COX RINGS OF EXTREMAL RATIONAL ELLIPTIC SURFACES

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ABSTRACT. In this paper we determine a minimal set of generators for the Cox rings of extremal rational elliptic surfaces. Moreover, we develop a technique for computing the ideal of relations between them which allows us to provide a presentation of the Cox ring in most cases.

INTRODUCTION

Let X be a smooth projective rational surface over $\mathbb{C}$ and $\pi : X \rightarrow \mathbb{P}^1$ be a relatively minimal jacobian elliptic fibration on X . In this paper we are interested in the Cox ring of X , which is the $\text{Pic}(X)$ -graded ring defined as follows (see Section 1):

$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

By [AL11] the Cox ring $\mathcal{R}(X)$ is a finitely generated algebra if and only if the Mordell-Weil group of π is finite, which means that π admits just a finite number of sections. The elliptic fibrations with such property are called *extremal*, after the work of Miranda and Persson who classified them in [MP86]. The aim of this paper is to give an explicit presentation for the Cox rings of such surfaces. Our main theorem is the following.

Theorem. *Let X be an extremal rational elliptic surface. Then its Cox ring is generated by sections in the following degrees:*

- (i) *the anticanonical class $-K_X$;*
- (ii) *classes of smooth irreducible rational curves C with $C^2 \in \{-2, -1, 0, 1\}$.*

We now give a short description of the content of the paper. Section 1 recalls the basic properties of elliptic fibrations and some preliminary results about Cox rings. In Section 2 we prove the main theorem, while Section 3 is devoted to elliptic surfaces of complexity one, that is, admitting an effective $\mathbb{C}^*$ -action. The Cox ring of such surfaces is computed by means of a technique developed in [Hau13, §3.3]. In Section 4 we provide a method for computing the ideal of relations of the Cox ring. This allows us to compute explicitly the Cox ring of all but three extremal rational elliptic surfaces. Finally, Section 5 deals with some applications of the previous results, as computing the Cox ring of the Cayley cubic and that of weak del Pezzo surfaces which are dominated by extremal rational elliptic surfaces.

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1. BASIC SETUP

1.1. Elliptic surfaces. We briefly recall some well-known facts about elliptic fibrations. A good and recent reference for these results is [SS10].

Let X be a smooth projective surface over $\mathbb{C}$ and $\pi : X \rightarrow \mathbb{P}^1$ be an *elliptic fibration*, that is, a morphism whose general fiber is a smooth curve of genus one. In what follows we will assume π to be jacobian, i.e., π admits a section, and relatively minimal, i.e., π does not contract (-1) -curves. Given a zero section P_0 , the set of sections of π forms a finitely generated abelian group, the *Mordell-Weil group* of π , which will be denoted by $MW(\pi)$.

The singular fibers of an elliptic fibration have been classified by Kodaira [Kod63] according to the types I_n ($n \geq 1$), II, III, IV, II^* , III^* , IV^* , I_n^* ($n \geq 0$). The intersection graph of some of the reducible fibers is represented in Figure 1, where each vertex represents a (-2) -curve and the number near to it is its multiplicity.

FIGURE 1. Some reducible fibers of an elliptic fibration

Rational elliptic surfaces with finite Mordell-Weil group have been classified by Miranda and Persson in [MP86]. Their result is summarized in Table 1.

TABLE 1. Extremal rational elliptic surfaces

Surface	Type	$MW(\pi)$	Surface	Type	$MW(\pi)$
X_{22}	$II^* II$	$\{0\}$	X_{431}	$IV^* I_3 I_1$	$\mathbb{Z}/3\mathbb{Z}$
X_{211}	$II^* 2I_1$	$\{0\}$	X_{222}	$I_2^* 2I_2$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
X_{411}	$I_4^* 2I_1$	$\mathbb{Z}/2\mathbb{Z}$	X_{141}	$I_1^* I_4 I_1$	$\mathbb{Z}/4\mathbb{Z}$
X_{9111}	$I_9 3I_1$	$\mathbb{Z}/3\mathbb{Z}$	X_{6321}	$I_6 I_3 I_2 I_1$	$\mathbb{Z}/6\mathbb{Z}$
X_{33}	$III^* III$	$\mathbb{Z}/2\mathbb{Z}$	$X_{11}(a)$	$2I_0^*$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
X_{321}	$III^* I_2 I_1$	$\mathbb{Z}/2\mathbb{Z}$	X_{5511}	$2I_5 2I_1$	$\mathbb{Z}/5\mathbb{Z}$
X_{8211}	$I_8 I_2 2I_1$	$\mathbb{Z}/4\mathbb{Z}$	X_{4422}	$2I_4 2I_2$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
X_{44}	$IV^* IV$	$\mathbb{Z}/3\mathbb{Z}$	X_{3333}	$4I_3$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$

A smooth projective rational surface with a jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ is known to be isomorphic to the blow-up of the projective plane at nine points, eventually infinitely near. In fact, X carries a unique elliptic fibration, induced by the pencil of cubics in $\mathbb{P}^2$ through the nine points. Since the class of a fiber of π is the anticanonical class, the sections of π are exactly the (-1) -curves. In Table 2 we give the equation of a pencil of cubic curves giving rise to any such surface.

TABLE 2. Pencils of cubics inducing extremal elliptic surfaces

Surface	Pencil of cubics
X_{22}	$(x_1^3 + x_0^2x_2) + tx_2^3 = 0$
X_{211}	$(x_1^3 + x_0^2x_2 + x_1^2x_2) + tx_2^3 = 0$
X_{411}	$(x_0^2x_1 + x_2^3 + x_1^2x_2) + tx_1x_2^2 = 0$
X_{9111}	$(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) + tx_0x_1x_2 = 0$
X_{33}	$x_0(x_0x_2 - x_1^2) + tx_2^3 = 0$
X_{321}	$x_0(x_0x_2 - x_1^2 + x_1x_2) + tx_2^3 = 0$
X_{8211}	$(x_0 - x_1)(x_0x_1 - x_2^2) + tx_0x_1x_2 = 0$
X_{44}	$x_1x_2(x_1 - x_2) + tx_0^3 = 0$
X_{431}	$x_1x_2(x_0 + x_1 + x_2) + tx_0^3 = 0$
X_{222}	$x_0x_1(x_0 - x_1) + t(x_1^3 + 2x_0x_1x_2 - 2x_1^2x_2 - x_0x_2^2 + x_1x_2^2) = 0$
X_{141}	$x_2(x_0x_1 - x_1^2 + x_0x_2) + tx_0x_1(x_0 - x_1) = 0$
X_{6321}	$(x_0 + x_1)(x_1 + x_2)(x_0 + x_2) + tx_0x_1x_2 = 0$
$X_{11}(a)$	$x_1x_2(x_1 - x_2) + t(x_1 - ax_2)x_0^2 = 0, a \in \mathbb{C} - \{0, 1\}$
X_{5511}	$(x_1 + x_2)(x_0 + x_1)(x_0 + x_1 + x_2) + tx_0x_1x_2 = 0$
X_{4422}	$(x_0 - x_2)(x_0 - 2x_1 + x_2)(x_0 + 2x_1 + x_2) + tx_0x_1x_2 = 0$
X_{3333}	$(x_0^3 + x_1^3 + x_2^3) + tx_0x_1x_2 = 0$

In what follows, it will be useful to know the intersection matrix for the curves with negative intersection on X . We recall that, in case $MW(\pi)$ is finite, two sections of π do not intersect by [Mir89, Proposition VII.3.2.]. The intersections between the torsion sections and the reducible fibers of extremal rational elliptic surfaces, which can be computed by means of the height formula [Shi90], are summarized in Table 3.

We will denote by Θ_i^j the i -th component, with the notation in Figure 1, of the j -th reducible fiber (once a numbering of the reducible fibers is chosen). In particular Θ_0^j will denote the component intersecting the zero section. If the Mordell–Weil group is isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$, $n \geq m$, we denote by P_1 a generator of $\mathbb{Z}/n\mathbb{Z}$ and by Q_1 a generator of $\mathbb{Z}/m\mathbb{Z}$. Moreover, P_j denotes the section which is j times P_1 with respect to the group law of the Mordell–Weil group. Observe that Table 3 gives the intersection numbers for P_1 and Q_1 ; the remaining intersection numbers can be deduced from these according to the group law. For example in the case X_{3333} the section corresponding to $P_i + Q_j$ intersects the n -th fiber in the component $\Theta_{ai+bj \pmod{3}}^n$, where $P_1 \cdot \Theta_a^n = 1$ and $Q_1 \cdot \Theta_b^n = 1$.

1.2. Cox rings. The Cox ring of a complete normal variety with finitely generated and free class group $\text{Cl}(X)$ can be defined as follows:

$$\mathcal{R}(X) := \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D)),$$

TABLE 3. Intersections between sections and reducible fibers

Surface	$MW(\pi)$	Intersection properties
X_{411}	$\mathbb{Z}/2\mathbb{Z}$	$P_1 \cdot \Theta_8 = 1$
X_{9111}	$\mathbb{Z}/3\mathbb{Z}$	$P_1 \cdot \Theta_3 = 1$
X_{33}, X_{321}	$\mathbb{Z}/2\mathbb{Z}$	$P_1 \cdot \Theta_6^1 = P_1 \cdot \Theta_1^2 = 1$
X_{8211}	$\mathbb{Z}/4\mathbb{Z}$	$P_1 \cdot \Theta_2^1 = P_1 \cdot \Theta_1^2 = 1$
X_{44}, X_{431}	$\mathbb{Z}/3\mathbb{Z}$	$P_1 \cdot \Theta_6^1 = P_1 \cdot \Theta_1^2 = 1$
X_{222}	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$P_1 \cdot \Theta_1^1 = P_1 \cdot \Theta_1^2 = P_1 \cdot \Theta_1^3 = 1$ $Q_1 \cdot \Theta_6^1 = Q_1 \cdot \Theta_1^1 = Q_1 \cdot \Theta_0^3 = 1$
X_{141}	$\mathbb{Z}/4\mathbb{Z}$	$P_1 \cdot \Theta_5^1 = P_1 \cdot \Theta_1^2 = 1$
X_{6321}	$\mathbb{Z}/6\mathbb{Z}$	$P_1 \cdot \Theta_1^i = 1, i = 1, 2, 3$
$X_{11}(a)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$P_1 \cdot \Theta_1^1 = P_1 \cdot \Theta_1^2 = 1$ $Q_1 \cdot \Theta_3^1 = Q_1 \cdot \Theta_3^2 = 1$
X_{5511}	$\mathbb{Z}/5\mathbb{Z}$	$P_1 \cdot \Theta_1^1 = P_1 \cdot \Theta_2^2 = 1$
X_{4422}	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$P_1 \cdot \Theta_1^i = P_1 \cdot \Theta_0^4 = 1, i = 1, 2, 3$ $Q_1 \cdot \Theta_2^1 = Q_1 \cdot \Theta_0^2 = Q_1 \cdot \Theta_1^j = 1, j = 3, 4$
X_{3333}	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$P_1 \cdot \Theta_1^i = P_1 \cdot \Theta_0^4 = 1, i = 1, 2, 3$ $Q_1 \cdot \Theta_1^i = Q_1 \cdot \Theta_2^2 = Q_1 \cdot \Theta_0^3 = 1, i = 1, 4$

where K is a subgroup of $\text{Div}(X)$ such that the class map $K \rightarrow \text{Cl}(X)$ is an isomorphism (see [ADHL15]). Given a homogeneous element f of $\mathcal{R}(X)$, we will denote its *degree* with $\deg(f) \in \text{Cl}(X)$ and we will say that f is the *defining section* of a divisor E if $E = \text{div}(f) + D$, where $D \in K$ is the representative for the class $\deg(f)$. Moreover, we will say that $\mathcal{R}(X)$ *has a generator in degree* $[D] \in \text{Cl}(X)$ if any minimal set of generators of $\mathcal{R}(X)$ contains an element of such degree.

If $\mathcal{R}(X)$ is a finitely generated algebra, then X is called a *Mori dream space*. By [AL11, Theorem 4.2, Corollary 5.4] a rational elliptic surface $\pi : X \rightarrow \mathbb{P}^1$ is a Mori dream space if and only if the Mordell-Weil group of π is finite, or equivalently if the effective cone of X is rational polyhedral. We observe that if X is an extremal elliptic surface, then the effective cone is generated by the classes of the negative curves of X , i.e., (-1) - and (-2) -curves, by [AL11, Proposition 1.1].

We now recall a standard result for proving that $\mathcal{R}(X)$ is not generated in degree $[D]$, where D is an effective divisor on X . Let E_1, E_2 be two disjoint curves of X not linearly equivalent to D and $x_1, x_2 \in \mathcal{R}(X)$ be their defining sections. By [Bea83, Lemma I.5] there is an exact cohomology sequence:

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \xrightarrow{g^*} H^0(X, D) \longrightarrow H^1(X, D - E_1 - E_2),$$

where $g^*(s, t) = sx_1 + tx_2$.

Lemma 1.1. *If g^* is surjective, then $\mathcal{R}(X)$ has no generators in degree $[D]$. In particular, this holds if $h^1(X, D - E_1 - E_2) = 0$.*

We finally state a proposition which is an immediate consequence of a result by Harbourne [Har97, Theorem III.1]. Observe that a nef divisor on a rational elliptic surface is linearly equivalent to an effective divisor by the Riemann-Roch theorem. Moreover, as a consequence of the following result, the linear system associated to a nef divisor is base point free if and only if it has no components in its base locus.

Proposition 1.2. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a rational elliptic surface and let D be a nef divisor of X .*

- (i) *If $-K_X \cdot D > 0$, then $h^1(X, D) = 0$.*
- (ii) *If $-K_X \cdot D \geq 2$, then $|D|$ is base point free.*
- (iii) *If $-K_X \cdot D = 1$, then $D \sim -aK_X + P$ for some positive integer a , where P is a section of π ; in this case $|D|$ contains P in its base locus.*
- (iv) *If $-K_X \cdot D = 0$, then $D \sim -aK_X$ for some non-negative integer a and $h^1(X, D) = a$.*

2. GENERATORS FOR THE COX RINGS OF EXTREMAL RATIONAL ELLIPTIC SURFACES

In this section we will determine a minimal set of generators for any extremal rational elliptic surface X . We start proving that the degrees of the generators of $\mathcal{R}(X)$ are at the boundary of the nef cone.

Proposition 2.1. *Let $\pi : X \rightarrow \mathbb{P}^1$ be an extremal rational elliptic surface and D be an ample divisor on X . Then $\mathcal{R}(X)$ is not generated in degree $[D]$.*

Proof. We first assume that X contains two (-2) -curves E_1, E_2 which either belong to distinct fibers of π and do not intersect the same section, or belong to the same fiber and have distance at least two (in the intersection graph). In this case the divisor $D - E_1 - E_2$ is nef and $-K_X \cdot (D - E_1 - E_2) = -K_X \cdot D > 0$. Thus $h^1(D - E_1 - E_2) = 0$ by Proposition 1.2. The existence of such pair of curves can be easily verified looking at Table 1 and Table 3 for any extremal rational elliptic surface except for the surface X_{3333} .

If $X = X_{3333}$ let E_1, E_2 be two disjoint (-2) -curves and let $B := -K_X + D - E_1 - E_2$. Since the only curve of negative self-intersection intersecting both E_1 and E_2 is a section and D is ample, then B is nef. Moreover, since $D - E_1$ is nef and $-K_X - E_2 \sim E'_2 + E''_2$, where E'_2 and E''_2 are the remaining components of the reducible fiber containing E_2 , then $B^2 = (D - E_1)^2 + 2(D - E_1) \cdot (E'_2 + E''_2) + (E'_2 + E''_2)^2 \geq 4 - 2 > 0$, so that B is big. Thus $h^1(D - E_1 - E_2) = 0$ by the Kawamata-Viehweg vanishing theorem.

In both cases the statement follows from Lemma 1.1. □

Remark 2.2. Observe that X_{3333} is obtained blowing up the nine base points of the pencil of cubics given in Table 2. The (-2) -curves of X are the proper transforms of the lines in $\mathbb{P}^2$ through three of the base points and the (-1) -curves are the exceptional divisors over such points. Thus, given two disjoint (-2) -curves, their images in $\mathbb{P}^2$ intersect at a base point of the pencil, so that both intersect the (-1) -curve corresponding to such point. This can be verified also computing the intersection numbers between sections and fibers as explained in Subsection 1.1.

Definition 2.3. A connected and reduced curve E of X is a *generalized (-1) -curve* if there exists a morphism to a smooth surface contracting E to a point, a *generalized (-2) -curve* if the intersection graph on its irreducible components is a Dynkin diagram.

It follows easily from the definition and Castelnuovo's criterion [Har77, Theorem 5.7] that a generalized (-1) -curve is either a (-1) -curve or a chain of (-2) -curves with a (-1) -curve at one extreme.

Lemma 2.4. *Let N be a connected and reduced divisor of X such that the intersection form on the irreducible components of its support is negative definite. Then N is either a generalized (-2) -curve or a generalized (-1) -curve.*

Proof. If N only contains (-2) -curves, then it is a generalized (-2) -curve since its support is properly contained in a fiber of π .

Now assume that N contains a (-1) -curve. We proceed by induction on the number k of irreducible components of N . If $k = 1$ the statement is obvious, so assume that $k > 1$. Let $f : X \rightarrow Y$ be the morphism which contracts a (-1) -curve E in N and let $N' = f(N)$. Since the intersection form on the support of N is negative definite, then E intersects exactly one other curve in the support of N , a (-2) -curve R with $E \cdot R = 1$. In fact, observe that if E intersects two distinct (-2) -curves F, G (which are necessarily disjoint), then $(2E + F + G)^2 = 0$. Similarly, one shows that E cannot intersect any (-1) -curve in N and that there is no (-2) -curve in N having intersection ≥ 2 with E . Thus $f(R) \subset N'$ is a (-1) -curve.

Observe that N' is still a connected and reduced divisor. Moreover, the intersection form on its components is negative definite since otherwise, by pull-back, one would obtain a curve with non-negative self-intersection whose support is contained in the support of N . Thus, by the induction hypothesis, N' is a generalized (-1) -curve. This implies that N is a generalized (-1) -curve. $\square$

2.1. Conic bundles. Let D be a *conic bundle*, that is, an effective divisor such that $D^2 = 0$, $-K_X \cdot D = 2$ and $|D|$ is base point free, so that the morphism associated to $|D|$ is a rational fibration. We now identify the conic bundles on X giving generators for the Cox ring.

Proposition 2.5. *Let D be an effective divisor on X such that $D^2 = 0$ and $|D|$ is base point free. Then either $D \sim aF$, $a > 0$, where F is a fiber of π , or D is a conic bundle. Moreover, $\mathcal{R}(X)$ is generated in degree $[D]$ exactly in the cases appearing in Table 4 (where the notation is as in Table 3).*

Proof. Since $D^2 = 0$ and $-K_X$ is nef, then by adjunction formula either $-K_X \cdot D = 0$ and $g(D) = 1$ or $-K_X \cdot D = 2$ and $g(D) = 0$. In the first case, by the Hodge index theorem, $D \sim aF$, where F is a fiber of π and a is a positive integer.

Observe that $\mathcal{R}(X)$ has a generator in degree $[F] = -K_X$ if and only if $H^0(X, -K_X)$ is not generated by sections of negative curves, i.e., if π has a unique reducible fiber. Similarly, if D is a conic bundle, $\mathcal{R}(X)$ has a generator in degree $[D]$ if and only if the rational fibration associated to $|D|$ has a unique reducible fiber (observe that any conic bundle has at least a reducible fiber by Kleiman criterion).

Let D be a reducible fiber of a conic bundle and let D^{red} be the sum of its integral components. Since $-K_X \cdot D = 2$, then D either contains two (-1) -curves or a (-1) -curve with multiplicity two in its support. Let E be a curve in the support of D^{red} such that $D^{red} - E$ is connected. Since the intersection form on the components of $D^{red} - E$ is negative definite, then $D^{red} - E$ is either a generalized (-1) -curve or a generalized (-2) -curve by Lemma 2.4. This implies that the support of D has the structure described by one of the diagrams in Figure 2, where the white vertices represent (-1) -curves and the remaining ones represent (-2) -curves. Moreover, D is the unique reduced divisor with the given support and with $D^2 = 0$: the multiplicity of each component is given by the number above each vertex in Figure 2. $\square$

TABLE 4. Generating conic bundles

X	D
$X_{22}, X_{211}, X_{411}, X_{9111}$	F
X_{22}, X_{211}	$2P_0 + 2\sum_{i=0}^5 \Theta_i + \Theta_6 + \Theta_8$
X_{411}	$P_0 + P_1 + \Theta_0 + \sum_{i=2}^6 \Theta_i + \Theta_8,$ $2P_0 + 2\Theta_0 + 2\sum_{i=2}^6 \Theta_i + \Theta_7 + \Theta_8,$ $2P_1 + 2\sum_{i=2}^6 \Theta_i + 2\Theta_8 + \Theta_0 + \Theta_1$
X_{8211}	$P_0 + P_1 + \sum_{i \neq 1} \Theta_i^1,$ $P_0 + P_3 + \sum_{i \neq 7} \Theta_i^1,$ $P_1 + P_2 + \sum_{i \neq 3} \Theta_i^1,$ $P_2 + P_3 + \sum_{i \neq 5} \Theta_i^1$
X_{9111}	$P_0 + P_2 + \sum_{i=0}^6 \Theta_i,$ $P_0 + P_1 + \sum_{i=3}^8 \Theta_i + \Theta_0,$ $P_1 + P_2 + \sum_{i=0}^3 \Theta_i + \sum_{i=6}^8 \Theta_i$
X_{33}, X_{321}	$P_0 + P_1 + \sum_{i=0}^6 \Theta_i^1$

FIGURE 2. Reducible fibers of conic bundles

2.2. Nef and big divisors. Let D be an effective divisor without components in its base locus, A, B be two smooth disjoint curves such that $D \cdot A = 0$ and s_A, s_B two sections defining A and B . Then we have the following diagram:

$$\begin{array}{ccccccc}
 & & & & H^0(X, D - B) & & \\
 & & & & \downarrow m_B & & \\
 0 & \longrightarrow & H^0(X, D - A) & \xrightarrow{m_A} & H^0(X, D) & \xrightarrow{r_A} & H^0(A, \mathcal{O}_A) \cong \mathbb{C}
 \end{array}$$

where m_A, m_B are the multiplication maps by s_A and s_B respectively, r_A is the restriction to A and the horizontal sequence is exact.

Lemma 2.6. *If $h^0(X, D - B) > 0$ and A is not in the base locus of $|D - B|$, then $H^0(X, D)$ is generated by $H^0(X, D - A)$, $H^0(X, D - B)$ and the defining sections of A and B .*

Proof. Observe that $D - A$ is effective and r_A is surjective, since otherwise $|D|$ would contain components in its base locus. Since A is not in the base locus of $|D - B|$, then $r_A \circ m_B$ is also surjective. Given $f \in H^0(X, D)$, let $r_A(f) = c \in H^0(A, \mathcal{O}_A)$ and let $g \in H^0(X, D - B)$ such that $r_A \circ m_B(g) = c$, then $f - m_B(g) \in H^0(X, D - A)$, proving the statement. $\square$

Let D be a nef and big divisor such that $|D|$ is base point free. In what follows we will denote by $D^\perp$ the set of negative curves orthogonal to D . Since $D^2 > 0$, the orthogonal complement of D in $\text{Pic}(X)$ is negative definite by the Hodge index theorem. Thus, if $D^\perp$ is not empty, the intersection form on the elements of $D^\perp$ is negative definite. By Lemma 2.4 each connected component of $D^\perp$ is either a generalized (-1) -curve or a generalized (-2) -curve.

Proposition 2.7. *Let D be a nef and big divisor such that $|D|$ is base point free and such that $D^\perp$ only contains (-1) -curves or only contains (-2) -curves. Then $\mathcal{R}(X)$ has no generators in degree $[D]$.*

Proof. In both cases we will explain how to choose A and B in order to apply Lemma 2.6.

We first assume that $D^\perp$ only contains (-1) -curves and let A be a (-1) -curve in $D^\perp$. If X contains another (-1) -curve B , then the divisor $D - B$ is nef. If $-K_X \cdot (D - B) = 1$, then $D = -aK_X + B + B'$ by Proposition 1.2, where B' is a (-1) -curve. Since π always has a fiber with more than two components, then there exists a (-2) -curve orthogonal to both B and B' (and thus to D), contradicting the hypothesis on D . Similarly, one can exclude the case $-K_X \cdot (D - B) = 0$. Thus $-K_X \cdot (D - B) \geq 2$ and $|D - B|$ is base point free by Proposition 1.2.

If X only contains a (-1) -curve $A \in D^\perp$, then $X = X_{22}$ or X_{211} . In both cases we can choose a (-2) -curve B which does not intersect A and intersects any other (-2) -curve in at most one point. Observe that $D - B$ is nef and $-K_X \cdot (D - B) = -K_X \cdot D \geq 2$, thus $|D - B|$ is base point free by Proposition 1.2.

Finally, if $D^\perp$ only contains (-2) -curves, let A be a (-2) -curve in $D^\perp$ and B be a smooth fiber of π . As before, $D - B$ is nef and its linear system is base point free by Proposition 1.2. $\square$

Proposition 2.8. *Let D be a nef and big divisor such that $|D|$ is base point free, B be an integral curve in $D^\perp$ and N be the connected component of $D^\perp$ which contains B . Then one of the following holds:*

- (i) *all the components in the base locus of $|D - B|$ are contained in N ;*
- (ii) *$D \sim -K_X + B + C$, where B and C are distinct (-1) -curves.*

Proof. Since $|D|$ is base point free, then $\varphi := \varphi_{|D|} : X \rightarrow Y$ is a morphism onto a normal surface. Moreover, since the only negative curves of X are (-1) - and (-2) -curves, then Y has at most ADE singularities. Let E be the exceptional divisor of φ over $\varphi(B)$. Observe that the support of E is N since it is the connected component of $D^\perp$ which contains B . If $E^2 = -1$, then E is a reduced divisor and N is a generalized (-1) -curve, so that $\varphi(B)$ is a smooth point. Otherwise, E is the fundamental cycle of an ADE singularity, so that $E^2 = -2$ and N is a generalized (-2) -curve (see [BHPvDV04, §3, Chapter III]).

We begin by showing that $E - B$ is contained in the base locus of $|D - B|$. To this aim, consider a non-trivial decomposition $E = E_1 + E_2$ where both E_1 and E_2 are effective divisors. We have that $E_i^2 \leq -2$ for some i since otherwise $E_1^2 = E_2^2 = -1$ and $E_1 \cdot E_2 = 0$, so that E would be disconnected. Hence from

$$-2 \leq E^2 = E_1^2 + E_2^2 + 2E_1 \cdot E_2,$$

we deduce $E_1 \cdot E_2 > 0$, so that some component C of the support of E_2 must have negative intersection with $D - E_1$, which implies that C is contained in the base locus of $|D - E_1|$. If we assume that the base locus of $|D - B|$ only contains a

proper subdivisor F of $E - B$, then we get a contradiction taking $E_1 = B + F$. This gives the claim.

We now show that $D - E$ is nef. Let C be a negative curve of X with $E \cdot C > 0$. First of all we observe that if $E^2 = -1$, then E is a generalized (-1) -curve and $E \cdot C' \leq 0$ for any curve C' contained in its support. The same holds if $E^2 = -2$ by [BHPVdV04, Corollary 3.6]. Hence in both cases C cannot be a component of the support N of E . We also have $D \cdot C > 0$, since otherwise C would be contained in some connected component of $D^\perp$, disjoint from N , giving $E \cdot C = 0$. Moreover, we have that

$$E \cdot C \leq 2.$$

In case $E^2 = -1$ this is clear since E is a reduced divisor. If $E^2 = -2$, then the support of E is contained in a fiber F of π . Moreover, by looking at the possible intersection graphs for E (see [BHPVdV04, §3, Chapter III]), we see that the multiplicity of a component of E is less than or equal to the multiplicity of the same curve as a component of the fiber F . Thus $F - E$ is effective, so that $E \cdot C \leq 2$.

Thus we have to consider only two cases. If $E \cdot C = 1$, then $(D - E) \cdot C \geq 0$ due to $D \cdot C > 0$. Assume now $E \cdot C = 2$. In this case $C^2 = -2$, since (-1) -curves do not intersect each other and intersect a fiber of π at one simple point. The divisor $E + C + K_X$ is linearly equivalent to an effective divisor by the Riemann-Roch theorem and the Serre's duality since $(E + C + K_X)^2 - K_X \cdot (E + C + K_X) = 0$. By the assumptions on D and Proposition 1.2 we have that $D \cdot (-K_X) \geq 2$, thus $D \cdot (E + C) \geq 2$. Hence

$$(D - E) \cdot C = D \cdot C - 2 = D \cdot (E + C) - 2 \geq 0,$$

proving that $D - E$ is nef.

We now study the base locus of $|D - E|$. We can assume $E^2 = -1$, since otherwise $-K_X \cdot (D - E) \geq 2$ so that $|D - E|$ would be base point free by Proposition 1.2. Moreover, by the same proposition, the linear system $|D - E|$ is base point free unless $D - E \sim -aK_X + C$, where C is a (-1) -curve of X , that is,

$$D \sim -aK_X + E + C.$$

In this case the unique component in the base locus of $|D - E|$ is C . So, either we are in case (i) of the statement or C is not contained in the support N of E and in particular $E \cdot C \geq 0$. Assume we are in this second case. Since $E^2 = -1$, then $D \cdot C = a + E \cdot C - 1 = D \cdot E = 0$, so that $a = 1$ and $E \cdot C = 0$. Observe that E , which is a generalized (-1) -curve, cannot contain (-2) -curves, since otherwise the last (-2) -curve in the chain would intersect D negatively contradicting the hypothesis. Thus we are in case (ii), concluding the proof. $\square$

2.3. Generators of $\mathcal{R}(X)$. We now determine a minimal generating set for the Cox ring of any extremal rational elliptic surface. We recall that K is the grading group of $\mathcal{R}(X)$; see Section 1.2.

Definition 2.9. Consider the following divisors:

- (i) an anticanonical divisor $-K_X$;
- (ii) smooth irreducible rational curves C with $C^2 \in \{-2, -1, 0, 1\}$.

For each such divisor consider the unique $D \in K$ which is linearly equivalent to it, choose a basis of $H^0(X, D)$ and let Ω be the union of all such bases. An element $f \in \Omega$ will be called a *distinguished section* of $\mathcal{R}(X)$.

Lemma 2.10. *Let $D = -K_X + A + B$, where A, B are two distinct (-1) -curves. Then $H^0(X, D)$ is generated by distinguished sections.*

Proof. By the Riemann-Roch theorem and Proposition 1.2 we have $h^0(X, D) = 3$. Observe that $H^0(X, D)$ contains the two-dimensional subspace $f_A f_B H^0(X, -K_X)$, where f_A, f_B are sections defining A, B respectively. So, it is enough to show that there exists a distinguished polynomial f in $H^0(X, D)$ such that $\operatorname{div}(f) + D$ contains neither A nor B in its support.

For any extremal rational elliptic surface having at least two sections A, B , there exists a chain of (-2) -curves inside one fiber of π which connects A and B . This can be checked looking at Table 1 and Table 3. This chain of curves, having A, B as external curves, defines a conic bundle D' on X (see Figure 2) such that $D - D'$ is linearly equivalent to an effective divisor C supported on negative curves. Thus we conclude taking $f = f_C g$, where g defines a smooth element of $|D'|$. Thus $H^0(X, D)$ is generated by sections of degree $[-K_X]$, $[D']$ and by sections defining negative curves. $\square$

Theorem 2.11. *Let $\pi : X \rightarrow \mathbb{P}^1$ be an extremal rational elliptic surface. Then the Cox ring $\mathcal{R}(X)$ is generated by distinguished sections.*

Proof. Let D be an effective divisor of X . We will prove the statement by induction on $d(D) := D \cdot H$, where H is an ample divisor on X .

If B is an integral component in the base locus of $|D|$, then $B^2 < 0$, since otherwise it would be nef and $h^0(X, B) > 1$ by the Riemann-Roch theorem and by Proposition 1.2 (iv). Thus B is either a (-1) -curve or a (-2) -curve. Moreover, the multiplication map

$$H^0(X, D - B) \xrightarrow{\cdot f_B} H^0(X, D)$$

by the section f_B defining B is an isomorphism. Since $d(D - B) < d(D)$, then $H^0(X, D)$ is generated by distinguished sections by induction.

If $D^2 = 0$ and $|D|$ is base point free, then either $D \sim -aK_X$ with $a \geq 1$, or D is a conic bundle by Proposition 2.5. In the first case $H^0(X, D)$ is generated by sections in degree $[-K_X]$, while in the second case it is generated by sections defining negative curves and by a section defining a smooth fiber of the conic bundle, the latter only in case (X, D) belongs to the list in the statement of Proposition 2.5. In both cases $H^0(X, D)$ is generated by distinguished sections.

Thus in what follows we can assume D to be nef, big and $-K_X \cdot D \geq 2$ by Proposition 1.2. Observe that we can also assume that D is not ample and that $D^\perp$ contains both (-1) -curves and (-2) -curves by Proposition 2.1, Proposition 2.7 and the induction hypothesis.

We now assume that $D^\perp$ has a unique connected component N . Observe that N is a generalized (-1) -curve by the previous assumptions and, since it is connected, the (-2) -curves in its support are components of a fiber F of π . Let A be a (-2) -curve in N .

If there exists a (-1) -curve B such that $B \cap N = \emptyset$, then $D - B$ is nef, so that $|D - B|$ contains no (-2) -curves in the base locus by Proposition 1.2. The curves A, B satisfy the hypothesis of Lemma 2.6, so that $H^0(X, D)$ is generated by sections in lower degrees $[D - A]$ and $[D - B]$, thus we conclude by the induction hypothesis. On the other hand, if any (-1) -curve intersects N (this includes the

case when all (-1) -curves are contained in N) there are two possible cases:

i) The external (-2) -curve of N intersects a (-1) -curve E . In this case let

$$Q = \sum_{i=1}^r N_i + E,$$

where N_i are the components of N . Observe that Q is a conic bundle. We now show that $D - Q$ is nef, i.e., it has non-negative intersection with any negative curve C . If C is contained in the support of Q , then $C \cdot Q = 0$, so that $(D - Q) \cdot C \geq 0$. If C is not contained in the support of Q and $C \cdot Q \leq 1$, then clearly $(D - Q) \cdot C \geq 0$ since $D \cdot C > 0$. Otherwise C is not in the support of Q , $C \cdot Q = 2$ and the union of C with the (-2) -curves in Q is the support of a fiber of π of type I_n , $n \geq 2$. Since $D \cdot C = D \cdot (-K_X) \geq 2$, again we obtain that $(D - Q) \cdot C \geq 0$. By Proposition 1.2 the curves A, Q satisfy the hypothesis of Lemma 2.6, thus we conclude by the induction hypothesis.

ii) The external (-2) -curve of N does not intersect any (-1) -curve. By looking at Table 3, we deduce that X is isomorphic to either X_{9111}, X_{22} or X_{211} . If X is isomorphic to X_{9111} , since any (-1) -curve intersects N and the external (-2) -curve of N does not intersect any (-1) -curve, we can assume that

$$N = P_0 + \sum_{i=0}^7 \Theta_i.$$

In case X is isomorphic to either X_{22} or X_{211} , let B be a smooth fiber of the conic bundle $2P_0 + 2\sum_{i=0}^5 \Theta_i + \Theta_6 + \Theta_8$. If $\Theta_7 \notin N$, then $D - B$ is nef and Lemma 2.6 can be applied as before taking A to be a (-2) -curve in N . Otherwise, if $\Theta_7 \in N$, then $N = P_0 + \sum_{i=0}^7 \Theta_i$ as in the previous case.

In both cases, when the support of N contains nine curves, the morphism $\varphi_{|D|}$ maps X onto a smooth rational surface S of Picard number one, so that S is $\nu_n(\mathbb{P}^2)$, where ν_n is the n -th Veronese embedding. This implies that $D \sim nL$, where L is the pull-back of a divisor generating $\text{Pic}(S)$. In this case $H^0(X, D) = \text{Sym}^n H^0(X, L)$, so that it is generated by elements in $H^0(X, L)$. Since a basis of $H^0(X, L)$ is given by distinguished sections of type (ii), then we conclude.

We now assume that $D^\perp$ has at least two connected components N_1, N_2 . Let $A \in N_1$ and $B \in N_2$. By Proposition 2.8, either A is not contained in the base locus of $|D - B|$ or $D \sim -K_X + A + B$. In the first case $H^0(X, D)$ is generated by $H^0(X, D - A)$, $H^0(X, D - B)$ and by defining sections of curves with negative self-intersection by Lemma 2.6, thus we conclude by the induction hypothesis. In the second case we conclude by Lemma 2.10. $\square$

Corollary 2.12. *A minimal set of generators for $\mathcal{R}(X)$ is given by the distinguished sections which define (-1) -curves, (-2) -curves, a smooth fiber for each conic bundle having a unique reducible fiber, a smooth fiber of π if it has a unique reducible fiber and smooth irreducible rational curves D with $D^2 = 1$ such that $D^\perp$ contains a generalized (-1) -curve of length nine.*

Proof. By Theorem 2.11 it is enough to find a set of distinguished sections which gives a minimal generating set of $\mathcal{R}(X)$. A distinguished section of degree $[-K_X]$ is contained in such a set if and only if the elliptic fibration has a unique reducible fiber. Similarly, the only conic bundles giving generators of $\mathcal{R}(X)$ are those with a unique reducible fiber, which have been classified in Proposition 2.5. Finally, if D

is a smooth irreducible rational curves D with $D^2 = 1$, then $\varphi := \varphi|_D : X \rightarrow \mathbb{P}^2$ is a birational morphism. In this case $H^0(X, D) = \varphi^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ is generated by sections defining negative curves unless the exceptional divisor of φ contracts a generalized (-1) -curve of length nine to one point. A divisor D with this property only exists for the surfaces X_{22}, X_{211} and X_{9111} (in the last case there are three such divisors, exchanged by the Mordell-Weil group). $\square$

3. ELLIPTIC SURFACES OF COMPLEXITY ONE

This section deals with extremal elliptic surfaces X which admit an action of the torus $T = \mathbb{C}^*$. These are the surfaces X_{22}, X_{33}, X_{44} and the surfaces $X_{11}(a)$, where $a \in \mathbb{C} - \{0, 1\}$ (see [Dui10, Proposition 9.2.17]). The action of the torus is given as follows, where $c \in T$:

$$\begin{aligned} X_{22}: & [x_0 : x_1 : x_2] \mapsto [cx_0 : x_1 : c^{-2}x_2], \\ X_{33}: & [x_0 : x_1 : x_2] \mapsto [cx_0 : x_1 : c^{-1}x_2], \\ X_{44}: & [x_0 : x_1 : x_2] \mapsto [cx_0 : x_1 : x_2], \\ X_{11}(a): & [x_0 : x_1 : x_2] \mapsto [cx_0 : x_1 : x_2]. \end{aligned}$$

We will compute the Cox ring of such surfaces by means of [Hau13, Theorem 3.18]. We recall that a fixed point for the T -action is called *elliptic* if it lies in the closure of infinitely many orbits. In order to compute the Cox ring we proceed in two steps, following the technique explained in [Hau13, §3.3]: first of all we produce a T -equivariant blow-up $\phi : X' \rightarrow X$ such that X' does not contain any elliptic fixed point, then we construct the intersection graph of the negative curves of X' , i.e., the *Orlik-Wagreich graph* of X' . This information is enough to give a presentation of the Cox ring of X' , from which one obtains a presentation for the Cox ring of X putting equal to 1 the variables corresponding to the exceptional divisors of ϕ . We will discuss in detail one example, the other ones being similar.

3.1. The surface X_{22} . The surface X_{22} contains one elliptic fixed point at the cusp of the fiber of type II, whose plane model is the curve of equation $x_1^3 + x_2x_0^2 = 0$, and a curve of fixed points which corresponds to the trivalent component of the fiber of type II*. Blowing-up the cusp three times we obtain a surface X'_{22} which does not contain elliptic points. The intersection graph of its negative curves is given in Figure 3. The four right-hand side vertices come from the blow-up of the cusp (the (-1) -curve there is pointwise fixed), the left-hand side is the graph of the fiber of type II*, while the central (-1) -curves are orbits whose self-intersections are uniquely determined by the following conditions on the Hirzebruch-Jung continued fractions:

$$[2, 2, 2, 2, 2, 1, 6] = 0, \quad [2, 1, 2] = 0, \quad [2, 2, 1, 3] = 0.$$

The partial quotients of the previous continued fractions give the orders of the isotropy subgroups of the T -action on the generic points of the negative curves of X'_{22} according to [Hau13, pg. 31]. This allows us to write down a presentation for $\mathcal{R}(X'_{22})$ by [Hau13, Theorem 3.18] and of $\mathcal{R}(X_{22})$ by putting equal to 1 the variables corresponding to the exceptional divisors of the blow-up $X'_{22} \rightarrow X_{22}$ (those corresponding to vertices with a white circle in Figure 3). The ring $\mathcal{R}(X_{22})$ is given in Table 5, where the variable T_7 is the defining section of the left trivalent vertex in the graph and $T_8, \dots, T_{13}, T_4, T_1, T_3, T_6, T_5, T_2$ are the defining sections of the remaining black vertices, ordered row by row from the left to the right.

 FIGURE 3. The surface X_{22}

We observe that the sections $x_1, x_0, x_2, x_1^3 + x_0^2 x_2$ define the images in $\mathbb{P}^2$ of the curves defined by T_2, T_3, T_1 and T_4 respectively. The unique relation of the Cox ring corresponds to the relation among the three cubics $x_1^3, x_0^2 x_2, x_1^3 + x_0^2 x_2$, which are elements of the pencil of cubics having $x_2 = 0$ as an inflectional tangent in $(1 : 0 : 0)$ and which have a cusp at $(0 : 0 : 1)$ whose principal tangent is $x_0 = 0$.

3.2. The surface X_{33} . The surface X_{33} contains one elliptic fixed point, the singular point of the fiber of type III, and a curve of fixed points, given by the trivalent component of the fiber of type III*. Blowing-up the elliptic point twice we obtain the surface X' whose intersection graph of negative curves is given in Figure 4. The ring $\mathcal{R}(X_{33})$ is thus computed as before and is given in Table 5 where the variable T_6 is the defining section of the left trivalent vertex in the graph and $T_1, T_{11}, T_{12}, T_{13}, T_2, T_5, T_4, T_7, T_8, T_9, T_{10}, T_3$ are the defining sections of the remaining black vertices, ordered row by row from the left to the right.

 FIGURE 4. The surface X_{33}

3.3. The surface X_{44} . The surface X_{44} contains one elliptic fixed point which is the singular point of the fiber of type IV. Blowing-up this point once we obtain the surface X'_{44} whose graph of negative curves is given in Figure 5. The ring $\mathcal{R}(X_{44})$ is thus computed as before and is given in Table 5 where the variable T_1 is the defining section of the left trivalent vertex in the graph and $T_5, T_6, T_7, T_2, T_8, T_9, T_{10}, T_3, T_{11}, T_{12}, T_{13}, T_4$ are the defining sections of the remaining black vertices, ordered row by row from the left to the right.

 FIGURE 5. The surface X_{44}

3.4. The surface $X_{11}(a)$. In this case we have a one parameter family of surfaces $X_{11}(a)$, $a \in \mathbb{C} - \{0, 1\}$. The surface $X_{11}(a)$ does not contain elliptic fixed points and its graph of negative curves is given in Figure 6 where the variables T_{12} and T_4 are the defining sections of the left fourvalent vertex and of the right fourvalent vertex respectively and $T_1, T_7, T_6, T_2, T_9, T_8, T_3, T_{11}, T_{10}, T_{13}, T_{14}, T_5$ are the defining sections of the remaining vertices, ordered row by row from the left to the right. In this case the Cox ring, which is given in Table 5, has two relations in the same degree. These relations correspond to the relations among the four lines $x_1 = 0$, $x_2 = 0$, $x_1 - x_2 = 0$, $x_1 - ax_2 = 0$ of the pencil of lines in $\mathbb{P}^2$ through $(1 : 0 : 0)$.

FIGURE 6. The surface $X_{11}(a)$

4. RELATIONS OF THE COX RING

In this section we will determine the ideal of relations $I(X)$ of $\mathcal{R}(X)$, where X is an extremal rational elliptic surface, following [BHKS, Construction 3.1].

4.1. Algebraic preliminaries. Let $\mathbb{C}[T, S]$ be a polynomial ring in $m+n$ variables $T_1, \dots, T_m, S_1, \dots, S_n$ graded by an abelian group $K_T \oplus K_S$, where K_S is free. We assume that the degrees of the S variables give a basis of K_S . In what follows $\mathbb{C}[T]$ denotes the polynomial ring in the first m variables graded by K_T , $\mathbb{C}[T, S^{\pm 1}]$ is the localization of $\mathbb{C}[T, S]$ with respect to all the S variables and $\mathbb{C}[T, S^{\pm 1}]_0$ is its degree zero part with respect to the K_S grading. Assume the following diagram of homomorphisms is given:

$$\begin{array}{ccccc}
 I & \longrightarrow & \mathbb{C}[T, S] & \xrightarrow{\psi} & R \\
 & & \downarrow \iota & \nwarrow \beta & \uparrow \sigma \\
 & & \mathbb{C}[T, S^{\pm 1}] & & \mathbb{C}[T] \\
 & & \uparrow \iota_0 & \nearrow \alpha & \uparrow \\
 & & \mathbb{C}[T, S^{\pm 1}]_0 & & J
 \end{array}$$

where ψ is a graded surjective homomorphism to a $K_T \oplus K_S$ -graded domain R with kernel I , σ is a K_T -graded homomorphism with kernel J , ι and ι_0 are the natural inclusions, $\sigma = \psi \circ \beta$ and

$$\iota_0 \circ \alpha(T_i) = \iota \circ \beta(T_i) = T_i \cdot m_i(S),$$

where $m_i(S) \in \mathbb{C}[S]$ is the unique monomial in the variables S such that $T_i \cdot m_i(S)$ has K_S degree equal to zero (the uniqueness follows from the hypothesis on the degrees of the S variables). We observe that α is an isomorphism.

Proposition 4.1. *Let $J' \subset \mathbb{C}[T, S]$ be the extension and contraction of the ideal $\alpha(J)$ (by ι_0 and ι). Then $J' \subset I$.*

Proof. Observe that $\beta(J) \subset I$ since $\sigma = \psi \circ \beta$. Moreover, from the equality

$$\beta(J) \cdot \mathbb{C}[T, S^{\pm 1}] = \alpha(J) \cdot \mathbb{C}[T, S^{\pm 1}]$$

we get that J' is contained in the saturation of I with respect to the variables S . Since R is a domain, then I is saturated, hence we get the statement. $\square$

4.2. Cox rings of blow-ups. Let $f : X \rightarrow Y$ be a blow-up morphism, where X and Y are smooth Mori dream spaces. Let $T_1, \dots, T_m, S_1, \dots, S_n$ be a minimal set of generators of $\mathcal{R}(X)$, where $S_1, \dots, S_n$ define the irreducible components of the exceptional divisor of f . The polynomial ring $\mathbb{C}[T, S]$ is graded by $\text{Cl}(X) = K_T \oplus K_S$, where $K_T = f^* \text{Cl}(Y)$ and K_S is the free abelian group generated by the degrees of the S variables.

In this setting we take $R = \mathcal{R}(X)$ and $\psi : \mathbb{C}[T, S] \rightarrow \mathcal{R}(X)$ to be the natural quotient homomorphism with kernel $I(X)$. Observe that, since $\beta(T_i)$ has degree zero with respect to the K_S -grading, then $\psi \circ \beta(T_i) = f^*(s_i)$ via the pull-back homomorphism $f^* : \mathcal{R}(Y) \rightarrow \mathcal{R}(X)$. Thus we obtain the following result.

Corollary 4.2. *Let J be the kernel of the K_T -graded homomorphism*

$$\phi : \mathbb{C}[T] \rightarrow \mathcal{R}(Y), \quad T_i \mapsto s_i.$$

Then the ideal J' defined before is contained in $I(X)$ and $J' = I(X)$ if $\dim J' = \dim I(X)$.

Proof. By definition $\sigma = f^* \circ \phi$. Since f^* is injective, the first claim in the statement follows from Proposition 4.1. By [BHKS, Proposition 3.3] J' is prime since J is prime. Thus the second claim follows from the fact that $I(X)$ is prime. $\square$

We recall that the quotient of an open subset of the affine variety $\text{Spec } \mathcal{R}(X)$ for the action of a torus of dimension $\text{rank } \text{Cl}(X)$ is isomorphic to X [ADHL15], thus $\dim I(X) = \text{rank } \text{Cl}(X) + 2$.

4.3. The ideal of relations of rational elliptic surfaces. We recall that any rational elliptic surface X is a blow-up of the projective plane at nine points, eventually infinitely near. Fix a blow-up morphism $X \rightarrow \mathbb{P}^2$, a natural basis for the Picard group of X is given by $e_0, e_1, \dots, e_9$, where e_0 is the pull-back of the hyperplane class of $\mathbb{P}^2$ and the e_i 's with $i \geq 1$ are the classes of the exceptional divisors with $e_i \cdot e_j = -\delta_{ij}$ (observe that the exceptional divisors with classes $e_1, \dots, e_9$ are not necessarily irreducible if there are infinitely near points). Thus in this case $K_T \cong \mathbb{Z}$ and $K_S \cong \mathbb{Z}^9$.

Example 4.3. We will apply Corollary 4.2 to compute the ideal of relations of the Cox ring of the surface $X = X_{411}$. We recall that X has an elliptic fibration with a unique reducible fiber of type I_4^* and Mordell-Weil group of order two. The surface can be obtained from the following pencil of cubic curves:

$$(x_0^2 x_1 + x_2^3 + x_1^2 x_2) + t x_1 x_2^2 = 0.$$

Let $l_1 : x_1 = 0$, $l_2 : x_2 = 0$ and $p = (1 : 0 : 0)$ be their intersection point. We call C the smooth cubic given by $t = 0$. Observe that the line l_2 is tangent to C in $q = (0 : 1 : 0)$ and the line l_1 is the inflectional tangent of C at p . The surface X is obtained blowing-up $\mathbb{P}^2$ five times at p (that is, at p and at four points infinitely near to it) and four times at q . We denote by $e_1, \dots, e_5$ the classes of the exceptional divisors over p and by $e_6, \dots, e_9$ those over q , where $e_i \cdot e_j = -\delta_{ij}$ and all the e_i 's are reducible except for e_5 and e_9 , which are taken to be the exceptional divisors of the last blow-up over p and q . By Theorem 2.11 a set of generators for the Cox ring of X is given by

- the sections T_1, T_2 defining the proper transforms of the lines l_1 and l_2 , whose degrees are:

$$[\Theta_1] = e_0 - e_1 - e_2 - e_3, \quad [\Theta_5] = e_0 - e_1 - e_6 - e_7;$$

- three sections T_3, T_4, T_5 defining smooth fibers of the conic bundles whose degrees are:

$$[\Theta_0 + \Theta_8 + \sum_{i=2}^6 \Theta_i + P_0 + P_1] = e_0 - e_6,$$

$$[2P_0 + 2\Theta_0 + 2\sum_{i=2}^6 \Theta_i + \Theta_7 + \Theta_8] = 2e_0 - e_6 - e_7 - e_8 - e_9,$$

$$[2P_1 + 2\Theta_8 + 2\sum_{i=2}^6 \Theta_i + \Theta_0 + \Theta_1] = 3e_0 - (e_1 + \dots + e_5) - 2e_6;$$

- the section T_6 defining the proper transform of the cubic C , of degree $3e_0 - \sum_{i=1}^9 e_i$;
- the sections $T_7, \dots, T_{15}$ defining the irreducible components of the exceptional divisors over p and q , whose degrees are:

$$[\Theta_0] = e_4 - e_5, \quad [\Theta_i] = e_{5-i} - e_{6-i}, \quad [P_0] = e_5 \quad (i = 2, 3, 4),$$

$$[\Theta_6] = e_7 - e_8, \quad [\Theta_7] = e_6 - e_7, \quad [\Theta_8] = e_8 - e_9, \quad [P_1] = e_9.$$

As a smooth fiber of the first conic bundle we can choose the proper transform of the line $x_0 = 0$. Moreover, the proper transform of the conic $x_0^2 + x_1x_2 = 0$ is smooth of degree $2e_0 - e_6 - e_7 - e_8 - e_9$ and the proper transform of the cubic $x_0^2x_1 + x_2^3 = 0$ is smooth of degree $3e_0 - (e_1 + \dots + e_5) - 2e_6$, thus their defining sections can be taken to be generators of $\mathcal{R}(X)$. With these choices, the sections $s_1, \dots, s_6$ defining the image of the curves defined by $T_1, \dots, T_6$ are the following:

$$s_1 = x_1, \quad s_2 = x_2, \quad s_3 = x_0, \quad s_4 = x_0^2 + x_1x_2, \\ s_5 = x_0^2x_1 + x_2^3, \quad s_6 = x_0^2x_1 + x_2^3 + x_1^2x_2.$$

A computation with Magma [BCP97] shows that a set of generators for the ideal J is:

$$T_1T_2 + T_3^2 - T_4, \quad T_1T_4 + T_2^3 - T_6, \quad T_1T_3^2 + T_2^3 - T_5,$$

and gives a set of generators for the ideal J' . This set is the one given in Table 5; in fact, $J' = I(X)$ since a computation with the same program shows that $\dim J' = 12 = \dim I(X)$.

Remark 4.4. The set of generators of the ideal J given above is induced by geometric relations among curves in $\mathbb{P}^2$. In fact, the three conics s_1s_2 , s_3^2 and s_4 belong to the pencil of conics whose tangent in $(0 : 1 : 0)$ is $x_2 = 0$ and whose tangent in $(0 : 0 : 1)$ is $x_1 = 0$, thus there exists a linear relation among them, which indeed corresponds to the first generator of J . The three cubics s_1s_4 , s_2^3 and s_6 belong to the pencil of cubics which pass through $(1 : 0 : 0)$ (resp $(0 : 0 : 1)$) and are osculating to the same cubic, $x_0^2x_1 + x_1^2x_2 + x_2^3 = 0$, with multiplicity 3 (resp. 6)

in this point. Thus there exists a linear relation among them, which corresponds to the second generator of J . Finally, the three cubics $s_1s_3^2$, s_2^3 and s_5 belong to the pencil of cubics with a cusp in $(0 : 1 : 0)$ whose principal tangent is $x_0 = 0$ and which are osculating with multiplicity 3 to the cubic $x_0^2x_1 + x_2^2$ in $(1 : 0 : 0)$. The linear relation between them corresponds to the third generator of J .

The procedure in the previous example allows us to compute the ideal of relations of the Cox ring of all the extremal rational elliptic surfaces, except for the surfaces X_{8211} , X_{9111} and X_{4422} , where we are not able to identify the ideal J' for computational reasons. The Magma code for these computations is available here

<http://www2.udec.cl/alaface/software/coxrs.html>

Theorem 4.5. *The Cox rings of all extremal rational elliptic surfaces are given in Table 5.*

Proof. In case the surface has complexity one, its Cox ring is computed in Section 3. Otherwise, the Cox rings are computed by means of Theorem 2.11 and Corollary 4.2. In order to apply Corollary 4.2, we recall that here $Y = \mathbb{P}^2$ and the birational morphism $f : X \rightarrow Y$ is chosen to be the blow-up of the base locus of one of the pencils of cubics given in Table 2. The sections $s_1, \dots, s_m \in \mathbb{C}[x_0, x_1, x_2]$ defining the homomorphism ϕ in Corollary 4.2 are the following:

X_{411}	$x_1, x_2, x_0, x_0^2 + x_1x_2, x_0^2x_1 + x_2^3, x_0^2x_1 + x_2^3 + x_1^2x_2$
X_{222}	$x_0 - x_1, x_0, x_1, x_1 - x_2, x_1^3 + 2x_0x_1x_2 - 2x_1^2x_2 - x_0x_2^2 + x_1x_2^2,$ $x_1^3 + 2x_0x_1x_2 - 2x_1^2x_2 - x_0x_2^2 + x_1x_2^2 + x_0x_1(x_0 - x_1)$
X_{141}	$x_0, x_1, x_0 - x_1, x_2, x_0x_1 - x_1^2 + x_0x_2$
X_{6321}	$x_0, x_1, x_2, x_0 + x_2, x_0 + x_1, x_1 + x_2, x_0 + x_1 + x_2, x_0x_1 + x_0x_2 + x_1x_2$
X_{5511}	$x_0, x_2, x_1, x_0 + x_1, x_0 + x_1 + x_2, x_1 + x_2$
X_{3333}	$x_2, x_1, x_0, x_0 + x_1 + x_2, x_0 + x_1 + \epsilon^2x_2,$ $x_0 + x_1 + \epsilon x_2, x_0 + \epsilon^2x_1 + x_2, x_0 + \epsilon^2x_1 + \epsilon^2x_2,$ $x_0 + \epsilon^2x_1 + \epsilon x_2, x_0 + \epsilon x_1 + x_2, x_0 + \epsilon x_1 + \epsilon^2x_2, x_0 + \epsilon x_1 + \epsilon x_2$

where ϵ is a primitive cube root of unity. $\square$

Remark 4.6. The ideal of relation of the surface X_{3333} is given up to the action of its automorphism group G , the Hessian group [AD09].

5. APPLICATIONS

Let X be an extremal rational elliptic surface. Any contraction of (-1) -curves and (-2) -curves gives a birational morphism $\varphi : X \rightarrow Y$, where Y is still a Mori dream space by [Oka, Theorem 1.1]. The morphism φ induces an isomorphism $\varphi_* : \mathbb{C}(X) \rightarrow \mathbb{C}(Y)$ between the fields of rational functions. This allows us to define a homomorphism

$$\varphi_* : \mathcal{R}(X) \rightarrow \mathcal{R}(Y), \quad f \mapsto \varphi_*f,$$

where $\deg(\varphi_*f) := \varphi_*(\deg(f))$. Moreover, in case the pull-back of divisors is well defined, for example if Y is smooth, one can consider the pull-back homomorphism $\varphi^* : \mathcal{R}(Y) \rightarrow \mathcal{R}(X)$. Since $\varphi_* \circ \varphi^* = \text{id}_{\mathcal{R}(Y)}$, then φ_* is surjective.

Let $f \in \mathcal{R}(X)$ be an element defining an integral component E of the exceptional divisor of φ . If $E \in K$, as we can always assume in our setting, then f can be taken

to be a constant equal to 1. The kernel of φ_* clearly contains the ideal I generated by the elements $f - 1$, where f is as before in degree E and 1 is in degree 0. Thus φ_* factorizes through a surjective homomorphism $\mathcal{R}' \rightarrow \mathcal{R}(Y)$, where $\mathcal{R}' = \mathcal{R}(X)/I$. It follows that $\mathcal{R}' \cong \mathcal{R}(Y)$ if $\dim \mathcal{R}' = \dim \mathcal{R}(Y)$ and $\text{Spec}(\mathcal{R}')$ is irreducible. The latter properties can be checked by means of a computer program; for example, with SINGULAR [DGPS15].

This criterion allows us to compute the Cox ring of several weak del Pezzo surfaces which can be obtained contracting (-1) -curves of extremal rational elliptic surfaces. We will give only two examples here.

Example 5.1. The Cayley's cubic surface Y is the only cubic surface in $\mathbb{P}^3$ with four ordinary double points:

$$Y : \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0.$$

The surface is constructed as follows. Let l_1, l_2, l_3, l_4 be the defining polynomials of four general lines of $\mathbb{P}^2$ whose unique linear relation is $l_1 + l_2 + l_3 + l_4 = 0$. Then the Cayley cubic Y is the image of the rational map $\mathbb{P}^2 \rightarrow \mathbb{P}^3$ defined by

$$(x_0 : x_1 : x_2) \mapsto (l_1 l_2 l_3 : l_1 l_2 l_4 : l_1 l_3 l_4 : l_2 l_3 l_4).$$

Indeed, the products of three of the above reducible cubics equal $l_i(l_1 l_2 l_3 l_4)^2$, for $i \in \{1, 2, 3, 4\}$, and thus their sum is zero by construction. Hence the minimal resolution $\tilde{Y}$ of Y is the blow-up of $\mathbb{P}^2$ along the six points of intersection of the four lines. In particular, $\tilde{Y}$ contains four (-2) -curves, corresponding to the four lines, and nine (-1) -curves corresponding to the six intersection points plus the strict transforms of three lines through pairs of these points.

Contracting the three (-1) -curves of X_{6321} corresponding to the generators T_{10}, T_{12}, T_{14} , with the notation in Table 5, it is easy to verify that one obtains a smooth surface whose configuration of negative curves is exactly the one of $\tilde{Y}$. Hence the Cox ring of $\tilde{Y}$ is obtained by that of X_{6321} by putting equal to one the variables T_{10}, T_{12}, T_{14} . In this case $\mathcal{R}(\tilde{Y})$ has 13 generators, which correspond to the negative curves, and the ideal of relations is

$$I(\tilde{Y}) = \left\langle \begin{array}{l} T_1 T_{11} T_{15} + T_3 T_{13} T_{17} - T_4 T_{16}, T_1 T_9 T_{15} + T_2 T_{13} T_{16} - T_5 T_{17}, \\ T_2 T_{11} T_{16} + T_3 T_9 T_{17} - T_6 T_{15}, T_1 T_9 T_{11} + T_6 T_{13} - T_7 T_{16} T_{17}, \\ T_3 T_9 T_{13} + T_5 T_{11} - T_7 T_{15} T_{16}, T_2 T_{11} T_{13} + T_4 T_9 - T_7 T_{15} T_{17}, \\ T_1 T_3 T_9^2 + T_2 T_7 T_{16}^2 - T_5 T_6, T_1 T_2 T_{11}^2 + T_3 T_7 T_{17}^2 - T_4 T_6, \\ T_1 T_7 T_{15}^2 + T_2 T_3 T_{13}^2 - T_4 T_5 \end{array} \right\rangle.$$

This gives an alternative proof of the result given in [HB03, §2]. The relations above are induced by the pencils of lines through $p_1, \dots, p_6$ and by the pencils of conics through four of the points (such that no three of them are collinear). In fact each such pencil clearly contains three sections which are polynomials in the generators of the Cox ring.

Example 5.2. In [Der14] Derenthal classified the weak del Pezzo surfaces whose Cox ring has only one relation and described their Cox rings. An easy check shows that, by contracting (-1) -curves on extremal rational elliptic surfaces, a surface for each of the types in [Der14, Theorem 2] can be obtained. This allows us to give an alternative computation of their Cox ring in several cases. For example, it is known that there are only two del Pezzo surfaces Y_0, Y_1 of degree one and type E_8 up to isomorphism [AN06]. These can be obtained by contracting the unique section in

the surface X_{22} and in the surface X_{211} respectively. Their Cox rings are generated by $T_1, \dots, T_{12}$ and the ideal of relations is:

$$I(Y_\lambda) = \langle T_1 T_3^2 + T_2^3 T_5^2 T_6 + T_4 T_8 T_9^2 T_{10}^3 T_{11}^4 T_{12}^5 + \lambda T_1 T_2^2 T_5^2 T_6^2 T_7^2 T_8^2 T_9^2 T_{10}^2 T_{11}^2 T_{12}^2 \rangle,$$

with $\lambda = 0, 1$.

Remark 5.3. In [Der14] and [HB03] the explicit description of the Cox ring of certain weak del Pezzo surfaces has been used to study the Manin's conjecture on the asymptotic behavior of the number of rational points of bounded height on such surfaces, following a method introduced by Salberger [Sal98].

6. THE COX RINGS

TABLE 5. The Cox rings of extremal rational elliptic surfaces

Surface	Degree matrix and $I(X)$
X_{22}	$\left(\begin{array}{cccc cccccccc} 1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right)$ $I(X) = \langle T_1 T_3^2 + T_2^3 T_5^2 T_6 - T_4 T_8 T_9^2 T_{10}^3 T_{11}^4 T_{12}^5 T_{13}^6 \rangle$
X_{211}	$\left(\begin{array}{cccc cccccccc} 1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right)$ $I(X) = \langle T_1 T_3^2 + T_2^3 T_5^2 T_6 - T_4 T_8 T_9^2 T_{10}^3 T_{11}^4 T_{12}^5 T_{13}^6 + T_1 T_2^2 T_5^2 T_6^2 T_7^2 T_8^2 T_9^2 T_{10}^2 T_{11}^2 T_{12}^2 T_{13}^2 \rangle$
X_{411}	$\left(\begin{array}{cccccc cccccccc} 1 & 1 & 1 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$ $I(X) = \left\langle \begin{array}{l} T_7^2 T_8^3 T_9^4 T_{10} T_{11}^2 T_{12}^3 T_{13}^4 T_{14}^4 T_{15}^4 T_1 T_2^4 - T_4 T_5 + T_3^2 T_6, \\ T_7^2 T_8^3 T_9^4 T_{10}^2 T_{14} T_{15}^2 T_2^3 - T_{10}^2 T_{11} T_6 + T_1 T_4, \quad T_{11}^2 T_{12}^3 T_{13}^4 T_{14}^2 T_{15}^2 T_1 T_2 - T_7^2 T_8 T_6 + T_{10} T_5, \\ T_7^4 T_8^4 T_9^4 T_{10} T_{14} T_{15}^2 T_2^3 - T_{11}^2 T_{12} T_5 + T_1 T_3^2, \quad T_{11}^4 T_{12}^4 T_{13}^3 T_{14}^2 T_{15}^2 T_1 T_2 - T_7^2 T_8 T_4 + T_{10} T_3^2 \end{array} \right\rangle$
X_{9111}	$\left(\begin{array}{cccccc cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & -1 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right)$

TABLE 5. (continued)

Surface	Degree matrix and $I(X)$
X_{222}	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ $I(X) = \left\langle \begin{array}{l} T_1 T_7^2 T_8^2 T_9^2 T_{10} - T_2 T_{14}^2 T_{15} + T_3 T_{12}^2 T_{13}, \quad T_1^2 T_4^2 T_8^2 T_9^2 T_{10} + T_2 T_5 T_{14}^2 - T_3 T_6 T_{12}^2, \\ T_1 T_4^2 T_{13} - T_2^2 T_3 T_{10} T_{11}^2 T_{14}^2 + T_6 T_7^2 T_8, \quad T_1 T_4^2 T_{15} - T_2 T_3 T_{10} T_{11}^2 T_{12}^2 + T_5 T_7^2 T_8, \\ T_1 T_2 T_3 T_8 T_9^2 T_{10}^2 T_{11}^2 + T_5 T_{13} - T_6 T_{15} \end{array} \right\rangle$
X_{141}	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $I(X) = \langle T_1 T_4 T_{14}^2 + T_2 T_3 T_9 T_{11} T_{13} - T_5 T_6^2 T_7, \quad T_1 T_6 T_7 T_8 T_{14} - T_2 T_{10} T_{11}^2 - T_3 T_{12} T_{13}^2 \rangle$
X_{6321}	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $I(X) = \left\langle \begin{array}{l} T_2 T_{11} T_{12} T_{16} + T_3 T_9 T_{10} T_{17} - T_6 T_{14} T_{15}, \quad T_1 T_{11} T_{12} T_{15} + T_3 T_{13} T_{14} T_{17} - T_4 T_{10} T_{16}, \\ T_1 T_9 T_{10} T_{15} + T_2 T_{13} T_{14} T_{16} - T_5 T_{12} T_{17}, \quad T_1 T_3 T_9 T_{15} T_{17} + T_2 T_4 T_{16}^2 - T_8 T_{12} T_{14}, \\ T_1 T_2 T_{11} T_{15} T_{16} + T_3 T_5 T_{17}^2 - T_8 T_{10} T_{14}, \quad T_3 T_9 T_{10} T_{13} T_{14} + T_5 T_{11} T_{12}^2 - T_7 T_{15} T_{16}, \\ T_1 T_9 T_{10} T_{11} T_{12} + T_6 T_{13} T_{14}^2 - T_7 T_{16} T_{17}, \quad T_1 T_6 T_{15}^2 + T_2 T_3 T_{13} T_{16} T_{17} - T_8 T_{10} T_{12}, \\ T_2 T_{11} T_{12} T_{13} T_{14} + T_4 T_9 T_{10}^2 - T_7 T_{15} T_{17}, \quad T_1^2 T_9 T_{11} T_{15}^2 - T_4 T_5 T_{16} T_{17} + T_8 T_{13} T_{14}^2, \\ T_1 T_2 T_{11}^2 T_{12}^2 + T_3 T_7 T_{17}^2 - T_4 T_6 T_{10} T_{14}, \quad T_3^2 T_9 T_{13} T_{17}^2 - T_4 T_6 T_{15} T_{16} + T_8 T_{11} T_{12}^2, \\ T_1 T_7 T_{15}^2 + T_2 T_3 T_{13}^2 T_{14}^2 - T_4 T_5 T_{10} T_{12}, \quad T_1 T_3 T_9^2 T_{10}^2 + T_2 T_7 T_{16} - T_5 T_6 T_{12} T_{14}, \\ T_2^2 T_{11} T_{13} T_{16}^2 - T_5 T_6 T_{15} T_{17} + T_8 T_9 T_{10}^2, \quad T_1 T_2 T_3 T_9 T_{11} T_{13} + T_4 T_5 T_6 - T_7 T_8 \end{array} \right\rangle$
$X_{11}(a)$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$ $I(X) = \langle (a-1)T_2 T_8 T_9^2 - T_3 T_{10} T_{11}^2 + T_5 T_{13} T_{14}^2, \quad (a-1)T_1 T_6 T_7^2 - aT_3 T_{10} T_{11}^2 + T_5 T_{13} T_{14}^2 \rangle$
X_{5511}	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $I(X) = \left\langle \begin{array}{l} T_2 T_{11} T_{13} T_{14} + T_4 T_7^2 T_8 - T_5 T_9 T_{10} T_{15}, \quad T_2 T_{11}^2 T_{12} + T_3 T_7 T_8 T_{15} - T_6 T_9 T_{10} T_{13}, \\ T_1 T_9^2 T_{10} + T_3 T_{13} T_{14} T_{15} - T_4 T_7 T_{11} T_{12}, \quad T_1 T_7 T_8 T_9 - T_5 T_{11} T_{12} T_{15} + T_6 T_{13}^2 T_{14}, \\ T_1 T_2 T_9 T_{11} + T_3 T_5 T_{15}^2 - T_4 T_6 T_7 T_{13} \end{array} \right\rangle$

TABLE 5. (continued)

Surface	Degree matrix and $I(X)$
X_{4422}	$\left(\begin{array}{cccccccccccc cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$
X_{3333}	$\left(\begin{array}{cccccccccccc cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$ $I(X) = \left\langle \begin{array}{l} (-2\epsilon - 1)T_1T_{20}T_{21} - T_5T_{15}T_{18} + T_6T_{14}T_{17} \\ (\epsilon - 1)T_1T_5T_{19}^2 + T_8T_{10}T_{13}T_{16} - T_9T_{11}T_{14}T_{17} \\ T_1T_2T_3 + 1/9(-\epsilon + 1)T_4T_9T_{11} + 1/9(\epsilon - 1)T_5T_7T_{12} \end{array} \right\rangle^G$

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