ON ANALYTIC PROPERTIES OF DEFORMATION SPACES OF KLEINIAN GROUPS

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ABSTRACT. Let G_0 be a non-elementary Kleinian group. We consider the deformation space of $D(G_0)$, the space of quasiconformal deformations of G_0 , and its complex analytic properties. We show some analytic structures of $D(G_0)$ which are improvements of results by Bers, Kra, Maskit and McMullen. In particular, we clarify that the structures for Kleinian groups with non-simply connected components are different from those for Kleinian groups without non-simply connected components.

1. INTRODUCTION AND MAIN RESULTS

A discrete subgroup G of $\mathrm{PSL}(2, \mathbb{C})$ is called a *Kleinian* group. It is well known that every Kleinian group G acts on the hyperbolic 3-space $\mathbb{H}^3$ properly discontinuously and it produces a hyperbolic 3-manifold $\mathbb{H}^3/G$. Hence, the deformation of Kleinian groups has its own interest from the viewpoint of the moduli of hyperbolic 3-manifolds as well as of complex analysis. Indeed, a series of works by Bers, Maskit and Kra ([2], [14]–[16], [20]) clarified some fundamental properties of quasiconformal deformation spaces of Kleinian groups. In this paper, we consider a global theory of the deformation of Kleinian groups. In particular, we consider complex analytic properties of deformation spaces of Kleinian groups which are improvements of the above works. The main tools in this article are holomorphic motions and Teichmüller theory.

Since every element of $\mathrm{PSL}(2, \mathbb{C})$ is represented by a 2×2 matrix with complex entries, we may consider a natural complex structure of spaces of Kleinian groups. Before stating our results, we begin with the holomorphic family of Kleinian groups. Let V be a complex manifold with a base point x_0 and G_0 be a Kleinian group. We denote by $\mathrm{Hom}(G_0, \mathrm{PSL}(2, \mathbb{C}))$, the set of homomorphisms from G_0 to $\mathrm{PSL}(2, \mathbb{C})$. We also denote by $\widetilde{\mathrm{Hom}}(G_0, \mathrm{PSL}(2, \mathbb{C}))$ the quotient space of $\mathrm{Hom}(G_0, \mathrm{PSL}(2, \mathbb{C}))$ by conjugation of $\mathrm{PSL}(2, \mathbb{C})$. Both spaces admit the natural complex structures coming from entries of matrices as above.

Suppose that a family of $\{\theta_x\}_{x \in V}$ of $\widetilde{\mathrm{Hom}}(G_0, \mathrm{PSL}(2, \mathbb{C}))$ satisfies the following conditions:

- (1) $\theta_{x_0} = id$.
- (2) For each $g \in G_0$, the map $V \ni z \mapsto \theta_z(g) \in \mathrm{PSL}(2, \mathbb{C})$ is holomorphic on V .

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Then, we call the triple $\mathcal{G} := (G_0, V, \{\theta_x\}_{x \in V})$ a *holomorphic family* of the Kleinian group G_0 over V . If θ_x is isomorphic for every $x \in V$, the family is called a holomorphic family of isomorphisms of G_0 over V . A holomorphic family $\mathcal{G}$ of G_0 is *order preserving* if $\theta_x(g)$ is of infinite order or the identity for every $g \in G_0$ of infinite order.

Note. In the above definition of holomorphic families, condition (2) means that there exist a neighbourhood U_z of $z \in V$ and representatives $\rho_x \in \text{Hom}(G_0, \text{PSL}(2, \mathbb{C}))$ of θ_x for $x \in U_z$ such that the map $U_z \ni x \mapsto \rho_x(g) \in \text{PSL}(2, \mathbb{C})$ is holomorphic for each $g \in G_0$.

Among results on the deformation of Kleinian groups, one important result is a theorem by L. Bers [2], which is on a local theory of the deformation of Kleinian groups. First of all, we recall the theorem of Bers [2] on holomorphic families over the unit disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ (See Corollary 2 in [24] for an infinite version of the theorem.)

Theorem 1.1. *Let G_0 be a non-elementary Kleinian group and $\{\theta_x\}_{x \in \Delta}$ be a holomorphic family of isomorphisms of G_0 over the unit disk Δ . Suppose that $\theta_x(G_0)$ is discrete for any $x \in \Delta$ and $\theta_x(g)$ is parabolic whenever $g \in G$ is parabolic. Then, for each $x \in \Delta$ there exists a quasiconformal mapping w_x of the Riemann sphere $\hat{\mathbb{C}}$ to itself such that it induces the isomorphism θ_x .*

We now present the main theorems in our paper; see the following sections for the relevant definitions.

In Theorem 1.1, the assumption that the holomorphic family consists of *isomorphisms* defined on the unit disk Δ , that is, simply connected, is crucial. If Kleinian groups satisfy a certain condition, we can relax the assumption as follows:

Theorem I. *Let K be an AB-removable compact subset of the unit disk Δ in $\mathbb{C}$. Suppose that G_0 is a Kleinian group whose components are all simply connected. Then, any order preserving holomorphic family $\mathcal{G} = (G_0, \Delta_K, \{\theta_x\}_{x \in \Delta_K})$ of G_0 over $\Delta_K := \Delta \setminus K$ can be extended to a holomorphic family $\tilde{\mathcal{G}} = (G_0, \Delta, \{\tilde{\theta}_x\}_{x \in \Delta})$ of G_0 over Δ .*

Immediately, we have the following corollary which is a generalization of the disk convexity of $\mathcal{QF}(\Gamma_0)$, the space of quasi-Fuchsian groups of a Fuchsian group Γ_0 , by McMullen [23].

Corollary 1.1. *Suppose that G_0 is a Kleinian group whose components are all simply connected. Let K be an AB-removable compact subset of Δ and φ be a holomorphic map from $\Delta_K \cup \partial\Delta$ to $\widehat{\text{Hom}}(G_0, \text{PSL}(2, \mathbb{C}))$ so that $\varphi(z)(\gamma)$ is parabolic whenever $\gamma \in G_0$ is parabolic. If $\varphi(\partial\Delta) \subset D(G_0)$, then φ can be extended to a holomorphic map $\tilde{\varphi}$ on Δ and $\tilde{\varphi}(\Delta) \subset D(G_0)$.*

Remark 1.1. (1) In Theorem I and Corollary 1.1, we do not assume that Kleinian groups are finitely generated.

- (2) The statement of the corollary does not hold if we replace Δ_K by an annulus. Namely, for a Fuchsian group Γ_0 , we can construct a holomorphic map $\varphi : U \rightarrow \mathcal{QF}(\Gamma_0)$ from any given annulus $U = \{z \in \Delta \mid 0 < r < |z| < 1\}$ with $r \in (0, 1)$ such that $\varphi(\partial U) \subset \mathcal{QF}(\Gamma_0)$, $\varphi(z)(\gamma)$ ($z \in U$) is parabolic whenever $\gamma \in \Gamma_0$ is parabolic but φ cannot be extended to a holomorphic map on the unit disk Δ (see Remark 4.1).

In Theorem I, we cannot drop the assumption that the Kleinian group has only simply connected components. In fact, we may show that Theorem I is not true if the Kleinian group has a non-simply connected component.

Theorem II. *Let G_0 be a finitely generated Kleinian group. Suppose that there exists a non-simply connected component of G_0 . Then, there exists a holomorphic family $\mathcal{G}$ of isomorphisms of G_0 over the punctured disk $\Delta^* := \{0 < |z| < 1\}$ such that it cannot be extended to a holomorphic family of G_0 over Δ .*

While Theorem II has its own interest, the theorem also helps us to show the following complex analytic properties of deformation spaces of Kleinian groups.

Theorem III. *Let G_0 be a finitely generated Kleinian group. Then*

- (1) *The Carathéodory pseudo distance on $D(G_0)$ is a distance.*
- (2) *If every component of G_0 is simply connected, then in $D(G_0)$ the Carathéodory distance is complete. In particular, $D(G_0)$ is H^∞ -convex.*
- (3) *If there exists a non-simply connected component of G_0 , then $D(G_0)$ is not H^∞ -convex. In particular, the Carathéodory distance is not complete in $D(G_0)$.*

It is well known that in Teichmüller space, the Kobayashi distance and the Teichmüller distance are the same. We show that it also holds for our deformation spaces.

Theorem IV. *Let G_0 be a Kleinian group. Then the Teichmüller distance $d_T^{D(G_0)}$ of $D(G_0)$ is equal to the Kobayashi distance $d_K^{D(G_0)}$ of $D(G_0)$. In particular, the Kobayashi distance of $D(G_0)$ is complete.*

The Kobayashi distance of $D(G_0)$ is complete for every Kleinian group G_0 while the Carathéodory distance is not complete in $D(G_0)$ if the Kleinian group G_0 has a non-simply connected component. The following is an immediate consequence.

Corollary 1.2. *Let G_0 be a finitely generated Kleinian group with a non-simply connected component. Then, the Kobayashi distance does not coincide with the Carathéodory distance in $D(G_0)$.*

2. FUNDAMENTAL NOTIONS AND TERMINOLOGIES

2.1. Holomorphic convexities. We begin with the holomorphic convexity of complex manifolds. Let M be a (connected) complex manifold and $\mathcal{O}(M)$ the space of holomorphic functions on M . For a subset $\mathcal{O}$ of $\mathcal{O}(M)$, M is called $\mathcal{O}$ -convex or *convex* for the space $\mathcal{O}$ if for any compact subset K of M , the $\mathcal{O}$ -convex hull $\hat{K}_{\mathcal{O}}$ of K is compact in M , where

$$\hat{K}_{\mathcal{O}} = \{p \in M \mid |f(p)| \leq \max_{z \in K} |f(z)| \text{ for all } f \in \mathcal{O}\}.$$

A complex manifold M is called *holomorphically convex* if it is $\mathcal{O}(M)$ -convex. A domain in $\mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy (cf. [8]). If $\mathcal{O}$ is $H^\infty(M)$, the space of bounded holomorphic functions on M , and if M is $H^\infty(M)$ -convex, then we call the manifold H^∞ -convex. Obviously, if M is H^∞ -convex, then it is holomorphically convex. It is known that if the

Carathéodory distance c_M of M is complete, then M is H^∞ -convex. Here, the Carathéodory (pseudo) distance c_M of M is defined by

$$(2.1) \quad c_M(x, y) = \sup_{f \in H_1^\infty(M)} \rho_\Delta(f(x), f(y))$$

for $x, y \in M$, where $H_1^\infty(M)$ is the set of holomorphic functions f on M with $\|f\|_\infty < 1$ and ρ_Δ is the Poincaré distance of the unit disk Δ . In general, the Carathéodory distance is a pseudo distance. Namely, $c_M(x, y) = 0$ does not necessarily imply $x = y$. The Carathéodory distance is not greater than the Kobayashi distance.

If M is a domain in $\mathbb{C}^n$ and it is convex for the space of polynomials in $\mathbb{C}^n$, then M is called *polynomial convex*. It is also easy to see that if M is polynomial convex, then it is H^∞ -convex.

2.2. Teichmüller spaces. In the following, we shall give a brief discussion on Teichmüller spaces. The reader may refer to [10], [11] or [17] for further details.

Let X_0 be a Riemann surface of type (g, n) with $2g - 2 + n > 0$. The Teichmüller space $T(X_0)$ of X_0 is the set of Teichmüller equivalence classes $[X, f]$ of marked Riemann surfaces (X, f) , where a marked Riemann surface is a pair (X, f) of a Riemann surface X and a quasiconformal homeomorphism f from X_0 onto X , and two marked Riemann surfaces (X, f) , (X', f') are Teichmüller equivalent if there exists a conformal mapping h from X to X' which is homotopic to $f' \circ f^{-1}$.

The Teichmüller space $T(X_0)$ admits a natural complex structure and it is realized as a contractible bounded domain in $\mathbb{C}^{3g-3+n}$. The Teichmüller space $T(X_0)$ also has a complete distance d_T called *the Teichmüller distance* defined by

$$d_T([X, f], [X', f']) = \inf_g \log K(g),$$

where the infimum is taken over all quasiconformal mappings g homotopic to $f' \circ f^{-1}$ and $K(g)$ is the maximal dilatation of g . It is known that the Teichmüller distance coincides with the Kobayashi distance with respect to the complex structure of $T(X_0)$. Moreover, in the Teichmüller space $T(X_0)$, the Carathéodory pseudo distance $c_{T(X_0)}$ is a distance and it is complete ([3], [26]).

The Teichmüller distance between $[X, f]$ and $[X', f']$ is actually achieved by a quasiconformal mapping $f_{X, X'}$ from X to X' called *the Teichmüller mapping*, whose Beltrami coefficient is $k|q|/q$, where $k \in [0, 1)$ is given by an equation

$$d_T([X, f], [X', f']) = \log \frac{1+k}{1-k},$$

and q is an integrable holomorphic quadratic differential on X with $\int_X |q| = 1$. Moreover, those q and k are uniquely determined by $[X', f']$.

Let $\text{Mod}(X_0)$ be the group of homotopy classes $[\omega]$ of quasiconformal self-maps ω of X_0 , which is called the *mapping class group*. The mapping class group $\text{Mod}(X_0)$ acts on $T(X_0)$ by

$$[\omega]([X, f]) = [X, f \circ \omega^{-1}] \quad ([\omega] \in \text{Mod}(X_0)).$$

It is easily seen that the action of $\text{Mod}(X_0)$ on $T(X_0)$ is isometric with respect to the Teichmüller distance d_T .

Teichmüller disks. Let q be a holomorphic quadratic differential on X_0 with $\int_{X_0} |q| = 1$. We consider a holomorphic map from Δ to the space of Beltrami coefficients by

$$(2.2) \quad \Delta \ni t \mapsto t \frac{|q|}{q}.$$

Since the Beltrami coefficient $t|q|/q$ determines a point in $T(X_0)$, the map yields a holomorphic map Φ_q from Δ to $T(X_0)$. Furthermore, the map Φ_q is isometric with respect to the hyperbolic distance on Δ and the Teichmüller distance on $T(X_0)$ and the image $\mathbb{D}[q] := \Phi_q(\Delta)$ in $T(X_0)$ is totally geodesic. The image $\mathbb{D}[q]$ is called a *Teichmüller disk*.

Jenkins-Strebel differentials. Let q be an integrable holomorphic quadratic differential on X_0 . Then a holomorphic germ

$$X_0 \ni z \mapsto \int^z \sqrt{q} \in \mathbb{C}$$

gives a flat structure on X_0 except for zeros of q . A horizontal line of the flat structure is called a horizontal trajectory of q . If an integrable holomorphic quadratic differential q has a horizontal trajectory which is a simple closed curve on X_0 and the set of simple closed horizontal trajectories is dense in X_0 , then the holomorphic quadratic differential q is called a *Jenkins-Strebel differential*. A Jenkins-Strebel differential q is called *single* if any two closed trajectory of q is homotopic to each other. It is known that for any non-trivial and non-peripheral simple closed curve α in X_0 there exists a single Jenkins-Strebel differential whose simple closed trajectories are homotopic to α .

Let q be a single Jenkins-Strebel differential. Then, the union of closed trajectories makes an annulus A_q in X_0 and the core curve α_q of A_q is homotopic to a closed trajectory of q . We consider the right Dehn twist $\tau_q \in \text{Mod}(X_0)$ about α_q . Then the following theorem is known:

Theorem 2.1 (Marden-Masur [19]). *Let q be a single Jenkins-Strebel differential of X_0 and A_q, α_q be the same ones as above. Then, the right Dehn twist $\tau_q \in \text{Mod}(X_0)$ preserves the Teichmüller disk $\mathbb{D}[q]$ given by q . Furthermore, $\Phi_q^{-1} \circ \tau_q \circ \Phi_q : \Delta \rightarrow \Delta$ is a parabolic Möbius transformation on Δ .*

2.3. Kleinian groups. For a Kleinian group G , the closure of the set of fixed points of elements of G with infinite order is called the *limit set* of G and it is denoted by $\Lambda(G)$. If the limit set $\Lambda(G)$ contains infinitely many points, then the Kleinian group G is called *non-elementary*. Throughout this paper, we assume that every Kleinian group is non-elementary.

The complement of the limit set $\Lambda(G)$ in $\hat{\mathbb{C}}$ is called the *region of discontinuity* of G and it is denoted by $\Omega(G)$. By the definition, $\Omega(G)$ is an open set in $\hat{\mathbb{C}}$ but it may be empty. A connected component of $\Omega(G)$ is called a *component* of G . It is known that $\Omega(G)$ is the maximal open set where the action of G is properly discontinuous. Hence, the quotient space $\Omega(G)/G$ consists of Riemann surfaces (or orbifolds) if $\Omega(G)$ is not empty. Furthermore, we have the following:

Theorem 2.2 (Ahlfors' finiteness theorem; cf. [22]). *Suppose that G is a finitely generated Kleinian group with non-empty region of discontinuity. Then $\Omega(G)/G$ is a finite union of hyperbolic Riemann surfaces of finite type.*

A Riemann surface is called *of finite type* if it is obtained by removing at most finitely many points from a compact Riemann surface.

Every element g of $\mathrm{PSL}(2, \mathbb{C})$ can be extended to the upper half space $\mathbb{H}^3 = \{(x, y, t) \mid x, y \in \mathbb{R}, t \in \mathbb{R}_+\}$ as a hyperbolic isometry. Hence, a Kleinian group G acts on the hyperbolic 3-space $\mathbb{H}^3$. Moreover, the action is properly discontinuous and we obtain a hyperbolic 3-manifold (or orbifold) $M(G) = \mathbb{H}^3/G$ and the boundary $\partial M(G)$ of $M(G)$ is $\Omega(G)/G$. While there are many aspects and results on hyperbolic 3-manifolds and Kleinian groups, we present the following theorem which is used later in this paper.

Theorem 2.3 (The loop theorem; cf. [20]). *Let M be a 3-manifold with boundary and B be a connected component of the boundary of M . Suppose that there exists a non-trivial loop α in B which is trivial in M . Then there exists a non-trivial simple loop β in B which is trivial in M .*

2.4. The deformation space of Kleinian groups. Let G_0 be a finitely generated non-elementary Kleinian group. The deformation space $D(G_0)$ of G_0 is the space of $\theta \in \widetilde{\mathrm{Hom}}(G_0, \mathrm{PSL}(2, \mathbb{C}))$ given by normalized quasiconformal mappings on $\hat{\mathbb{C}}$. Namely, $\theta \in \widetilde{\mathrm{Hom}}(G_0, \mathrm{PSL}(2, \mathbb{C}))$ belongs to $D(G_0)$ if and only if there exists a quasiconformal self-map w of $\hat{\mathbb{C}}$ fixing $0, 1$ and ∞ such that θ is represented by an isomorphism of G_0 to $\mathrm{PSL}(2, \mathbb{C})$ given by

$$G_0 \ni g \mapsto wgw^{-1} \in \mathrm{PSL}(2, \mathbb{C}).$$

In [16] and [21], it is shown that $D(G_0)$ is a complex manifold and it is holomorphically convex. Recently, Earle and Marden [5] discuss the deformation spaces with holomorphic families of Riemann surfaces.

On the space $D(G_0)$, we may define the Teichmüller distance $d_T^{D(G_0)}$:

$$d_T^{D(G_0)}(\theta_1, \theta_2) = \inf_{w_1, w_2} \log K(w_2 \circ w_1^{-1}) \quad (\theta_1, \theta_2 \in D(G_0)),$$

where w_i ($i = 1, 2$) are taken over all quasiconformal self-maps of $\hat{\mathbb{C}}$ giving θ_i . A standard argument of quasiconformal mappings shows that the Teichmüller distance $d_T^{D(G_0)}$ is complete in $D(G_0)$.

Let X_0 be a hyperbolic Riemann surface of finite type and Γ_0 be a Fuchsian group acting on $\mathbb{H}$ with $X_0 = \mathbb{H}/\Gamma_0$. We denote by $\mathcal{QF}(\Gamma_0)$ the set of all conjugacy classes of isomorphisms of Γ_0 to $\mathrm{PSL}(2, \mathbb{C})$ given by a normalized quasiconformal mapping of $\hat{\mathbb{C}}$. The space $\mathcal{QF}(\Gamma_0)$ is called *the space of quasi-Fuchsian groups* for Γ_0 . It is actually the deformation space $D(\Gamma_0)$ and it is regarded as a subset of $\widetilde{\mathrm{Hom}}(\Gamma_0, \mathrm{PSL}(2, \mathbb{C}))$, the space of representations of the Fuchsian group Γ_0 to $\mathrm{PSL}(2, \mathbb{C})$.

The simultaneous uniformization theorem by Bers ([11]) implies that $\mathcal{QF}(\Gamma_0)$ can be identified with $T(X_0) \times T(\overline{X_0})$, the product space of the Teichmüller spaces, where $\overline{X_0}$ is the complex conjugate of the Riemann surface X_0 . Moreover, it has a natural complex structure. Under the complex structure, McMullen ([23]) shows the following disk convexity:

Theorem 2.4. *Let φ be a holomorphic map from $\overline{\Delta}$ to $\widetilde{\mathrm{Hom}}(\Gamma_0, \mathrm{PSL}(2, \mathbb{C}))$ so that $\varphi(z)(\gamma)$ is parabolic for every parabolic element $\gamma \in \Gamma_0$. Suppose that $\varphi(\partial\Delta) \subset \mathcal{QF}(\Gamma_0)$. Then, $\varphi(\Delta) \subset \mathcal{QF}(\Gamma_0)$.*

3. HOLOMORPHIC MOTIONS

In this section, we shall give some fundamental facts on holomorphic motions used in this paper.

Let E be a subset of $\hat{\mathbb{C}}$ containing at least three points and V be a connected complex manifold with a base point x_0 . A map $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ is called a *holomorphic motion* of E over V if it satisfies the following conditions:

- (1) $\phi(x_0, \cdot)$ is the identity on E ;
- (2) for each $x \in V$, $\phi(x, \cdot)$ is injective on E ;
- (3) for each $z \in E$, $\phi(\cdot, z)$ is a holomorphic function on V .

A holomorphic motion $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ is called *normalized* if E contains $0, 1$ and ∞ and $\phi(x, \cdot)$ fixes $0, 1$ and ∞ for any $x \in V$. From now on, we assume that a holomorphic motion is always normalized since every holomorphic motion is conjugate to a normalized one by Möbius transformations.

A striking theorem on holomorphic motions is the following theorem by Slodkowski [27]:

Theorem 3.1 (Slodkowski). *Let E be a subset of $\hat{\mathbb{C}}$ containing at least three points. Then every holomorphic motion of E over the unit disk Δ in $\mathbb{C}$ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ .*

Furthermore, Earle, Kra and Krushkal' ([4]) showed that Theorem 3.1 is still true for holomorphic motions which are Möbius group equivariant as follows;

Theorem 3.2. *Let G be a group of Möbius transformations and $E \subset \hat{\mathbb{C}}$ a closed set invariant for G . Suppose that a holomorphic motion $\phi : \Delta \times E \rightarrow \hat{\mathbb{C}}$ is G -equivariant on E , that is, there exists an isomorphism $\rho_\lambda : G \rightarrow \text{PSL}(2, \mathbb{C})$ for each $\lambda \in \Delta$ such that*

$$\phi(\lambda, g(z)) = \rho_\lambda(g)(\phi(\lambda, z)) \quad (z \in E).$$

Then ϕ extends a holomorphic motion $\tilde{\phi} : \Delta \times \hat{\mathbb{C}}$ of $\hat{\mathbb{C}}$ over Δ which is G -equivariant on $\hat{\mathbb{C}}$.

On the other hand, it is known that then there exist a simply connected complex manifold V with $\dim V \geq 2$ and a holomorphic motion of a finite set over V such that it cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over V which is a consequence of a theorem by Hubbard ([9]). See also [12] for an easier example.

We consider conditions for holomorphic motions of E over V to be extended to holomorphic motions of $\hat{\mathbb{C}}$ over V when $\dim V = 1$ and V is not simply connected. In particular, we consider $V = \Delta_K := \Delta \setminus K$ for an AB -removable compact subset K of Δ as in our main theorems. We say that a compact subset K of Δ is *AB-removable* if any bounded holomorphic function in Δ_K is extended to a holomorphic function in Δ . Obviously, K is nowhere dense and Δ_K is connected.

The concept of “monodromy” of a holomorphic motion was studied in [1]; see Section 2.4 of that paper. For the reader’s convenience, we include here a brief discussion.

We assume that E is a finite set and it contains $n(> 3)$ points including $0, 1$ and ∞ . Let $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion. Consider a holomorphic universal covering $\pi : \tilde{V} \rightarrow V$ with the cover transformation group G_V and take a point $\tilde{x}_0 \in \tilde{V}$ so that $\pi(\tilde{x}_0) = x_0$. Then $\Phi = \pi^*(\phi) : \tilde{V} \times E \rightarrow \hat{\mathbb{C}}$ is a holomorphic

motion of E over $\tilde{V}$ with $\tilde{x}_0$ as the base point. Then, there exists a base point preserving continuous map $f : \tilde{V} \rightarrow M(\mathbb{C})$ such that

$$(3.1) \quad \Phi(x, z) = w^{f(x)}(z) \quad \text{for each } x \in \tilde{V} \text{ and each } z \in E,$$

where $M(\mathbb{C})$ is the space of Beltrami coefficients on $\mathbb{C}$,

$$w^{f \circ g(\tilde{x}_0)}(z) = \Phi(g(\tilde{x}_0), z) = \phi(\pi \circ g(\tilde{x}_0), z) = \phi(x_0, z) = z.$$

Therefore, $w^{f \circ g(\tilde{x}_0)}$ keeps every point of E fixed.

It is not hard to see that the homotopy class for $w^{f \circ g(\tilde{x}_0)}$ relative to E is well defined, that is, it does not depend on the choice of continuous mappings f . The map $w^{f \circ g(\tilde{x}_0)}$ is a quasiconformal self-map of the Riemann surface $X_E := \hat{\mathbb{C}} \setminus E$. Therefore, it represents a mapping class of X_E , and by Lemma 2.12 in [1], we have a homomorphism $\sigma_\phi : \pi_1(V, x_0) \rightarrow \text{Mod}(0, n)$ given by

$$(3.2) \quad \sigma_\phi(c) = [w^{f \circ g_c(\tilde{x}_0)}]$$

where $\text{Mod}(0, n)$ is the mapping class group of the n -times punctured sphere, $g_c \in G_V$ is an element corresponding to $c \in \pi_1(V, x_0)$, and $[w]$ denotes the mapping class of X_E for w . We call the homomorphism σ_ϕ the *monodromy* of the holomorphic motion ϕ of the finite set E . The monodromy is called *trivial* if it maps every element of $\pi_1(V, x_0)$ to the identity of $\text{Mod}(0, n)$.

If ϕ extends to a holomorphic motion of $\hat{\mathbb{C}}$, then it represents a holomorphic family of quasiconformal mappings with the parameter space V . This gives us the following (cf. [1])

Proposition 3.1. *Let $\phi : V \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion of a finite set E . If ϕ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over V , then the monodromy σ_ϕ is trivial.*

If $V = \Delta_K := \Delta \setminus K$ for an AB -removable compact subset K of Δ , we can say more ([1]).

Proposition 3.2. *Let $\phi : \Delta_K \times E \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion of E over Δ_K . Then, the holomorphic motion ϕ can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ if and only if the monodromy of $\phi|_{E'}$ is trivial for any finite subset E' of E , where $\phi|_{E'}$ is a holomorphic motion of E' over Δ_K defined by $\phi|_{E'}(x, z) = \phi(x, z)$ for $(x, z) \in \Delta_K \times E'$.*

By using the above proposition, we can show the following:

Proposition 3.3 ([1]). *Let K be an AB -removable compact subset of Δ and E be a closed set containing $0, 1$ and ∞ .*

- (1) *If every connected component of the complement of E is either simply connected or conformally equivalent to the punctured disk, then every holomorphic motion of E over Δ_K can be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ_K .*
- (2) *Suppose that E consists of more than five points. If some component of the complement of E is neither simply connected nor conformally equivalent to the punctured disk, then there exists a holomorphic motion of E over Δ_K such that it cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ_K .*

Proof. While the proof is given in [1], we will show a sketch of the proof of (2) since it shows the difficulty of the proof of Theorem II.

Let K be an AB -removable compact subset of Δ . We may assume that K contains the origin. Let E be a closed set satisfying the conditions of (2) and Ω a connected component of E^c which is neither simply connected nor conformally equivalent to the punctured disk. Since E contains at least five points, there exists a simply connected domain D such that $\partial D \subset \Omega$, D contains at least two points of E , say z_1, z_2 , and $D^c \cap E$ contains at least three points. We may assume that $0, 1$ and ∞ are not in D . We take z_0 in $D \setminus E$.

Let $h : \Delta \rightarrow D$ be a Riemann map with $h(0) = z_0$. Then, there exists a positive number $r < 1$ such that $h(\{r < |x| < 1\}) \cap E = \emptyset$. We construct a holomorphic motion ϕ_0 of E over Δ^* as follows for $x \in \Delta^*$:

$$(3.3) \quad \phi_0(x, z) = \begin{cases} z, & z \in E \setminus D, \\ h(xh^{-1}(z)/r), & z \in E \cap D. \end{cases}$$

Then, we can see that the monodromy of ϕ_0 about the origin is not trivial. Hence, it follows from Proposition 3.2 that ϕ_0 cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over Δ_K . $\square$

4. PROOF OF THEOREM I

Let $\mathcal{G} = (G_0, \Delta_K, \{\theta_x\}_{x \in \Delta_K})$ be an order preserving holomorphic family of G_0 over Δ_K . First, we show that the family is a holomorphic family of isomorphisms.

Indeed, if there exists $x \in \Delta_K$ such that θ_x is not an isomorphism, then the kernel Γ_0 of θ_x is a non-trivial normal subgroup of G_0 . Since G_0 is non-elementary, Γ_0 contains a loxodromic element g . Noting that the trace function $f(z) := \text{Tr}^2 \theta_z(g)$ ($z \in \Delta_K$) is invariant for the conjugation of $\text{PSL}(2, \mathbb{C})$ and $\theta_{x_0} = id$, we see that f is holomorphic and $f(x) = 4$ but $f(x_0) \neq 4$. Thus, f is a non-constant holomorphic function on Δ_K . Hence it is an open map and we see that there exists $y \in \Delta_K$ near x such that $\theta_y(g)$ is an elliptic element of finite order. It is a contradiction. Thus, the family is a holomorphic family of isomorphisms.

By the same argument, we see that every θ_x ($x \in \Delta_K$) is type preserving, that is, $\theta_x(g)$ is parabolic if and only if $g \in G_0$ is parabolic. Indeed, if $\theta_x(g)$ is parabolic for some loxodromic element $g \in G_0$, then we verify that there exists a point $y \in \Delta_K$ such that $\theta_y(g)$ is an elliptic element of finite order. It is absurd because θ_y is order preserving (and isomorphic).

Therefore, from the Sullivan-Riley theorem ([25], [28]) we see that every $G_x := \theta_x(G_0)$ is discrete ($x \in \Delta_K$). By Theorem 1.1, we see that the family $\{G_x\}_{x \in \Delta_K}$ has the following property: for a simply connected neighbourhood U of a point x in Δ_K , there exists a quasiconformal mapping w_y for each $y \in U$ such that $\theta_y \circ \theta_x^{-1}$ is conjugate to the isomorphism:

$$G_0 \ni g \mapsto w_y \circ g \circ w_y^{-1} \in \text{PSL}(2, \mathbb{C}).$$

Consider two loxodromic elements $\gamma_1, \gamma_2 \in G_0$. We may assume that the attracting fixed point of γ_1 is 0, the repelling fixed point of γ_1 is 1 and the attracting fixed point of γ_2 is ∞ . For $x \in \Delta_K$, we take a simply connected domain U in Δ_K containing x and the base point x_0 . Since U is conformally equivalent to Δ , it follows from Theorem 1.1 that there exists a normalized quasiconformal map w_x^U for each $x \in U$ such that it gives the conjugacy class of θ_x . The normalized quasiconformal

map w_x^U does not depend only on $x \in \Delta_K$. However, we may show that $w_x^U|_{\Lambda(G_0)}$ depends only on x .

Lemma 4.1. *The map $w_x^U|_{\Lambda(G_0)}$ does not depend on the choice of U . It depends only on $x \in \Delta_K$.*

Proof. We consider another simply connected domain V in Δ_K containing x and x_0 . Then, we may also take a normalized quasiconformal map w_x^V giving the conjugacy class of θ_x . Hence, there exists $A \in \mathrm{PSL}(2, \mathbb{C})$ such that w_x^U and $A \circ w_x^V$ define the same isomorphism of G_0 . In particular, we have

$$w_x^U \circ \gamma_i \circ (w_x^U)^{-1} = A \circ w_x^V \circ \gamma_i \circ (w_x^V)^{-1} \circ A^{-1} \quad (i = 1, 2).$$

Because of the normalization of fixed points of γ_1 and γ_2 as above, we see that A must fix the three points, $0, 1$ and ∞ and A is the identity. So, w_x^V defines the same isomorphism ρ_x of G_0 as w_x^U . Therefore, we have

$$w_x^U|_{\Lambda(G_0)} = w_x^V|_{\Lambda(G_0)}. \quad \square$$

Thus, $\phi(x, z) := w_x^U(z)$ is independent of the choice of U and it is well defined on $\Delta_K \times \Lambda(G_0)$. Obviously, $\phi(x, \cdot) = w_x^U(\cdot)$ is injective on $\Lambda(G_0)$ and $\phi(x_0, \cdot) = id$.

Let L_0 be the set of loxodromic fixed points of G_0 . For a loxodromic Möbius transformation g , we denote by $\alpha(g)$ the attractive fixed point of g . Since the quasiconformal map w_x^U sends the attractive fixed point of $\gamma \in G_0$ to that of $\gamma_x := w_x^U \circ \gamma \circ (w_x^U)^{-1}$, we verify that for each loxodromic transformation $\gamma \in G_0$, $\phi(x, \alpha(\gamma)) = w_x^U(\alpha(\gamma))$ is holomorphic on Δ_K and $\phi(\cdot, \cdot)$ is a holomorphic motion of L_0 over Δ_K .

It is known that $\overline{L_0} = \Lambda(G_0)$. Thus, the holomorphic motion ϕ of L_0 can be extended to a holomorphic motion of $\Lambda(G_0)$ over Δ_K (cf. [18]). We use the same letter ϕ for the extended holomorphic motion of $\Lambda(G_0)$. We also see that the holomorphic motion ϕ is G_0 -equivariant, namely

$$\phi(x, g(z)) = \rho_x(g)(\phi(x, z))$$

holds for every $g \in G_0$ and for every $z \in \Lambda(G_0)$. Indeed, if $z = \alpha(\gamma) \in L_0$ for some $\gamma \in G_0$, then $g(z) = \alpha(g\gamma g^{-1})$ for $g \in G_0$. Thus, we obtain

$$\phi(x, g(z)) = \alpha(\rho_x(g\gamma g^{-1})) = \rho_x(g)(\alpha(\rho_x(\gamma))) = \rho_x(g)(\phi(x, z)),$$

for $z \in L_0$. Since L_0 is dense in $\Lambda(G_0)$, the holomorphic motion ϕ is G_0 -equivariant on $\Lambda(G_0)$ because of the continuity of ϕ . Thus, it follows from Proposition 3.3 (1) and Theorem 2 in [24] that ϕ can be extended to a holomorphic motion $\tilde{\phi}$ of $\hat{\mathbb{C}}$ over Δ . We show that $\tilde{\phi}$ is still a G_0 -equivariant holomorphic motion of $\Lambda(G_0)$ over Δ .

Let x be a point in K . It suffices to show that $\tilde{\phi}(x, \cdot)$ is G_0 -equivariant on $\Lambda(G_0)$. Since an AB -removable set is nowhere dense, we may take a sequence $\{x_n\}_{n=1}^\infty \subset \Delta_K$ converging to x . Taking a subsequence if necessary, we see that there exist normalized quasiconformal maps w_n and w such that

- (1) w_n and w_n^{-1} uniformly converges to w and w^{-1} as $n \rightarrow \infty$ on $\hat{\mathbb{C}}$, respectively;
- (2) $\tilde{\phi}(x_n, z) = w_n(z)$ and $\tilde{\phi}(x, z) = w(z)$.

From (2), we have

$$w_n(g(z)) = \tilde{\phi}(x_n, g(z)) = \rho_{x_n}(g)(\tilde{\phi}(x_n, z)) = \rho_{w_n}(g)(w_n(z)),$$

for every $z \in \Lambda(G_0)$. Then, we see that $\rho_{w_n}(g) = w_n \circ g \circ w_n^{-1}$ converges to a homeomorphism on $\Lambda(G_0)$ by (1). Since $\rho_{w_n}(g) \in \mathrm{PSL}(2, \mathbb{C})$ and $\Lambda(G_0)$ contains more than three points, the limit $\rho(g) = w \circ g \circ w^{-1}$ also belongs to $\mathrm{PSL}(2, \mathbb{C})$ (cf. [17] §2.2–2.3). Obviously, ρ is an isomorphism of G_0 and we have

$$(4.1) \quad \tilde{\phi}(x, g(z)) = \rho(g)(\tilde{\phi}(x, z)),$$

for every $g \in G_0$ and $z \in \Lambda(G_0)$. Because of (4.1), the map $\rho(g)$ does not depend on $\{x_n\}$. Thus, $\tilde{\phi}(x, \cdot)$ is G_0 -equivariant on $\Lambda(G_0)$ for $x \in K$.

Since $\tilde{\phi}$ is a G_0 -equivariant holomorphic motion of $\Lambda(G_0)$ over Δ , we obtain a G_0 -equivariant holomorphic motion of $\hat{\mathbb{C}}$ over Δ from Theorem 3.2 and the proof is completed.

Remark 4.1. It is known that Teichmüller space $T(X)$ of a compact Riemann surface X of genus $g > 1$ is regarded as a bounded domain containing the origin in $\mathbb{C}^{3g-3}$. Thus, we may take an annulus $A \subset \Delta \hookrightarrow T(X) \subset \mathbb{C}^{3g-3}$. Since $T(X)$ is a subset of $\mathcal{QF}(\Gamma)$ for a Fuchsian group Γ which represents X , it is easy to see that there exists a holomorphic map $\varphi : A \cup \partial A \rightarrow T(X) \subset \mathcal{QF}(\Gamma)$ such that it cannot be extended to a holomorphic map from Δ to $\mathcal{QF}(\Gamma)$. For example, for $A = \{1 < |z| < 2\}$, we take $\epsilon > 0$ so small that $\varphi(z) = (\epsilon z^{-1}, 0, \dots, 0) \in \mathbb{C}^{3g-3}$ belongs to $T(X)$ for any $z \in A \cup \partial A$. Every $\varphi(z) \in T(X)$ induces a type preserving isomorphism. However, since $T(X)$ is a bounded domain, the map φ cannot be extended to a holomorphic map on Δ .

5. PROOF OF THEOREM II

Let Ω be a non-simply connected component of $\Omega(G_0)$ of a finitely generated Kleinian group G_0 . As in the argument in the proof of Proposition 3.3, we may construct a holomorphic motion of $\Lambda(G_0)$ over Δ^* which cannot be extended to a holomorphic motion of $\hat{\mathbb{C}}$. However, it is not G_0 -equivariant at all. The difficulty is to make a G_0 -equivariant holomorphic motion.

For the universal covering $\pi : \Delta \rightarrow \Omega$, we consider

$$\Gamma_\Omega := \{\gamma \in \mathrm{PSL}(2, \mathbb{C}) \mid \pi \circ \gamma = g \circ \pi, \text{ for some } g \in G_0\}.$$

The group Γ_Ω is called the Fuchsian equivalent of G_0 on Ω and it is seen that

$$\Delta/\Gamma_\Omega = \Omega/G_0^\Omega =: X_\Omega,$$

where G_0^Ω is the set of $g \in G_0$ which fix Ω . Since G_0 is of finitely generated, X_Ω is a Riemann surface of finite type (Theorem 2.2). The main idea of the proof is to construct a holomorphic family of the Kleinian group G_0 parametrized by Δ with invariance under the action of some parabolic cyclic group. In fact, the parabolic cyclic group is determined by a Dehn twist on X_Ω .

Since π is the universal covering map of Ω , for a non-trivial simple closed geodesic c in Ω we obtain an element $g_c \in \Gamma_\Omega \setminus \{id\}$ which represents c . The transformation $g_c \in \Gamma_\Omega$ defines a non-trivial closed curve α_c in X_Ω . On the other hand, the curve c is not represented by any non-trivial element in G_0 since it is a closed curve in Ω . Hence, the closed curve α_c in X_Ω is trivial in the hyperbolic 3-manifold $M := (\mathbb{H}^3 \cup \Omega(G_0))/G_0$. Thus, from Theorem 2.3 we have a non-trivial simple closed curve β in X_Ω which is trivial in M .

We denote by $\tilde{\beta}$ a connected component of the lift of β in Ω . Since β is trivial in M , there exists an annular neighbourhood $A_{\tilde{\beta}}$ of $\tilde{\beta}$ such that the canonical projection $\pi_{\Omega} : \Omega \rightarrow X_{\Omega}$ is injective on $A_{\tilde{\beta}}$.

Now, we consider the right Dehn twist τ_{β} about β . We may assume that τ_{β} is the identity on $X_{\Omega} \setminus \pi_{\Omega}(A_{\tilde{\beta}})$. Therefore, we can lift this to a self-map $\tau_{\tilde{\beta}}$ of Ω . Actually, $\tau_{\tilde{\beta}}$ is the product of the right Dehn twists about $\pi_{\Omega}^{-1}(\beta) (= \bigcup_{g \in G_0} g(\tilde{\beta}))$. Moreover, we see that

$$g \circ \tau_{\tilde{\beta}}(z) = \tau_{\tilde{\beta}} \circ g(z)$$

for any $z \in \Omega$ and any $g \in G_0^{\Omega}$.

Let q_{β} be a single Jenkins-Strebel differential on X_{Ω} whose closed trajectories are homotopic to β . We denote by $\Phi_{\beta} : \Delta \rightarrow \mathbb{D}[q_{\beta}]$ the holomorphic map induced by the map

$$\Delta \ni t \mapsto t \frac{|q_{\beta}|}{q_{\beta}}.$$

It follows from Theorem 2.1 that the Teichmüller disk $\mathbb{D}[q_{\beta}]$ in $T(X_{\Omega})$ given by q_{β} is preserved by the action of the Dehn twist τ_{β} about β and a parabolic Möbius transformation γ on $\Delta (\simeq \mathbb{D}[q_{\beta}])$ gives the action of τ_{β} . More precisely, for every $z \in \Delta$

$$(5.1) \quad \Phi_{\beta}(\gamma(z)) = [\tau_{\beta}](\Phi_{\beta}(z))$$

holds, where $[\omega]$ means the mapping class of a quasiconformal self-map ω of X_{Ω} .

Let $f_{\beta} : X_{\Omega} \rightarrow X_{\Omega}$ be a quasiconformal map by a Beltrami coefficient $\gamma(0)|q_{\beta}|/q_{\beta}$. Since the map represents the right Dehn twist about β , it is lifted to a quasiconformal map $\tilde{f}_{\beta} : \Omega \rightarrow \Omega$ with

$$\tilde{f}_{\beta} \circ g = g \circ \tilde{f}_{\beta}$$

for every $g \in G_0^{\Omega}$.

The quasiconformal map $\tilde{f}_{\beta}$ on Ω is extended to a quasiconformal map on $\Omega(G_0)$ to itself defined by $g \circ \tilde{f}_{\beta} \circ g^{-1}$ on $g(\Omega)$ for $g \in G_0$ and the identity on $\Omega(G_0) \setminus \bigcup_{g \in G_0} g(\Omega)$. We use the same letter $\tilde{f}_{\beta}$ for the extended map. Then, we have $\tilde{f}_{\beta} \circ g = g \circ \tilde{f}_{\beta}$ for any $g \in G_0$ on $\Omega(G_0)$. From Maskit's self-map theorem ([21]), we conclude that $\tilde{f}_{\beta}$ is extended to a quasiconformal map on $\hat{\mathbb{C}}$ which commutes with any $g \in G_0$. We also use the same letter $\tilde{f}_{\beta}$ for the extended map and denote by μ the Beltrami coefficient of $\tilde{f}_{\beta}$. Obviously, the extended map $\tilde{f}_{\beta}$ also satisfies

$$(5.2) \quad \tilde{f}_{\beta} \circ g = g \circ \tilde{f}_{\beta}$$

for every $g \in G_0$. Therefore, the Beltrami coefficient μ is G_0 -compatible. Namely, the Beltrami coefficient μ satisfies

$$\mu(g(z))\overline{g'(z)}g'(z)^{-1} = \mu(z) \quad \text{a.e.}$$

for every $g \in G_0$. Therefore, a holomorphic map

$$\Delta \ni t \mapsto t\mu$$

defines a holomorphic family of Beltrami coefficients which are G_0 -compatible.

From this family, we construct a holomorphic family of normalized quasiconformal mappings w_t on $\hat{\mathbb{C}}$ ($t \in \Delta$). The holomorphic family $\{w_t\}_{t \in \Delta}$ induces a holomorphic family $\{\theta_t\}_{t \in \Delta}$ of isomorphisms of G_0 over Δ defined by

$$\theta_t(g) = w_t g w_t^{-1} \quad (g \in G_0).$$

We may take $w_0 = id$ and $w_{\gamma(0)} = \tilde{f}_\beta$. Then, we have $\theta_0 = \theta_{\gamma(0)}$ because of (5.2). By the same reason, we conclude that

$$\theta_t = \theta_{\gamma(t)}$$

for every $t \in \Delta$. Therefore, the holomorphic family $\mathcal{G} := (G_0, \Delta, \{\theta_t\}_{t \in \Delta})$ of G_0 over Δ is projected to a holomorphic family $\hat{\mathcal{G}}$ of G_0 over the punctured disk $\Delta^* = \Delta / \langle \gamma \rangle = \{0 < |z| < 1\}$.

The holomorphic family $\hat{\mathcal{G}}$ of G_0 over Δ^* cannot be extended to a holomorphic family of G_0 over Δ . Indeed, if it can be extended to a holomorphic family $\hat{\mathcal{G}}_0$ of G_0 over Δ , then it consists of quasiconformal deformations of G_0 . However, if we take $x \in (0, 1)$ so that $x \rightarrow 0$, $\theta_x(G_0)$ gives a squeezing deformation of X_Ω about β . Thus, the hyperbolic lengths of the geodesics homotopic to β tends to zero as $x \rightarrow 0$ (see [19]). Thus, the maximal dilatations of quasiconformal mappings which give isomorphisms from G_0 to $\theta_x(G_0)$ ($0 < x < 1/2$) are unbounded. It is a contradiction. $\square$

6. THE HOLOMORPHIC CONVEXITY OF DEFORMATION SPACES OF KLEINIAN GROUPS

In this section, we shall prove Theorem III.

Let G_0 be a finitely generated Kleinian group. We give a proof of the first statement of the theorem. We show that in $D(G_0)$, the Carathéodory pseudo distance $c_{D(G_0)}$ is a distance. Namely, we show that $c_{D(G_0)}(x, y) > 0$ if $x \neq y$.

For each $\gamma \in G_0$, we consider a function $f_\gamma(z) = (\text{Tr } \rho_z(\gamma))^2$, where $(\text{Tr } (\alpha))^2$ stands for the square of the trace of $\alpha \in \text{PSL}(2, \mathbb{C})$ and ρ_z is the isomorphism given by $z \in D(G_0)$. The function f_γ is holomorphic on $D(G_0)$ and if γ is loxodromic, then the range is in $\mathbb{C} \setminus [0, 4]$ because the isomorphisms ρ_z are type preserving for $z \in D(G_0)$. Thus, $F_\gamma := \varphi \circ f_\gamma$ is a bounded holomorphic function for a conformal map φ from $\mathbb{C} \setminus [0, 4]$ into Δ .

Take any two distinct points x, y in $D(G_0)$. Then the two points x and y give two different isomorphisms ρ_x and ρ_y of $\text{Hom}(G_0, \text{PSL}(2, \mathbb{C}))$, respectively. Moreover, the isomorphisms ρ_x, ρ_y are not conjugate to each other since they are distinct points in $D(G_0)$. Hence, we may find a loxodromic element $\gamma \in G_0$ such that $F_\gamma(x) \neq F_\gamma(y)$ and we verify that $c_{D(G_0)}(x, y) > 0$. So, the first statement of the theorem is proved.

Next, we show that the Carathéodory distance on $D(G_0)$ is complete if every component of G_0 is simply connected.

Let X_0 be a hyperbolic Riemann surface of type (g, n) . Bers showed that the Teichmüller space $T(X_0)$ of X_0 is biholomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+n}$ (cf. [17], [11]). Since the Carathéodory distance in $T(X_0)$ is complete (cf. [3], [26]), the Teichmüller space $T(X_0)$ is H^∞ -convex. In fact, it is polynomial convex ([26]).

Suppose that every component of G_0 is simply connected. It follows from Ahlfors' finiteness theorem (Theorem 2.2) that there are finitely many non-equivalent components, say $\Omega_1, \dots, \Omega_n$, of G_0 and $X_j := \Omega_j / G_{0,j}$ is a Riemann surface of finite type, where $G_{0,j}$ is a stabilized subgroup of Ω_j ($j = 1, \dots, n$). Since every Ω_j is simply connected, $D(G_0)$ is biholomorphically equivalent to the product of Teichmüller spaces $T(X_1) \times \dots \times T(X_n)$, from a theorem of Kra and Maskit ([16]).

Now, we consider the Carathéodory distance. Generally, in the product manifold $M = M_1 \times \dots \times M_n$ of complex manifolds $M_1, \dots, M_n$, the Carathéodory distance c_M satisfies

$$c_M(a, b) \geq \max_{j=1, \dots, n} \{c_{M_j}(a_j, b_j)\}$$

for $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in M$. Actually, it is a direct consequence of the decreasing property of the Carathéodory pseudo distance (cf. [13]). Therefore, if the Carathéodory distances c_{M_j} are complete, so is the Carathéodory distance c_M and M is H^∞ -convex.

Applying the above arguments for $D(G_0)$ which is biholomorphically equivalent to the product of Teichmüller spaces $T(X_1), \dots, T(X_n)$, we conclude that the Carathéodory distance $c_{D(G_0)}$ is complete and we obtain the second statement of the theorem.

Finally, we show that the last statement of the theorem, that is, $D(G_0)$ is not H^∞ -convex and the Carathéodory distance is not complete if G_0 has a non-simply connected component.

Suppose that G_0 has a non-simply connected component. Then, from Theorem II we may construct a holomorphic family of G_0 over the punctured disk Δ^* but the holomorphic family cannot be extended to a holomorphic family of G_0 over Δ . In fact, we construct a holomorphic map $F : \Delta^* \rightarrow D(G_0)$ which cannot be extended to a holomorphic map on Δ .

Put $K = F(\{|t| = 1/2\})$. Then K is a compact subset of $D(G_0)$. For every bounded holomorphic function f on $D(G_0)$, $f \circ F$ is a bounded holomorphic function on Δ^* . Therefore, it can be extended to a holomorphic function on Δ . From the maximum principle, we have

$$|f \circ F(t)| \leq \max_{|t|=1/2} |f \circ F(t)|$$

for all t with $0 < |t| \leq 1/2$. Hence, $F(\{0 < |t| \leq 1/2\})$ is contained in H^∞ -convex hull of K . Since F cannot be extended to a holomorphic map on Δ , $F(\{0 < |t| \leq 1/2\})$ is not a compact subset of $D(G_0)$. Therefore, $D(G_0)$ is not H^∞ -convex and the Carathéodory distance is not complete.

Remark 6.1. As for the complex structure of $D(G_0)$, Kra and Maskit [16] showed that $D(G_0)$ is biholomorphically equivalent to a domain in $\mathbb{C}^n$ and the domain is holomorphically convex. We have seen in Theorem III that if every component of G_0 is simply connected, then $D(G_0)$ is H^∞ -convex, which is a stronger property than the holomorphic convexity. On the other hand, we show that the situation is completely different when G_0 has a non-simply connected component.

7. PROOF OF THEOREM IV

We show that the Teichmüller distance $d_T^{D(G_0)}$ of $D(G_0)$ coincides with the Kobayashi distance as the Teichmüller distance d_T of the Teichmüller space. The proof is done by using the same argument as in Earle-Kra-Krushkal' ([4]) for Teichmüller spaces.

Let $\mathcal{M}(G_0)$ be the space of G_0 -compatible Beltrami coefficients on $\hat{\mathbb{C}}$. For $\rho_1, \rho_2 \in D(G_0)$, we may take an extremal quasiconformal map $w_{1,2}$ for the Teichmüller distance. Thus, we have

$$d_T^{D(G_0)}(\rho_1, \rho_2) = \log K(w_{1,2}) = \log \frac{1 + \|\mu_{1,2}\|_\infty}{1 - \|\mu_{1,2}\|_\infty},$$

where $\mu_{1,2}$ is the Beltrami coefficient of $w_{1,2}$. We consider a holomorphic map $\Phi : \Delta \rightarrow D(G_0)$ defined by

$$\Phi(t) = \left[t \frac{\mu_{1,2}}{\|\mu_{1,2}\|} \right],$$

where $[\mu]$ ($\mu \in \mathcal{M}(G_0)$) is an element of $D(G_0)$ given by a quasiconformal map whose Beltrami coefficient is μ . Then, from the definition of the Kobayashi distance, we have

$$(7.1) \quad d_K^{D(g)}(\rho_1, \rho_2) \leq \rho_\Delta(0, \|\mu\|_\infty) = d_T^{D(G_0)}(\rho_1, \rho_2).$$

For every holomorphic map $f : \Delta \rightarrow D(G)$ with $f(0) = \rho_1$ and $f(t_0) = \rho_2$, we have a holomorphic map $g : \Delta \rightarrow \mathcal{M}(G_0)$ such that

$$f = \Phi \circ g.$$

It is a consequence of Theorem 3.2. Hence, we have

$$\rho_\Delta(0, t_0) \geq d_{\mathcal{M}(G_0)}(g(0), g(t_0)),$$

where $d_{\mathcal{M}(G_0)}$ is the Kobayashi distance of $\mathcal{M}(G_0)$. On the other hand, it is shown that it is the same as the Teichmüller distance of $\mathcal{M}(G_0)$ by using the same argument as in [4] Proposition 1. Therefore, by taking the infimum of the left-hand side over all holomorphic maps f , we have the opposite inequality of (7.1). This completes the proof of the theorem.

Remark 7.1. As we have stated in the previous section, the deformation space $D(G_0)$ is biholomorphically equivalent to a product of Teichmüller spaces if every component is simply connected. In this case, the coincidence of the two distances is known. In fact, they are the same distance on product Teichmüller spaces with an arbitrary index set ([7]). Indeed, the method in the proof of Proposition 7.28 in [7] also shows that the the infinitesimal Teichmüller metric and the infinitesimal Kobayashi metric are equal on $D(G_0)$, which was suggested by the referee.

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