-1} Y^{\leq 2} \partial u + b_2(u) |\partial u|^{p-2} (Y^{\leq 1} \partial u)^2 \\ &\quad + b_3(u) |\partial u|^{p-1} Y u Y^{\leq 1} \partial u + b_4(u) |\partial u|^p Y u Y u + b_5(u) |\partial u|^p Y^2 u. \end{aligned}$$

By Lemma 2.2,

$$(4.4) \quad |\partial u| \lesssim \frac{\|u\|_{E_2}}{\langle r \rangle}, \quad |Yu| \lesssim \|u\|_{E_2}.$$

Moreover, since $u(0) \in L_x^2$ and $\partial_t u \in L_t^\infty L_x^2$, we have $u(t) \in L_x^2$ for any t , which ensures that

$$(4.5) \quad \|u(t)\|_{L_x^\infty} \leq C \|\nabla u\|_{L_x^2} + C \|\nabla^2 u\|_{L^2} \leq C \|u\|_{E_2}.$$

Here, the constant C is independent of the L^2 norm of $u(t)$. By the boundedness of u , smoothness of b_i and (4.4), we see that

$$|Y^{\leq 2} F_p(u)| \lesssim |\partial u|^{p-1} (|Y^{\leq 2} \partial u| + |Y^{\leq 2} u|/\langle r \rangle) + |\partial u|^{p-2} |Y^{\leq 1} \partial u|^2.$$

The first term can be dealt with as follows, by (4.4), Lemma 2.2 and Lemma 3.4 with $\mu_1 = (p-2)/2$,

$$\begin{aligned} & \| |\partial u|^{p-1} (|Y^{\leq 2} \partial u| + |Y^{\leq 2} u|/\langle r \rangle) \|_{L_t^1 L_x^2} \\ & \lesssim \|\langle r \rangle \partial u\|_{L_t^\infty L_x^\infty}^{p-2} \|\langle r \rangle^{(3-p)/2} \partial u\|_{L_t^2 L_x^\infty} \|\langle r \rangle^{-(p-1)/2} \left(|Y^{\leq 2} \partial u| + \frac{|Y^{\leq 2} u|}{\langle r \rangle} \right) \|_{L_t^2 L_x^2} \\ & \lesssim \|u\|_{LE_2}^p. \end{aligned}$$

Similarly, for the second term, we get

$$\begin{aligned} \| |\partial u|^{p-2} |Y^{\leq 1} \partial u|^2 \|_{L_t^1 L_x^2} & \lesssim \|\langle r \rangle \partial u\|_{L_t^\infty L_x^\infty}^{p-2} \|\langle r \rangle^{-(p-2)/2} Y^{\leq 1} \partial u\|_{L_t^2 L_x^4}^2 \\ & \lesssim \|u\|_{E_2}^{p-2} \|\langle r \rangle^{-(p-2)/2-1/2} Y^{\leq 2} \partial u\|_{L_t^2 L_x^2}^2 \\ & \lesssim \|u\|_{LE_2}^p. \end{aligned}$$

In conclusion, we see that there exists a constant C_2 such that

$$(4.6) \quad \sum_{|a| \leq 2} \|Y^a F_p(u)\|_{L_t^1 L_x^2} \leq C_2 \|u\|_{LE_2}^p \leq C_2 (2C_1 \varepsilon)^p \leq \varepsilon$$

for $\varepsilon \leq \varepsilon_0$ with

$$C_2 (2C_1)^p \varepsilon_0^p \leq 1.$$

This finishes the proof of (4.3) and so is the uniform boundedness (4.2).

Convergence of the sequence $\{u_k\}$: Notice that

$$F_p(u) - F_p(v) = \sum_{\alpha} a_{\alpha}(u) (|\partial u|^p - |\partial v|^p) + |\partial v|^p (a_{\alpha}(u) - a_{\alpha}(v)).$$

Let $\varepsilon_0 > 0$ be such that $2C_4 \varepsilon_0 \leq 1$. Since a_{α} are smooth functions, it is clear from (4.5), (4.2) that

$$|a_{\alpha}(u) - a_{\alpha}(v)| \lesssim |u - v|, \quad |a_{\alpha}(u)| \lesssim 1, \quad \forall u, v \in \{u_k\}.$$

Based on this observation, (4.4) and the fact $p > 2$, together with Lemma 2.2 and Lemma 3.4 with $\mu_1 = (p-2)/2$, we see that

$$\begin{aligned} & \|F_p(u) - F_p(v)\|_{L_t^1 L_x^2} \\ & \lesssim \|(|\partial u| + |\partial v|)^{p-1} (|\partial(u-v)| + |u-v|/\langle r \rangle)\|_{L_t^1 L_x^2} \\ & \lesssim (\|\langle r \rangle \partial u\|_{L_t^\infty L_x^\infty} + \|\langle r \rangle \partial v\|_{L_t^\infty L_x^\infty})^{p-2} \\ & \quad \times (\|\langle r \rangle^{1/2-(p-2)/2} \partial u\|_{L_t^2 L_x^\infty} + \|\langle r \rangle^{1/2-(p-2)/2} \partial v\|_{L_t^2 L_x^\infty}) \|u - v\|_{LE} \\ & \lesssim (\|u\|_{LE_2} + \|v\|_{LE_2})^{p-1} \|u - v\|_{LE}, \end{aligned}$$

that is, there exists a constant C_3 such that for any $u, v \in \{u_k\}$,

$$\|F_p(u) - F_p(v)\|_{L_t^1 L_x^2} \leq C_3(\|u\|_{LE_2} + \|v\|_{LE_2})^{p-1} \|u - v\|_{LE} \leq \|u - v\|_{LE} / (2C_1) ,$$

$$\|u_{k+1} - u_k\|_{LE} \leq C_1 \|F_p(u_k) - F_p(u_{k-1})\|_{L_t^1 L_x^2} \leq \frac{1}{2} \|u_k - u_{k-1}\|_{LE}$$

provided that $\varepsilon \leq \varepsilon_0$ with

$$2C_1 C_3 (4C_1)^{p-1} \varepsilon_0^{p-1} \leq 1 .$$

Together with the uniform boundedness (4.2), we find a unique global solution $u \in L_t^\infty H^3 \cap Lip_t H^2$ with $\|u\|_{LE_2} \leq 2C_1 \varepsilon$.

Regularity of the solution: To complete the proof, it remains to prove the regularity of the solution $u \in C_t H^3 \cap C_t^1 H^2$. Since $L_t^\infty H^3 \cap Lip_t H^2 \subset C_t H^2 \cap C_t^1 H^1$, it suffices to prove $\partial u \in C_t H^2$. For any $t_1 > t_0 \geq 0$, we set w to be the solution to the homogeneous wave equation

$$\square_g w = 0, w(t_0) = u(t_0), \partial_t w(t_0) = \partial_t u(t_0) ,$$

and so

$$\square_g(u - w) = F_p(u), (u - w)(t_0) = 0, \partial_t(u - w)(t_0) = 0.$$

It is well known that $w \in C_t H^3 \cap C_t^1 H^2$. Now, by Lemma 3.5,

$$\begin{aligned} \|u(t_1) - u(t_0)\|_{E_2} &\leq \|u(t_1) - w(t_1)\|_{E_2} + \|w(t_1) - u(t_0)\|_{E_2} \\ &\leq \sum_{|a| \leq 2} \|Y^a F_p(u)\|_{L_t^1([t_0, t_1]; L_x^2)} + o(t_1 - t_0) . \end{aligned}$$

Since $Y^a F_p(u) \in L_t^1 L_x^2$ (see (4.6)), it is clear that $\|Y^a F_p(u)\|_{L_t^1([t_0, t_1]; L_x^2)} = o(t_1 - t_0)$ and so is $\|u(t_1) - u(t_0)\|_{E_2}$, which completes the proof of the regularity.

5. HIGH DIMENSIONAL WAVE EQUATIONS

In this section, for high dimensional wave equations, we will prove Theorem 1.2, based on Lemmas 2.2 and 3.5, for given f and g such that (1.7) is satisfied.

We will proceed as in Section 4 to give the proof and we give only the proof of the uniform boundedness of the iteration series u_k .

Based on (4.1), we shall argue inductively to prove that

$$(5.1) \quad \|u_{k+1}\|_{LE_m} \leq 2C_1 \varepsilon ,$$

for which it suffices to show

$$(5.2) \quad \sum_{|a| \leq m} \|Y^a F_p(u)\|_{L_T^1 L_x^2} \leq \varepsilon ,$$

for any u with $\|u\|_{LE_m} \leq 2C_1 \varepsilon \leq 1$.

We are assuming $a_\alpha(u) = a_\alpha$, and so $|Y^{\leq m} F_p(u)|$ is controlled by terms of the following form:

$$(5.3) \quad |\partial u|^{p-j} \prod_{i=1}^j |Y^{a_i} \partial u| , 1 \leq a_i \leq a_{i+1}, \sum_{i=1}^j a_i \leq m .$$

Here, we have abused the notation a_i with the order $|a_i|$. We will consider two cases separately: 1) $j \leq 1$, 2) $j \geq 2$.

Case 1) ($j \leq 1$). In this case, by Lemma 2.2, we see that

$$\begin{aligned} \|\partial u\|^{p-1} Y^{\leq m} \partial u \|_{L_t^1 L_x^2} &\leq \|\langle r \rangle^{(n-1)/2} \partial u\|_{L_t^\infty L_x^\infty}^{p-2} \|\langle r \rangle^{(n-1)/2 - (n-1)(p-1)/4} \partial u\|_{L_t^2 L_x^\infty} \\ &\quad \times \|\langle r \rangle^{-(n-1)(p-1)/4} Y^{\leq m} \partial u\|_{L_t^2 L_x^2} \\ &\lesssim \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2}^{p-2} \|\langle r \rangle^{-(n-1)(p-1)/4} Y^{\leq m} \partial u\|_{L_t^2 L_x^2}^2. \end{aligned}$$

Case 2) ($j \geq 2$). Since $a_1 \geq 1$ and $\sum a_i \leq m$, we get $a_i \in [1, [n/2]]$ and

$$c_i = m - a_i \in [1, [n/2]] .$$

Fix $\varepsilon_i \in (0, c_i)$, such that

$$(5.4) \quad \sum \varepsilon_i = -\frac{n}{2}(j-1) + \sum c_i .$$

Notice that it is possible since

$$-\frac{n}{2}(j-1) + \sum c_i = -\frac{n}{2}(j-1) + mj - \sum a_i \geq (j-1)(m - \frac{n}{2}) > 0 .$$

We can define $p_i \in (2, \infty)$ through

$$n(\frac{1}{2} - \frac{1}{p_i}) = c_i - \varepsilon_i \in (0, n/2) .$$

Here, we notice that (5.4) ensures that $\sum 1/p_i = 1/2$.

Now, we set $d_i = (n-1)(p-j)/(2j) + (n-1)(1/2 - 1/p_i)$ for $i < j$, and $d_j = -\sum_{i=1}^{j-1} d_i$. Then

$$d_j - (n-1)(p-j)/(2j) - (n-1)(1/2 - 1/p_j) = -(n-1)(p-1)/2 .$$

We could control $\|\partial u\|^{p-j} \prod_{i=1}^j Y^{a_i} \partial u\|_{L_t^1 L_x^2}$ as follows by applying Lemma 2.2 and Hölder's inequality:

$$\begin{aligned} &\leq \|\langle r \rangle^{(n-1)/2} \partial u\|_{L_t^\infty L_x^\infty}^{p-j} \prod_{i=1}^{j-2} \|\langle r \rangle^{d_i - (n-1)(p-j)/(2j)} Y^{a_i} \partial u\|_{L_t^\infty L_x^{p_i}} \\ &\quad \times \|\langle r \rangle^{d_{j-1} - (n-1)(p-j)/(2j) - (n-1)(p-1)/4} Y^{a_{j-1}} \partial u\|_{L_t^2 L_x^{p_{j-1}}} \\ &\quad \times \|\langle r \rangle^{d_j - (n-1)(p-j)/(2j) + (n-1)(p-1)/4} Y^{a_j} \partial u\|_{L_t^2 L_x^{p_j}} \\ &\lesssim \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2}^{p-j} \prod_{i=1}^{j-2} \|\langle r \rangle^{(n-1)(1/2 - 1/p_i)} Y^{a_i} \partial u\|_{L_t^\infty L_x^{p_i}} \\ &\quad \times \|\langle r \rangle^{(n-1)(1/2 - 1/p_{j-1}) - (n-1)(p-1)/4} Y^{a_{j-1}} \partial u\|_{L_t^2 L_x^{p_{j-1}}} \\ &\quad \times \|\langle r \rangle^{(n-1)(1/2 - 1/p_j) - (n-1)(p-1)/4} Y^{a_j} \partial u\|_{L_t^2 L_x^{p_j}} \\ &\lesssim \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2}^{p-j} \prod_{i=1}^{j-2} \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2} \\ &\quad \times \|\langle r \rangle^{-(n-1)(p-1)/4} Y^{\leq m} \partial u\|_{L_t^2 L_x^2}^2 \\ &\lesssim \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2}^{p-2} \|\langle r \rangle^{-(n-1)(p-1)/4} Y^{\leq m} \partial u\|_{L_t^2 L_x^2}^2 . \end{aligned}$$

In summary, we have proved that

$$\|Y^{\leq m} F_p(u)\|_{L_t^1 L_x^2} \lesssim \|Y^{\leq m} \partial u\|_{L_t^\infty L_x^2}^{p-2} \|\langle r \rangle^{-(n-1)(p-1)/4} Y^{\leq m} \partial u\|_{L_t^2 L_x^2}^2 .$$

When $p > 1 + 2/(n-1)$, by Lemma 3.4 with $\mu_1 = (n-1)(p-1)/4 - 1/2 > 0$, it is controlled by $\|u\|_{LE_m}^p$ and so is (5.2), with small enough ε . For the remaining situation, that is, $n = 3$ and $p = 2$, we have $(n-1)(p-2)/4 = 1/2$, and we get from (3.13) in Lemma 3.4 that

$$\|Y^{\leq m} F_p(u)\|_{L_T^1 L_x^2} \lesssim \ln(2+T) \|u\|_{LE_m}^2,$$

which yields (5.2), for $T \leq \exp(c/\varepsilon)$ with small enough ε and c .

6. RADIAL GLASSEY CONJECTURE

In this section, we give the proof for Theorem 1.3 by using Lemma 2.1 and Lemma 3.7.

We set $u_0 \equiv 0$ and recursively define u_{k+1} to be the solution to the linear equation

$$(6.1) \quad \square_g u_{k+1} = c_1 |\partial_t u_k|^p + c_2 |\nabla u_k|^p \equiv G_p(u_k), \quad u(0, x) = f, \quad \partial_t u(0, x) = g.$$

As in Section 5, we give only the proof of the uniform boundedness of the iteration series u_k with respect to $\widetilde{LE}_1$, which could be reduced to the proof of

$$(6.2) \quad \|u\|_{\widetilde{LE}_1} \leq 2C_1 \varepsilon \Rightarrow \|G_p(u)\|_{\widetilde{LE}_1^*} \leq \varepsilon$$

by Lemma 3.7 with $m = 1$.

Recall from (2.2) in Lemma 2.1 with $s = 1$ that we have for radial functions u (see e.g. [10, Lemma 2.4]),

$$(6.3) \quad \|r^{(n-2)/2} \langle r \rangle^{1/2} \partial u\|_{L_x^\infty} \lesssim \|r^{(n-2)/2} \langle r \rangle^{1/2} \partial u\|_{L_r^\infty L_\omega^2} \lesssim \|\nabla^{\leq 1} \partial u\|_{L^2}.$$

So

$$\begin{aligned} \|G_p(u)\|_{\widetilde{LE}_1^*} &\lesssim \|r^{1/2-\mu} \langle r \rangle^{\mu'} \nabla^{\leq 1} G_p(u)\|_{L^2 L^2} \\ &\lesssim \|r^{1/2-\mu} \langle r \rangle^{\mu'} |\partial u|^{p-1} \nabla^{\leq 1} \partial u\|_{L^2 L^2} \\ &\lesssim \|r^{(n-2)/2} \langle r \rangle^{1/2} \partial u\|_{L_{t,x}^\infty}^{p-1} \|r^{1/2-\mu-(n-2)(p-1)/2} \langle r \rangle^{\mu'-(p-1)/2} \nabla^{\leq 1} \partial u\|_{L^2 L^2} \\ &\lesssim \|u\|_{LE_1}^p \end{aligned}$$

provided that $1/2 - \mu - (n-2)(p-1)/2 \geq -1/2 + \mu$, $1/2 - \mu - (n-2)(p-1)/2 + \mu' - (p-1)/2 \leq -1/2 + \mu - \mu'$, which is true if we set

$$\mu = \frac{1}{2} - \frac{n-2}{4}(p-1), \quad \mu' = \frac{p-1}{4}.$$

Then by choosing $\varepsilon > 0$ small enough, we get (6.2), and we have the proof for Theorem 1.3.

7. GLOBAL EXISTENCE FOR RADIAL DATA

In this section, we give the proof for Theorem 1.4, filling some gaps between Theorems 1.2 and 1.3, by using Lemma 2.1 and Lemma 3.7, as well as Lemmas 2.2 and 2.3.

7.1. Dimension $n = 5$. At first, let us give the proof for $n = 5$ and $p \in [2, 3)$. We give only the proof of the uniform boundedness here, which, by (3.23), amounts to the proof of

$$(7.1) \quad \|\nabla^{\leq 2} \partial u(0)\|_{L_x^2} \leq \varepsilon \Rightarrow \|\partial^{\leq 2} \partial u(0)\|_{L_x^2} \lesssim \varepsilon ,$$

$$(7.2) \quad \|\partial^{\leq 2} u\|_{\widetilde{LE}} \lesssim \varepsilon \Rightarrow \|\partial^{\leq 2} G_p(u)\|_{\widetilde{LE}^* + L_t^1 L_x^2} \leq \varepsilon ,$$

for radial functions and small enough ε .

For (7.1), we observe from the equation that

$$|\partial^{\leq 2} \partial u(0)| \lesssim |\nabla^{\leq 2} \partial u(0)| + |\partial u(0)|^{p-1} |\nabla^{\leq 1} \partial u(0)| + |\partial u(0)|^{2p-1} .$$

Since $p \in [2, 3)$, by Sobolev embeddings $H^1 \subset L^2 \cap L^{10/3}$ and $H^2 \subset L^2 \cap L^{10}$, we see that

$$\|\partial^{\leq 2} \partial u(0)\|_{L^2} \lesssim \|\nabla^{\leq 2} \partial u(0)\|_{L^2} + \|\nabla^{\leq 2} \partial u(0)\|_{L^2}^p + \|\nabla^{\leq 2} \partial u(0)\|_{L^2}^{2p-1} ,$$

which completes the proof of (7.1).

Recall from Lemma 2.2 with $k = 1$ that

$$\|\langle r \rangle^{(n-1)/n} u\|_{L_x^{2n/(n-2)}} \lesssim \|Y^{\leq 1} u\|_{L_x^2} ,$$

from which, together with (2.2) with $s = 1$, and Hölder's inequality, yields

$$(7.3) \quad \|r^{1/4} \langle r \rangle^{3/4} u\|_{L_r^4 L_\omega^3} \lesssim \|Y^{\leq 1} u\|_{L_x^2}, n = 5.$$

Here, we introduced the mixed-norm $L_r^\rho L_\omega^s$ with respect to the polar coordinates $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$

$$\|f\|_{L_r^\rho L_\omega^s} = \left(\int \|f(r \cdot)\|_{L_\omega^s}^\rho r^{n-1} dr \right)^{1/\rho} ,$$

with obvious modification for $\rho = \infty$. A cutoff argument gives us that for any $m_1, m_2 \in \mathbb{R}$, we have

$$(7.4) \quad \|r^{1/4+m_1} \langle r \rangle^{3/4+m_2} u\|_{L_r^4 L_\omega^3} \lesssim \|r^{m_1} \langle r \rangle^{m_2} Y^{\leq 1} u\|_{L^2} + \|r^{m_1-1} \langle r \rangle^{m_2} u\|_{L^2}, n = 5.$$

Actually, introducing a radial cutoff function $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi = 1$ for $|x| \leq 2$ and $\phi = 0$ for $|x| \geq 3$, and applying (7.3), we get

$$\begin{aligned} \|r^{1/4+m_1} \langle r \rangle^{3/4+m_2} \phi u\|_{L^4} &\lesssim \|r^{1/4} \langle r \rangle^{3/4} (\phi r^{m_1} u)\|_{L^4} \\ &\lesssim \sum_{|a| \leq 1} \|Y^a (r^{m_1} \phi u)\|_{L^2(r \leq 3)} \\ &\lesssim \|r^{m_1-1} u\|_{L^2(r \leq 3)} + \|r^{m_1} Y u\|_{L^2(r \leq 3)} , \end{aligned}$$

and for $j \geq 1$,

$$\begin{aligned} \|r^{1/4+m_1} \langle r \rangle^{3/4+m_2} u\|_{L^4(j+1 \leq r \leq j+2)} &\lesssim j^{m_1+m_2} \|r^{1/4} \langle r \rangle^{3/4} u\|_{L^4(j+1 \leq r \leq j+2)} \\ &\lesssim j^{m_1+m_2} \sum_{|a| \leq 1} \|Y^a u\|_{L^2(j \leq r \leq j+3)} . \end{aligned}$$

The above two inequalities together imply (7.4) if we l^4 -sum over $j \geq 0$.

Also, by Lemma 2.1 with $s = 2$, we have for radial functions u (see e.g., [10, Lemma 2.4]),

$$(7.5) \quad \|r^{1/2} \langle r \rangle^{3/2} \partial u\|_{L_x^\infty} \lesssim \|r^{1/2} \langle r \rangle^{3/2} \partial u\|_{L_r^\infty L_\omega^2} \lesssim \|\nabla^{\leq 2} \partial u\|_{L^2}, n = 5 .$$

It is easy to see that there are two kinds of typical terms in $\partial^{\leq 2} G_p(u)$, i.e., $|\partial u|^{p-1} \partial^{\leq 2} \partial u$ and $|\partial u|^{p-2} |\partial^2 u|^2$.

The first term can be dealt with as follows by (7.5):

$$\begin{aligned}
& \| |\partial u|^{p-1} \partial^{\leq 2} \partial u \|_{\widetilde{LE}^*} \\
& \lesssim \| r^{1/2} \langle r \rangle^{3/2} \partial u \|_{L_t^\infty L_x^\infty}^{p-1} \| r^{1/2-\mu} \langle r \rangle^{\mu'} (r^{1/2} \langle r \rangle^{3/2})^{-(p-1)} \partial^{\leq 2} \partial u \|_{L_t^2 L_x^2} \\
& \lesssim \| \nabla^{\leq 2} \partial u \|_{L_x^\infty L_x^2}^{p-1} \| r^{-1/2+\mu} \langle r \rangle^{-\mu'} \partial^{\leq 2} \partial u \|_{L_t^2 L_x^2} \\
& \lesssim \| \partial^{\leq 2} u \|_{\widetilde{LE}}^p,
\end{aligned}$$

if

$$(7.6) \quad \mu = (3-p)/4, \mu' - \mu = p - 3/2.$$

Similarly, for the second term, by (7.5), we get

$$\begin{aligned}
& \| |\partial u|^{p-2} |\partial^2 u|^2 \|_{L_t^1 L_x^2} \\
& \lesssim \| r^{1/2} \langle r \rangle^{3/2} \partial u \|_{L_t^\infty L_x^\infty}^{p-2} \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \partial^2 u \|_{L_t^2 L_x^4}^2 \\
& \lesssim \| \nabla^{\leq 2} \partial u \|_{L_t^\infty L_x^2}^{p-2} \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \partial^2 u \|_{L_t^2 L_x^4}^2.
\end{aligned}$$

Here, for the last term, we distinguish two cases: $\partial^2 u = \partial \partial_t u$ and $\partial^2 u = \nabla^2 u$. For the first case, since u is radial, $\partial_t u$ is also radial and $\nabla \partial_t u = \frac{x}{r} \partial_r \partial_t u$. By (7.6) and (7.4), we have

$$\begin{aligned}
& \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \partial \partial_t u \|_{L_t^2 L_x^4} \\
& \leq \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \partial_{t,r} \partial_t u \|_{L_t^2 L_r^4 L_\omega^3} \\
& \leq \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 1} \partial_{t,r} \partial_t u \|_{L_t^2 L_x^2} + \| r^{-(p+3)/4} \langle r \rangle^{-3(p-1)/4} \partial_{t,r} \partial_t u \|_{L_t^2 L_x^2} \\
& \leq \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \partial^{\leq 2} \partial_t u \|_{L_t^2 L_x^2} + \| r^{-(p+3)/4} \langle r \rangle^{-3(p-1)/4} \partial \partial_t u \|_{L_t^2 L_x^2} \\
& \lesssim \| \partial^{\leq 2} u \|_{\widetilde{LE}}.
\end{aligned}$$

For the second case, it is a little more involved, due to the fact that ∇u is nonradial. Anyway, in this case, we observe that $|\nabla^2 u| \lesssim |\partial_r^2 u| + |\partial_r u|/r$. As a consequence,

$$\begin{aligned}
& \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \nabla \nabla u \|_{L_x^4} \\
& \lesssim \| r^{-(p-2)/4} \langle r \rangle^{-3(p-2)/4} \partial_r^2 u \|_{L_r^4 L_\omega^3} + \| r^{-(p-2)/4-1} \langle r \rangle^{-3(p-2)/4} \partial_r u \|_{L_r^4 L_\omega^3} \\
& \lesssim \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 1} \partial_r^2 u \|_{L_x^2} + \| r^{-(p-1)/4-1} \langle r \rangle^{-3(p-1)/4} \partial_r^2 u \|_{L_x^2} \\
& \quad + \| r^{-(p-1)/4-1} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 1} \partial_r u \|_{L_x^2} + \| r^{-(p-1)/4-2} \langle r \rangle^{-3(p-1)/4} \partial_r u \|_{L_x^2} \\
& \lesssim \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 3} u \|_{L_x^2} + \| r^{-(p-1)/4-1} \langle r \rangle^{-3(p-1)/4} \nabla^2 u \|_{L_x^2} \\
& \quad + \| r^{-(p-1)/4-2} \langle r \rangle^{-3(p-1)/4} \nabla u \|_{L_x^2} \\
& \lesssim \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 3} u \|_{L_x^2} + \| r^{-(p-1)/4-1} \langle r \rangle^{-3(p-1)/4} \nabla^2 u \|_{L_x^2} \\
& \quad + \| r^{-(p-1)/4} \Delta(\langle r \rangle^{-3(p-1)/4} \nabla u) \|_{L_x^2} \\
& \lesssim \| r^{-(p-1)/4} \langle r \rangle^{-3(p-1)/4} \nabla^{\leq 3} u \|_{L_x^2} + \| r^{-(p-1)/4-1} \langle r \rangle^{-3(p-1)/4} \nabla^2 u \|_{L_x^2},
\end{aligned}$$

where, in the second to the last inequality, we have used Lemma 2.3 with $k = 2$.³

³The argument used here is inspired by an unpublished work of the author, in collaboration with Kunio Hidano and Kazuyoshi Yokoyama.

In conclusion, we have proved

$$\| |\partial u|^{p-1} \partial^{\leq 2} \partial u \|_{\widetilde{LE}^*} + \| |\partial u|^{p-2} |\partial \partial u|^2 \|_{L_t^1 L_x^2} \lesssim \| \partial^{\leq 2} u \|_{\widetilde{LE}}^p ,$$

which completes the proof of (7.2).

7.2. Even spatial dimension $n \geq 4$. In this subsection, we turn to the proof for the even spatial dimension $n \geq 4$. As before, we give only the proof of

$$(7.7) \quad \| \nabla^{\leq n/2} \partial u(0) \|_{L_x^2} \leq \varepsilon, \| \partial u(0) \|_{L_x^\infty} \lesssim 1 \Rightarrow \| \partial^{\leq n/2} \partial u(0) \|_{L_x^2} \lesssim \varepsilon ,$$

$$(7.8) \quad \| \partial^{\leq n/2} u \|_{\widetilde{LE}} \lesssim \varepsilon \Rightarrow \| \partial^{\leq n/2} G_p(u) \|_{\widetilde{LE}^*} \leq \varepsilon ,$$

for radial functions and small enough ε .

For (7.7), by equation and a tedious calculation, we can find that

$$\begin{aligned} & | \partial^a \partial u(0) | \lesssim | \partial u(0) |^{|a|(p-1)+1} \\ & + \sum_{|a_i| \geq 1, \sum |a_i| \leq |a|} | \partial u(0) |^{(|a| - \sum |a_i|)(p-1) - m + 1} \prod_{i=1}^m | \nabla^{a_i} \partial u(0) | . \end{aligned}$$

The L^2 norm of the first term on the right is trivial. For the remaining terms, since $\partial u(0)$ is bounded, we need only to control the L^2 norm of $\prod_{i=1}^m | \nabla^{a_i} \partial u(0) |$. For this purpose, we introduce $q_i \in [2, \infty)$, satisfying

$$\sum \frac{1}{q_i} = \frac{1}{2}, \frac{n}{q_i} \geq |a_i| ,$$

which is possible since $\sum |a_i| \leq |a| \leq n/2$. Then by Sobolev embeddings $H^{n/2-|a_i|} \subset L^{q_i}$ and Hölder's inequality, we get

$$\begin{aligned} \| \prod_{i=1}^m | \nabla^{a_i} \partial u(0) | \|_{L^2} & \lesssim \prod_{i=1}^m \| \nabla^{a_i} \partial u(0) \|_{L^{q_i}} \\ & \lesssim \prod_{i=1}^m \| \nabla^{a_i} \partial u(0) \|_{H^{n/2-|a_i|}} \\ & \lesssim \| \partial^{\leq n/2} \partial u(0) \|_{L_x^2}^m , \end{aligned}$$

which yields the desired result (7.7).

By Lemma 2.1, we have for radial functions,

$$(7.9) \quad \| r^\epsilon \langle r \rangle^{(n-1)/2-\epsilon} \partial u \|_{L_x^\infty} \lesssim \| r^\epsilon \langle r \rangle^{(n-1)/2-\epsilon} \partial u \|_{L_r^\infty L_\omega^2} \lesssim \| \partial u \|_{H^{n/2}}, \forall \epsilon > 0 .$$

Since $G_p(u) = c_1 |\partial_t u|^p + c_2 |\nabla u|^p$, $|\partial^{\leq n/2} G_p(u)|$ is controlled by terms of the following form:

$$(7.10) \quad | \partial u |^{p-j} \prod_{i=1}^j | \partial^{a_i} \partial u | , 1 \leq a_i \leq a_{i+1}, \sum_{i=1}^j a_i \leq n/2 .$$

We will consider two cases separately: 1) $j \leq 1$, 2) $j \geq 2$.

Case 1) $j \leq 1$. In this case, with $\mu = \frac{1}{4}$, $\mu' = \frac{(n-1)(p-1)-1}{4}$, we set $\epsilon = 1/(2(p-1))$. By (7.9), we have

$$\begin{aligned} & \| |\partial u|^{p-1} \partial^{\leq n/2} \partial u \|_{\widetilde{LE}^*} \\ & \leq \| r^\epsilon \langle r \rangle^{(n-1)/2-\epsilon} \partial u \|_{L_t^\infty L_x^\infty}^{p-1} \| r^{1/2-\mu-\epsilon(p-1)} \langle r \rangle^{\mu'-[(n-1)/2-\epsilon](p-1)} \partial^{\leq n/2} \partial u \|_{L_t^2 L_x^2} \\ & \lesssim \| \partial^{\leq n/2} \partial u \|_{L_t^\infty L_x^2}^{p-1} \| \partial^{\leq n/2} u \|_{\widetilde{LE}}. \end{aligned}$$

Case 2) $j \geq 2$. Since $a_1 \geq 1$ and $\sum a_i \leq n/2$, we get $a_i \in [1, n/2 - 1]$ and

$$c_i = n/2 - a_i \in [1, n/2 - 1] \in (0, n/2) .$$

Fix $\varepsilon_i \in [0, c_i)$, such that

$$(7.11) \quad \sum \varepsilon_i = -\frac{n}{2}(j-1) + \sum c_i .$$

Notice that it is possible since

$$-\frac{n}{2}(j-1) + \sum c_i = -\frac{n}{2}(j-1) + j\frac{n}{2} - \sum a_i \geq 0 .$$

We can define $p_i \in (2, \infty)$ through

$$n \left(\frac{1}{2} - \frac{1}{p_i} \right) = c_i - \varepsilon_i \in \left(0, \frac{n}{2} \right) .$$

Here, we notice that (7.11) ensures that $\sum 1/p_i = 1/2$. Setting $\epsilon = 1/(2(p-j))$, we have

$$\frac{n-1-2\epsilon}{2}(p-j) + \sum_{i=1}^j (n-1) \left(\frac{1}{2} - \frac{1}{p_i} \right) = 2\mu' .$$

Then, applying Lemma 2.2, (7.9) and Hölder's inequality, we get

$$\begin{aligned} & \| |\partial_t u|^{p-j} \prod_{i=1}^j \partial^{a_i} \partial u \|_{\widetilde{LE}^*} \\ & \leq \| r^\epsilon \langle r \rangle^{(n-1)/2-\epsilon} \partial u \|_{L_t^\infty L_x^\infty}^{p-j} \prod_{i=1}^{j-1} \| \langle r \rangle^{(n-1)(1/2-1/p_i)} \partial^{a_i} \partial u \|_{L_t^\infty L_x^{p_i}} \\ & \quad \times \| r^{1/2-\mu-\epsilon(p-j)} \langle r \rangle^{\mu'-(n-1-2\epsilon)(p-j)/2-\sum_{i=1}^{j-1}(n-1)(1/2-1/p_i)} \partial^{a_j} \partial u \|_{L_t^2 L_x^{p_j}} \\ & \lesssim \| \nabla^{\leq n/2} \partial u \|_{L_t^\infty L_x^2}^{p-j} \| r^{-1/2+\mu} \langle r \rangle^{-\mu'+(n-1)(1/2-1/p_j)} \partial^{a_j} \partial u \|_{L_t^2 L_x^{p_j}} \\ & \quad \times \prod_{i=1}^{j-1} \| Y^{\leq c_i} \partial^{a_i} \partial u \|_{L_t^\infty L_x^2} \\ & \lesssim \| \partial^{\leq n/2} \partial u \|_{L_t^\infty L_x^2}^{p-1} \| r^{-1/2+\mu} \langle r \rangle^{-\mu'+(n-1)(1/2-1/p_j)} \partial^{a_j} \partial u \|_{L_t^2 L_x^{p_j}} , \end{aligned}$$

where in the last inequality, we have used the fact that $|Y^a \partial u| \lesssim |\nabla^{\leq a} \partial u|$ for radial functions. It remains to prove

$$(7.12) \quad \| r^{-1/2+\mu} \langle r \rangle^{-\mu'+(n-1)(1/2-1/p_j)} \partial^{a_j} \partial u \|_{L_t^2 L_x^{p_j}} \lesssim \| \partial^{\leq n/2} u \|_{\widetilde{LE}} .$$

In fact, by Lemmas 2.2 and 2.3, we know that

$$\begin{aligned}
& \|r^{-1/2+\mu}\langle r \rangle^{-\mu'+(n-1)(1/2-1/p_j)}\partial^{a_j}\partial u\|_{L_x^{p_j}} \\
& \lesssim \|Y^{\leq c_j}(r^{-1/2+\mu}\langle r \rangle^{-\mu'}\partial^{a_j}\partial u)\|_{L_x^2} \\
& \lesssim \|r^{-1/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}\partial u\|_{L_x^2} + \sum_{k=1}^{c_j} \sum_{|b|\leq c_j-k} \|r^{-1/2+\mu-k}\langle r \rangle^{-\mu'}\partial^{\leq a_j+b}\partial u\|_{L_x^2} \\
& \lesssim \|r^{-1/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}\partial u\|_{L_x^2} + \|r^{-3/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2-1}\partial u\|_{L_x^2} \\
& \quad + \sum_{3\leq 2l+1\leq c_j} \sum_{|b|\leq c_j-2l} \|r^{-1/2+\mu-2l-1}\langle r \rangle^{-\mu'}\partial^{\leq a_j+b}\partial u\|_{L_x^2} \\
& \quad + \sum_{2\leq 2l\leq c_j} \sum_{|b|\leq c_j-2l} \|r^{-1/2+\mu-2l}\langle r \rangle^{-\mu'}\partial^{\leq a_j+b}\partial u\|_{L_x^2} \\
& \lesssim \|r^{-1/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}\partial u\|_{L_x^2} + \|r^{-3/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}u\|_{L_x^2} \\
& \quad + \sum_{3\leq 2l+1\leq c_j} \sum_{|b|\leq c_j-2l} \|r^{-1/2+\mu-1}\Delta^l(\langle r \rangle^{-\mu'}\partial^{\leq a_j+b}\partial u)\|_{L_x^2} \\
& \quad + \sum_{2\leq 2l\leq c_j} \sum_{|b|\leq c_j-2l} \|r^{-1/2+\mu}\Delta^l(\langle r \rangle^{-\mu'}\partial^{\leq a_j+b}\partial u)\|_{L_x^2} \\
& \lesssim \|r^{-1/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}\partial u\|_{L_x^2} + \|r^{-3/2+\mu}\langle r \rangle^{-\mu'}\partial^{\leq n/2}u\|_{L_x^2}.
\end{aligned}$$

In summary, we have proved that

$$\|\partial^{\leq n/2}G_p(u)\|_{\widetilde{LE}^*} \lesssim \|\partial^{\leq n/2}\partial u\|_{L_t^\infty L_x^2}^{p-1} \|\partial^{\leq n/2}u\|_{\widetilde{LE}}$$

which is sufficient to prove (7.8), with small enough ε .

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