ON THE THIRD HOMOLOGY OF SL_2 AND WEAK HOMOTOPY INVARIANCE

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ABSTRACT. The goal of the paper is to achieve - in the special case of the linear group SL_2 - some understanding of the relation between group homology and its $\mathbb{A}^1$ -invariant replacement. We discuss some of the general properties of the $\mathbb{A}^1$ -invariant group homology, such as stabilization sequences and Grothendieck-Witt module structures. Together with very precise knowledge about refined Bloch groups, these methods allow us to deduce that in general there is a rather large difference between group homology and its $\mathbb{A}^1$ -invariant version. In other words, weak homotopy invariance fails for SL_2 over many families of non-algebraically closed fields.

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1. INTRODUCTION

In this paper, we investigate the difference between group homology and its $\mathbb{A}^1$ -invariant version. It is well-known that K-theory and hence also the homology of the infinite general linear group GL is $\mathbb{A}^1$ -invariant, i.e., for any regular ring R and any $n \geq 0$, the map $GL(R) \rightarrow GL(R[T_1, \dots, T_n])$ given by inclusion of constants induces an isomorphism in homology. This, however, is only a stable phenomenon: the examples of Krstić and McCool [KM97] show that H_1 of SL_2 is not $\mathbb{A}^1$ -invariant because there exist many non-elementary matrices in $SL_2(R[T])$ if R is not a field.

Received by the editors October 18, 2013 and, in revised form, April 25, 2014.

2010 *Mathematics Subject Classification.* Primary 20G10; Secondary 14F42.

Key words and phrases. Weak homotopy invariance, group homology.

In light of this failure of the strong form of $\mathbb{A}^1$ -invariance, one can combine the separate polynomial rings into the simplicial object $R[\Delta^\bullet]$ with n -simplices

$$R[\Delta^n] = R[T_0, \dots, T_n] / \left(\sum_{i=0}^n T_i = 1 \right) \cong R[T_1, \dots, T_n]$$

and ask if the inclusion of constants $SL_2(R) \rightarrow SL_2(R[\Delta^\bullet])$ induces an isomorphism on group homology. This is the question of “weak $\mathbb{A}^1$ -invariance” or “weak homotopy invariance”. The attribute “weak” is justified by the fact that if the strong form of $\mathbb{A}^1$ -invariance above is satisfied, i.e., if the map $SL_2(R) \rightarrow SL_2(R[T_1, \dots, T_n])$ induces an isomorphism on homology for all $n \geq 0$, then the spectral sequence computing the homology of $SL_2(R[\Delta^\bullet])$ collapses and induces an isomorphism $SL_2(R) \rightarrow SL_2(R[\Delta^\bullet])$. More generally, it makes sense to consider, for an arbitrary linear algebraic group G , the homology of the polynomial singular resolution $BG(R[\Delta^\bullet])$ as an $\mathbb{A}^1$ -invariant replacement of group homology; cf. Definition 2.2 below. As mentioned earlier, weak homotopy invariance asks if (or when) the natural change-of-topology morphism $BG(R) \rightarrow BG(R[\Delta^\bullet])$ induces an isomorphism on homology. In this paper, we will investigate some properties of this change-of-topology morphism and prove the following main result which exhibits situations in which even weak homotopy invariance fails. We refer to Theorem 7.4 and Theorem 7.6 for precise formulations.

Theorem 1. (1) *Let k be an infinite perfect field of characteristic $\neq 2$ which is finitely generated over its prime field. Let ℓ be an odd prime such that $[k(\zeta_\ell) : k]$ is even, where ζ_ℓ is a primitive ℓ -th root of unity. Then the kernel of the change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}/\ell) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell)$$

is not finitely generated. In particular, weak homotopy invariance with finite coefficients can fail for fields which are not algebraically closed.

(2) *For k a field complete with respect to a discrete valuation, with finite residue field $\bar{k}$ of order $q = p^f$ (p odd), the change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}[1/2])$$

factors through $K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2]$, and its kernel is isomorphic to the pre-Bloch group $\mathcal{P}(\bar{k}) \otimes \mathbb{Z}[1/2]$ which is cyclic of order $(q+1)^{n'}$, where n' denotes the odd part of the positive integer n .

There are several ingredients coming together. On the side of $\mathbb{A}^1$ -invariant group homology, we can use $\mathbb{A}^1$ -homotopy theory to produce stabilization results and Grothendieck-Witt module structures. On the other hand, the computations in [Hut13a, Hut13b, Hut13c] allow one to understand very explicitly the structure of $H_3(SL_2(k), \mathbb{Z}[1/2])$. In particular, the above theorem follows from a comparison of the $k^\times / (k^\times)^2$ -module structures on both sides: the module structure on the $\mathbb{A}^1$ -invariant group homology factors through an action of the Grothendieck-Witt ring of k , but the corresponding statement for $H_3(SL_2(k), \mathbb{Z}[1/2])$ is false for fields with non-trivial valuations. In the case of number fields there is a further contrast between the two sides: the $\mathbb{A}^1$ -invariant group homology is finitely generated, while the residue maps of [Hut13a] show that the group homology itself is not finitely generated.

Our main theorem also has consequences for the comparison of unstable K-theories. There are (at least) two natural definitions of unstable K-theory associated to SL_2 and a field k , one via Quillen's plus-construction definition $\pi_i BSL_2(k)^+$ and one via the Karoubi-Villamayor definition $\pi_i BSL_2(k[\Delta^\bullet])$. From Theorem 1 and the work of Asok and Fasel [AF12a, Theorem 3], we can deduce that these two definitions of unstable K-theory fail to be equivalent for many families of non-algebraically closed fields. This is discussed in detail in Section 8; cf. in particular Proposition 8.2 and Proposition 8.4.

The methods we use to establish Theorem 1 above only apply to fields which are not quadratically closed: the change-of-topology morphism is injective for quadratically closed fields of characteristic 0; cf. Corollary 4.3. Moreover, over algebraically closed fields of characteristic 0, weak homotopy invariance holds in degree 3 with finite coefficients; cf. Corollary 4.4. Note that weak homotopy invariance with finite coefficients over algebraically closed fields is a necessary ingredient in Morel's approach to the Friedlander-Milnor conjecture [Mor11].

We would like to mention some questions that could be pursued in further research: first, the relation between homotopy invariance and weak homotopy invariance deserves further study. The behaviour of the spectral sequence associated to the bisimplicial set $BSL_2(k[\Delta^\bullet])$ plays a major role, and the differentials of the spectral sequence relate such disparate phenomena as counterexamples to homotopy invariance, scissors congruence groups and surjective stabilization for symplectic groups. Second, it would be interesting to see what happens to (weak) homotopy invariance for homology groups beyond the metastable range as well as for higher rank groups.

Structure of the paper: We review the definition and basic properties of the $\mathbb{A}^1$ -invariant version of group homology in Section 2. In Section 3 we prove some stabilization results which are needed in Section 4 and Section 7. In Section 4 we review the relationship between $H_3(SL_2(k), \mathbb{Z})$ and $K_3^{\text{ind}}(k)$ and apply these results to the change-of-topology morphism in the case of quadratically closed fields of characteristic 0. The Grothendieck-Witt module structures on $\mathbb{A}^1$ -invariant homology of SL_2 are investigated in Section 5. Some details on (refined) Bloch groups and their module structures are provided in Section 6. In Section 7, we use the results of the preceding sections to prove our main result: that the kernel of the change-of-topology morphism is often very large. In Section 8, we discuss the consequences of our results for comparison of unstable K-theories. Finally, in Section 9 we conclude with some remarks on the cokernel of the change-of-topology morphism.

2. GROUP HOMOLOGY MADE $\mathbb{A}^1$ -INVARIANT: DEFINITION

In this section, we recall the construction which enforces $\mathbb{A}^1$ -invariance in group homology, i.e., replaces group homology by something representable in $\mathbb{A}^1$ -homotopy theory. The crucial definition is the singular resolution of a linear algebraic group; cf. [Jar83]:

Definition 2.1. Let k be a field. There is a standard simplicial k -algebra $k[\Delta^\bullet]$ with n -simplices given by

$$k[\Delta^n] = k[X_0, \dots, X_n] / \left(\sum X_i - 1 \right)$$

and face and degeneracy maps given by

$$d_i(X_j) = \begin{cases} X_j & j < i \\ 0 & j = i \\ X_{j-1} & j > i \end{cases}, \quad s_i(X_j) = \begin{cases} X_j & j < i \\ X_i + X_{i+1} & j = i \\ X_{j+1} & j > i \end{cases}.$$

To a linear algebraic group G , we can then associate a simplicial group $G(k[\Delta^\bullet])$. Recall that the classifying space of the simplicial group $G(k[\Delta^\bullet])$ is defined to be the diagonal of the bisimplicial set $BG(k[\Delta^\bullet])$. One simplicial direction is given by the usual classifying space construction, and the other one is given by the simplicial algebra $k[\Delta^\bullet]$ above. We will abuse notation and denote the diagonal of the bisimplicial by $BG(k[\Delta^\bullet])$ as well. This will cause no confusion, as (except in Section 9) we will never actually deal with the bisimplicial set, only with its diagonal.

Definition 2.2. We call the homology of $BG(k[\Delta^\bullet])$ the *group homology made $\mathbb{A}^1$ -invariant*. There is a natural inclusion $G(k) \hookrightarrow G(k[\Delta^\bullet])$ which identifies $G(k)$ with the set of zero-simplices $G(k[X_0]/(X_0 - 1))$ in $G(k[\Delta^\bullet])$. We refer to this as the *change-of-topology morphism*.

The natural coefficients for the $\mathbb{A}^1$ -invariant version of group homology are $G(k[\Delta^\bullet])$ -modules, where a $G(k[\Delta^\bullet])$ -module $M_\bullet$ is a simplicial abelian group with a simplicial action of $G(k[\Delta^\bullet])$. While studying these objects and their relation to continuous representations of topological groups would be very interesting, we will only look at trivial coefficients.

Definition 2.3. Let M be an abelian group with trivial G -action. Let $M_\bullet$ be the corresponding constant simplicial group, viewed as a trivial $G(k[\Delta^\bullet])$ -module. We say that the group G has *weak homotopy invariance in degree n over the field k with M -coefficients*, if the change-of-topology map induces an isomorphism

$$H_n(G(k), M) \xrightarrow{\cong} H_n(BG(k[\Delta^\bullet]), M_\bullet).$$

Remark 2.4. The map $G(k) \rightarrow G(k[\Delta^\bullet])$ is called change-of-topology because we can view $G(k[\Delta^\bullet])$ as the discrete group $G(k)$ equipped with a topology coming from algebraic simplices. There is a strong structural similarity between weak homotopy invariance over an algebraically closed field with finite coefficients and the Friedlander-Milnor conjecture; cf. [Knu01, Section 5]:

- (1) Weak homotopy invariance asks if for an algebraic group G , the change-of-topology $G(k) \rightarrow G(k[\Delta^\bullet])$ (from the discrete group to the group topologized by algebraic simplices) induces an isomorphism in group homology.
- (2) Milnor's form of the Friedlander-Milnor conjecture asks if for a complex Lie group G , the change-of-topology $G^\delta \rightarrow G$ (from the discrete group to the group with the analytic topology) induces an isomorphism in group homology with finite coefficients.
- (3) Friedlander's generalized isomorphism conjecture asks if for an algebraically closed field k and a linear group G , the change-of-topology from the discrete group $G(k)$ to the group G "with the étale topology" induces an isomorphism in group homology with finite coefficients.

In fact, it is the main result of [Mor11] that the above questions about different change-of-topology morphisms are equivalent. Calling the morphism $G(k) \rightarrow G(k[\Delta^\bullet])$ a change-of-topology is supposed to underline these similarities.

The change-of-topology morphism and in particular its effect on third homology with trivial coefficients is the centre of interest in the present work.

3. GROUP HOMOLOGY MADE $\mathbb{A}^1$ -INVARIANT: STABILIZATION

In this section, we will develop some stabilization results for the $\mathbb{A}^1$ -invariant homology of special linear and symplectic groups. These results will be helpful in understanding the change-of-topology morphism.

3.1. Algebraic groups and fibre bundles in $\mathbb{A}^1$ -homotopy theory. We start with some recollection of statements from $\mathbb{A}^1$ -homotopy theory. Recall that Morel and Voevodsky defined $\mathbb{A}^1$ -homotopy theory in [MV99] as the $\mathbb{A}^1$ -localization of the category of simplicial Nisnevich sheaves on the category $\mathbf{Sm}_S$ of smooth schemes over a base scheme S . This paved the way to use methods of algebraic topology in the study of algebraic varieties.

As part of an explicit fibrant replacement functor, Morel and Voevodsky [MV99, Section 2.3] considered the $\mathbb{A}^1$ -singular resolution for simplicial sheaves: to any simplicial sheaf X on the site $\mathbf{Sm}_S$ of smooth schemes over the base scheme S , the $\mathbb{A}^1$ -singular resolution is given as

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)(U) = \mathrm{Hom}(U \times \Delta^{\bullet}, X).$$

Because a field is a henselian local ring, the simplicial group $G(k[\Delta^{\bullet}])$ used in Definition 2.2 above is the simplicial group $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)(k)$ of sections over $\mathrm{Spec} k$ of (the Nisnevich sheafification of) the $\mathbb{A}^1$ -singular resolution $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)$ of G . The same assertion holds for the corresponding classifying space, i.e.,

$$(B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G))(k) \cong B(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)(k)).$$

On the left-hand side, we have the sections over k of the classifying space construction [MV99, Section 4.1] applied to the simplicial sheaf of groups $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)$; on the right-hand side, we have the classifying space associated to the sections of the simplicial resolution of G ; these two simplicial sets are weakly equivalent. The right-hand side space $B(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)(k))$ is the one we are interested in studying because its homology is group homology of G made $\mathbb{A}^1$ -invariant; the left-hand side space $(B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G))(k)$ is the one we can study using $\mathbb{A}^1$ -homotopy theory.

The stabilization results will be consequences of fibre sequences associated to principal bundles under algebraic groups. The corresponding theory was developed by Morel [Mor12, Sections 8 and 9] and Wendt [Wen11]. We only recall the particular fibre sequences we will need. Note that the term fibre sequence used here is the one from [Hov99, Definition 6.2.6]: a fibre sequence of simplicial sets is a sequence $X \rightarrow Y \rightarrow Z$ with an action of ΩZ on X , which in the homotopy category is isomorphic to the sequence $p^{-1}(x) \rightarrow E \xrightarrow{p} B$ where $x \in B$ is a chosen base point, p is a fibration of fibrant objects and $p^{-1}(x)$ has the standard action of ΩB .

Proposition 3.1. (1) For each $n \geq 1$, there is a fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k) \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k) \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k).$$

(2) For each $n \geq 1$, there is a fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^{2n} \setminus \{0\})(k) \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Sp_{2n-2})(k) \rightarrow B \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Sp_{2n})(k).$$

Proof. The fibre sequences in the statement are deloopings of the fibre sequences established in [Wen11, Theorem 6.8]. For further details on the proof, cf. loc. cit. Note also that the hypothesis on the field k being infinite, which appears in [Wen11], is superfluous in the cases cited because these groups are special in the sense of Serre, i.e., all G -torsors are Zariski-locally trivial. Alternatively, the first fibre sequence can also be deduced from [Mor12, Theorem 8.1, Theorem 8.9]. $\square$

We remark that the spaces appearing in the above fibre sequences are $\mathbb{A}^1$ -connected so that the choice of base points does not matter.

The next result recalls a consequence of Morel's computation of low-dimensional homotopy groups of spheres.

Proposition 3.2. *The simplicial set $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k)$ is $(n-2)$ -connected and for $n \geq 2$ there is an isomorphism*

$$\pi_{n-1}(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(\mathrm{Spec} k)) \cong K_n^{MW}(k),$$

where $K_n^{MW}(k)$ denotes the Milnor-Witt K -theory of k ; cf. [Mor12, Section 3].

Proof. By [Mor12, Theorem 6.40], the space $\mathbb{A}^n \setminus \{0\}$ is $\mathbb{A}^1$ -($n-2$)-connected and there is an isomorphism of Nisnevich sheaves of abelian groups

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong K_n^{MW}.$$

By [Mor12, Theorem 8.9], the space $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})$ is $\mathbb{A}^1$ -local and hence there are isomorphisms

$$\pi_i^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k) \cong \pi_i(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k)).$$

This implies the claims about the homotopy groups of the simplicial set. The above argument (with slightly more details) can also be found in [Wen11, Proposition 6.6]. $\square$

Remark 3.3. An elementary matrix $e_{ij}(\lambda) \in SL_n(k)$ is in the path-component of the identity in $SL_n(k[\Delta^\bullet])$, the path being given by $e_{ij}(\lambda T) \in SL_n(k[T]) \cong SL_n(k[\Delta^1])$. Since $SL_n(k) = E_n(k)$ is generated by such elementary matrices, we have $\pi_0 SL_n(k[\Delta^\bullet]) = \{I\}$, i.e., the simplicial set $SL_n(k[\Delta^\bullet])$ is connected. The same is true for $Sp_{2n}(k[\Delta^\bullet])$, or more generally for the simplicial replacement of any algebraic group whose k -points are generated by unipotent elements; cf. [Jar83].

As a consequence, the classifying spaces $BSL_n(k[\Delta^\bullet])$ and $BSp_{2n}(k[\Delta^\bullet])$ are simply connected.

3.2. Stabilization theorems. We first provide an analogue of the stabilization results of Hutchinson and Tao [HT10] for the homology of SL_n made $\mathbb{A}^1$ -invariant:

Proposition 3.4. *Let k be an infinite field. Then the homomorphisms*

$$H_i(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow H_i(BSL_n(k[\Delta^\bullet]), \mathbb{Z})$$

induced by the standard inclusion $SL_{n-1}(k) \hookrightarrow SL_n(k)$ are isomorphisms for $i \leq n-2$. There is an exact sequence

$$\begin{aligned} H_n(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) &\rightarrow H_n(BSL_n(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_n^{MW}(k) \\ &\rightarrow H_{n-1}(BSL_{n-1}(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow H_{n-1}(BSL_n(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow 0. \end{aligned}$$

Proof. By Proposition 3.1 (1), we have a fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k) \rightarrow B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k) \rightarrow B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k).$$

By Remark 3.3, $B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}SL_{n-1}(k)$ is simply-connected. Since

$$\pi_{n-1}(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k)) = \pi_n(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k)),$$

Proposition 3.2 implies that the pair $(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k))$ is $(n-1)$ -connected. By the relative Hurewicz theorem [GJ99, Corollary III.3.12] we have a vanishing of relative homology groups of the pair:

$$\tilde{H}_i(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k)) = 0, i < n.$$

Moreover, the first non-vanishing relative homology group can be identified as

$$\begin{aligned} & H_n(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k)) \\ & \cong \pi_n(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k)) \\ & \cong \pi_{n-1}(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})(k)). \end{aligned}$$

Therefore, using Proposition 3.2, we find that the above relative homology group $H_n(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k))$ can be identified with $K_n^{MW}(k)$. The claims then are direct consequences of the long exact sequence for relative homology of the pair $(B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_n)(k), B\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(SL_{n-1})(k))$. $\square$

Next, we consider the symplectic groups. This is the main result we will use in the later development.

Proposition 3.5. *Let k be an infinite field. Then the homomorphisms*

$$H_i(BSp_{2n-2}(k[\Delta^{\bullet}]), \mathbb{Z}) \rightarrow H_i(BSp_{2n}(k[\Delta^{\bullet}]), \mathbb{Z})$$

induced by the standard inclusion $Sp_{2n-2}(k) \hookrightarrow Sp_{2n}(k)$ are isomorphisms for $i \leq 2n-2$. There is an exact sequence

$$\begin{aligned} & H_{2n}(BSp_{2n-2}(k[\Delta^{\bullet}]), \mathbb{Z}) \rightarrow H_{2n}(BSp_{2n}(k[\Delta^{\bullet}]), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k) \\ & \rightarrow H_{2n-1}(BSp_{2n-2}(k[\Delta^{\bullet}]), \mathbb{Z}) \rightarrow H_{2n-1}(BSp_{2n}(k[\Delta^{\bullet}]), \mathbb{Z}) \rightarrow 0. \end{aligned}$$

Proof. The proof is the same as for Proposition 3.4 above, only using the symplectic versions of Proposition 3.1 and Remark 3.3. $\square$

Remark 3.6. Similar stabilization results can be proved for GL_n and some inclusions of orthogonal groups.

3.3. $H_{2n}(Sp_{2n}(k), \mathbb{Z})$ surjects onto $K_{2n}^{MW}(k)$. For fields k of characteristic not equal to 2, it is shown in [HT10, Lemma 3.5, Theorem 3.9] that there is a natural homomorphism of graded $\mathbb{Z}[k^{\times}]$ -algebras

$$\sigma_n = T_n \circ \epsilon_n : H_n(SL_n(k), \mathbb{Z}) \rightarrow K_n^{MW}(k), \quad n \geq 0.$$

Here the algebra structure on $(H_n(SL_n(k), \mathbb{Z}))_{n \geq 0}$ comes from the external product

$$H_n(SL_n(k), \mathbb{Z}) \otimes H_m(SL_m(k), \mathbb{Z}) \rightarrow H_{n+m}(SL_{n+m}(k), \mathbb{Z})$$

induced by the block matrix homomorphism $SL_n(k) \times SL_m(k) \rightarrow SL_{n+m}(k)$.

Thus, for any $n \geq 0$, the inclusion $Sp_{2n}(k) \rightarrow SL_{2n}(k)$ induces a natural map $H_{2n}(Sp_{2n}(k), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k)$.

Lemma 3.7. *For any field k of characteristic not equal to 2, the natural map $H_{2n}(Sp_{2n}(k), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k)$ is surjective.*

Proof. Since $Sp_2(k) = SL_2(k)$, the map $H_2(Sp_2(k), \mathbb{Z}) \rightarrow K_2^{MW}(k)$ is an isomorphism; cf. [HT10, Theorem 3.10]. There are natural group homomorphisms $Sp_{2n}(k) \times Sp_{2m}(k) \rightarrow Sp_{2n+2m}(k)$, and hence, for any n a homomorphism

$$\underbrace{Sp_2(k) \times \cdots \times Sp_2(k)}_n \rightarrow Sp_{2n}(k).$$

These maps induce a commutative diagram

$$\begin{array}{ccc} H_2(Sp_2(k), \mathbb{Z}) \otimes \cdots \otimes H_2(Sp_2(k), \mathbb{Z}) & \longrightarrow & H_{2n}(Sp_{2n}(k), \mathbb{Z}) \\ \downarrow \cong & & \downarrow \\ H_2(SL_2(k), \mathbb{Z}) \otimes \cdots \otimes H_2(SL_2(k), \mathbb{Z}) & \longrightarrow & H_{2n}(SL_{2n}(k), \mathbb{Z}) \\ \downarrow \cong & & \downarrow \\ K_2^{MW}(k) \otimes \cdots \otimes K_2^{MW}(k) & \longrightarrow & K_{2n}^{MW}(k) \end{array}$$

Finally, since $K_n^{MW}(k)$ is additively generated by products $[a_1] \cdots [a_n]$, $[a_i] \in K_1^{MW}(k)$, the lowest horizontal arrow in this diagram is a surjection. $\square$

3.4. Injective stabilization for symplectic groups. We will provide an improvement of the stabilization result Proposition 3.5 for the symplectic groups, using the surjectivity statement of Lemma 3.7. The main ingredient is the following comparison between stabilization morphisms for discrete and simplicial groups.

Proposition 3.8. *Let k be a field of characteristic 0.*

(1) *There is a commutative diagram*

$$\begin{array}{ccc} H_n(SL_n(k), \mathbb{Z}) & \longrightarrow & H_n(BSL_n(k[\Delta^\bullet]), \mathbb{Z}) \\ & \searrow T_n \circ \epsilon_n & \swarrow \\ & K_n^{MW}(k), & \end{array}$$

where the top morphism is the change of topology, the left descending morphism is the one of [HT10] and the right descending morphism is the one from the stabilization sequence of Proposition 3.4.

(2) *There is a commutative diagram*

$$\begin{array}{ccc} H_{2n}(Sp_{2n}(k), \mathbb{Z}) & \longrightarrow & H_{2n}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \\ & \searrow T_n \circ \epsilon_n \circ \iota & \swarrow \\ & K_{2n}^{MW}(k), & \end{array}$$

where the top morphism is the change of topology, the left descending morphism is the one of Lemma 3.7 and the right descending morphism is the one from the stabilization sequence of Proposition 3.5.

Proof. We first show that (1) implies (2). Consider the following commutative diagram:

$$\begin{array}{ccccc} H_{2n}(Sp_{2n}(k), \mathbb{Z}) & \longrightarrow & H_{2n}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) & \longrightarrow & K_{2n}^{MW}(k) \\ \downarrow & & \downarrow & & \downarrow = \\ H_{2n}(SL_{2n}(k), \mathbb{Z}) & \longrightarrow & H_{2n}(BSL_{2n}(k[\Delta^\bullet]), \mathbb{Z}) & \longrightarrow & K_{2n}^{MW}(k) \end{array}$$

The right square is part of a commutative ladder of stabilization sequences arising as in [AF12a, Section 3, after Lemma 3.1] from the inclusion $Sp_{2n} \hookrightarrow SL_{2n}$ and the subsequent isomorphism of quotients

$$SL_{2n}/SL_{2n-1} \cong Sp_{2n}/Sp_{2n-2}.$$

The left square is simply induced by the respective inclusions of groups and change-of-topology maps. Our goal is to show that the top composition $H_{2n}(Sp_{2n}(k), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k)$ is the map from Lemma 3.7. But this is a consequence of the commutativity and (1).

To show (1), we use the stabilization results above, in a manner analogous to the arguments in [AF12b, Section 3]. There is a case distinction, based on the parity of n . In the case $n = 2i + 1$, we consider the following diagram:

$$\begin{array}{ccccc} & & K_{2i+2}^{MW}(k) & & \\ & & \downarrow \alpha & \searrow 0 & \\ H_{2i+1}(SL_{2i+1}(k), \mathbb{Z}) & \xrightarrow{\iota_{2i+1}} & H_{2i+1}(BSL_{2i+1}(k[\Delta^\bullet]), \mathbb{Z}) & \xrightarrow{\sigma} & K_{2i+1}^{MW}(k) \\ \cong \downarrow & & \downarrow & \nearrow \text{dotted} & \\ H_{2i+1}(SL_{2i+2}(k), \mathbb{Z}) & \xrightarrow{\iota_{2i+2}} & H_{2i+1}(BSL_{2i+2}(k[\Delta^\bullet]), \mathbb{Z}) & & \end{array}$$

The left vertical map is the stabilization map of [HT10], which is an isomorphism by [HT10, Corollary 6.12]. The middle column is a part of the stabilization exact sequence of Proposition 3.4. The morphism σ is the map we want to compare with $T_n \circ \epsilon_n$. Note that the map α is the one from the stabilization sequence for the inclusion $SL_{2i+1} \hookrightarrow SL_{2i+2}$, while the morphism σ is the one induced from the stabilization sequence for the inclusion $SL_{2i} \hookrightarrow SL_{2i+1}$. The composition $\sigma \circ \alpha : K_{2i+2}^{MW}(k) \rightarrow K_{2i+1}^{MW}(k)$ is then the same as the corresponding composition in the stabilization sequences for $\mathbb{A}^1$ -homotopy groups of [Wen11, Theorem 6.8]. This morphism has been identified as 0 in [AF12b, Lemma 3.5], hence σ extends through $H_{2i+1}(BSL_{2i+2}(k[\Delta^\bullet]), \mathbb{Z})$ as claimed. The question reduces to commutativity of the same diagram for $n = \infty$, i.e., the stable case.

In the case $n = 2i$, we consider the following diagram:

$$\begin{array}{ccccc}
 & & K_{2i+1}^{MW}(k) & & 0 \\
 & & \downarrow & \nearrow \beta & \downarrow \\
 I^{2i+1}(k) & \cdots \cdots \cdots = & \alpha & \cdots \cdots \cdots & I^{2i+1}(k) \\
 \downarrow \gamma & & \downarrow & & \downarrow \\
 H_{2i}(SL_{2i}(k), \mathbb{Z}) & \xrightarrow{\iota_{2i}} & H_{2i}(BSL_{2i}(k[\Delta^\bullet]), \mathbb{Z}) & \xrightarrow{\sigma} & K_{2i}^{MW}(k) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_{2i}(SL_{2i+1}(k), \mathbb{Z}) & \xrightarrow{\iota_{2i+1}} & H_{2i}(BSL_{2i+1}(k[\Delta^\bullet]), \mathbb{Z}) & \cdots \cdots \cdots & K_{2i}^{MW}(k) \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0
 \end{array}$$

The right vertical column is one of the standard exact sequences for Milnor-Witt K-theory; cf. [Mor04, Section 5]. The left vertical column is the exact sequence from the stabilization theorem of [HT10, Corollary 6.13]. The middle column is the stabilization exact sequence of Proposition 3.4. The diagram without the dotted arrows is commutative, and the only square is obviously commutative.

Note that the map α is the one from the stabilization sequence for the inclusion $SL_{2i} \hookrightarrow SL_{2i+1}$, while the morphism σ is the one induced from the stabilization sequence for the inclusion $SL_{2i-1} \hookrightarrow SL_{2i}$. The composition $\sigma \circ \alpha : K_{2i+1}^{MW}(k) \rightarrow K_{2i}^{MW}(k)$ is then the same as the corresponding composition in the stabilization sequences for $\mathbb{A}^1$ -homotopy groups of [Wen11, Theorem 6.8]. This morphism has been identified as η in [AF12b, Lemma 3.5], hence α factors through β as claimed.

The composition $\sigma \circ \iota_{2i} \circ \gamma$ has been identified in [HT10, Corollary 6.13] as the canonical inclusion $I^{2i+1}(k) \hookrightarrow K_{2i+1}^{MW}(k)$ arising from $\eta : K_{2i+1}^{MW}(k) \rightarrow K_{2i}^{MW}(k)$. Hence, we have the dotted equality arrow in the diagram. This means that on the subgroup $I^{2i+1} \hookrightarrow H_{2i}(SL_{2i}(k), \mathbb{Z})$, the two maps - change of topology plus stabilization and $T_{2i} \circ \epsilon_{2i}$ - agree. To show that they agree on all of $H_{2i}(SL_{2i}(k), \mathbb{Z})$, we need to consider the bottom row. The dotted arrow exists, since $K_{2i+1}^{MW}(k) \rightarrow K_{2i}^{MW}(k)$ has been identified with η .

There is another map $H_{2i}(SL_{2i+1}(k), \mathbb{Z}) \rightarrow K_{2i}^M(k)$ induced from $T_{2i} \circ \epsilon_{2i} : H_{2i}(SL_{2i}(k), \mathbb{Z}) \rightarrow K_{2i}^{MW}(k)$ modulo I^{2i+1} . It now suffices to identify this map with the bottom composition in the big diagram.

To show that the bottom composition agrees with $T_{2i} \circ \epsilon_{2i}$ modulo I^{2i+1} , it suffices to check this after stabilization to SL_∞ . We have a commutative diagram

$$\begin{array}{ccc}
 H_{2i}(SL_{2i+1}(k), \mathbb{Z}) & \longrightarrow & H_{2i}(BSL_{2i+1}(k[\Delta^\bullet]), \mathbb{Z}) \\
 \downarrow \cong & & \downarrow \cong \\
 H_{2i}(SL_\infty(k), \mathbb{Z}) & \xrightarrow{\cong} & H_{2i}(BSL_\infty(k[\Delta^\bullet]), \mathbb{Z})
 \end{array}$$

The left isomorphism is a case of [HT10, Theorem 1.1], the right isomorphism is a case of Proposition 3.4, and the bottom isomorphism is a consequence of homotopy invariance of algebraic K-theory. After stabilization, we see that both maps

$H_{2i}(SL_\infty(k), \mathbb{Z}) \rightarrow K_{2i}^M(k)$ and $H_{2i}(BSL_\infty(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_{2i}^M(k)$ factor through Suslin's homomorphism $H_{2i}(BGL_\infty(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_{2i}^M(k)$ and the respective inclusion; cf. also the discussion after [AF12a, Lemma 3.1]. The result is proved. $\square$

The following is now an obvious consequence of Proposition 3.5, Lemma 3.7 and Proposition 3.8.

Corollary 3.9. *Let k be a field of characteristic 0.*

- (1) *In the stabilization sequence for the special linear groups, cf. Proposition 3.4, the morphism*

$$H_{2n}(BSL_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k)$$

is surjective. Hence, the standard inclusion $SL_{2n-1} \hookrightarrow SL_{2n}$ induces an isomorphism

$$H_{2n-1}(BSL_{2n-1}(k[\Delta^\bullet]), \mathbb{Z}) \xrightarrow{\cong} H_{2n-1}(BSL_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \cong H_{2n-1}(BSL_\infty(k)).$$

- (2) *In the stabilization sequence for the symplectic groups, cf. Proposition 3.5, the morphism*

$$H_{2n}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k)$$

is surjective. Hence, the standard inclusion $Sp_{2n-2} \hookrightarrow Sp_{2n}$ induces an isomorphism

$$H_{2n-1}(BSp_{2n-2}(k[\Delta^\bullet]), \mathbb{Z}) \xrightarrow{\cong} H_{2n-1}(BSp_{2n}(k[\Delta^\bullet]), \mathbb{Z}) \cong H_{2n-1}(BSp_\infty(k)).$$

Remark 3.10. We recall the computation of the homotopy sheaf $\pi_2^{\mathbb{A}^1}(SL_2)$ from [AF12a, Theorem 3]. For an infinite perfect field of characteristic $\neq 2$, there is an exact sequence

$$0 \rightarrow T'_4(k) \rightarrow \pi_2^{\mathbb{A}^1}(SL_2)(\mathrm{Spec} k) \rightarrow KSp_3(k) \rightarrow 0.$$

The group $T'_4(k)$ sits in an exact sequence

$$I^5(k) \rightarrow T'_4(k) \rightarrow S'_4(k) \rightarrow 0,$$

where $I^5(k)$ is the fifth power of the fundamental ideal of the Witt ring $W(k)$ and there is a surjection $K_4^M(k)/12 \twoheadrightarrow S'_4(k)$.

Comparing with our stabilization result above, we find that the group $T'_4(k)$ lies in the kernel of the Hurewicz homomorphism

$$\pi_3^{\mathbb{A}^1}(BSL_2)(k) \cong \pi_3(BSL_2(k[\Delta^\bullet])) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}),$$

at least if the field k has characteristic 0.

Remark 3.11. We want to point out that a stabilization result like Proposition 3.5 cannot be true for the discrete groups over arbitrary fields. Assume that there were isomorphisms

$$H_i(Sp_{2n-2}(k), \mathbb{Z}) \cong H_i(Sp_{2n}(k), \mathbb{Z})$$

for $i \leq 2n - 2$ and exact sequences

$$H_{2n}(Sp_{2n}(k), \mathbb{Z}) \rightarrow K_{2n}^{MW}(k) \rightarrow H_{2n-1}(Sp_{2n-2}(k), \mathbb{Z}) \rightarrow H_{2n-1}(Sp_{2n}(k), \mathbb{Z}) \rightarrow 0.$$

Then by Lemma 3.7 $H_3(SL_2(k), \mathbb{Z}) \cong H_3(Sp_\infty(k), \mathbb{Z})$. Together with the stabilization result of Corollary 3.9 above and homotopy invariance for the infinite symplectic group [Kar73], this would imply $H_3(SL_2(k), \mathbb{Z}) \cong H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$. But there are fields for which such an isomorphism cannot exist, by our Theorem 1,

which we will prove in the rest of the paper. The exact form of the optimal stabilization result for the symplectic groups seems not yet known.

Remark 3.12. As a final remark, we want to point out that the stabilization theorem of [HT10] has been recently generalized to local rings with infinite residue fields (of arbitrary characteristic) by Marco Schlichting. Using his work, it is possible to drop the characteristic 0 requirements in the preceding result.

4. THE THIRD HOMOLOGY OF SL_2 AND INDECOMPOSABLE K_3

Suslin has shown in [Sus90, Corollary 5.2] that for any field k , the Hurewicz homomorphism $K_3(k) \rightarrow H_3(GL_\infty(k))$ induces an isomorphism

$$H_3(SL_\infty(k)) \cong \frac{K_3(k)}{\{-1\} \cdot K_2(k)}.$$

It follows that there is a natural induced surjective map

$$H_3(SL_\infty(k)) \rightarrow K_3^{\text{ind}}(k) := \frac{K_3(k)}{K_3^M(k)}.$$

Let γ denote the induced composite map

$$H_3(SL_2(k)) \rightarrow H_3(SL_\infty(k)) \rightarrow K_3^{\text{ind}}(k).$$

Lemma 4.1. *Let k be an infinite field.*

(1) γ is surjective.

(2) γ induces an isomorphism

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \cong K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2].$$

(3) If $k^\times = (k^\times)^2$, then γ induces an isomorphism

$$H_3(SL_2(k), \mathbb{Z}) \cong K_3^{\text{ind}}(k).$$

Proof. (1) This is [HT09, Lemma 5.1].

(2) This is [Mir08, Proposition 6.4 (ii)].

(3) This is [Mir08, Proposition 6.4 (iii)]. □

We also note that since the map γ factors through the stabilization homomorphism we have

Lemma 4.2. *Let k be a field of characteristic 0. Then there is a natural surjective homomorphism $\gamma' : H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(k)$ giving rise to a commutative triangle*

$$\begin{array}{ccc} H_3(SL_2(k), \mathbb{Z}) & & \\ \downarrow & \searrow \gamma & \\ H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) & \xrightarrow{\gamma'} & K_3^{\text{ind}}(k). \end{array}$$

Proof. There is a natural commutative square

$$\begin{array}{ccc} H_3(SL_2(k), \mathbb{Z}) & \longrightarrow & H_3(SL_\infty(k), \mathbb{Z}) \\ \downarrow & & \downarrow \cong \\ H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) & \longrightarrow & H_3(BSL_\infty(k[\Delta^\bullet]), \mathbb{Z}). \end{array}$$

In this diagram, the right-hand vertical map is an isomorphism by Corollary 3.9. □

Combining Lemma 4.1 (3) and Lemma 4.2, we immediately deduce:

Corollary 4.3. *Let k be a field of characteristic 0 satisfying $k^\times = (k^\times)^2$. Then the change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is injective. The image of this map is isomorphic to $K_3^{\text{ind}}(k)$ and is a direct summand of $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$.

Corollary 4.4. *Let k be an algebraically closed field of characteristic 0. Then weak homotopy invariance holds over k with finite coefficients, i.e.,*

$$H_3(SL_2(k), \mathbb{Z}/\ell) \cong H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell) \cong 0.$$

Proof. By Corollary 4.3, it suffices to show that $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell) \cong 0$. We have $H_2(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \cong K_2^{MW}(k)$ by [Mor12, p.185]. But since k is algebraically closed, the group $K_2^{MW}(k) \cong K_2^M(k)$ is uniquely divisible. The universal coefficient theorem then implies that it suffices to show that $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell = 0$. By Corollary 3.9, it suffices to show vanishing of $H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell$. But then we have the surjection

$$KSp_3(k) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell \rightarrow H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell,$$

and unique divisibility of $KSp_3(k)$ implies the required vanishing. The latter unique divisibility follows from the rigidity theorem of [Kar83], the algebraic vs. topological comparison of [Kar84, Theorem 1] and the fact that $BSp(\mathbb{C})$ is 3-connected. $\square$

Remark 4.5. In line with Remark 3.12, Schlichting's extension of the stabilization theorem of [HT10] implies weak homotopy invariance for H_3 also in positive characteristics. The unique divisibility of KSp_3 for algebraically closed fields in positive characteristics follows again from [Kar83].

5. ACTIONS OF MULTIPLICATIVE GROUPS AND GROTHENDIECK-WITT RINGS

In this section, we will discuss the comparison of natural $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structures on the homology groups $H_\bullet(SL_2(k), \mathbb{Z})$ and $H_\bullet(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$. The two main statements are that the change-of-topology morphism is equivariant for these additional module structures, and that on the $\mathbb{A}^1$ -homotopy side, the module structure on $H_\bullet(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$ descends to a $GW(k)$ -module structure. This is achieved by analysing the action of units on the $\mathbb{A}^1$ -homotopy type of $\mathbb{P}^1$.

5.1. Review of Grothendieck-Witt rings. We first recall the definition and notation for Grothendieck-Witt rings, collated from various sources ([Lam05], [HT10] and [Mor12, Section 3]).

For a field k , we have the ring $\mathbb{Z}[k^\times/(k^\times)^2]$, which is the integral group ring over the group $k^\times/(k^\times)^2$ of square classes of k . For $a \in k^\times$ we will let $\langle a \rangle \in \mathbb{Z}[k^\times/(k^\times)^2]$ denote the square class of a . We let $\mathcal{I}_k$ denote the augmentation ideal with generators $\langle\langle a \rangle\rangle := \langle a \rangle - 1$.

The Grothendieck-Witt ring $GW(k)$ of the field k is the group completion of the set of isometry classes of non-degenerate symmetric bilinear forms over k . The addition and multiplication operations are given by the orthogonal sum and tensor product of symmetric bilinear forms, respectively. For $a \in k$, the 1-dimensional form $(x, y) \mapsto axy$ is denoted by $\langle a \rangle \in GW(k)$. The dimension function provides

an augmentation $\dim : GW(k) \rightarrow \mathbb{Z}$, and the augmentation ideal $I(k)$ is called the *fundamental ideal*. It is generated by the Pfister forms $\langle\langle a \rangle\rangle := \langle a \rangle - 1$.

There are well-known presentations of the Grothendieck-Witt ring of field, dating back to the original work of Witt. For example, Theorem 4.3 in [Lam05, Chapter II] asserts that, as an abelian group, $GW(k)$ is generated by the square classes $\langle a \rangle$, subject to the additional family of relations

$$\langle a \rangle + \langle b \rangle - \langle a + b \rangle + \langle ab(a + b) \rangle, a, b, a + b \in k^\times.$$

However, we note that these relations can be re-written (in the group ring) as

$$\langle b \rangle \langle\langle c \rangle\rangle \langle\langle 1 - c \rangle\rangle,$$

where $c = a/(a + b)$. Thus the additive subgroup generated by these relations is equal to the *ideal* of $\mathbb{Z}[k^\times/(k^\times)^2]$ generated by the expressions $\langle\langle c \rangle\rangle \langle\langle 1 - c \rangle\rangle$. We deduce:

Proposition 5.1. *There is a homomorphism of augmented rings*

$$\mathbb{Z}[k^\times/(k^\times)^2] \rightarrow GW(k) : \langle a \rangle \mapsto \langle a \rangle.$$

This homomorphism is surjective, and its kernel is the ideal, $\mathcal{J}_k$, generated by the ‘Steinberg elements’ $\langle\langle a \rangle\rangle \langle\langle 1 - a \rangle\rangle \in \mathcal{I}_k^2$.

Recall from [Mor12, Definition 3.1] that the Milnor-Witt K-theory $K_\bullet^{MW}(k)$ of the field k is defined to be the graded associative ring generated by symbols $[a], a \in k^\times$ of degree 1 and η of degree -1 with the following relations:

- (1) $[a][1 - a] = 0$ for each $a \in k^\times \setminus \{1\}$,
- (2) $[ab] = [a] + [b] + \eta[a][b]$ for each $a, b \in k^\times$,
- (3) $[a]\eta = \eta[a]$ for each $a \in k^\times$,
- (4) $\eta h = 0$ for $h = \eta[-1] + 2$.

There is an augmentation $K_0^{MW}(k) \rightarrow K_0^{MW}(k)/\eta \cong \mathbb{Z}$. The morphism

$$GW(k) \rightarrow K_0^{MW}(k) : \langle a \rangle \mapsto 1 + \eta[a]$$

induces an isomorphism of augmented rings.

5.2. Module structures and equivariance.

Construction 5.2. We start by describing some actions of the multiplicative group on SL_2 and $\mathbb{P}^1$.

An action of $\mathbb{G}_m$ on SL_2 : recall the standard exact sequence of algebraic groups,

$$1 \rightarrow SL_2 \hookrightarrow GL_2 \xrightarrow{\det} \mathbb{G}_m \rightarrow 1.$$

The determinant $\det : GL_2 \rightarrow \mathbb{G}_m$ has a splitting given by

$$\mathbb{G}_m \rightarrow GL_2 : u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The splitting followed by the conjugation action of GL_2 on SL_2 induces an action of $\mathbb{G}_m$ on SL_2 . Evaluating on R -points for R any k -algebra, the action of $k^\times$ on $SL_2(R)$ is evidently described as

$$k^\times \times SL_2(R) \rightarrow SL_2(R) : (u, A) \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

From the above description, it is clear that for a k -algebra homomorphism $R \rightarrow S$, the induced homomorphism $SL_2(R) \rightarrow SL_2(S)$ is $k^\times$ -equivariant.

An action of $k^\times$ on $\mathbb{P}^1(k)$: the projection

$$GL_2 \rightarrow \mathbb{P}^1 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c : d]$$

induces an action of $k^\times$ on $\mathbb{P}^1(k)$. This action is given by

$$k^\times \times \mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k) : (u, [x, y]) \mapsto [ux : y].$$

An action of $k^\times$ on $BSL_2(k[\Delta^\bullet])$: the action of $\mathbb{G}_m$ on SL_2 above induces an action of $k^\times$ on the simplicial sets $SL_2(k[\Delta^\bullet])$. This follows from the $k^\times$ -equivariance of maps induced from ring homomorphisms. This action is an action by group automorphisms (as opposed to just automorphisms as a space), hence it induces an action of $k^\times$ on $BSL_2(k[\Delta^\bullet])$.

From the above constructions, the following is immediate:

Lemma 5.3. *The following morphisms are $k^\times$ -equivariant, for the module structures defined above:*

- (i) the Hurewicz homomorphism $\pi_\bullet(BSL_2(k[\Delta^\bullet])) \rightarrow H_\bullet(BSL_2(k[\Delta^\bullet]))$,
- (ii) the identification $\pi_\bullet(BSL_2(k[\Delta^\bullet])) \cong \pi_{\bullet}^{\mathbb{A}^1}(BSL_2)(k)$,
- (iii) the loop-space isomorphism $\pi_{\bullet+1}^{\mathbb{A}^1}(BSL_2)(k) \cong \pi_{\bullet}^{\mathbb{A}^1}(SL_2)(k)$,
- (iv) the projection $\pi_{\bullet}^{\mathbb{A}^1}(SL_2)(k) \rightarrow \pi_{\bullet}^{\mathbb{A}^1}(\mathbb{P}^1)(k)$ (which happens to be an isomorphism for $\bullet \geq 2$).

Proposition 5.4. *Assume that k is an infinite perfect field of characteristic $\neq 2$. Then the action of $k^\times$ on $SL_2(k[\Delta^\bullet])$ induces a $GW(k)$ -module structure on $\pi_3(BSL_2(k[\Delta^\bullet]))$ and hence on $H_3(BSL_2(k[\Delta^\bullet]))$.*

Proof. We explain how the first assertion can be deduced from the results of [AF12a]. Consider the action of $\mathbb{G}_m$ on BSp_{2n} defined in [AF12a], Section 4, just before Lemma 4.7. Evaluating this action on $\text{Spec } k$ for $n = 1$ provides exactly the action of $k^\times$ on $BSL_2(k[\Delta^\bullet])$ described above. In [AF12a, Lemma 4.7, Corollary 4.9], the action of the sheaf $\mathbb{G}_m$ on the space BSp_2 is described more explicitly: in the notation of loc. cit. there is an exact sequence

$$0 \rightarrow T'_4(k) \rightarrow \pi_2^{\mathbb{A}^1}(Sp_2)(k) \rightarrow GW_3^2(k) \rightarrow 0.$$

Note that $\pi_2^{\mathbb{A}^1}(Sp_2)(k) \cong \pi_3 BSL_2(k[\Delta^\bullet])$ by the work of Moser [Mos] and that $GW_3^2(k)$ is just different notation for $KSp_3(k)$. The action of $k^\times$ on T'_4 is trivial by [AF12a, Corollary 4.9], and the action of $k^\times$ on $KSp_3(k)$ is such that $u \in k^\times$ acts via multiplication by $\langle u \rangle \in GW(k)$ on KSp_3 . In particular, the computations in [AF12a, Section 4] cited above imply that the action of the sheaf $\mathbb{G}_m$ on the space BSL_2 actually extends to an action of the sheaf GW . Evaluation on $\text{Spec } k$ then implies the first claim.

For the second assertion, recall from Remark 3.3 that $BSL_2(k[\Delta^\bullet])$ is simply-connected. Therefore, the Hurewicz homomorphism

$$\pi_3(BSL_2(k[\Delta^\bullet])) \rightarrow H_3(BSL_2(k[\Delta^\bullet]))$$

is surjective, and by Lemma 5.3, it is also equivariant. Therefore, the $GW(k)$ -module structure on π_3 descends to a $GW(k)$ -module structure on H_3 . $\square$

Lemma 5.5. *The conjugation action of $k^\times$ on $SL_2(k)$ descends to an action of $\mathbb{Z}[k^\times/(k^\times)^2]$ on $H_\bullet(SL_2(k), \mathbb{Z})$. The natural change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is equivariant for the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structures.

Proof. For any unit $u \in k^\times$, the conjugation action of u^2 is the same as conjugating with $\text{diag}(u, u^{-1}) \in SL_2(k)$. The squares therefore act via inner automorphisms, hence trivially on the homology. Homology groups are abelian groups, so the action of $k^\times/(k^\times)^2$ can be extended to the group ring linearly.

The change-of-topology morphism is induced from the inclusion of bisimplicial sets $BSL_2(k) \rightarrow BSL_2(k[\Delta^\bullet])$, where the first bisimplicial set is constant in the $\Delta^\bullet$ -direction. The degree-wise morphisms $BSL_2(k) \rightarrow BSL_2(k[\Delta^n])$ are induced from the inclusion of the constants $k \hookrightarrow k[\Delta^n]$. Equivariance for the $k^\times$ -module structures is then clear. The $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure on $H_3(SL_2(k))$ has been described above. The $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure on $H_3(BSL_2(k[\Delta^\bullet]))$ comes from the $GW(k)$ -module structure in the previous proposition composed with $\mathbb{Z}[k^\times/(k^\times)^2] \rightarrow GW(k)$. The corresponding equivariance is then also clear. $\square$

We state an obvious corollary:

Corollary 5.6. *Assume that k is an infinite perfect field of characteristic $\neq 2$. We have the following factorization of the change-of-topology morphism:*

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(SL_2(k), \mathbb{Z}) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

6. BLOCH GROUPS AND SPECIALIZATION HOMOMORPHISMS

We review the relationship between $H_3(SL_2(k), \mathbb{Z})$ and the refined Bloch group, $\mathcal{RB}(k)$, and we use this relationship to compute lower bounds for the kernel of the map

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k).$$

We begin by recalling that for any infinite field k there is a natural surjective homomorphism $H_3(SL_2(k), \mathbb{Z}) \rightarrow K_3^{\text{ind}}(k)$ ([HT09, Lemma 5.1]) which induces an isomorphism

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \cong K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2]$$

([Mir08, Proposition 6.4 (ii)]).

Recall from Section 5 that $\mathcal{I}_k \subseteq \mathbb{Z}[k^\times/(k^\times)^2]$ is the augmentation ideal and $\mathcal{J}_k \subseteq \mathbb{Z}[k^\times/(k^\times)^2]$ is the kernel of $\mathbb{Z}[k^\times/(k^\times)^2] \rightarrow GW(k)$. Thus, if M is a $\mathbb{Z}[k^\times/(k^\times)^2]$ -module we have

$$\mathcal{I}_k M = \ker(M \rightarrow M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z})$$

and

$$\mathcal{J}_k M = \ker(M \rightarrow M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k)).$$

In particular, the natural map

$$M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \rightarrow M \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z}$$

(induced by $GW(k) \rightarrow \mathbb{Z}$) is an isomorphism if and only if $\mathcal{J}_k M = \mathcal{I}_k M$.

For a field k with at least 4 elements, the *scissors congruence group* or *pre-Bloch group*, $\mathcal{P}(k)$, is the $\mathbb{Z}$ -module with generators $[a]$, $a \in k^\times$, subject to the relations

- (1) $[1] = 0$, and
(2)

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \text{ for } x, y \neq 1.$$

The *refined pre-Bloch group*, $\mathcal{RP}(k)$, is the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module with generators $[a]$, $a \in k^\times$, subject to the relations

- (1) $[1] = 0$, and
(2)

$$[x] - [y] + \langle x \rangle \left[\frac{y}{x}\right] - \langle x^{-1} - 1 \rangle \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \langle 1-x \rangle \left[\frac{1-x}{1-y}\right] \text{ for } x, y \neq 1.$$

We let

$$S_2(k) := \frac{k^\times \otimes_{\mathbb{Z}} k^\times}{\langle \{x \otimes y + y \otimes x \mid x, y \in k^\times\} \rangle},$$

the second (graded) symmetric power. We let $x \circ y$ denote the image of $x \otimes y$ in $S_2(k)$. We endow $S_2(k)$ with the trivial $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure.

The *refined Bloch group*, $\mathcal{RB}(k)$, of the field k (with at least 4 elements) is the kernel of the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module homomorphism Λ :

$$\begin{aligned} \Lambda = (\lambda_1, \lambda_2) : \mathcal{RP}(k) &\rightarrow \mathcal{I}_k^2 \oplus S_2(k), \\ [a] &\mapsto (\langle\langle a \rangle\rangle, \langle\langle 1-a \rangle\rangle, a \circ (1-a)). \end{aligned}$$

The following is the main result (Theorem 4.3 (1)) of [Hut13b]:

Proposition 6.1. *For an infinite field k there is a natural complex*

$$0 \rightarrow \mathrm{Tor}(\mu_k, \mu_k) \rightarrow H_3(SL_2(k), \mathbb{Z}) \rightarrow \mathcal{RB}(k) \rightarrow 0$$

of $\mathbb{Z}[k^\times/(k^\times)^2]$ -modules which is exact except possibly at the middle term where the homology is annihilated by 4.

For $1 \neq x \in k^\times$, the element $[x] + [1-x] \in \mathcal{P}(k)$ is independent of x and has order dividing 6 ([Sus90, Lemma 1.3, Lemma 1.5]). We denote this constant element $[x] + [1-x]$ by $\mathcal{C}_k$. Furthermore, by [Sus90, Lemma 1.2], for $x \in k^\times$ the elements $\psi(x) := [x] + [x^{-1}] \in \mathcal{P}(k)$ satisfy $2\psi(x) = 0$ and $\psi(xy) = \psi(x) + \psi(y)$. We denote by $\mathcal{S}_k$ the group $\{\psi(x) \mid x \in k^\times\} \subset \mathcal{P}(k)$ and by $\tilde{\mathcal{P}}(k)$ the group $\mathcal{P}(k)/\mathcal{S}_k$. Observe that the natural map $\mathcal{P}(k) \rightarrow \tilde{\mathcal{P}}(k)$ induces an isomorphism $\mathcal{P}(k) \otimes \mathbb{Z}[1/2] \cong \tilde{\mathcal{P}}(k) \otimes \mathbb{Z}[1/2]$.

Similarly, we let $\psi_1(x)$ denote the element $[x] + \langle -1 \rangle [x^{-1}] \in \mathcal{RP}(k)$. Note that these elements are not generally of finite order. Let $\widetilde{\mathcal{RP}}(k)$ denote the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module obtained by taking the quotient of $\mathcal{RP}(k)$ modulo the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module generated by the set $\{\psi_1(x) \mid x \in k^\times\}$ and let $\widetilde{\mathcal{RB}}(k)$ denote the image of the composite map $\mathcal{RB}(k) \rightarrow \mathcal{RP}(k) \rightarrow \widetilde{\mathcal{RP}}(k)$. Then ([Hut13a, Lemma 4.1]) we have:

Lemma 6.2. *The natural map $\mathcal{RB}(k) \rightarrow \widetilde{\mathcal{RB}}(k)$ is surjective with kernel annihilated by 4. In particular, it induces an isomorphism $\mathcal{RB}(k) \otimes \mathbb{Z}[1/2] \cong \widetilde{\mathcal{RB}}(k) \otimes \mathbb{Z}[1/2]$.*

Now suppose that k is an infinite field with surjective valuation $v : k^\times \rightarrow \Gamma$, where Γ is a totally ordered additive abelian group, and corresponding residue field $\bar{k}$. Let $\phi : \Gamma \rightarrow \mathbb{Z}/2$ be a group homomorphism. For an abelian group A , we let $A[\phi]$ denote A endowed with the $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure

$$\langle x \rangle \cdot a := (-1)^{\phi(v(x))} a \quad \text{for all } x \in k^\times, a \in A.$$

Then we have ([Hut13b, section 4.3]):

Proposition 6.3. *There is a natural surjective $\mathbb{Z}[k^\times/(k^\times)^2]$ -module homomorphism $S_{v,\phi} = S_\phi : \widehat{\mathcal{RP}}(k) \rightarrow \tilde{\mathcal{P}}(\bar{k})[\phi]$ determined by the formula*

$$S_\phi([a]) = \begin{cases} [\bar{a}], & v(a) = 0, \\ \mathcal{C}_{\bar{k}}, & v(a) > 0, \\ -\mathcal{C}_{\bar{k}}, & v(a) < 0. \end{cases}$$

Furthermore, if $\phi \neq 0$ the image of the induced composite homomorphism

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow \mathcal{RB}(k) \rightarrow \tilde{\mathcal{P}}(\bar{k})[\phi]$$

contains $4 \cdot \tilde{\mathcal{P}}(\bar{k})$.

The following corollary, which follows from the case $\phi = 0$ in Proposition 6.3, will be needed below:

Corollary 6.4. *Let k be a field with valuation and corresponding residue field $\bar{k}$. There is a natural surjective homomorphism $\mathcal{P}(k) \rightarrow \tilde{\mathcal{P}}(\bar{k})$.*

Proof. When $\phi = 0$, $\tilde{\mathcal{P}}(\bar{k})$ has the trivial $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure, and hence the homomorphism S_ϕ factors through $\mathcal{RP}(k)_{k^\times} = \mathcal{P}(k)$. $\square$

Corollary 6.5. *Let k be a field with surjective valuation $v : k^\times \rightarrow \Gamma$ and residue field $\bar{k}$. Suppose that*

- (1) $\Gamma/2\Gamma \neq 0$ and
- (2) $16 \cdot \tilde{\mathcal{P}}(\bar{k}) \neq 0$.

Then $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}) \neq 0$.

Proof. Let $\phi : \Gamma \rightarrow \mathbb{Z}/2$ be a non-zero homomorphism. Let $y \in \tilde{\mathcal{P}}(\bar{k})$ with $16y \neq 0$. There exists $x \in H_3(SL_2(k), \mathbb{Z})$ with $S_\phi(x) = 4y$.

Choose $\pi \in \mathcal{O}_v$ with $\phi(v(\pi)) = 1$. So $\langle \pi \rangle y = -y$, and hence $\langle\langle \pi \rangle\rangle y = \langle\langle \pi^{-1} \rangle\rangle y = -2y$.

But $v(1-\pi) = 0 = v(\pi-1)$, and hence $v(1-\pi^{-1}) = v(\pi^{-1})$ and $\phi(v(1-\pi^{-1})) = 1$ also. Thus $\langle\langle 1-\pi^{-1} \rangle\rangle y = -2y$ also. It follows that

$$S_\phi(\langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle x) = \langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle S_\phi(x) = (-2) \cdot (-2) \cdot 4y = 16y \neq 0,$$

and hence $0 \neq \langle\langle \pi^{-1} \rangle\rangle \langle\langle 1-\pi^{-1} \rangle\rangle x \in \mathcal{J}_k H_3(SL_2(k), \mathbb{Z})$. $\square$

The following is Theorem 6.19 in [Hut13a]:

Proposition 6.6. *Let k be a local field with finite residue field $\bar{k}$ of odd order. If $\mathbb{Q}_3 \subset k$, suppose that $[k : \mathbb{Q}_3]$ is odd. Then there is an isomorphism of $\mathbb{Z}[k^\times/(k^\times)^2]$ -modules*

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \cong (K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2]) \oplus (\mathcal{P}(\bar{k}) \otimes \mathbb{Z}[1/2]).$$

In this isomorphism, the map from $H_3(SL_2(k), \mathbb{Z}[1/2])$ to the second factor is induced by S_ϕ , where ϕ is the non-trivial homomorphism $\Gamma = \mathbb{Z} \rightarrow \mathbb{Z}/2$.

Remark 6.7. If k is a finite field with q elements, then $\mathcal{P}(k)$ has order $q + 1$ and $\mathcal{P}(k) \otimes \mathbb{Z}[1/2]$ is cyclic ([Hut13b, Lemma 7.4]) of order $(q + 1)'$. Here, n' denotes the odd part of the integer n : $n = 2^a n'$ with $a \geq 0$ and n' odd.

More generally we have the following ([Hut13c, Theorem 6.47]):

Proposition 6.8. *Let $k_0, k_1, \dots, k_n = k$ be a sequence of fields satisfying:*

- (1) *For each $i \in \{1, \dots, n\}$ there is a complete discrete value v_i on k_i with residue field k_{i-1} .*
- (2) *k_0 is either finite or real-closed or quadratically closed.*
- (3) *$\text{char}(k_0) \neq 2$.*
- (4) *Either $\text{char}(k) = 3$ or $\text{char}(k_0) \neq 3$ or k contains a primitive cube root of unity.*

Then there is a natural split short exact sequence

$$0 \rightarrow \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} \rightarrow H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2] \rightarrow 0.$$

Remark 6.9. The direct sum decomposition occurring here is the eigenspace decomposition for the group of characters on $k^\times / (k^\times)^2$ which restrict to the trivial character on $k_0^\times / (k_0^\times)^2$ (see [Hut13c, section 6]).

To spell this out, let k be complete with respect to a discrete valuation v with residue field $\bar{k}$ of characteristic not equal to 2. Let $U := \{a \in k^\times \mid v(a) = 0\}$. By Hensel's Lemma $u \in U$ is square if and only if $\bar{u} \in \bar{k}^\times$ is a square. Thus $U/U^2 \cong \bar{k}^\times / (\bar{k}^\times)^2$, and if $\pi \in k^\times$ is a uniformizer there is a natural (split) short exact sequence

$$1 \rightarrow \bar{k}^\times / (\bar{k}^\times)^2 \rightarrow k^\times / (k^\times)^2 \rightarrow \pi^{\mathbb{Z}/2} \rightarrow 1,$$

where the first injection is obtained by choosing an inverse image x in U of a given element $\bar{x} \in \bar{k}$.

Now let $\mathcal{X}_k := \text{Hom}(k^\times / (k^\times)^2, \mu_2)$. As noted, the conditions on k in the proposition (completeness of v_i and $\text{char}(k_0) \neq 2$) ensure that there are natural injective maps

$$k_{i-1}^\times / (k_{i-1}^\times)^2 \rightarrow k_i^\times / (k_i^\times)^2,$$

and hence there are surjective restriction homomorphisms

$$\mathcal{X}_{k_i} \rightarrow \mathcal{X}_{k_{i-1}}.$$

For each $i \leq n$, let

$$W_i := \ker(\mathcal{X}_k \rightarrow \mathcal{X}_{k_i}) = \{\chi \in \mathcal{X}_k \mid \chi|_{k_i^\times} = 1\}.$$

For each $i < n$ and $\chi \in W_i \setminus W_{i+1}$, the χ -eigenspace of $H_3(SL_2(k), \mathbb{Z}[1/2])$ – which we will denote $H_3(SL_2(k), \mathbb{Z}[1/2])_\chi$ – is isomorphic to $\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2]$, and, by definition, the square class $\langle a \rangle$ acts as multiplication by $\chi(a)$ on this factor.

Lemma 6.10. *Let k be as in Proposition 6.8. Let M be a $\mathbb{Z}[k^\times / (k^\times)^2] \otimes \mathbb{Z}[1/2]$ -module. Let $i < n$ and $\chi \in W_i \setminus W_{i+1}$. Then $\mathcal{J}_k M_\chi = M_\chi$.*

Proof. Let $\mathcal{O}_n = \{a \in k \mid v_n(a) \geq 0\}$ and for $j < n$ define recursively $\mathcal{O}_j = \{a \in \mathcal{O}_{j+1} \mid v_j(\pi_j(a)) \geq 0\}$, where $\pi_j : \mathcal{O}_{j+1} \rightarrow k_j$ is the natural surjection. Let $x \in \mathcal{O}_{i+1}$ with $v_{i+1}(\pi_{i+1}(x)) = 1$. Then the group (of order 2) $k_{i+1}^\times / ((k_{i+1}^\times)^2 \cdot k_i^\times)$ is generated by the class of x , and hence $\chi(\langle x \rangle) = \chi(\langle x^{-1} \rangle) = -1$. Since

$$v_{i+1}(\pi_{i+1}(x - 1)) = 0,$$

it follows that the class of $x-1$ represents an element - possibly trivial - of $k_i^\times/(k_i^\times)^2$, and hence $\chi(\langle x-1 \rangle) = 1$. Hence

$$\chi(\langle 1-x^{-1} \rangle) = \chi(\langle x^{-1} \rangle)\chi(\langle x-1 \rangle) = -1 \cdot 1 = -1.$$

Thus $\langle\langle x^{-1} \rangle\rangle$ and $\langle\langle 1-x^{-1} \rangle\rangle$ both act on M_χ as multiplication by -2 , and hence $\langle\langle x^{-1} \rangle\rangle \langle\langle 1-x^{-1} \rangle\rangle \in \mathcal{J}_k$ acts on M_χ as multiplication by 4 . $\square$

Corollary 6.11. *If k is as in Proposition 6.8, then*

(1)

$$\begin{aligned} H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) &\cong H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} \mathbb{Z} \\ &\cong K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2] \end{aligned}$$

and

(2) *there is a natural short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} &\rightarrow H_3(SL_2(k), \mathbb{Z}[1/2]) \\ &\rightarrow H_3(SL_2(k), \mathbb{Z}[1/2]) \otimes_{\mathbb{Z}[k^\times/(k^\times)^2]} GW(k) \rightarrow 0. \end{aligned}$$

$$(3) \quad \mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \cong \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}.$$

Proof. The second statement follows from the first by Proposition 6.8, and the third is an immediate consequence of the second.

To prove the first isomorphism of statement (1), we must show that

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) = \mathcal{I}_k H_3(SL_2(k), \mathbb{Z}[1/2]).$$

Now

$$\begin{aligned} \mathcal{I}_k H_3(SL_2(k), \mathbb{Z}[1/2]) &= \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}} \\ &= \bigoplus_{i=0}^{n-1} \left(\bigoplus_{\chi \in W_i \setminus W_{i+1}} H_3(SL_2(k), \mathbb{Z}[1/2])_\chi \right) \end{aligned}$$

by Proposition 6.8 and the remark which follows it. The result follows by Lemma 6.10. $\square$

Remark 6.12. Let k satisfy the hypotheses of Proposition 6.8. If k_0 is finite or quadratically closed, then the Witt ring $W(k_0)$ of k_0 is 2-torsion. An easy induction using Springer's Theorem on Witt rings of fields complete with respect to a discrete valuation ([Lam05, Chapter VI, Theorem 1.4]) implies that $W(k)$ is 2-torsion and hence that $GW(k) \otimes \mathbb{Z}[1/2] = \mathbb{Z}[1/2]$.

However, in the case that k_0 is real closed, then – by Springer's Theorem again – the fundamental ideal of $W(k)$ contains a free abelian group of rank 2^n and the map $GW(k) \otimes \mathbb{Z}[1/2] \rightarrow \mathbb{Z}[1/2]$ has a large kernel.

The following is a special case of [Hut13a, Theorem 5.1]:

Proposition 6.13. *Let $\mathcal{O}$ be a unique factorization domain with field of fractions k . Let P be a set of representatives of the orbits of prime elements of $\mathcal{O}$ under the action, by multiplication, of $\mathcal{O}^\times$. For each $p \in P$ there is a discrete valuation $v_p : k^\times \rightarrow \mathbb{Z}$ with corresponding residue field $\bar{k}_p$. Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/2$ be the non-zero homomorphism.*

Then the specialization homomorphisms induce a well-defined surjective map of $\mathbb{Z}[k^\times/(k^\times)^2]$ -modules

$$S = \sum_{p \in P} S_{p, \phi} : H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow \bigoplus_{p \in P} \mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2].$$

Corollary 6.14. *Let $\mathcal{O}$ be a unique factorization domain with field of fractions k . Then the map induced by S ,*

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow \bigoplus_{p \in P} \mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2],$$

is surjective.

More generally, the map

$$\mathcal{J}_k(H_3(SL_2(k), \mathbb{Z}) \otimes A) \rightarrow \bigoplus_{p \in P} (\mathcal{P}(\bar{k}_p) \otimes A)$$

is surjective for any commutative $\mathbb{Z}[1/2]$ -algebra A .

Proof. We will use the notation

$$\mathcal{P}(\mathcal{O}) := \bigoplus_{p \in P} \mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2].$$

By Proposition 6.13, it is enough to show that $\mathcal{J}_k \mathcal{P}(\mathcal{O}) = \mathcal{P}(\mathcal{O})$.

Let $x \in \mathcal{P}(\mathcal{O})$. There exist primes $p_1, \dots, p_t \in P$ such that $x = \sum_{i=1}^t x_i$ with $x_i \in \mathcal{P}(\bar{k}_{p_i}) \otimes \mathbb{Z}[1/2]$.

Recall that $\langle a \rangle \in \mathbb{Z}[k^\times/(k^\times)^2]$ acts as multiplication by $(-1)^{v_{p_i}(a)}$ on $\mathcal{P}(\bar{k}_{p_i})$. Choose $a \in \mathcal{O}$ with the property that $v_{p_i}(a)$ is odd for $1 \leq i \leq t$. Let $b = 1/a$. Then $v_{p_i}(b) = v_{p_i}(1 - b)$ is odd for all i . It follows that $\langle\langle b \rangle\rangle \langle\langle 1 - b \rangle\rangle x_i = (-2)^2 x_i = 4x_i$ for all i and hence that

$$x = \langle\langle b \rangle\rangle \langle\langle 1 - b \rangle\rangle \left(\frac{x}{4} \right) \in \mathcal{J}_k \mathcal{P}(\mathcal{O}).$$

□

Remark 6.15. With a little more care, one can show that the image of the map

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}) \rightarrow \bigoplus_{p \in P} \tilde{\mathcal{P}}(\bar{k}_p)$$

contains $\bigoplus_{p \in P} 16 \cdot \tilde{\mathcal{P}}(\bar{k}_p)$.

7. ON THE FAILURE OF WEAK HOMOTOPY INVARIANCE

This section sums up our insights into the failure of weak homotopy invariance for the third homology of SL_2 .

7.1. Module structures. We show how the failure of weak homotopy invariance derives from the fact that the square class module structure on $\mathbb{A}^1$ -invariant group homology descends to a Grothendieck-Witt module structure, by the results of Section 5, while the one on group homology typically does not, by the results of Section 6.

We begin by noting that the kernel of the change-of-topology morphism is non-trivial for a quite general class of fields.

Theorem 7.1. *Let k be a field of characteristic $\neq 2$, with surjective valuation $v : k^\times \rightarrow \Gamma$ and residue field $\bar{k}$. Suppose that*

- (1) $\Gamma/2\Gamma \neq 0$ and
- (2) $16 \cdot \tilde{\mathcal{P}}(\bar{k}) \neq 0$.

Then the kernel of the natural change-of-topology morphism

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is not trivial.

Furthermore, if ℓ is an odd prime and if $\mathcal{P}(\bar{k}) \otimes \mathbb{Z}/\ell \neq 0$, then the same statement holds with $\mathbb{Z}$ replaced by $\mathbb{Z}/\ell$.

Proof. The first statement is a direct consequence of Corollaries 5.6 and 6.5 above.

For the second statement, the universal coefficient theorem together with the snake lemma implies a commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & \ker \phi & \longrightarrow & \ker \psi \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_3(SL_2(k), \mathbb{Z}) \otimes \mathbb{Z}/\ell & \longrightarrow & H_3(SL_2(k), \mathbb{Z}/\ell) \\ & & \downarrow \phi & & \downarrow \psi \\ 0 & \longrightarrow & H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \otimes \mathbb{Z}/\ell & \longrightarrow & H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell) \end{array}$$

But the group $\ker \phi$ contains $\mathcal{J}_k(H_3(SL_2(k), \mathbb{Z}) \otimes \mathbb{Z}/\ell)$, since $GW(k) \otimes \mathbb{Z}/\ell = (\mathbb{Z}[k^\times/(k^\times)^2] \otimes \mathbb{Z}/\ell)/\text{Image}(\mathcal{J}_k)$. This, in turn, maps onto the non-zero group $\mathcal{P}(\bar{k}) \otimes \mathbb{Z}/\ell = \tilde{\mathcal{P}}(\bar{k}) \otimes \mathbb{Z}/\ell$, thus proving the second claim. $\square$

Next, we observe (Theorem 7.4 below) that when the field k is small, the kernel of the change-of-topology morphism is always large.

Lemma 7.2. *Let k be a global field. Let ℓ be an odd prime such that $[k(\zeta_\ell) : k]$ is even, where ζ_ℓ denotes a primitive ℓ -th root of unity. Then there are infinitely many finite places v of k satisfying $\ell \nmid q_v + 1$, where q_v is the cardinality of the residue field, $\bar{k}_v$, at v .*

Proof. Let $L = k(\zeta_\ell)$. By the Chebotarev density theorem (see [Neu99, Chapter VII, Theorem 13.4] for number fields and [Ros02, Theorem 9.13A] for global function fields) there are infinitely many primes of k (not dividing ℓ) whose Frobenius in $\text{Gal}(L/k)$ has order 2. It follows that, for such a prime v , $\zeta_\ell \notin \bar{k}_v = \mathbb{F}_{q_v}$, but $\zeta_\ell \in \mathbb{F}_{q_v^2}$. Thus, for such v , $\ell \nmid q_v - 1$, but $\ell \mid q_v^2 - 1$. $\square$

Remark 7.3. Of course, for any given number field k , $[k(\zeta_\ell) : k] = \ell - 1$ for all but finitely many odd primes ℓ .

If k is a global field of positive characteristic, there are infinitely many odd primes ℓ such that $[k(\zeta_\ell) : k]$ is even.

Theorem 7.4. *Let k be an infinite perfect field which is finitely generated over its prime field, and assume that $\text{char } k \neq 2$. Then the kernel of the natural change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}[1/2])$$

is not finitely generated.

Furthermore, if ℓ is an odd prime for which $[k(\zeta_\ell) : k]$ is even, then the same statement holds with $\mathbb{Z}[1/2]$ replaced by $\mathbb{Z}/\ell$.

Proof. Let $A = \mathbb{Z}[1/2]$ or $A = \mathbb{Z}/\ell$ for an odd prime ℓ .

Now the field must contain a subfield, k_0 say, isomorphic either to $\mathbb{Q}$ or to $\mathbb{F}_p(x)$ where $p = \text{char}(k) > 0$. Let d denote the transcendence degree of k over k_0 . We will prove the result, together with the statement that $\mathcal{P}(k) \otimes A$ is not finitely generated, by induction on d .

Suppose first that $d = 0$. Then k is a global field. In this case (fixing an infinite prime in the function field case), the ring of integers $\mathcal{O}_k$ is a Dedekind domain with finite class group. Hence there exists $a \in \mathcal{O}_k$ for which $\mathcal{O} := \mathcal{O}_k[a^{-1}]$ is a unique factorization domain. Now for any prime p of $\mathcal{O}$, $\mathcal{P}(\bar{k}_p) \otimes \mathbb{Z}[1/2]$ is cyclic of order $(q_p + 1)'$, where n' denotes the prime-to-2 part of the number n (Remark 6.7). This, together with Lemma 7.2, implies that $\mathcal{P}(\bar{k}_p) \otimes A$ is non-zero for infinitely many primes p . The claim then follows from Corollary 5.6 and Corollary 6.14.

We also observe that for any field k , there is an exact sequence

$$\mathcal{P}(k) \rightarrow S_2(k) \rightarrow K_2(k) \rightarrow 0,$$

where the first map sends $[a]$ to $a \circ (1 - a)$ and the second sends $a \circ b$ to $\{a, b\}$. However, when k is a global field $K_2(k)$ is a torsion group while $S_2(k)$ modulo torsion is a free abelian group of infinite rank. It follows that $\mathcal{P}(k)$ maps onto a free abelian group of infinite rank, and hence $\mathcal{P}(k) \otimes A$ is not finitely generated.

Now suppose $d > 0$ and the result is known for fields of smaller transcendence degree over k_0 . Then for any discrete valuation p on k , $\bar{k}_p$ is an infinite finitely generated field of smaller transcendence degree. By the proof of Corollary 6.5 the kernel of the change-of-topology morphism surjects onto $\mathcal{P}(\bar{k}_p) \otimes A$. By induction, $\mathcal{P}(\bar{k}_p) \otimes A$ is already not finitely generated. Also by Corollary 6.4 we have the surjection $\mathcal{P}(k) \otimes A \rightarrow \mathcal{P}(\bar{k}_p) \otimes A$ and the result follows. $\square$

Remark 7.5. The induction step could also be proved by noting that the conditions on the field k in the theorem imply that there is a subring $\mathcal{O}$ of k with the following properties: $\mathcal{O}$ is a unique factorization domain with field of fractions k and for which the set P of association classes of prime elements is infinite.

Finally, we note that in the case of local fields, we can describe the exact structure of the change-of-topology kernel over $\mathbb{Z}[1/2]$.

Theorem 7.6. *Let k be a field satisfying the conditions of Proposition 6.8. Then the change-of-topology morphism factors through $K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2]$. Its kernel is isomorphic to*

$$\bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}.$$

In particular, if k is complete with respect to a discrete valuation with residue field k_0 which is either finite of odd order or real-closed or quadratically closed, then this kernel is isomorphic to $\mathcal{P}(k_0) \otimes \mathbb{Z}[1/2]$.

Proof. Recall that $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2])$ is contained in the kernel of the change-of-topology morphism by Corollary 5.6. But

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) = \ker (H_3(SL_2(k), \mathbb{Z}[1/2]) \rightarrow K_3^{\text{ind}}(k) \otimes \mathbb{Z}[1/2])$$

by Corollary 6.11. Since the map to $K_3^{\text{ind}}(k)$ factors through the change-of-topology morphism by Lemma 4.2, it follows that $\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2])$ is equal to the change-of-topology kernel.

Finally,

$$\mathcal{J}_k H_3(SL_2(k), \mathbb{Z}[1/2]) \cong \bigoplus_{i=0}^{n-1} (\mathcal{P}(k_i) \otimes \mathbb{Z}[1/2])^{\oplus 2^{n-i-1}}$$

by Corollary 6.11 (3). □

7.2. Number fields: Finite generation. In the case when k is a number field, we can deduce the failure of weak homotopy invariance in another way which is of independent interest - via simple size considerations: the group $H_3(SL_2(k), \mathbb{Z})$ is not finitely-generated while $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$ is. This last fact is a consequence of finite generation results in symplectic K-theory:

Proposition 7.7. (i) *Let k be a non-archimedean local field of characteristic $\neq 2$, and let ℓ be an odd prime different from the characteristic. Then the group $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}/\ell)$ is finite.*

(ii) *Let k be a number field. Then the homology group $H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}[1/2])$ is a finitely generated $\mathbb{Z}[1/2]$ -module.*

Proof. By Corollary 3.9, it suffices to prove the statements for

$$H_3(Sp_\infty(k), \mathbb{Z}) \cong H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z}).$$

The simplicial set $BSp_\infty(k[\Delta^\bullet])$ is simply connected because Sp_∞ is $\mathbb{A}^1$ -connected. Therefore, the Hurewicz theorem implies a surjection

$$\pi_3^{\mathbb{A}^1}(BSp_\infty)(\text{Spec } k) \cong \pi_3(BSp_\infty(k[\Delta^\bullet])) \rightarrow H_3(BSp_\infty(k[\Delta^\bullet]), \mathbb{Z}).$$

We are thus reduced to show finite generation for $\pi_3^{\mathbb{A}^1}(BSp_\infty)(k) \cong KSp_3(k)$. The symplectic K-theory assertion is proved in Proposition 7.8, resp. Proposition 7.10, below. □

We now recall a finite generation result for KSp_3 over number fields. We have to apologize for different notation for hermitian K-groups and Grothendieck-Witt groups appearing in the proof. This is mainly due to different existing notation in the literature. For the definition of the Karoubi-style notation U and V , and the relevant periodicity exact sequences, we refer to the fundamental paper of Karoubi; cf. [Kar73]. For the more contemporary notation for Grothendieck-Witt groups, we refer to the work of Schlichting; cf. [Sch09].

Proposition 7.8. *Let k be a number field. Then $KSp_3(k) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ is a finitely generated $\mathbb{Z}[1/2]$ -module.*

Proof. To determine $KSp_3(k)$, we use the computations of Hornbostel; cf. [Hor02]. Note that in loc.cit., the group $KSp_3(k)$ is denoted by ${}_{-1}K_3^h(k)$. Let $A = \mathcal{O}_{k,S}$ be a ring of S -integers for a finite set S of places containing all the infinite places and all the places lying above 2. Using [Hor02, Corollary 4.15], we obtain an exact sequence

$$\cdots \rightarrow \bigoplus_{\mathfrak{p}} {}_{-1}U_3(A/\mathfrak{p}) \rightarrow {}_{-1}K_3^h(A) \rightarrow {}_{-1}K_3^h(k) \rightarrow \bigoplus_{\mathfrak{p}} {}_{-1}U_2(A/\mathfrak{p}) \rightarrow \cdots.$$

The U -theory groups of finite fields are determined in [Hor02, Corollary 4.17]:

$${}_{-1}U_2(\mathbb{F}_q) \cong \mathbb{Z}/2, \quad \text{and} \quad {}_{-1}U_3(\mathbb{F}_q) \cong Gr_4,$$

where Gr_4 is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. In particular, tensoring the above exact sequence with $\mathbb{Z}[1/2]$ yields an isomorphism ${}_{-1}K_3^h(A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \cong {}_{-1}K_3^h(k) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

We are very grateful to the referee for generously providing the following argument for finite generation of $KSp_3(A)$ for a ring of S -integers $A = \mathcal{O}_{k,S}$. First note that finite generation for algebraic K-theory is proved in [Qui73]. There are Karoubi periodicity sequences of the form

$$K_i(A) \rightarrow GW_i^j(A) \rightarrow GW_{i-1}^{j-1}(A) \rightarrow K_{i-1}(A).$$

We first use the sequence for $j = 2$ and $i = 3$, in which case $GW_3^2(A) = KSp_3(A)$ in our notation:

$$K_3(A) \rightarrow KSp_3(A) \rightarrow GW_2^1(A).$$

By the above-mentioned finite generation for algebraic K-theory, it suffices to show finite generation for $GW_2^1(A)$. Now we use the Karoubi periodicity sequence for $j = 1$ and $i = 2$:

$$K_2(A) \rightarrow GW_2^1(A) \rightarrow GW_1^0(A),$$

which reduces us to show finite generation for $GW_1^0(A) = KO_1(A)$. By [Bas74, Corollary 4.7.6], there is an exact sequence

$$K_1(A) \rightarrow KO_1(A) \rightarrow {}_2\text{Pic}(A) \oplus \text{Hom}(\text{Spec } A, \mathbb{Z}/2) \rightarrow 0.$$

As we are interested in results with $\mathbb{Z}[1/2]$ -coefficients, the last term is 2-torsion and hence not relevant. Nevertheless, it is well known that S -class groups are finite, which implies finite generation of the 2-torsion term. The finite generation for $KO_1(A)[1/2]$ then follows from finite generation of $K_1(A)$, which is the well-known finite generation of S -unit groups from number theory. $\square$

Remark 7.9. Alternatively, one can deduce finite generation of $KSp_n(\mathcal{O}_{K,S})$ as follows: the existence of the Borel-Serre compactification of $BSp_{2n}(\mathcal{O}_{k,S})$ implies that its homology is finitely generated. The elementary subgroup $Ep_{2n}(\mathcal{O}_{k,S})$ is a perfect finite-index subgroup of $Sp_{2n}(\mathcal{O}_{k,S})$. Applying Serre's theory of classes of abelian groups to the universal cover of $BSp_{\infty}(\mathcal{O}_{k,S})^+$ implies finite generation for symplectic K-groups.

Proposition 7.10. *Let k be a non-archimedean local field of characteristic $\neq 2$, and let ℓ be an odd prime different from the characteristic. Then $KSp_3(k) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell$ is finite.*

Proof. We denote by $\mathcal{O}$ the valuation ring, and by $\mathcal{O}/\mathfrak{m}$ the residue field. Using the localization sequence for symplectic K-theory as in Proposition 7.8, it suffices to prove the assertion for ${}_{-1}U_2(\mathcal{O}/\mathfrak{m})$ and ${}_{-1}K_3^h(\mathcal{O})$. As in Proposition 7.8, ${}_{-1}U_2(\mathbb{F}_q)$ is a finite 2-group, hence it does not contribute. The valuation ring is a complete discrete valuation ring, therefore we have an isomorphism

$${}_{-1}K_3^h(\mathcal{O}) \otimes \mathbb{Z}/\ell \cong {}_{-1}K_3^h(\mathcal{O}/\mathfrak{m}) \otimes \mathbb{Z}/\ell.$$

The group on the right-hand side is a finite group. $\square$

Theorem 7.11. *Let k be a number field. Then the kernel of the natural change-of-topology morphism*

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is not finitely generated. The same also holds with $\mathbb{Z}/\ell$ -coefficients, with ℓ an odd prime for which $[k(\zeta_\ell) : k]$ is even.

Proof. Let A denote either $\mathbb{Z}[1/2]$ or $\mathbb{Z}/\ell$, where ℓ is an odd prime for which $[k(\zeta_\ell) : k]$ is even. By the proof of Theorem 7.4, the group $H_3(SL_2(k), A)$ is not finitely generated while, by Proposition 7.7, $H_3(BSL_2(k[\Delta^\bullet]), A)$ is finitely generated. Thus the kernel of

$$H_3(SL_2(k), A) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), A)$$

is not finitely generated. $\square$

8. ON THE COMPARISON OF UNSTABLE K-THEORIES

In this section, we briefly discuss some further consequences of our results for the comparison of “unstable K-theories”. There are two possible definitions of unstable K-theory associated to a linear algebraic group G and a commutative unital ring R that we want to discuss. For simplicity, we restrict to $G = SL_n$ and $R = k$, avoiding subtleties in more general settings.

- On the one hand, following Quillen [Qui69], one can define a version of unstable K-theory using the plus-construction applied to a classifying space:

$$K_i^Q(SL_n, k) := \pi_i BSL_n(k)^+.$$

The plus-construction here is taken with respect to the maximal perfect subgroup $E_n(k) = SL_n(k)$.

- On the other hand, following Karoubi-Villamayor [KV69], one can define a version of unstable K-theory as homotopy of the classifying space of the simplicial replacement:

$$KV_i(SL_n, k) := \pi_i BSL_n(k[\Delta^\bullet]).$$

There is a natural comparison morphism $BSL_n(k)^+ \rightarrow BSL_n(k[\Delta^\bullet])$ which is induced from the inclusion of constants $BSL_n(k) \rightarrow BSL_n(k[\Delta^\bullet])$ via the universal property of the plus-construction. Applying homotopy groups, this induces a comparison morphism $K_i^Q(SL_n, k) \rightarrow KV_i(SL_n, k)$.

It is well known (and primarily a consequence of homotopy invariance of algebraic K-theory) that the comparison morphism becomes a weak equivalence $BSL(k)^+ \rightarrow BSL(k[\Delta^\bullet])$ in the limit $n \rightarrow \infty$. Our results now show that the comparison morphism $BSL_2(k)^+ \rightarrow BSL_2(k[\Delta^\bullet])$ fails to be a weak equivalence for large classes of fields.

Proposition 8.1. *Let k be a field satisfying the conditions of Theorem 1. Then the natural map*

$$BSL_2(k)^+ \rightarrow BSL_2(k[\Delta^\bullet])$$

is not a weak equivalence.

Proof. The inclusion of constants $BSL_2(k) \rightarrow BSL_2(k[\Delta^\bullet])$ factors through the map in the statement by the universal property of the plus-construction. Assume that the induced map $BSL_2(k)^+ \rightarrow BSL_2(k[\Delta^\bullet])$ is a weak equivalence; in particular, it induces an isomorphism in homology. Since the plus-construction map $BSL_2(k) \rightarrow BSL_2(k)^+$ also induces an isomorphism in homology, this implies that the change-of-topology map

$$H_\bullet(BSL_2(k), \mathbb{Z}) \rightarrow H_\bullet(BSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

is an isomorphism. For fields k satisfying the conditions of Theorem 1 this yields a contradiction, hence the map $BSL_2(k)^+ \rightarrow BSL_2(k[\Delta^\bullet])$ cannot be a weak equivalence. $\square$

With a slightly more careful analysis, we can even make a stronger statement, showing that the respective K_3 -groups cannot be isomorphic.

Proposition 8.2. *Let k be a number field. Then the kernel of the natural map $K_3^Q(SL_2, k) \rightarrow KV_3(SL_2, k)$ is not finitely generated.*

Proof. We argue via finite generation as in the proof of Theorem 7.11.

(1) We first note that the finite generation argument of Proposition 7.8 more generally proves that the group $\pi_3^{\mathbb{A}^1}(BSL_2)(k) \otimes \mathbb{Z}[1/2]$ is finitely generated if k is a number field. From [AF12a, Theorem 3], there is an exact sequence

$$0 \rightarrow T'_4(k) \rightarrow \pi_2^{\mathbb{A}^1}(SL_2)(\text{Spec } k) \rightarrow KSp_3(k) \rightarrow 0,$$

where the last term $KSp_3(k)$ is finitely generated by Proposition 7.8. The group $T'_4(k)$ sits in an exact sequence

$$I^5(k) \rightarrow T'_4(k) \rightarrow S'_4(k) \rightarrow 0,$$

where $I^5(k)$ is the fifth power of the fundamental ideal of the Witt ring $W(k)$ and there is a surjection $K_4^M(k)/12 \twoheadrightarrow S'_4(k)$.

For k be a number field, [BT73, Theorem II.2.1(3)] implies $K_n^M(k) \cong \mathbb{Z}/2^{r_1}$, where $n \geq 3$ and r_1 is the number of real completions. So $K_4^M(k)$ is finite, hence also $S'_4(k)$ is finite.

The structure of the Witt ring and the powers of the fundamental ideal are also explicitly known for number fields, by the Hasse-Minkowski local-global principle. By [Lam05, Corollary VI.3.9], the Witt ring $W(k)$ splits as $W(k) \cong \mathbb{Z}^r \oplus W(k)_{\text{tor}}$, where r is the number of real places of k and the subgroup of torsion elements satisfies $8(W(k)_{\text{tor}}) = 0$. For $n \geq 3$, the n -th power of the fundamental ideal $I^n(k) = 2^{n-1}I(k)$ is free abelian. In particular, I^5 is a free abelian group of rank r , in particular finitely generated.

We conclude that for k a number field, the group $T'_4(k)$ is finitely generated. By the exact sequence of Asok and Fasel above, the group $\pi_2^{\mathbb{A}^1}(SL_2)(k)$ is finitely generated as well, which seems to be an interesting finite-generation result in $\mathbb{A}^1$ -homotopy.

(2) Since $SL_2(k) = E_2(k)$, the space $BSL_2(k)^+$ is simply connected. Therefore, the Hurewicz homomorphism $\pi_3 BSL_2(k)^+ \rightarrow H_3(BSL_2(k), \mathbb{Z})$ is surjective. As in

the proof of Theorem 7.4, the group $H_3(BSL_2(k), \mathbb{Z})$ is not finitely generated, and therefore $\pi_3 BSL_2(k)^+$ is not finitely generated.

(3) From (1) and (2) above, we conclude that the kernel of the comparison map $K_3^Q(SL_2, k) \rightarrow KV_3(SL_2, k)$ is not finitely generated. $\square$

Remark 8.3. Together with the surjection

$$\pi_3^{\mathbb{A}^1}(BSL_2)(k) \otimes \mathbb{Z}[1/2] \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}[1/2]),$$

the above argument provides another way of proving Theorem 7.11 without resorting to the improved stability results of Corollary 3.9.

Proposition 8.4. *Assume that k is a field satisfying the conditions of Theorem 1. Then the comparison map $K_3^Q(SL_2, k) \rightarrow KV_3(SL_2, k)$ is not injective.*

Proof. We saw earlier in Section 5 that the $k^\times$ -action on $BSL_2(k[\Delta^\bullet])$ induces a $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure on $\pi_3 BSL_2(k[\Delta^\bullet])$ which descends to a $GW(k)$ -module structure, by Proposition 5.4. As in Section 5, the conjugation with diagonal matrices $\text{diag}(u, 1)$ also induces a $\mathbb{Z}[k^\times/(k^\times)^2]$ -module structure on $\pi_3 BSL_2(k)^+$, and the comparison map

$$\pi_3 BSL_2(k)^+ \rightarrow \pi_3 BSL_2(k[\Delta^\bullet])$$

is $\mathbb{Z}[k^\times/(k^\times)^2]$ -equivariant. We have a commutative diagram

$$\begin{array}{ccc} \ker H & \longrightarrow & \ker H \otimes GW(k) \\ \downarrow & & \downarrow \\ \pi_3 BSL_2(k)^+ & \longrightarrow & \pi_3 BSL_2(k)^+ \otimes GW(k) \\ \downarrow H & & \downarrow \\ H_3(BSL_2(k), \mathbb{Z}) & \longrightarrow & H_3(BSL_2(k), \mathbb{Z}) \otimes GW(k) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

We omitted that all tensors above are over $\mathbb{Z}[k^\times/(k^\times)^2]$ for typographical reasons. The map $\ker H \rightarrow \ker H \otimes GW(k)$ is surjective since $\mathbb{Z}[k^\times/(k^\times)^2] \rightarrow GW(k)$ is surjective. Assuming $\pi_3 BSL_2(k)^+$ is a $GW(k)$ -module, then $\pi_3 BSL_2(k)^+ \rightarrow \pi_3 BSL_2(k)^+ \otimes GW(k)$ is an isomorphism. But then $\ker H$ and $\ker H \otimes GW(k)$ have isomorphic images in $\pi_3 BSL_2(k)^+$ and $\pi_3 BSL_2(k)^+ \otimes GW(k)$, respectively. This implies that $H_3(BSL_2(k), \mathbb{Z})$ already has a $GW(k)$ -module structure, contradicting the conclusion of Theorem 1. $\square$

The above results imply that there is no unique natural definition of unstable K-theory.

9. THE COKERNEL OF THE CHANGE-OF-TOPOLOGY MORPHISM

The main results in this article concern estimates of the kernel of the change-of-topology morphism from $H_3(SL_2(k), A)$ to $H_3(BSL_2(k[\Delta^\bullet]), A)$. In this section, we discuss the cokernel of this morphism. In order to do this, we treat some aspects of the spectral sequence associated to the bisimplicial set $BSL_2(k[\Delta^\bullet])$. This spectral

sequence computes the homology of the diagonal simplicial set $dBSL_2(k[\Delta^\bullet])$ and has the form

$$E_{p,q}^1 = H_q(BSL_2(k[\Delta^p]), \mathbb{Z}) \Rightarrow H_{p+q}(dBSL_2(k[\Delta^\bullet]), \mathbb{Z}),$$

with differentials $d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$. Here, the differentials

$$d_{p,q}^1 = \sum_{i=0}^p (-1)^i H_q(d_i) : H_q(BSL_2(k[\Delta^p]), \mathbb{Z}) \rightarrow H_q(BSL_2(k[\Delta^{p-1}]), \mathbb{Z})$$

are induced from the simplicial structure of the simplicial algebra $k[\Delta^\bullet]$.

We will assume in this section that the underlying field k is infinite of characteristic $\neq 2$.

Remark 9.1. For infinite k , homotopy invariance for homology of $SL_2(k[T])$ (cf. [Knu01, Theorem 4.3.1]) implies $E_{1,q}^2 \cong 0$ for all $q > 1$.

Since $H_0(SL_2(k[\Delta^n]), \mathbb{Z}) \cong \mathbb{Z}$ for all n , we also have

$$E_{p,0}^2 \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 9.2. The differential $d_{2,1}^2$ is trivial: by the stabilization results, there is a natural commutative diagram

$$\begin{array}{ccc} H_2(SL_2(k)) & \longrightarrow & H_2(BSL_2(k[\Delta^\bullet])) \\ \cong \downarrow & & \downarrow \cong \\ H_2(Sp_\infty(k)) & \xrightarrow{\cong} & H_2(BSp_\infty(k[\Delta^\bullet])) \end{array}$$

The right vertical isomorphism is a consequence of Proposition 3.5, and the left vertical isomorphism is a consequence of Matsumoto's computation of H_2 . The lower isomorphism follows from homotopy invariance for the infinite symplectic group [Kar73]. By the vanishing in Remark 9.1, we have an exact sequence

$$H_1(SL_2(k[\Delta^2]), \mathbb{Z}) / (\text{im } d_{3,1}^1) \xrightarrow{d_{2,1}^2} H_2(SL_2(k), \mathbb{Z}) \rightarrow H_2(BSL_2(k[\Delta^\bullet]), \mathbb{Z}) \rightarrow 0,$$

which proves the claim.

Proposition 9.3. *There is a short exact sequence*

$$\begin{aligned} 0 \rightarrow H_3(SL_2(k)) / (\text{im } d_{2,2}^2 + \text{im } d_{3,1}^3) &\rightarrow H_3(BSL_2(k[\Delta^\bullet])) \\ &\rightarrow H_1(SL_2(k[\Delta^2])) / (\text{im } d_{3,1}^1) \rightarrow 0 \end{aligned}$$

Moreover, the natural change-of-topology $H_3(SL_2(k)) \rightarrow H_3(BSL_2(k[\Delta^\bullet]))$ factors through the injection above.

Proof. By the vanishing in Remark 9.1, we have $E_{1,2}^\infty = E_{3,0}^\infty = 0$. The exact sequence claimed is the one induced from the spectral sequence

$$0 \rightarrow E_{0,3}^\infty \rightarrow H_3(BSL_2(k[\Delta^\bullet])) \rightarrow E_{2,1}^\infty \rightarrow 0.$$

We prove $E_{2,1}^\infty = H_1(SL_2(k[\Delta^2])) / (\text{im } d_{3,1}^1)$. We have $H_1(SL_2(k[T])) = 0$, since $SL_2(k[T])$ is perfect. Hence the differential $d_{2,1}^1$ is trivial. No differential except $d_{3,1}^1$ hits the $(2,1)$ -entry. By Remark 9.2, the differential $d_{2,1}^2$ is also trivial. This proves the claim.

We identify $E_{0,3}^\infty$. All differentials starting at $(0, 3)$ are trivial. Therefore, $E_{0,3}^\infty$ is the quotient of $H_3(SL_2(k))$ by all differentials hitting it. The differentials $d_{1,3}^1$ and $d_{4,0}^1$ are trivial by Remark 9.1. Only the differentials $d_{2,2}^2$ and $d_{3,1}^2$ remain.

The last statement is obvious, since the natural change-of-topology morphism includes $SL_2(k)$ as 0-simplices. Therefore, the natural map factors as

$$H_\bullet(SL_2(k), \mathbb{Z}) \rightarrow E_{0,\bullet}^\infty \rightarrow H_\bullet(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

□

Remark 9.4. This is a good place to point out that the above result implies that the spectral sequence

$$E_{p,q}^1 = H_q(BSL_2(k[\Delta^p]), \mathbb{Z}) \Rightarrow H_{p+q}(dBSL_2(k[\Delta^\bullet]), \mathbb{Z})$$

does not degenerate at the E^2 -page. Moreover, the differentials $d_{2,2}^2$ and $d_{3,1}^3$ provide a natural relation between the counterexamples to homotopy invariance for homology of SL_2 and the kernel of the natural change-of-topology morphism

$$H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(BSL_2(k[\Delta^\bullet]), \mathbb{Z}).$$

In the cases discussed in Theorem 7.4 and Theorem 7.6, the above differentials induce non-trivial morphisms from homotopy-invariance-counterexamples to pre-Bloch groups of residue fields.

The explicit computation of such differentials is very complicated, for various reasons. First of all, our knowledge of the groups $H_2(SL_2(k[\Delta^2]))$ and $H_1(SL_2(k[\Delta^3]))$ is very limited. The constructions of [KM97] show that these groups tend to be very large, but do not give a precise description of their structure. Second, the essential step in the computation of $d_{2,2}^2$ needs explicit lifts of null-homologous cycles in $\mathcal{Z}_2(SL_2(k[T]))$ to 3-chains. While the amalgam decomposition of $SL_2(k[T])$ can in principle be used to compute such things, the computations quickly become very complicated.

Proposition 9.5. *Let k be an infinite field of characteristic $\neq 2$. There is a commutative diagram with exact columns*

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H_3(SL_2(k))/(\text{im } d_{2,2}^2 + \text{im } d_{3,1}^3) & \longrightarrow & H_3(Sp_\infty(k)) \\ \downarrow & & \downarrow \\ H_3(BSL_2(k[\Delta^\bullet])) & \xrightarrow{s} & H_3(BSp_\infty(k[\Delta^\bullet])) \\ \downarrow & & \downarrow \\ E_{2,1}^\infty & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array}$$

The map s above is surjective. In particular, we have a surjection

$$t : E_{2,1}^\infty \twoheadrightarrow \text{coker}(H_3(SL_2(k)) \rightarrow H_3(Sp_\infty(k))).$$

If k is of characteristic 0, then s and t are isomorphisms.

Proof. We first note that the inclusions of groups $SL_2 \hookrightarrow Sp_\infty$ induce a morphism of the bisimplicial object $BSL_2(k[\Delta^\bullet]) \rightarrow BSp_\infty(k[\Delta^\bullet])$. The spectral sequence computing the homology of the diagonal of a bisimplicial object is compatible with morphisms of bisimplicial objects. The exact column on the left is a consequence of Proposition 9.3. By [Kar73], the homology of the infinite symplectic group has $\mathbb{A}^1$ -invariance for regular rings in which 2 is invertible. Therefore the spectral sequence associated to the bisimplicial object $BSp_\infty(k[\Delta^\bullet])$ collapses and produces the exact column on the right. The whole diagram is commutative by the above-mentioned compatibility of the spectral sequences with the stabilization morphism.

Surjectivity of s is a consequence of stabilization in Proposition 3.5, and the characteristic 0 isomorphism is a consequence of our improved stabilization result in Corollary 3.9. An application of the snake-lemma then proves the last claim. Note in particular that the top horizontal morphism is induced from the standard inclusion $SL_2 \rightarrow Sp_\infty$, hence it is really the stabilization morphism. $\square$

Remark 9.6. It is interesting to see that the surjectivity of the change-of-topology map can be completely translated into a question on stabilization of the homology of linear groups, namely the question of surjective stabilization for the morphism $H_3(SL_2) \rightarrow H_3(Sp_\infty)$. The known stabilization results for the symplectic groups do not seem to decide surjective stabilization in this range.

ACKNOWLEDGEMENTS

The authors would like to thank Aravind Asok for some discussions on the computations in [AF12a, AF12b], and Jens Hornbostel and Marco Schlichting for some discussions on finiteness properties of symplectic K-theory of number fields. The authors are very grateful to the anonymous referee whose detailed report, helpful remarks and insightful suggestions helped to greatly improve the paper in form and content.

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