BEHAVIOR OF HOLOMORPHIC MAPPINGS ON p -COMPACT SETS IN A BANACH SPACE

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ABSTRACT. We study the behavior of holomorphic mappings on p -compact sets in Banach spaces. We show that the image of a p -compact set by an entire mapping is a p -compact set. Some results related to the localization of p -compact sets in the predual of homogeneous polynomials are also obtained. Finally, the “size” of p -compactness of the image of the unit ball by p -compact linear operators is studied.

1. INTRODUCTION

Motivated by the well-known Grothendieck characterization of compact sets [10], Sinha and Karn [16] introduced the notion of p -compactness. Their work provided motivation for many papers related to the p -compactness notion in a number of different directions. We mention [1], [2], [4], [5], [7], [8], [11], [16], and [17].

In this paper we study the behavior of holomorphic mappings on p -compact sets. It is a simple fact that the image of a compact set by a continuous function is a compact set, but we shall observe that the analogous result is not always true if one considers p -compact sets. With this in mind, it is natural to ask what happens if we consider holomorphic mappings and, in particular, polynomials. We show that the image of a p -compact set by an entire mapping is always p -compact.

The paper is organized as follows: In section 2 we establish our notation and definitions. In section 3 we study the behavior of holomorphic mappings on p -compact sets. First we show that the image of a p -compact set by a polynomial is always a p -compact set. In fact, this result is already known, with a proof by the first author and P. Rueda [2] that uses tensor product methods. However, the machinery that we develop here allows us to obtain a positive answer for entire mappings. In section 4 we get a result related to the localization of p -compact sets in symmetric tensor products of a Banach space with itself, which runs parallel to a result that Ryan [14] obtained for compact sets. Furthermore, we answer in the negative the following question: Is the image of a p -compact set, by an m -homogeneous polynomial, p/m -compact? To do this we give an example of a homogeneous polynomial P and a p -compact set K whose image $P(K)$ is not a

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q -compact set for any $1 \leq q < p$. As a converse, we show that given a p -compact subset of the image of an m -homogeneous polynomial, in general it is not possible to find a p -compact set whose image contains the given p -compact set. In section 5 we show that the “size” (in the sense that we will make precise there) of p -compactness of the image of the unit ball by p -compact linear operators between ℓ_p spaces can be arbitrarily large.

2. NOTATION AND DEFINITIONS

The symbol $\mathbb{C}$ represents the field of all complex numbers, $\mathbb{N}$ represents the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The letters X and Y will always represent complex Banach spaces. The letter U denotes a nonvoid open subset of X and the symbol B_X represents the open unit ball of X . The symbol $\overline{B}_X$ represents the closed unit ball of X .

Given a subset M of X , the symbol $f|_M$ will denote the restriction of a mapping $f : U \rightarrow Y$ to the subset $M \subset U$, the symbol I_M will denote the identity mapping on M , and for any topology τ on X , $\overline{M}^\tau$ will denote the τ -closure of M in X .

For each integer $n \in \mathbb{N}$, let $\mathcal{L}(X_1, \dots, X_n; Y)$ be the Banach space of all continuous n -linear mappings $A : X_1 \times \dots \times X_n \rightarrow Y$, endowed with the sup norm $\|A\| = \sup\{\|A(x_1, \dots, x_n)\| : \|x_i\| \leq 1, i = 1, \dots, n\}$. When $X_1 = \dots = X_n = X$ we write $\mathcal{L}(^n X; Y)$ instead of $\mathcal{L}(\underbrace{X, \dots, X}_{n \text{ times}}; Y)$, and in particular, $\mathcal{L}(^1 X; Y) = \mathcal{L}(X; Y)$.

The subspace of all compact operators of $\mathcal{L}(X; Y)$ will be denoted by $\mathcal{K}(X; Y)$.

Given a continuous n -linear mapping $A \in \mathcal{L}(^n X; Y)$, the map $P : X \rightarrow Y$, defined by $P(x) = A(\underbrace{x, \dots, x}_{n \text{ times}})$ for every $x \in X$, is said to be a continuous n -

homogeneous polynomial. $\mathcal{P}(^n X; Y)$ will denote the vector space of all continuous n -homogeneous polynomials from X into Y , which is a Banach space with norm $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$. The constant functions from X into Y will be denoted by $\mathcal{P}(^0 X; Y)$. When $Y = \mathbb{C}$ we will write $\mathcal{P}(^n X)$ instead of $\mathcal{P}(^n X; \mathbb{C})$.

If A is a multilinear mapping and P is the polynomial generated by A , we write $P = \widehat{A}$. Conversely, given a polynomial $P \in \mathcal{P}(^n X; Y)$, there is a unique symmetric continuous n -linear mapping $\check{P} \in \mathcal{L}(^n X; Y)$ such that $P(x) = \check{P}(\underbrace{x, \dots, x}_{n \text{ times}})$. It is well

known that the correspondence $A \longleftrightarrow \widehat{A}$ is a topological isomorphism between $\mathcal{L}^s(^n X; Y)$, the space of all symmetric continuous n -linear mappings from X to Y , and $\mathcal{P}(^n X; Y)$ (see, for example, [13, Theorem 2.2]).

A mapping $P : X \rightarrow Y$ is said to be a continuous polynomial if it can be represented as a sum $P = P^0 + P^1 + \dots + P^n$, with $P^j \in \mathcal{P}(^j X; Y)$ for $j = 0, 1, \dots, n$. By $\mathcal{P}(X; Y)$ we will denote the vector space of all continuous polynomials from X into Y .

Let X be a Banach space, let $1 \leq p \leq \infty$, with the conjugate index p' given by $\frac{1}{p} + \frac{1}{p'} = 1$ (where $p' = 1$ if $p = \infty$), let $\ell_p(X)$ ($1 \leq p < \infty$) (resp., $\ell_\infty(X)$) denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $\sum_{n=1}^\infty \|x_n\|^p < \infty$ (resp., $(x_n)_n$ is bounded), and let $c_0(X)$ denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $x_n \rightarrow 0$ in X . Then a set $K \subset X$ is said to be relatively p -compact if there is a sequence $(x_n)_{n=1}^\infty$ in $\ell_p(X)$ ($((x_n)_{n=1}^\infty$ in $c_0(X) \subset \ell_\infty(X)$ if $p = \infty$) such that $K \subset \left\{ \sum_{n=1}^\infty a_n x_n : (a_n)_{n=1}^\infty \in \overline{B}_{\ell_{p'}} \right\}$. A relatively p -compact and closed set

will be called p -compact. We denote this last set by $p\text{-co}\{(x_n)_{n=1}^\infty\}$ and we will call it a fundamental p -compact set since these sets form a fundamental system of p -compact sets of X .

From Grothendieck's characterization of compact sets (see [10], [12, Proposition 1.e.2]), a subset K of a Banach space X is relatively compact if and only if there is a sequence $(x_n)_{n=1}^\infty$ in $c_0(X)$ such that $K \subset \{\sum_{n=1}^\infty a_n x_n : \sum_{n=1}^\infty |a_n| \leq 1\}$. Thus, by the above definition one can consider compact sets as ∞ -compact. Also, note that p -compact sets are q -compact if $1 \leq p \leq q \leq \infty$.

We refer to [9] or [13] for the properties of polynomials and holomorphic mappings in infinite dimensional spaces, and to [12] for the theory of Banach spaces.

3. THE STABILITY OF p -COMPACT SETS

First of all we observe that the image of a p -compact set by a continuous function need not be a p -compact set. In fact, we have the following example.

Example 3.1. Let $1 < p < \infty$ and let $(a_n)_{n=1}^\infty \in c_0 \setminus \ell_p$. Let $f : \mathbb{R} \rightarrow \ell_p$ be defined by

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ (n+1)(1-nt)a_{n+1}e_{n+1} + n((n+1)t-1)a_n e_n & \text{if } t \in [\frac{1}{n+1}, \frac{1}{n}] \text{ for } n \in \mathbb{N}, \\ a_1 e_1 & \text{if } t \geq 1, \end{cases}$$

where $(e_n)_{n=1}^\infty$ is the canonical basis of ℓ_p . Observe that f is continuous and that $f([0, 1]) \supset (a_n e_n)_{n=1}^\infty$. By [1, Example 1] we know that the set $(a_n e_n)_{n=1}^\infty$ is a relatively compact set in ℓ_p which is not p -compact. Consequently $f([0, 1])$ cannot be a p -compact set in ℓ_p . But, since the interval $[0, 1]$ is a p -compact subset of $\mathbb{R}$ (see e.g. [1, Proposition 2]), we are done.

On the other hand, contrary to the non-linear case, it is easy to show that the image of a p -compact set by a continuous linear operator is a p -compact set.

In [2], using linearization of homogeneous polynomials and tensor products, the first author and P. Rueda show that the image of a p -compact set by a homogeneous polynomial is also p -compact. For a more general result see [3, Theorem 3.5]. Below we obtain the same result in a different way. Although our proof is longer than that in [2], the proof that we will present has the advantage of yielding, as a bonus, a proof showing that the same is true for entire mappings.

Lemma 3.2. *Let X and Y be Banach spaces, and let $n \in \mathbb{N}$, $P \in \mathcal{P}(^n X; Y)$ and $1 \leq p \leq \infty$. If $(x_k)_{k=1}^\infty \in \ell_p(X)$, then the set $P\left(\left\{\sum_{k=1}^\infty \alpha_k x_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}\right\}\right)$ is a p -compact subset of Y .*

Proof. Since ∞ -compact sets are compact, we shall give a proof for the case $p > 1$. Minor modifications yield a proof for the $p = 1$ case.

Let $n \in \mathbb{N}$ and $P \in \mathcal{P}(^n X; Y)$. Let $(x_k)_{k=1}^\infty$ be a sequence in $\ell_p(X)$. To show that the image $P\left(\left\{\sum_{k=1}^\infty \alpha_k x_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}\right\}\right)$ is p -compact, let us take an element $x \in \left\{\sum_{k=1}^\infty \alpha_k x_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}\right\}$ with $x = \sum_{k=1}^\infty \alpha_k x_k$ for some $(\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}$.

Let $h \in \mathbb{N}$ be any integer. By the Leibniz formula we have that

$$\begin{aligned} P\left(\sum_{k=1}^h \alpha_k x_k\right) &= \sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \frac{n!}{l_1! \dots l_h!} \alpha_1^{l_1} \dots \alpha_h^{l_h} \overset{\vee}{P}(x_1^{l_1}, \dots, x_h^{l_h}) \\ &= \sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \left(\frac{n!}{l_1! \dots l_h!}\right)^{1/p'} \alpha_1^{l_1} \dots \alpha_h^{l_h} \left(\frac{n!}{l_1! \dots l_h!}\right)^{1/p} \overset{\vee}{P}(x_1^{l_1}, \dots, x_h^{l_h}), \end{aligned}$$

where $\overset{\vee}{P}$ is the continuous symmetric n -linear mapping corresponding to the n -homogeneous polynomial P . Note that, by the multinomial theorem,

$$\begin{aligned} (1) \quad & \left(\sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \left[\left(\frac{n!}{l_1! \dots l_h!}\right)^{1/p'} |\alpha_1^{l_1}| \dots |\alpha_h^{l_h}| \right]^{p'} \right)^{1/p'} \\ &= \left(\sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \frac{n!}{l_1! \dots l_h!} |\alpha_1^{p'}|^{l_1} \dots |\alpha_h^{p'}|^{l_h} \right)^{1/p'} \\ &= \left(\sum_{k=1}^h |\alpha_k|^{p'} \right)^{n/p'} \leq \|(\alpha_k)_{k \in \mathbb{N}}\|_{p'}^n \leq 1 \end{aligned}$$

and analogously

$$\begin{aligned} (2) \quad & \left(\sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \left[\left(\frac{n!}{l_1! \dots l_h!}\right)^{1/p} \|\overset{\vee}{P}(x_1^{l_1}, \dots, x_h^{l_h})\| \right]^p \right)^{1/p} \\ &\leq \left(\sum_{\substack{l_1+\dots+l_h=n \\ l_j \in \{0,1,\dots,n\}}} \frac{n!}{l_1! \dots l_h!} \|\overset{\vee}{P}\|^p (||x_1||^p)^{l_1} \dots (||x_h||^p)^{l_h} \right)^{1/p} \\ &= \|\overset{\vee}{P}\| \left(\sum_{k=1}^h ||x_k||^p \right)^{n/p} \leq \|\overset{\vee}{P}\| \left(\sum_{k=1}^{\infty} ||x_k||^p \right)^{n/p} < \infty. \end{aligned}$$

Now we claim that if $(\alpha_k)_{k=1}^{\infty} \in \overline{B}_{\ell_{p'}}$, we have

$$(3) \quad P\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) = \sum_{\substack{\sum_{j=1}^{\infty} l_j=n \\ l_j \in \{0,1,\dots,n\}}} \frac{n!}{l_1! \dots l_j! \dots} \prod_{j=1}^{\infty} \alpha_j^{l_j} \overset{\vee}{P}(x_1^{l_1}, \dots, x_j^{l_j}, \dots).$$

Let

$$\mathbb{A}_n := \left\{ (l_j)_{j=1}^{\infty} \in \{0, \dots, n\}^{\mathbb{N}} : \sum_{j=1}^{\infty} l_j = n \right\}$$

and let J be a finite subset of $\mathbb{A}_n$. Then there exists an integer $h \in \mathbb{N}$ such that

$$J \subset \mathbb{B}_n^h := \left\{ (l_j)_{j=1}^h \in \{0, \dots, n\}^h : \sum_{j=1}^h l_j = n \right\} \times \{0\}^{\mathbb{N}}.$$

Thus

$$\begin{aligned} & \sum_{l \in J} \frac{n!}{l_1! \dots l_j! \dots} \prod_{j=1}^{\infty} |\alpha_j|^{l_j} \left\| \check{P}(x_1^{l_1}, \dots, x_j^{l_j}, \dots) \right\| \\ & \leq \sum_{\substack{l_1 + \dots + l_h = n \\ l_j \in \{0, 1, \dots, n\}}} \frac{n!}{l_1! \dots l_h!} |\alpha_1|^{l_1} \dots |\alpha_h|^{l_h} \left\| \check{P} \right\| \|x_1\|^{l_1} \dots \|x_h\|^{l_h} \\ & = \left\| \check{P} \right\| \left(\sum_{k=1}^h |\alpha_k| \|x_k\| \right)^n \leq \left\| \check{P} \right\| \left(\sum_{k=1}^{\infty} |\alpha_k|^{p'} \right)^{n/p'} \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^{n/p} \\ & \leq \left\| \check{P} \right\| \|(x_k)_{k=1}^{\infty}\|_p^n < \infty, \end{aligned}$$

which shows that the series on the right-hand side of (3) is absolutely summable. In particular, the sequence $\left(\sum_{l \in \mathbb{B}_n^h} \binom{n}{l} \alpha^l \check{P}(x^l) \right)_{h \in \mathbb{N}}$ converges to $\sum_{l \in \mathbb{A}_n} \binom{n}{l} \alpha^l \check{P}(x^l)$, where we use the notation: $\alpha = (\alpha_k)_{k=1}^{\infty}$, $x = (x_k)_{k=1}^{\infty}$, and

$$\binom{n}{l} = \frac{n!}{\prod_{j=1}^{\infty} l_j!}, \quad \alpha^l = \prod_{j=1}^{\infty} \alpha_j^{l_j}, \quad \check{P}_n(x^l) = \check{P}_n(x_1^{l_1}, \dots, x_j^{l_j}, \dots).$$

Since the sequence $\left(P \left(\sum_{k=1}^h \alpha_k x_k \right) \right)_{h \in \mathbb{N}}$ converges to $P \left(\sum_{k=1}^{\infty} \alpha_k x_k \right)$ and

$$P \left(\sum_{k=1}^h \alpha_k x_k \right) = \sum_{l \in \mathbb{B}_n^h} \binom{n}{l} \alpha^l \check{P}(x^l),$$

it follows that

$$P \left(\sum_{k=1}^{\infty} \alpha_k x_k \right) = \sum_{l \in \mathbb{A}_n} \binom{n}{l} \alpha^l \check{P}(x^l) = \sum_{l \in \mathbb{A}_n} \binom{n}{l}^{1/p'} \alpha^l \binom{n}{l}^{1/p} \check{P}(x^l).$$

Noting that $\mathbb{A}_n$ is a countable set and by using (1) and (2), respectively, we have

$$\left(\binom{n}{l}^{1/p'} \alpha^l \right)_{l \in \mathbb{A}_n} \in \overline{B}_{\ell_{p'}} \quad \text{and} \quad \left(\binom{n}{l}^{1/p} \check{P}(x^l) \right)_{l \in \mathbb{A}_n} \in \ell_p(Y),$$

which completes the proof. $\square$

Corollary 3.3. *Let X and Y be Banach spaces and let $P \in \mathcal{P}(X; Y)$. If $K \subset X$ is a p -compact set, then the set $P(K) \subset Y$ is a p -compact set.*

Proof. Let K be a p -compact subset of X . Then, there is a sequence $(x_k)_{k=1}^{\infty} \in \ell_p(X)$ such that $K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k : (\alpha_k)_{k=1}^{\infty} \in \overline{B}_{\ell_{p'}} \right\}$. By the previous lemma the set $P(K)$ is a p -compact subset of Y , for every homogeneous polynomial $P \in \mathcal{P}(^n X; Y)$ and for every $n \in \mathbb{N}$. Since a finite sum of p -compact sets is p -compact, we are done. $\square$

Now we are going to use our “quantitative” version of the proof that the image of a p -compact set by a polynomial is also p -compact in order to prove the analogue of Corollary 3.3 for entire functions.

Let X be a Banach space and let U be an open subset of X . We denote by $\mathcal{H}(U; Y)$ the space of all holomorphic functions f from U into Y . When $Y = \mathbb{C}$ we write $\mathcal{H}(U)$ instead of $\mathcal{H}(U; \mathbb{C})$.

We will need the following.

Lemma 3.4. *Let X and Y be Banach spaces, and let U be a balanced open subset of X and $s > 0$ satisfying $sB_X \subset U$. Consider a sequence $(x_k)_{k=1}^\infty$ in $\ell_p(X)$ with p -norm $\|(x_k)_{k=1}^\infty\|_p < \frac{s}{e}$. Finally, let $\epsilon > 0$ and $R > e$ such that*

$$(1 + \epsilon)^{1/p} \left(\frac{R}{e} \right)^{1/p'} \|(x_k)_{k=1}^\infty\|_p < \frac{s}{e}.$$

We denote $M = 2 \left(\frac{e}{R-e} \right)^{1/p'} \frac{1}{\epsilon^{1/p}} + 2$. Then, for every $K \subset p\text{-co} \{(x_k)_{k=1}^\infty\}$ and every $f \in \mathcal{H}(U; Y)$ bounded on sB_X it holds that $f(K)$ is relatively p -compact. Moreover, there exists a sequence (y_j) in $\ell_p(Y)$ (that depends on f) such that

$$f(K) \subset p\text{-co} \{(y_j)_{j=1}^\infty\}$$

and

$$\|(y_j)_{j=1}^\infty\|_p \leq M \|f\|_{sB_X}.$$

Observe that M depends neither on f nor on (y_j) .

Proof. As in the polynomial case, we give a proof for the case $p > 1$ since the proof for the case $p = 1$ is similar.

We will show that the set

$$f \left(\left\{ \sum_{k=1}^\infty \alpha_k x_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}} \right\} \right)$$

is relatively p -compact.

Let $f = \sum_{n=0}^\infty P_n$ be the Taylor series expansion of f about 0, with $P_n \in \mathcal{P}(^n X; Y)$ for every $n \in \mathbb{N}_0$.

Let $x \in \left\{ \sum_{k=1}^\infty \alpha_k x_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}} \right\}$. By (3) we have that

$$f(x) - f(0) = \sum_{n=1}^\infty P_n(x) = \sum_{n=1}^\infty \sum_{l \in \mathbb{A}_n} \binom{n}{l} \alpha^l \check{P}_n(x^l),$$

where we use the same notation as in Lemma 3.2.

Since we have chosen $\epsilon > 0$ and $R > e$ such that

$$(4) \quad (1 + \epsilon)^{1/p} \left(\frac{R}{e} \right)^{1/p'} e \left(\sum_{k=1}^\infty \|x_k\|^p \right)^{1/p} < s,$$

we can write

$$f(x) - f(0) = \sum_{n=1}^\infty \sum_{l \in \mathbb{A}_n} \left(\left(\frac{e}{R} \right)^{n/p'} \binom{n}{l} \alpha^l \right) \left(\binom{n}{l}^{1/p} \left(\frac{R}{e} \right)^{n/p'} \check{P}_n(x^l) \right).$$

Note that as in (1) we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} \left(\left(\frac{e}{R} \right)^{n/p'} \binom{n}{l}^{1/p'} |\alpha|^l \right)^{p'} \right)^{1/p'} &= \left(\sum_{n=1}^{\infty} \left(\frac{e}{R} \right)^n \sum_{l \in \mathbb{A}_n} \binom{n}{l} |\alpha^{p'}|^l \right)^{1/p'} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\frac{e}{R} \right)^n \cdot 1 \right)^{1/p'} = \left(\frac{e}{R-e} \right)^{1/p'}. \end{aligned}$$

The polarization formula gives $\|\check{P}_n\| \leq e^n \|P_n\|$ for every $n \in \mathbb{N}$ (see e.g. [9, Proposition 1.8]). And by (2) we obtain

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} \left(\binom{n}{l}^{1/p} \left(\frac{R}{e} \right)^{n/p'} \|\check{P}_n(x^l)\| \right)^p \right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} \binom{n}{l} \left(\left(\frac{R}{e} \right)^{p/p'} \right)^n \|\check{P}_n(x^l)\|^p \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\left(\frac{R}{e} \right)^{p/p'} \right)^n \|\check{P}_n\|^p \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^n \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} \left(\left(\frac{R}{e} \right)^{p/p'} \right)^n (e^n \|P_n\|)^p \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^n \right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{(1+\epsilon)^n} \|P_n\|^p \left((1+\epsilon)^{1/p} \left(\frac{R}{e} \right)^{1/p'} e \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^{1/p} \right)^{B_X} \right)^{1/p} \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{(1+\epsilon)^n} \|P_n\|_{sB_X}^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} \frac{1}{(1+\epsilon)^n} \|f\|_{sB_X}^p \right)^{1/p} \\ &= \|f\|_{sB_X} \left(\sum_{n=1}^{\infty} \frac{1}{(1+\epsilon)^n} \right)^{1/p} = \frac{1}{\epsilon^{1/p}} \|f\|_{sB_X} < \infty, \end{aligned}$$

where $\|P_n\|_{sB_X} \leq \|f\|_{sB_X}$ holds because of Cauchy's inequalities. Now

$$f(x) = \frac{1}{2} y_0 + \sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} a_{n,l} y_{n,l},$$

where $y_0 = 2f(0)$,

$$a_{n,l} = \frac{1}{2} \left(\frac{R-e}{e} \right)^{1/p'} \left(\frac{e}{R} \right)^{n/p'} \binom{n}{l}^{1/p'} |\alpha|^l$$

and

$$y_{n,l} = 2 \left(\frac{e}{R-e} \right)^{1/p'} \binom{n}{l}^{1/p} \left(\frac{R}{e} \right)^{n/p'} \check{P}_n(x^l),$$

for $l \in \mathbb{A}_n$ and $n = 1, 2, \dots$. Since

$$\left(\left(\frac{1}{2} \right)^{p'} + \sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} |a_{n,l}|^{p'} \right)^{1/p'} \leq \frac{1}{2} + \frac{1}{2} = 1$$

and

$$(\|y_0\| + \sum_{n=1}^{\infty} \sum_{l \in \mathbb{A}_n} \|y_{n,l}\|^p)^{1/p} \leq 2\|f\|_{sB_X} + 2\left(\frac{e}{R-e}\right)^{1/p'} \frac{1}{\varepsilon^{\frac{1}{p}}} \|f\|_{sB_X} < \infty,$$

the proof is complete. $\square$

Theorem 3.5. *Let X and Y be Banach spaces, and let U be a balanced open subset of X . If $f \in \mathcal{H}(U; Y)$ and*

$$K := \left\{ \sum_{k=1}^{\infty} \alpha_k x_k : (\alpha_k)_{k=1}^{\infty} \in \overline{B}_{\ell_{p'}} \right\} \subset U$$

is the fundamental p -compact set generated by the sequence $(x_k)_{k=1}^{\infty} \in \ell_p(X)$, then $f(K)$ is a p -compact subset of Y .

Proof. Again, we will provide a proof for the case $p > 1$, the case $p = 1$ being similar.

Let us take $R > 1$ and $s > 0$ such that $R^2 K + sB_X \subset U$ and $\|f\|_{R^2 K + sB_X} < +\infty$. Let $k_0 \in \mathbb{N}$ such that $(\sum_{k=k_0}^{\infty} \|x_k\|^p)^{\frac{1}{p}} < \frac{s}{e}$.

Let $\eta : R^2 B_{\ell_{p'}^{k_0}} \rightarrow X$ be defined by $\eta(\lambda_1, \dots, \lambda_{k_0}) = \sum_{k=1}^{k_0} \lambda_k x_k$ for every element $(\lambda_1, \dots, \lambda_{k_0}) \in R^2 B_{\ell_{p'}^{k_0}}$. For fixed $x \in X$ so that $\eta(R^2 B_{\ell_{p'}^{k_0}}) + x \subset U$ we define $g_x : R^2 B_{\ell_{p'}^{k_0}} \rightarrow Y$ by $g_x(\lambda) = f(\eta(\lambda) + x)$ for every $\lambda \in R^2 B_{\ell_{p'}^{k_0}}$. By the Cauchy integral formula we have $g_x(\lambda) = \sum_{\alpha \in \mathbb{N}_0^{k_0}} c_{\alpha}(x) \lambda^{\alpha}$ with

$$c_{\alpha}(x) = \frac{1}{(2\pi i)^{k_0}} \int_{|\lambda_1|=r_1} \cdots \int_{|\lambda_{k_0}|=r_{k_0}} \frac{g_x(\lambda)}{\lambda^{\alpha+1}} d\lambda_1 \dots d\lambda_{k_0},$$

for $r = (r_1, \dots, r_{k_0})$, $r_j > 0$, $j = 1, \dots, k_0$, $\|r\|_{p'} < R^2$, where $\mathbf{1} = (1, \dots, 1)$. Therefore

$$(5) \quad \|c_{\alpha}(x)\| \leq \frac{1}{r^{\alpha}} \sup_{|\lambda|=r} \|f(\eta(\lambda) + x)\|.$$

Clearly, if $K_1 = p\text{-co}\{x_1, \dots, x_{k_0}\}$, then $\eta(t\overline{B}_{\ell_{p'}^{k_0}}) = tK_1$ for every $0 < t < R^2$. From [9, Lemma 1.38] we know that for $\alpha \in \mathbb{N}_0^{k_0}$

$$\sup_{z \in \ell_{p'}^{k_0}} |z^{\alpha}| = \left(\frac{\alpha^{\alpha}}{|\alpha|^{|\alpha|}} \right)^{\frac{1}{p'}}$$

and this supremum is attained at

$$z_{0,\alpha} = \left(\left(\frac{\alpha_1}{|\alpha|} \right)^{\frac{1}{p'}}, \dots, \left(\frac{\alpha_{k_0}}{|\alpha|} \right)^{\frac{1}{p'}} \right) \in S_{\ell_{p'}^{k_0}}.$$

By taking $r = z_{0,\alpha}$ in (5) we get

$$(6) \quad \|c_{\alpha}\|_{sB_X} (R^2 z_{0,\alpha})^{\alpha} \leq \|f\|_{R^2 K_1 + sB_X} < +\infty$$

for every $\alpha \in \mathbb{N}_0^{k_0}$. In other words every function $(Rz_{0,\alpha})^{\alpha} c_{\alpha}$ is holomorphic and bounded on $R^2 K_1 + sB_X$. Hence, we can apply Lemma 3.4 (for $(\sum_{k=k_0}^{\infty} \|x_k\|^p)^{\frac{1}{p}} < \frac{s}{e}$ and all holomorphic and bounded functions on sB_X) to obtain a constant $M > 0$

such that for any given function $(Rz_{0,\alpha})^\alpha c_\alpha$, there exists a sequence $(y_{j,\alpha})_{j=1}^\infty \in \ell_p(Y)$ satisfying

$$(i) \quad (Rz_{0,\alpha})^\alpha c_\alpha(p\text{-co}\{x_{k_0+1}, x_{k_0+2}, \dots\}) \subset p\text{-co}\{(y_{j,\alpha})_{j=1}^\infty\}$$

and

$$(ii) \quad \|(y_{j,\alpha})_{j=1}^\infty\|_p \leq M \|(Rz_{0,\alpha})^\alpha c_\alpha\|_{sB_X} \leq \frac{M}{R^\alpha} \|f\|_{R^2K+sB_X} \quad \text{by (6)}$$

for every $\alpha \in \mathbb{N}_0^{k_0}$.

Observe that we have

$$\begin{aligned} & \left(\sum_{\alpha \in \mathbb{N}_0^{k_0}} \sum_{j=1}^\infty \|y_{j,\alpha}\|^p \right)^{\frac{1}{p}} \leq \sum_{\alpha \in \mathbb{N}_0^{k_0}} \left(\sum_{j=1}^\infty \|y_{j,\alpha}\|^p \right)^{\frac{1}{p}} = \sum_{\alpha \in \mathbb{N}_0^{k_0}} \|(y_{j,\alpha})_{j=1}^\infty\|_p \\ & \leq \sum_{\alpha \in \mathbb{N}_0^{k_0}} \frac{M}{R^\alpha} \|f\|_{R^2K+sB_X} = \left(\sum_{h=0}^\infty \frac{1}{R^h} \right)^{k_0} M \|f\|_{R^2K+sB_X} \\ & = \left(\frac{R}{R-1} \right)^{k_0} M \|f\|_{R^2K+sB_X} < +\infty. \end{aligned}$$

Hence $(y_{j,\alpha})_{(j,\alpha) \in \mathbb{N} \times \mathbb{N}_0^{k_0}} \in \ell_p(\mathbb{N}_0^{k_0+1}, Y)$.

Consider now $(\lambda_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}$. We write $\sum_{k=1}^\infty \lambda_k x_k = \sum_{k=1}^{k_0} \lambda_k x_k + x$, where $x = \sum_{k=k_0+1}^\infty \lambda_k x_k$. So

$$f\left(\sum_{k=1}^\infty \lambda_k x_k\right) = g_x(\lambda) = \sum_{\alpha \in \mathbb{N}_0^{k_0}} c_\alpha(x) \lambda^\alpha = \sum_{\alpha \in \mathbb{N}_0^{k_0}} (Rz_{0,\alpha})^\alpha c_\alpha(x) \left(\frac{\lambda}{Rz_{0,\alpha}}\right)^\alpha,$$

where $\lambda = (\lambda_1, \dots, \lambda_{k_0}) \in \overline{B}_{\ell_{p'}^{k_0}}$.

By (i), for every $\alpha \in \mathbb{N}_0^{k_0}$, there exists $(\beta_{j,\alpha}) \in \overline{B}_{\ell_{p'}}$ such that $(Rz_{0,\alpha})^\alpha c_\alpha(x) = \sum_{j=1}^\infty \beta_{j,\alpha} y_{j,\alpha}$. Thus

$$(7) \quad f\left(\sum_{k=1}^\infty \lambda_k x_k\right) = \sum_{\alpha \in \mathbb{N}_0^{k_0}} \left(\sum_{j=1}^\infty \beta_{j,\alpha} \frac{\left(\frac{R}{R-1}\right)^{\frac{k_0}{p'}}}{\left(\frac{R}{R-1}\right)^{\frac{k_0}{p'}}} y_{j,\alpha} \right) \frac{1}{R^\alpha} \left(\frac{\lambda}{z_{0,\alpha}}\right)^\alpha.$$

But $|\lambda|^\alpha \leq |z_{0,\alpha}|^\alpha$, so $\frac{|\lambda|^\alpha}{|z_{0,\alpha}|^\alpha} \leq 1$. Thus

$$\begin{aligned} & \sum_{\alpha \in \mathbb{N}_0^{k_0}} \frac{1}{\left(\frac{R}{R-1}\right)^{k_0}} \sum_{j=1}^\infty |\beta_{j,\alpha}|^{p'} \left(\frac{1}{R^\alpha}\right)^{p'} \left(\frac{|\lambda|}{|z_{0,\alpha}|}\right)^{\alpha p'} \\ & \leq \frac{1}{\left(\frac{R}{R-1}\right)^{k_0}} \sum_{\alpha \in \mathbb{N}_0^{k_0}} \sum_{j=1}^\infty |\beta_{j,\alpha}|^{p'} \left(\frac{1}{R^\alpha}\right)^{p'} \leq \frac{1}{\left(\frac{R}{R-1}\right)^{k_0}} \sum_{\alpha \in \mathbb{N}_0^{k_0}} \left(\frac{1}{R^\alpha}\right)^{p'} \\ & \leq \frac{1}{\left(\frac{R}{R-1}\right)^{k_0}} \sum_{\alpha \in \mathbb{N}_0^{k_0}} \frac{1}{R^\alpha} = \frac{1}{\left(\frac{R}{R-1}\right)^{k_0}} \left(\sum_{h=0}^\infty \frac{1}{R^h}\right)^{k_0} = 1, \end{aligned}$$

i.e., $\left(\frac{1}{\left(\frac{R}{R-1}\right)^{\frac{k_0}{p'}}} \beta_{j,\alpha} \frac{1}{R^\alpha} \left(\frac{\lambda}{z_{0,\alpha}}\right)^\alpha\right)_{(j,\alpha) \in \mathbb{N} \times \mathbb{N}_0^{k_0}} \in \overline{B}_{\ell_{p'}}$.

Hence by (7)

$$f(K) \subset p\text{-co}\left\{\left(\left(\frac{R}{R-1}\right)^{\frac{k_0}{p'}} y_{j,\alpha}\right)_{(j,\alpha) \in \mathbb{N} \times \mathbb{N}_0^{k_0}}\right\}$$

and $\left(\left(\frac{R}{R-1}\right)^{\frac{k_0}{p'}} y_{j,\alpha}\right)_{(j,\alpha) \in \mathbb{N} \times \mathbb{N}_0^{k_0}} \in \ell_p(Y)$, i.e., $f(K)$ is p -compact. $\square$

Corollary 3.6. *Let X and Y be Banach spaces and let $f \in \mathcal{H}(X; Y)$. If $K \subset X$ is a p -compact set, then the set $f(K) \subset Y$ is a p -compact set.*

Remark 3.7. Let U be an open balanced subset of a complex Banach space X (or even U an open balanced and convex set). We do not know if given a p -compact set K contained in U , there necessarily exists a sequence $(x_k)_{k=1}^\infty$ contained in $\ell_p(X) \cap U$ such that $K \subset p\text{-co}\{(x_k)_{k=1}^\infty\}$. If this were the case, then by Theorem 3.5 the image of any p -compact set under any holomorphic function on U would again be p -compact.

4. LOCALIZATION OF p -COMPACT SETS OF SYMMETRIC TENSOR PRODUCTS OF A BANACH SPACE

We now give a result related to the localization of p -compact sets of symmetric tensor products of a Banach space. Let $\widehat{\bigotimes_{s,m,\pi} X}$ denote the complete m -fold symmetric tensor product of X with the projective norm topology π , and let $q_m : X \rightarrow \widehat{\bigotimes_{s,m,\pi} X}$ be the m -homogeneous polynomial defined by $q_m(x) := x \otimes \cdots \otimes x$, for every $x \in X$. In his thesis R. Ryan [14] showed that given a compact subset K of $\widehat{\bigotimes_{s,m,\pi} X}$, there is a sequence $(z_l)_{l=1}^\infty \in c_0(X)$ such that $K \subset \overline{\Gamma\{(q_m(z_l))_{l=1}^\infty\}}$ (see [14, Proposition 2.5]), which is related to the localization of compact sets in $\widehat{\bigotimes_{s,m,\pi} X}$. In the p -compact case we have the following similar result which extends Ryan's work beyond the $p = \infty$ case.

Proposition 4.1. *Let X be a Banach space and let $m \in \mathbb{N}$. Then, given a p -compact subset K of $\widehat{\bigotimes_{s,m,\pi} X}$, there is a sequence $(z_l)_{l=1}^\infty \in \ell_{mp}(X)$ such that $K \subset p\text{-co}\{(q_m(z_l))_{l=1}^\infty\}$.*

Proof. (a) Let $K \subset \widehat{\bigotimes_{s,m,\pi} X}$ be a p -compact subset. Then, there is a sequence $(\tau_k)_{k=1}^\infty \in \ell_p(\widehat{\bigotimes_{s,m,\pi} X})$ such that $K \subset \left\{\sum_{k=1}^\infty \lambda_k \tau_k : (\lambda_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}\right\}$. Let $\tau \in K$ be any element. Then, there is a sequence $(\lambda_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}$ such that $\tau = \sum_{k=1}^\infty \lambda_k \tau_k$. By [14, Proposition 2.5], for each $\tau_k \in \widehat{\bigotimes_{s,m,\pi} X}$ we can find a representation of the form

$$\tau_k = \sum_{i=1}^\infty \lambda_{k,i} \underbrace{x_{k,i} \otimes \cdots \otimes x_{k,i}}_{m \text{ times}}, \quad x_{k,i} \in X, \lambda_{k,i} \in \mathbb{C},$$

with $\|x_{k,i}\| = 1$ for every $i \in \mathbb{N}$, such that

$$\mu_k := \sum_{i=1}^\infty |\lambda_{k,i}| \leq \pi(\tau_k) + \frac{1}{2^k}.$$

(See, also, [15, Proposition 2.8] and the formulas given for π in [15, page 22].) Now, for each k using the corresponding representation and writing

$$\begin{aligned}\tau &= \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{\infty} \frac{\lambda_{k,i}}{\mu_k} \mu_k^{1/m} x_{k,i} \otimes \cdots \otimes \mu_k^{1/m} x_{k,i} \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_k \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/p'} \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \otimes \cdots \otimes \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i},\end{aligned}$$

we see that

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left| \lambda_k \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/p'} \right|^{p'} &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |\lambda_k|^{p'} \left| \frac{\lambda_{k,i}}{\mu_k} \right| \\ &\leq \sum_{k=1}^{\infty} |\lambda_k|^{p'} \sum_{i=1}^{\infty} \left| \frac{\lambda_{k,i}}{\mu_k} \right| \leq \sum_{k=1}^{\infty} |\lambda_k|^{p'} \leq 1,\end{aligned}$$

and that

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left\| \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \right\|^{mp} &\leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left| \frac{\lambda_{k,i}}{\mu_k} \right| \mu_k^p \|x_{k,i}\|^{mp} \\ &\leq \sum_{k=1}^{\infty} \mu_k^p \sum_{i=1}^{\infty} \left| \frac{\lambda_{k,i}}{\mu_k} \right| \leq \sum_{k=1}^{\infty} \mu_k^p \leq \sum_{k=1}^{\infty} \left(\pi(\tau_k) + \frac{1}{2^k} \right)^p < \infty.\end{aligned}$$

Therefore,

$$\left(\lambda_k \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/p'} \right)_{(k,i) \in \mathbb{N} \times \mathbb{N}} \in \overline{B}_{\ell_{p'}}, \quad \left(\left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \right)_{(k,i) \in \mathbb{N} \times \mathbb{N}} \in \ell_{mp}(X),$$

and also

$$\left(\left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \otimes \cdots \otimes \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \right)_{(k,i) \in \mathbb{N} \times \mathbb{N}} \in \ell_p(\widehat{\bigotimes_{s,m,\pi} X}).$$

Therefore, since the double series considered above are all absolutely convergent, by a reordering we obtain sequences

$$\begin{aligned}(\gamma_l)_{l=1}^{\infty} &:= \left(\lambda_k \left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/p'} \right)_{(k,i) \in \mathbb{N} \times \mathbb{N}} \in \overline{B}_{\ell_{p'}}, \\ (z_l)_{l=1}^{\infty} &:= \left(\left(\frac{\lambda_{k,i}}{\mu_k} \right)^{1/(mp)} \mu_k^{1/m} x_{k,i} \right)_{(k,i) \in \mathbb{N} \times \mathbb{N}} \in \ell_{mp}(X).\end{aligned}$$

Thus we have shown the existence of a sequence $(z_l)_{l=1}^{\infty} \in \ell_{mp}(X)$ such that

$$K \subset p\text{-co}\{(z_l \otimes \cdots \otimes z_l)_{l=1}^{\infty}\} = p\text{-co}\{(q_m(z_l))_{l=1}^{\infty}\},$$

as we desired. $\square$

In the study of the images of p -compact sets by polynomials, note that the image of a p -summable sequence by an m -homogeneous polynomial is p/m -summable. This fact naturally gives rise to the following question: Must the image of a p -compact set under an m -homogeneous polynomial be p/m -compact? We will show

below that this is not always the case. More precisely, we shall give an example of an m -homogeneous polynomial having the following properties:

(i) The image of a certain p -summable sequence by this polynomial is zero, although

(ii) The image of a p -compact set generated by a certain sequence under this polynomial is not even q -compact for any $1 \leq q < p$.

Example 4.2. For every $m \in \mathbb{N}$, there exist an m -homogeneous continuous polynomial $P : \ell_p \longrightarrow \ell_p$ and a p -compact subset K of ℓ_p such that $P(K)$ is (p -compact but) not q -compact for any $1 \leq q < p$.

Proof. Let us define an m -homogeneous polynomial $P : \ell_p \longrightarrow \ell_p$ by

$$P(x) := x_1^{m-1} \sum_{j=1}^{\infty} x_{2j} e_{2j}, \quad x = (x_n)_{n=1}^{\infty} \in \ell_p,$$

where $(e_h)_{h=1}^{\infty}$ is the canonical basis of ℓ_p . Note that, for every $h \in \mathbb{N}$, $P(e_h) = 0$ and

$$(*) \quad P(e_1 + a_h e_{2h}) = a_h e_{2h}, \quad \text{for every } h \in \mathbb{N} \text{ and } a_h \in \mathbb{C}.$$

At this point note that the image of a sequence $\{b_n e_n\}_{n=1}^{\infty}$ with $(b_n)_{n=1}^{\infty} \in \ell_p$, which generates a p -compact set, under P is zero. But now we will see that the image of a p -compact set generated by a certain p -summable sequence can have a quite different nature.

If for any $p \geq 1$ we let $a_h := \frac{1}{\sqrt[p]{h(\log h)^2}}$, $h = 3, 4, \dots$, then $(a_h)_{h=3}^{\infty} \in \ell_p$, whereas $(a_h)_{h=3}^{\infty} \notin \ell_q$ for every $0 < q < p$. Indeed, if $\sum_{n=1}^{\infty} a_h^q < \infty$, then it would follow that $\lim_{h \rightarrow \infty} h a_h^q = 0$. However $\lim_{h \rightarrow \infty} \frac{h}{h^{q/p} (\log h)^{2q}} = \infty$. Now, as the sequence $(2e_1, \frac{2}{\sqrt[p]{h(\log h)^2}} e_{2h} : h = 3, 4, \dots)$ belongs to $\ell_p(\ell_p)$, by definition

$$K := p\text{-co} \left\{ 2e_1, \frac{2}{\sqrt[p]{h(\log h)^2}} e_{2h} : h = 3, 4, \dots \right\}$$

is p -compact. By Lemma 3.2 we know that $P(K)$ is p -compact. We claim that $P(K)$ is not q -compact for any $1 \leq q < p$. Indeed, the sequence $(e_1 + \frac{2}{\sqrt[p]{h(\log h)^2}} e_{2h} : h = 3, 4, \dots)$ is included in K and by $(*)$ we have that

$$P\left(e_1 + \frac{1}{\sqrt[p]{h(\log h)^2}} e_{2h}\right) = \frac{1}{\sqrt[p]{h(\log h)^2}} e_{2h}, \quad \text{for every } h = 3, 4, \dots$$

If we define a sequence $(\alpha_n)_{n=1}^{\infty}$ by

$$\alpha_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{\sqrt[p]{\frac{n+4}{2}(\log \frac{n+4}{2})^2}} & \text{if } n \text{ is even,} \end{cases}$$

we have that $(\alpha_n)_{n=1}^{\infty} \in \ell_p \setminus \ell_q$ for every $q < p$, so that by the proof of [1, Example 1] we see that the image $P\left(\left(e_1 + \frac{1}{\sqrt[p]{h(\log h)^2}} e_{2h}\right)_{h=3}^{\infty}\right) = \{(\alpha_n e_n)_{n=1}^{\infty}\}$ is not q -compact for any $q < p$. Hence, since $P(K) \supset P\left(\left(e_1 + \frac{1}{\sqrt[p]{h(\log h)^2}} e_{2h}\right)_{h=3}^{\infty}\right)$, we conclude that $P(K)$ cannot be q -compact for any $q < p$. $\square$

Now we consider the question of the pulling back of p -compact subsets of the image. That is, if $P : X \longrightarrow Y$ is an m -homogeneous polynomial and $K \subset P(X)$ is

a p -compact set, does there exist a p -compact subset $L \subset X$ such that $K \subset P(L)$? The following shows that the answer in general is negative.

Proposition 4.3. *Let $m \in \mathbb{N}$, $m \geq 2$, and let $q_m : \ell_p \longrightarrow \widehat{\bigotimes_{s,m,\pi} \ell_p}$ be defined by $q_m(x) := \underbrace{x \otimes \cdots \otimes x}_{m \text{ times}}$, $x \in \ell_p$. There is a p -compact subset K of $\widehat{\bigotimes_{s,m,\pi} \ell_p}$ such that a set $L \subset \ell_p$ is r -compact and $K \subset q_m(L)$ if and only if $r \geq pm$.*

Proof. Let us consider a sequence of strictly positive numbers $(a_h)_{h=1}^\infty \in \ell_p \setminus \ell_q$, where $0 < q < p < \infty$, and take

$$K := \{a_h \underbrace{e_h \otimes \cdots \otimes e_h}_{m \text{ times}} : h \in \mathbb{N}\} \cup \{0\},$$

where $(e_h)_{h=1}^\infty$ is the canonical basis of ℓ_p . Since $\pi(a_h e_h \otimes \cdots \otimes e_h) = a_h$, K is p -compact in $\widehat{\bigotimes_{s,m,\pi} \ell_p}$.

We claim that if a subset $L \subset \ell_p$ is r -compact and $K \subset q_m(L)$, then $r \geq pm$. To see this, note that $q_m^{-1}(a_h e_h \otimes \cdots \otimes e_h) = \{\lambda \sqrt[m]{a_h} e_h : \lambda \in \mathbb{C}, \lambda^m = 1\}$. Hence, since for every $h \in \mathbb{N}$, $q_m(\sqrt[m]{a_h} e_h) = a_h e_h \otimes \cdots \otimes e_h$, there is a $\lambda_h \in \mathbb{C}$ with $\lambda_h^m = 1$ such that $\lambda_h \sqrt[m]{a_h} e_h \in L$. Since we can take L to be balanced, it follows that $\sqrt[m]{a_h} e_h \in L$, for every $h \in \mathbb{N}$. But, since $(\sqrt[m]{a_h})_{h=1}^\infty \in \ell_{pm} \setminus \ell_k$, $0 < k < pm$, by a modification of the proof of [1, Example 1] we see that $(\sqrt[m]{a_h} e_h)_{h=1}^\infty \cup \{0\}$ cannot be k -compact for any $0 < k < pm$, so that the set L cannot be k -compact for $0 < k < pm$. Therefore $r \geq pm$.

Now, the rest of the proof is a consequence of Proposition 4.1. $\square$

We end this section with the following observations.

Remarks 4.4. (a) Let $m \in \mathbb{N}$ with $m \leq p$, and let K be a subset of $\widehat{\bigotimes_{s,m,\pi} X}$. If there is a sequence $(a_n)_{n=1}^\infty \in \ell_p(X)$ such that $K \subset r\text{-co}\{(a_n \otimes \cdots \otimes a_n)_{n=1}^\infty\}$ for any $r \geq 1$, then K is a 1-compact subset of $\widehat{\bigotimes_{s,m,\pi} X}$.

(b) Let K be a subset of $\widehat{\bigotimes_{s,m,\pi} X}$. If there is a sequence $(a_n)_{n=1}^\infty \in \ell_p(X)$, $p \geq 1$, such that $K \subset r\text{-co}\{(a_n \otimes \cdots \otimes a_n)_{n=1}^\infty\}$ for any $r \geq p$, then K is a p -compact subset of $\widehat{\bigotimes_{s,m,\pi} X}$.

5. ON p -COMPACT LINEAR OPERATORS BETWEEN ℓ_p SPACES

The definition of p -compact sets leads naturally to the notion of p -compact operators which is introduced in [16].

Definition 5.1. An operator $T \in \mathcal{L}(X; Y)$ is called p -compact if there exists a sequence $(y_n)_{n=1}^\infty \in \ell_p(Y)$ ($(y_n)_{n=1}^\infty \in c_0(Y)$ if $p = \infty$) such that

$$T(\overline{B}_X) \subset \left\{ \sum_{n=1}^\infty \alpha_n y_n : (\alpha_n)_{n=1}^\infty \in \overline{B}_{\ell_{p'}} \right\}.$$

Let $\mathcal{K}_p(X; Y)$ denote the subspace of all p -compact operators of $\mathcal{L}(X; Y)$. Clearly $\mathcal{K}_p(X; Y) \subset \mathcal{K}(X; Y)$. Note also that for $1 \leq p < q \leq \infty$ we have $\mathcal{K}_p(X; Y) \subset \mathcal{K}_q(X; Y)$ (see [16, Proposition 4.3]) and below we will see that this inclusion is in general strict.

On the other hand if $1 < p < q < \infty$, by Pitt's Theorem we know that $\mathcal{L}(\ell_q; \ell_p) = \mathcal{K}(\ell_q; \ell_p)$, that is, every continuous linear operator from ℓ_q into ℓ_p is compact. But

the following proposition, which is a modification of [1, Example 1], shows that in the p -compact case there are linear continuous operators from ℓ_q to ℓ_p , $1 \leq p < q$, which are not p -compact. More precisely we have the following.

Proposition 5.2. *There is a linear operator $T \in \mathcal{L}(\ell_q; \ell_p)$ which is not r -compact for $1 \leq r \leq p$, if*

- (a) $1 \leq p \leq q \leq pq'$, or
- (b) $1 < p \leq q \leq \min\{p^2, pp'\}$.

Furthermore, in each case the corresponding operator is q -compact if, in addition, $p < q \leq q'$.

Proof. We shall only prove the case (a), the other case being similar.

First, note that if $1 \leq p = q$, then the identity operator from ℓ_p to ℓ_p is not compact, so that it is not r -compact for any $1 \leq r$.

Now, let $1 \leq p < q \leq pq'$ and let $T: \ell_q \rightarrow \ell_p$ be defined by

$$T((x_n)_{n=1}^\infty) := (a_n x_n)_{n=1}^\infty, \quad (x_n)_{n=1}^\infty \in \ell_q,$$

where $(a_n)_{n=1}^\infty \in \ell_q \setminus \ell_p$. As $pq \geq q$ we have that $(a_n)_{n=1}^\infty \in \ell_{pq}$, i.e. $(|a_n|^p)_{n=1}^\infty \in \ell_q$. On the other hand, by hypothesis, $pq' \geq q$, thus $(x_n)_{n=1}^\infty \in \ell_{pq'}$, i.e. $(|x_n|^p)_{n=1}^\infty \in \ell_{q'}$. Hence, by Hölder's inequality, $(|a_n x_n|^p)_{n=1}^\infty \in \ell_1$. Therefore T is well defined.

It is sufficient to prove the claim for $r = p$. Suppose, for a contradiction, that $T \in \mathcal{K}_p(\ell_q; \ell_p)$. Then there is a sequence $(z_k)_{k=1}^\infty \in \ell_p(\ell_p)$, each $z_k = \sum_{l=1}^\infty z_l^k e_l \in \ell_p$, such that

$$T(\overline{B}_{\ell_q}) \subset p\text{-co}\{(z_k)_{k=1}^\infty\} = \left\{ \sum_{k=1}^\infty \alpha_k z_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}} \right\},$$

where $(e_l)_{l=1}^\infty$ is the canonical basis. Hence, in particular, we have that

$$\{(T(e_n))_{n=1}^\infty\} = \{(a_n e_n)_{n=1}^\infty\} \subset \left\{ \sum_{k=1}^\infty \alpha_k z_k : (\alpha_k)_{k=1}^\infty \in \overline{B}_{\ell_{p'}} \right\}.$$

Here we actually show that this last inclusion cannot be true; that is, the set $\{(a_n e_n)_{n=1}^\infty\}$ cannot be p -compact (note that this set is q -compact since it belongs to $\ell_q(\ell_p)$). To see this, let us fix any $n \in \mathbb{N}$. Then there is a sequence $(\lambda_k^n)_{k=1}^\infty \in \overline{B}_{\ell_{p'}}$ so that

$$a_n e_n = \sum_{k=1}^\infty \lambda_k^n z_k = \sum_{k=1}^\infty \sum_{l=1}^\infty \lambda_k^n z_l^k e_l,$$

from which it follows that $a_n = \sum_{k=1}^\infty \lambda_k^n z_n^k$ and $\sum_{k=1}^\infty \lambda_k^n z_l^k = 0$ if $l \neq n$. Now, by Hölder's inequality we get

$$\begin{aligned} \sum_{n=1}^\infty |a_n|^p &= \sum_{n=1}^\infty \left| \sum_{k=1}^\infty \lambda_k^n z_n^k \right|^p \\ &\leq \sum_{n=1}^\infty \left(\sum_{k=1}^\infty |\lambda_k^n|^{p'} \right)^{p/p'} \sum_{k=1}^\infty |z_n^k|^p \leq \sum_{n=1}^\infty \sum_{k=1}^\infty |z_n^k|^p < \infty, \end{aligned}$$

which is a contradiction since $(a_n)_{n=1}^\infty \notin \ell_p$. Hence the set $\{(a_n e_n)_{n=1}^\infty\}$ is not p -compact, so T is not a p -compact operator. Consequently, T is not an r -compact operator for any $1 \leq r \leq p$.

Finally observe that T is q -compact if, in addition, $q' \geq q > p$. □

The above proposition can be seen as a partial complement to a recent result of J. M. Delgado and C. Piñeiro [6, Corollary 3.8]: (i) $\mathcal{L}(\ell_r; \ell_1) = \mathcal{K}_2(\ell_r; \ell_1)$ for every $r \geq 2$ and (ii) if $p > 2$, then $\mathcal{L}(\ell_r; \ell_1) = \mathcal{K}_p(\ell_r; \ell_1)$ for every $r > p'$. Actually, Proposition 5.2 shows, in particular, that $\mathcal{L}(\ell_2; \ell_1) \neq \mathcal{K}_1(\ell_2, \ell_1)$, which is a case that is not covered by the results given in [6].

Also we note that, in [6, Proposition 3.10], Delgado and Piñeiro assert that if $1 \leq p$, $1 < q$ and $\mathcal{L}(X; \ell_q) = \mathcal{K}_p(X; \ell_q)$, then X must be finite dimensional. In particular, this shows that $\mathcal{L}(c_0; \ell_q) \neq \mathcal{K}_p(c_0; \ell_q)$ when $1 \leq p$, $1 < q$. We do not know of any concrete example of a non- p -compact linear operator for the case $X = c_0$.

The sequence $(x_n)_{n=1}^\infty$ in the definition of p -compact set is not unique, so following [11] it is natural to consider

$$\mathfrak{m}_p(K; X) := \inf\{\|(x_n)_{n=1}^\infty\|_p : K \subset p\text{-co}\{(x_n)_{n=1}^\infty\}\}.$$

If $K \subset X$ is a p -compact set, then it follows from Hölder's inequality that $\|x\| \leq \mathfrak{m}_p(K; X)$, for every $x \in K$. Hence, as noted in [11], in this sense $\mathfrak{m}_p(K; X)$ measures the size of K as a p -compact subset of X .

In [8, Proposition 3.15] it is shown that the norm k_p on $\mathcal{K}_p(X; Y)$, defined by

$$k_p(T) := \inf \left\{ \left(\sum_{n=1}^{\infty} \|y_n\|^p \right)^{1/p} : (y_n)_{n=1}^\infty \in \ell_p(Y) \text{ and } T(\overline{B}_X) \subset p\text{-co}\{(y_n)_{n=1}^\infty\} \right\},$$

coincides with the ideal norm defined by Sinha and Karn in [16]. It is easy to see that $\|T\| \leq k_p(T)$ for any $T \in \mathcal{K}_p(X; Y)$. Note also that, according to the definitions, $\mathfrak{m}_p(T(\overline{B}_X); Y) = k_p(T)$. Below we will see that the size of p -compactness of the image of the unit ball can be arbitrarily large. That is, given any arbitrarily large positive number R , we can find a p -compact linear operator from ℓ_q to ℓ_p , with norm ≤ 1 , such that its ideal norm is bigger than R .

Proposition 5.3. *Given any number $R > 0$ there is an operator $T \in \mathcal{K}_p(\ell_q; \ell_p)$ with $\|T\| \leq 1$, but $k_p(T) > R$, where $1 \leq q \leq p < \infty$.*

Proof. Let $1 \leq q \leq p < \infty$ and let $R > 0$ be any number. Let us take any element $(a_n)_{n=1}^\infty \in \overline{B}_{\ell_\infty} \cap \ell_p$ such that $\|(a_n)_{n=1}^\infty\|_p > R$. Let $T : \ell_q \rightarrow \ell_p$ be defined by

$$T((\beta_n)_{n=1}^\infty) := (a_n \beta_n)_{n=1}^\infty = \sum_{n=1}^{\infty} \beta_n (a_n e_n), \quad (\beta_n)_{n=1}^\infty \in \ell_q.$$

Now, if we take $(\beta_n)_{n=1}^\infty \in \ell_q$ with $\|(\beta_n)_{n=1}^\infty\|_q \leq 1$, then it follows that

$$\|T((\beta_n)_{n=1}^\infty)\|_p = \left(\sum_{n=1}^{\infty} |a_n \beta_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |\beta_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |\beta_n|^q \right)^{1/q} \leq 1$$

which shows that $\|T\| \leq 1$. Note that $(a_n e_n)_{n=1}^\infty \in \ell_p(\ell_p)$ and that

$$T(\overline{B}_{\ell_q}) = \left\{ \sum_{n=1}^{\infty} \beta_n (a_n e_n) : (\beta_n)_{n=1}^\infty \in \overline{B}_{\ell_q} \right\}$$

so that T is p -compact.

Now, if we take a sequence $(x_n)_{n=1}^\infty \in \ell_p(\ell_p)$ such that $T(\overline{B}_{\ell_q}) \subset p\text{-co}\{(x_n)_{n=1}^\infty\}$, hence in particular $\{T(e_n) : n \in \mathbb{N}\} \subset p\text{-co}\{(x_n)_{n=1}^\infty\}$, then the calculation in the

proof of Proposition 5.2 shows that

$$\sum_{n=1}^{\infty} |a_n|^p \leq \sum_{n=1}^{\infty} \|x_n\|_p^p = \|(x_n)_{n=1}^{\infty}\|_p^p$$

and hence $\|(a_n)_{n=1}^{\infty}\|_p \leq \|(x_n)_{n=1}^{\infty}\|_p$. Therefore $k_p(T) \geq \|(a_n)_{n=1}^{\infty}\|_p > R$. $\square$

The above result also shows the difficulty of getting a factorization of p -compact operators of norm ≤ 1 through p -compact operators of norm ≤ 1 (see the factorization results on p -compact operators in [4] and [16]).

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