AMPLE GROUP ACTION ON AS-REGULAR ALGEBRAS AND NONCOMMUTATIVE GRADED ISOLATED SINGULARITIES

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ABSTRACT. In this paper, we introduce a notion of ampleness of a group action G on a right noetherian graded algebra A , and show that it is strongly related to the notion of A^G to be a graded isolated singularity introduced by the second author of this paper. Moreover, if S is a noetherian AS-regular algebra and G is a finite ample group acting on S , then we will show that $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } \nabla S * G)$ where ∇S is the Beilinson algebra of S . We will also explicitly calculate a quiver $Q_{S,G}$ such that $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } kQ_{S,G})$ when S is of dimension 2.

1. INTRODUCTION

In commutative ring theory, regular rings, Gorenstein rings, and Cohen-Macaulay rings are fundamental classes of rings that are studied. Since these classes of rings can be characterized by homological algebra, noncommutative (graded) generalizations of these rings, namely, AS-regular algebras, AS-Gorenstein algebras, and AS-Cohen-Macaulay algebras, have been defined and intensively studied in noncommutative algebraic geometry. On the other hand, although isolated singularities are also an important class of rings in commutative ring theory, its notion is geometric in nature, so they have not been generalized nor studied well in noncommutative algebraic geometry. Recently, the second author of this paper introduced a notion of graded isolated singularity for (noncommutative) graded algebras, which agrees with the usual notion of isolated singularity if the algebra is commutative and generated in degree 1, and found some nice properties of such algebras in [19]. This paper continues to study (noncommutative) graded isolated singularities.

In commutative ring theory and representation theory of algebras, the fixed subalgebra S^G of a polynomial algebra $S = k[x_1, \dots, x_d]$ by a finite subgroup $G \leq \text{GL}(d, k)$ has been intensively studied. In particular, it is known that S^G is Gorenstein if and only if $G \leq \text{SL}(d, k)$, and S^G is an isolated singularity if and only if G acts freely on $\mathbb{A}^d \setminus \{0\}$. Since an AS-regular algebra S is a noncommutative generalization of a polynomial algebra, it is natural to study a fixed subalgebra S^G of an AS-regular algebra S by a finite group G acting on S . In [8], Jørgensen and Zhang introduced the notion of homological determinant for a graded algebra

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automorphism of S , which is the same as the usual determinant if S is a polynomial algebra generated in degree 1, and showed that, for a noetherian AS-regular algebra S , if every element of G has homological determinant 1, then S^G is AS-Gorenstein. In this paper, we will introduce a notion of ampleness of G for S , and show that, for a noetherian AS-regular algebra S of dimension $d \geq 2$ and a finite group G consisting of elements of homological determinant 1, G is ample for S if and only if S^G is a graded isolated singularity and $S * G \cong \underline{\text{End}}_{S^G}(S)$ (Theorem 3.10). If S is a polynomial algebra generated in degree 1, then $S * G \cong \underline{\text{End}}_{S^G}(S)$ in the above setting by [5, Theorem 3.2], so G is ample for S if and only if S^G is an isolated singularity in the usual sense (Corollary 3.11).

Let $S = k[x_1, \dots, x_d]$ be a (weighted) polynomial algebra and $G \leq \text{GL}(d, k)$ a finite subgroup acting on S . The derived category $\mathcal{D}^b(\text{tails } S^G)$ and the graded stable category $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$ have been studied in two situations, namely, when S is generated in degree 1 in [5], and when $G = \langle \text{diag}(\xi^{\deg x_1}, \dots, \xi^{\deg x_d}) \rangle$ is a finite cyclic group where $\xi \in k^\times$ is a primitive r -th root of unity so that $S^G = S^{(r)}$ is the r -th Veronese algebra of S in [18] and [1, Section 5] (see [13] when S is AS-regular). One of the motivations of this paper is to integrate both situations to find a common theory. As one of such common theories, we show in this paper that $\mathcal{D}^b(\text{Proj } S^G) := \mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } \nabla S * G)$ when S is a noetherian AS-regular algebra of dimension $d \geq 2$ and G is a finite ample group acting on S where ∇S is the Beilinson algebra of S (Theorem 3.14). As a main ingredient of the proof, we also show that if S is a noetherian AS-regular algebra, then $S * G$ is a noetherian AS-regular algebra over kG as introduced in [11] (Corollary 3.6). If G is ample, then S^G is a graded isolated singularity, so the noncommutative projective scheme $\text{Proj } S^G$ associated to S^G is smooth by definition. It follows that $\text{Proj } S^G$ itself is the minimal resolution of $\text{Proj } S^G$, so the above derived equivalence can be thought of as a noncommutative projective version of (algebraic) McKay correspondence (see [13] for details).

In the last section of this paper, we study the case when S is a noetherian AS-regular algebra of dimension 2. If $S = k[x, y]$ is the polynomial algebra and $G \leq \text{SL}(2, k)$, then it is known that S^G is an isolated singularity (so G is ample for S if $\deg x = \deg y = 1$). In this paper, we show that if $S \not\cong k\langle x, y \rangle / (xy \pm yx)$, then every finite group G consisting of elements of homological determinant 1 is ample for S (Theorem 4.5), and, in this case, we explicitly calculate a quiver $Q_{S, G}$ such that $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } kQ_{S, G})$ (Theorem 4.8). We will continue to study $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$ in the subsequent paper.

1.1. Terminology and notation. Throughout this paper, we fix a field k . Unless otherwise stated, a graded algebra means an $\mathbb{N}$ -graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ over k . The group of graded k -algebra automorphisms of A is denoted by $\text{GrAut } A$. We denote by $\text{GrMod } A$ the category of graded right A -modules, and by $\text{grmod } A$ the full subcategory consisting of finitely generated modules. Morphisms in $\text{GrMod } A$ are right A -module homomorphisms of degree zero. Graded left A -modules are identified with graded A^o -modules where A^o is the opposite graded algebra of A . For $M \in \text{GrMod } A$ and $n \in \mathbb{Z}$, we define $M_{\geq n} = \bigoplus_{i \geq n} M_i \in \text{GrMod } A$, and $M(n) \in \text{GrMod } A$ by $M(n) = M$ as an ungraded right A -module with the new grading $M(n)_i = M_{n+i}$. The rule $M \mapsto M(n)$ is a k -linear autoequivalence for $\text{GrMod } A$ and $\text{grmod } A$, called the shift functor. For $M, N \in \text{GrMod } A$, we write

the vector space $\text{Ext}_A^i(M, N) = \text{Ext}_{\text{GrMod } A}^i(M, N)$ and the graded vector space

$$\underline{\text{Ext}}_A^i(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Ext}_A^i(M, N(n)).$$

For a graded right (resp. left) A -module M , we denote by $DM := \underline{\text{Hom}}_k(M, k)$ the dual graded vector space of M . Note that DM has a graded left (resp. right) A -module structure. Let $\sigma \in \text{GrAut } A$ be a graded algebra automorphism. For a graded right A -module $M \in \text{GrMod } A$, we define a new graded right A -module $M_\sigma \in \text{GrMod } A$ by $M_\sigma = M$ as a graded vector space with the new right action $m * a := m\sigma(a)$ for $m \in M$ and $a \in A$. If M is a graded A - A bimodule, then M_σ is also a graded A - A bimodule by this new right action.

We say that $M \in \text{GrMod } A$ is torsion if, for any $m \in M$, there exists $n \in \mathbb{N}$ such that $mA_{\geq n} = 0$. We denote by $\text{Tors } A$ the full subcategory of $\text{GrMod } A$ consisting of torsion modules. We define the torsion functor $\underline{\Gamma}_m : \text{GrMod } A \rightarrow \text{Tors } A$ by

$$\underline{\Gamma}_m(M) = \lim_{n \rightarrow \infty} \underline{\text{Hom}}_A(A/A_{\geq n}, M).$$

We define the local cohomology modules of $M \in \text{GrMod } A$ by

$$\underline{H}_m^i(M) = R^i \underline{\Gamma}_m(M) = \lim_{n \rightarrow \infty} \underline{\text{Ext}}_A^i(A/A_{\geq n}, M),$$

and the depth of M by

$$\text{depth } M = \inf R \underline{\Gamma}_m(M) = \inf \{i \in \mathbb{N} \mid \underline{H}_m^i(M) \neq 0\}.$$

We write $\text{Tails } A$ for the quotient category $\text{GrMod } A / \text{Tors } A$. The quotient functor is denoted by $\pi : \text{GrMod } A \rightarrow \text{Tails } A$. The objects in $\text{Tails } A$ will be denoted by script letters, like $\mathcal{M} = \pi M$. The shift functor on $\text{GrMod } A$ induces an autoequivalence $(n) : \mathcal{M} \mapsto \mathcal{M}(n)$ on $\text{Tails } A$ which we also call the shift functor. For $\mathcal{M}, \mathcal{N} \in \text{Tails } A$, we write the vector space $\text{Ext}_A^i(\mathcal{M}, \mathcal{N}) = \text{Ext}_{\text{Tails } A}^i(\mathcal{M}, \mathcal{N})$ and the graded vector space

$$\underline{\text{Ext}}_A^i(\mathcal{M}, \mathcal{N}) := \bigoplus_{n \in \mathbb{Z}} \text{Ext}_A^i(\mathcal{M}, \mathcal{N}(n)).$$

When A is right noetherian, we define $\text{tails } A := \text{grmod } A / \text{tors } A$ where $\text{tors } A = \text{Tors } A \cap \text{grmod } A$. We define the global dimension of $\text{tails } A$ by

$$\text{gldim}(\text{tails } A) = \sup \{i \mid \text{Ext}_A^i(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \text{tails } A\}.$$

It is easy to see that if A has finite global dimension, then $\text{tails } A$ has finite global dimension.

Definition 1.1 ([19, Definition 2.2]). A right noetherian graded algebra A is called a graded isolated singularity if $\text{gldim}(\text{tails } A) < \infty$.

If A is a commutative graded algebra finitely generated in degree 1, then A is a graded isolated singularity if and only if A is an isolated singularity in the usual sense.

If $A_0 = k$, then we say that A is connected graded. Let A be a noetherian connected graded algebra. Then we view $k = A/A_{\geq 1} \in \text{GrMod } A$ as a graded A -module.

Definition 1.2. A connected graded algebra A is called an AS-Gorenstein (resp. AS-regular) algebra of dimension d and of Gorenstein parameter ℓ if

- $\text{injdim}_A A = \text{injdim}_{A^e} A = d < \infty$ (resp. $\text{gldim } A = d < \infty$), and

$$\bullet \text{ } \underline{\text{Ext}}_A^i(k, A) \cong \underline{\text{Ext}}_{A^o}^i(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

If A is a noetherian AS-Gorenstein algebra of dimension d and of Gorenstein parameter ℓ , then $\underline{H}_m^i(A) = 0$ for all $i \neq d$, and $\underline{H}_m^d(A) \cong DA(\ell)$ as a graded right and left module. We say that $M \in \text{grmod } A$ is graded maximal Cohen-Macaulay if $\underline{H}_m^i(M) = 0$ for all $i \neq d$. We denote by $\text{CM}^{\mathbb{Z}}(A)$ the full subcategory of $\text{grmod } A$ consisting of graded maximal Cohen-Macaulay modules.

Let A be a graded algebra, $\sigma \in \text{GrAut } A$, and $M, N \in \text{GrMod } A$. A k -linear graded map $f : M \rightarrow N$ is called σ -linear if $f : M \rightarrow N_\sigma$ is a graded A -module homomorphism. If A is AS-Gorenstein, then by [8, Lemma 2.2], $\sigma : A \rightarrow A$ induces a σ -linear map $\underline{H}_m^i(\sigma) : \underline{H}_m^i(A) \rightarrow \underline{H}_m^i(A)$. Moreover, there exists a constant $c \in k^\times$ such that $\underline{H}_m^d(\sigma) : \underline{H}_m^d(A) \rightarrow \underline{H}_m^d(A)$ is equal to $cD(\sigma^{-1}) : DA(\ell) \rightarrow DA(\ell)$. The constant c^{-1} is called the homological determinant of σ , and we denote $\text{hdet } \sigma = c^{-1}$ (see [8, Definition 2.3]). By [8, Lemma 2.5], hdet defines a group homomorphism $\text{GrAut } A \rightarrow k^\times$. We define the homological special linear group on A by the kernel of hdet

$$\text{HSL}(A) = \{\sigma \in \text{GrAut } A \mid \text{hdet } \sigma = 1\}.$$

If S is a commutative AS-regular algebra of dimension d , then $S = k[x_1, \dots, x_d]$ is a polynomial algebra. If it is generated in degree 1, then $\text{GrAut } S = \text{GL}(d, k)$ and hdet coincides with the usual determinant of a matrix so that $\text{HSL}(S) = \text{SL}(d, k)$.

Theorem 1.3 ([8, Theorem 3.3]). *If A is a noetherian AS-Gorenstein algebra of dimension d and of Gorenstein parameter ℓ , and $G \leq \text{HSL}(A)$ is a finite subgroup, then the fixed subalgebra A^G is a noetherian AS-Gorenstein algebra of dimension d and of Gorenstein parameter ℓ .*

1.2. Conventions on group actions. In this subsection, we give some conventions on group actions on algebras used in this paper. Let A be a graded algebra and $G \leq \text{GrAut } A$ a finite subgroup. We will always assume that $\text{char } k$ does not divide $|G|$ so that $e := \frac{1}{|G|} \sum_{g \in G} 1 * g \in A * G$ is a well-defined idempotent. Note that this condition is equivalent to the condition that kG is semi-simple. Note also that A^G and $A * G$ are graded by $(A^G)_i = A^G \cap A_i$ and $(A * G)_i = A_i \otimes_k kG$ for $i \in \mathbb{N}$.

The following lemma is well known. These identifications of algebras and modules play essential roles in this paper.

Lemma 1.4. *Let A be a graded algebra and $G \leq \text{GrAut } A$ a finite subgroup. Suppose that $\text{char } k$ does not divide $|G|$ so that $e := \frac{1}{|G|} \sum_{g \in G} 1 * g \in A * G$ is well defined.*

- (1) *The map $\varphi : A^G \rightarrow e(A * G)e$ defined by $\varphi(c) = e(c * 1)e$ is an isomorphism of graded algebras.*
- (2) *The map $\psi : A \rightarrow (A * G)e$ defined by $\psi(a) = (a * 1)e$ is an isomorphism of graded right A^G -modules where the right A^G -module structure on $(A * G)e$ is given by identifying A^G with $e(A * G)e$ via φ .*
- (3) *The map $\phi : A \rightarrow e(A * G)$ defined by $\phi(a) = e(a * 1)$ is an isomorphism of graded left A^G -modules where the left A^G -module structure on $e(A * G)$ is given by identifying A^G with $e(A * G)e$ via φ .*

We also endow a right $A * G$ -module structure on A by identifying A with $e(A * G)$ via the above ϕ so that, for $a \in A$, $b * g \in A * G$, $a \cdot (b * g) = g^{-1}(ab)$. By Lemma 1.4,

$$A^G \cong e(A * G)e \cong \underline{\text{End}}_{A * G}(e(A * G)) \cong \underline{\text{End}}_{A * G}(A)$$

as algebras. We will see in this paper that

$$A * G \cong \underline{\text{End}}_{A^G}((A * G)e) \cong \underline{\text{End}}_{A^G}(A)$$

as algebras in some nice situations (algebraic McKay correspondence).

2. AMPLENESS

We first review the notion of ampleness introduced in [2]. An algebraic triple $(\mathcal{C}, \mathcal{O}, s)$ consists of a k -linear abelian category $\mathcal{C}$, an object $\mathcal{O} \in \mathcal{C}$, and a k -linear autoequivalence $s \in \text{Aut}_k \mathcal{C}$. Given a right noetherian graded algebra A , the noncommutative projective scheme associated to A is defined by the algebraic triple $\text{Proj } A = (\text{tails } A, \mathcal{A}, (1))$. It is important to find a better homogeneous coordinate ring of $\text{Proj } A$, that is, to find a better graded algebra B (e.g. $\text{gldim } B < \infty$) such that $\text{tails } B \cong \text{tails } A$. For a suitable choice of an ample pair $(\mathcal{O}, s)$ for $\text{tails } A$ defined below, the graded algebra $B = B(\text{tails } A, \mathcal{O}, s)$ constructed in Definition 2.2 may be a better homogeneous coordinate ring than A .

Definition 2.1 ([2]). Let $(\mathcal{C}, \mathcal{O}, s)$ be an algebraic triple. We say that the pair $(\mathcal{O}, s)$ is ample for $\mathcal{C}$ if

(A1) for every object $\mathcal{M} \in \mathcal{C}$, there are positive integers $r_1, \dots, r_p \in \mathbb{N}^+$ and an epimorphism $\bigoplus_{i=1}^p s^{-r_i} \mathcal{O} \rightarrow \mathcal{M}$ in $\mathcal{C}$, and

(A2) for every epimorphism $\mathcal{M} \rightarrow \mathcal{N}$ in $\mathcal{C}$, there is an integer n_0 such that the induced map $\text{Hom}_{\mathcal{C}}(s^{-n} \mathcal{O}, \mathcal{M}) \rightarrow \text{Hom}_{\mathcal{C}}(s^{-n} \mathcal{O}, \mathcal{N})$ is surjective for every $n \geq n_0$.

Definition 2.2. We define the graded algebra associated to an algebraic triple $(\mathcal{C}, \mathcal{O}, s)$ by

$$B(\mathcal{C}, \mathcal{O}, s) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(\mathcal{O}, s^i \mathcal{O}).$$

This gives a functor from the category of algebraic triples to the category of graded algebras. Note that $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(\mathcal{O}, s^i \mathcal{F})$ has a natural graded right $B(\mathcal{C}, \mathcal{O}, s)$ -module structure for any object $\mathcal{F} \in \mathcal{C}$.

The χ -condition below plays an essential role in noncommutative algebraic geometry.

Definition 2.3 ([2]). We say that a right noetherian graded algebra A satisfies χ_i if $\underline{\text{Ext}}_A^j(T, N)$ are finite dimensional for all $T \in \text{tors } A$, $N \in \text{grmod } A$ and $0 \leq j \leq i$.

It is known that every noetherian AS-Gorenstein algebra satisfies χ_i for all $i \in \mathbb{N}$. The theorem below justifies the notion of ampleness.

Theorem 2.4 ([2, Corollary 4.6 (1)]). *Let $(\mathcal{C}, \mathcal{O}, s)$ be an algebraic triple. If $\mathcal{O} \in \mathcal{C}$ is a noetherian object, $\dim_k \text{Hom}_{\mathcal{C}}(\mathcal{O}, \mathcal{M}) < \infty$ for all $\mathcal{M} \in \mathcal{C}$, and $(\mathcal{O}, s)$ is ample for $\mathcal{C}$, then $B := B(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ is a locally finite right noetherian graded algebra satisfying χ_1 , and the functor*

$$\mathcal{C} \rightarrow \text{tails } B; \quad \mathcal{F} \mapsto \pi \left(\bigoplus_{i \in \mathbb{N}} \text{Hom}_{\mathcal{C}}(\mathcal{O}, s^i \mathcal{F}) \right)$$

induces an equivalence of algebraic triples $(\mathcal{C}, \mathcal{O}, s) \rightarrow (\text{tails } B, \mathcal{B}, (1)) = \text{Proj } B$.

2.1. Tails of a graded endomorphism algebra. In this subsection, we will consider a Morita type question, that is, finding a condition on $M \in \text{grmod } A$ such that $\text{tails } \underline{\text{End}}_A(M) \cong \text{tails } A$. Since $B(\text{grmod } A, M, (1)) = \underline{\text{End}}_A(M)$ as graded algebras, the answer is given by the ampleness of $(\mathcal{M}, (1))$ for tails A .

Let A, B be graded algebras and M a graded A - B bimodule. We write

$$\begin{aligned} - \otimes_A M &: \text{Tails } A \rightarrow \text{Tails } B, \\ \underline{\text{Hom}}_B(M, -) &: \text{Tails } B \rightarrow \text{Tails } A \end{aligned}$$

for the functors induced by the functors

$$\begin{aligned} - \otimes_A M &: \text{GrMod } A \rightarrow \text{GrMod } B, \\ \underline{\text{Hom}}_B(M, -) &: \text{GrMod } B \rightarrow \text{GrMod } A \end{aligned}$$

by abuse of notation.

Theorem 2.5. *Let A be a noetherian AS-Gorenstein graded isolated singularity of dimension $d \geq 2$, and $M \in \text{CM}^{\mathbb{Z}}(A)$. If M contains A as a direct summand, then $\underline{\text{Hom}}_A(M, -) : \text{tails } A \rightarrow \text{tails } \underline{\text{End}}_A(M)$ is an equivalence functor.*

Proof. Note that $A \in \text{tails } A$ is a noetherian object. Since A is a locally finite right noetherian graded algebra satisfying χ_1 , we have that $\dim_k \text{Hom}_A(A, \mathcal{M}) < \infty$ for all $\mathcal{M} \in \text{tails } A$ by [2, Corollary 4.6 (2)]. We will now show that $(\mathcal{M}, (1))$ is ample for tails A . Since $\mathcal{M}$ contains A as a direct summand, it is easy to see that the condition (A1) is satisfied. Note that every exact sequence $0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow 0$ in tails A is induced by an exact sequence $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$ in $\text{grmod } A$. Consider the exact sequence

$$\underline{\text{Hom}}_A(M, L) \rightarrow \underline{\text{Hom}}_A(M, N) \rightarrow \underline{\text{Ext}}_A^1(M, K).$$

Since A is a noetherian AS-Gorenstein graded isolated singularity and $M \in \text{CM}^{\mathbb{Z}}(A)$, it follows that $\underline{\text{Ext}}_A^1(M, K)$ is finite dimensional over k by [19, Lemma 5.7], so $\underline{\text{Ext}}_A^1(\mathcal{M}, \mathcal{K})$ is right bounded by [2, Corollary 7.3 (2)]. It follows that

$$\text{Ext}_A^1(\mathcal{M}(-n), \mathcal{K}) = \underline{\text{Ext}}_A^1(\mathcal{M}, \mathcal{K})_n = 0$$

for all $n \gg 0$. Since

$$\text{Hom}_A(\mathcal{M}(-n), \mathcal{L}) \rightarrow \text{Hom}_A(\mathcal{M}(-n), \mathcal{N}) \rightarrow \underline{\text{Ext}}_A^1(\mathcal{M}(-n), \mathcal{K})$$

is exact, $\text{Hom}_A(\mathcal{M}(-n), \mathcal{L}) \rightarrow \text{Hom}_A(\mathcal{M}(-n), \mathcal{N})$ is surjective for all $n \gg 0$, so the condition (A2) is also satisfied, hence $(\mathcal{M}, (1))$ is ample for tails A .

Since $\text{depth}_A M = d \geq 2$,

$$\underline{\text{End}}_A(M) = B(\text{grmod } A, M, (1)) \cong B(\text{tails } A, \mathcal{M}, (1))$$

by [13, Lemma 3.3]. By Theorem 2.4, $\underline{\text{End}}_A(M)_{\geq 0} \cong B(\text{tails } A, \mathcal{M}, (1))_{\geq 0}$ is locally finite right noetherian and the functor

$$\pi \circ \underline{\text{Hom}}_A(\mathcal{M}, -) : \text{tails } A \rightarrow \text{tails } \underline{\text{End}}_A(M)_{\geq 0}$$

is an equivalence functor. Since $\underline{\text{End}}_A(M)$ is locally finite and left bounded by [2, Proposition 3.1], $\underline{\text{End}}_A(M) / \underline{\text{End}}_A(M)_{\geq 0}$ is finite dimensional over k , so $\underline{\text{End}}_A(M)$ is finitely generated as a graded right $\underline{\text{End}}_A(M)_{\geq 0}$ -module. It follows that $\underline{\text{End}}_A(M)$ is right noetherian, so the natural functor $\text{tails } \underline{\text{End}}_A(M) \rightarrow \text{tails } \underline{\text{End}}_A(M)_{\geq 0}$ is an equivalence functor by [2, Proposition 2.5 (2)], hence the functor

$$\pi \circ \underline{\text{Hom}}_A(\mathcal{M}, -) : \text{tails } A \rightarrow \text{tails } \underline{\text{End}}_A(M)$$

is an equivalence functor. For $N \in \text{grmod } A$, there exists $n \in \mathbb{Z}$ such that

$$\underline{\text{Hom}}_A(\mathcal{M}, \mathcal{N})_{\geq n} \cong \underline{\text{Hom}}_A(M, N)_{\geq n}$$

in $\text{grmod } \underline{\text{End}}_A(M)$ by [2, Corollary 7.3 (2)], so the above equivalence functor is induced by the functor $\underline{\text{Hom}}_A(M, -) : \text{grmod } A \rightarrow \text{grmod } \underline{\text{End}}_A(M)$. $\square$

Corollary 2.6. *Let S be a noetherian AS-regular algebra of dimension $d \geq 2$, and $G \leq \text{HSL}(S)$. If S^G is a graded isolated singularity, then $\underline{\text{Hom}}_{S^G}(S, -) : \text{tails } S^G \rightarrow \text{tails } \underline{\text{End}}_{S^G}(S)$ is an equivalence functor.*

Proof. Since $G \leq \text{HSL}(S)$, S^G is a noetherian AS-Gorenstein graded isolated singularity of dimension $d \geq 2$ by Theorem 1.3. Since $S \in \text{CM}^{\mathbb{Z}}(S^G)$ contains S^G as a direct summand, the result follows from Theorem 2.5. $\square$

We will consider another situation. Recall that if $e \in A$ is an idempotent, then $\underline{\text{End}}_A(eA) \cong eAe$ as graded algebras, so we will now find a condition on $e \in A$ such that $\text{tails } eAe \cong \text{tails } A$.

The following lemma is known to the experts. The proof is similar to that of Lemma 2.9.

Lemma 2.7. *Let A be a right noetherian graded algebra and $e \in A$ an idempotent such that eAe is a right noetherian graded algebra. If eA is finitely generated as a graded left eAe -module and $A/(e) \in \text{Tors } A$, then*

$$- \otimes_A Ae : \text{Tails } A \rightleftarrows \text{Tails } eAe : - \otimes_{eAe} eA$$

are equivalence functors quasi-inverse to each other. Moreover, if $Ae \in \text{grmod } eAe$, then

$$- \otimes_A Ae : \text{tails } A \rightleftarrows \text{tails } eAe : - \otimes_{eAe} eA$$

are equivalence functors quasi-inverse to each other.

For the purpose of this paper, we modify the χ -condition as follows.

Definition 2.8. We say that a graded algebra A satisfies $\tilde{\chi}_i$ if $\underline{\text{Ext}}_A^j(T, N) \in \text{Tors } A$ for all $T \in \text{GrMod } A^e$ such that $T_A \in \text{Tors } A$, $N \in \text{GrMod } A$ and $0 \leq j \leq i$.

Lemma 2.9. *Let A be a right noetherian graded algebra and $e \in A$ an idempotent such that eAe is a right noetherian graded algebra. If A satisfies $\tilde{\chi}_1$, eA is finitely generated as a graded left eAe -module and $A/(e) \in \text{Tors } A$, then*

$$\underline{\text{Hom}}_A(eA, -) : \text{Tails } A \rightleftarrows \text{Tails } eAe : \underline{\text{Hom}}_{eAe}(Ae, -)$$

are equivalence functors quasi-inverse to each other. Moreover, if $Ae \in \text{grmod } eAe$, then

$$\underline{\text{Hom}}_A(eA, -) : \text{tails } A \rightleftarrows \text{tails } eAe : \underline{\text{Hom}}_{eAe}(Ae, -)$$

are equivalence functors quasi-inverse to each other.

Proof. For $M \in \text{GrMod } eAe$,

$$\underline{\text{Hom}}_A(eA, \underline{\text{Hom}}_{eAe}(Ae, M)) \cong \underline{\text{Hom}}_{eAe}(eA \otimes_A Ae, M) \cong \underline{\text{Hom}}_{eAe}(eAe, M) \cong M$$

in $\text{GrMod } eAe$.

Let $\psi : Ae \otimes_{eAe} eA \rightarrow A$ be a map defined by $\psi(ae \otimes eb) = aeb$. It is easy to see that ψ is a graded A - A bimodule homomorphism with $C := \text{Im } \psi = AeA = (e)$ the two-sided ideal of A generated by e . Let $K := \text{Ker } \psi$ so that

$$0 \rightarrow K \rightarrow Ae \otimes_{eAe} eA \rightarrow A \rightarrow A/(e) \rightarrow 0$$

is an exact sequence in $\text{GrMod } A^e$. By applying $(-)e$ to the above exact sequence, we have

$$0 \rightarrow Ke \rightarrow Ae \otimes_{eAe} eAe \xrightarrow{\sim} Ae \rightarrow 0 \rightarrow 0,$$

so $Ke = 0$. Since $A/(e) \in \text{Tors } A$ and K can be viewed as a graded right $A/(e)$ -module, it follows that $K \in \text{Tors } A$. For $N \in \text{GrMod } A$, exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow Ae \otimes_{eAe} eA \rightarrow C \rightarrow 0, \\ 0 \rightarrow C \rightarrow A \rightarrow A/(e) \rightarrow 0 \end{aligned}$$

in $\text{GrMod } A^e$ induce exact sequences

$$\begin{aligned} 0 \rightarrow \underline{\text{Hom}}_A(C, N) \rightarrow \underline{\text{Hom}}_A(Ae \otimes_{eAe} eA, N) \rightarrow \underline{\text{Hom}}_A(K, N), \\ 0 \rightarrow \underline{\text{Hom}}_A(A/(e), N) \rightarrow \underline{\text{Hom}}_A(A, N) \rightarrow \underline{\text{Hom}}_A(C, N) \rightarrow \underline{\text{Ext}}_A^1(A/(e), N) \end{aligned}$$

in $\text{GrMod } A$. Since A satisfies $\tilde{\chi}_1$, we have that

$$\underline{\text{Hom}}_A(K, N), \underline{\text{Hom}}_A(A/(e), N), \underline{\text{Ext}}_A^1(A/(e), N) \in \text{Tors } A,$$

so

$$\begin{aligned} \pi \underline{\text{Hom}}_{eAe}(Ae, \underline{\text{Hom}}_A(eA, N)) &\cong \pi \underline{\text{Hom}}_A(Ae \otimes_{eAe} eA, N) \\ &\cong \pi \underline{\text{Hom}}_A(C, N) \cong \pi \underline{\text{Hom}}_A(A, N) \cong \pi N \end{aligned}$$

in $\text{Tails } A$.

Since $\underline{\text{Hom}}_A(eA, -) \cong (-)e \cong - \otimes_A Ae : \text{GrMod } A \rightarrow \text{GrMod } eAe$, it follows that $\underline{\text{Hom}}_{eAe}(Ae, -) \cong - \otimes_{eAe} eA : \text{Tails } eAe \rightarrow \text{Tails } A$ by Lemma 2.7, so if $Ae \in \text{grmod } eAe$, then

$$\underline{\text{Hom}}_A(eA, -) : \text{tails } A \cong \text{tails } eAe : \underline{\text{Hom}}_{eAe}(Ae, -)$$

are equivalence functors quasi-inverse to each other by Lemma 2.7 again. $\square$

In the second statement of the above lemma, we may replace the condition $\tilde{\chi}_1$ by the condition χ_1 .

2.2. Ampleness of a group action. Following the previous subsections, we will define a notion of ampleness of a group action on a graded algebra. Let A be a graded algebra, and $r \in \mathbb{N}^+$. The r -th Veronese algebra of A is defined by $A^{(r)} := \bigoplus_{i \in \mathbb{N}} A_{ri}$, and the r -th quasi-Veronese algebra of A is defined by

$$A^{[r]} := \begin{pmatrix} A^{(r)} & A(1)^{(r)} & \cdots & A(r-1)^{(r)} \\ A(-1)^{(r)} & A^{(r)} & \cdots & A(r-2)^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A(-r+1)^{(r)} & A(-r+2)^{(r)} & \cdots & A^{(r)} \end{pmatrix}$$

where the multiplication of $A^{[r]}$ is given by $(a_{ij})(b_{ik}) = (\sum_k a_{kj}b_{ik})$ (see [11]).

Theorem 2.10. *Let A be a right noetherian connected graded algebra of depth $A \geq 2$ satisfying χ_1 , and $r \in \mathbb{N}^+$. Then $(\mathcal{A}, (r))$ is ample for tails A if and only if*

$$(-)e : \text{tails } A^{[r]} \rightarrow \text{tails } e(A^{[r]})e$$

is an equivalence functor where

$$e = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \begin{pmatrix} A^{(r)} & A(1)^{(r)} & \cdots & A(r-1)^{(r)} \\ A(-1)^{(r)} & A^{(r)} & \cdots & A(r-2)^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A(-r+1)^{(r)} & A(-r+2)^{(r)} & \cdots & A^{(r)} \end{pmatrix} = A^{[r]}$$

is an idempotent.

Proof. By [13, Theorem 3.5], $(\mathcal{A}, (r))$ is ample for tails A if and only if $(-)^{(r)} : \text{grmod } A \rightarrow \text{grmod } A^{(r)}$ induces an equivalence functor $(-)^{(r)} : \text{tails } A \rightarrow \text{tails } A^{(r)}$. There exists an equivalence functor $Q : \text{grmod } A \rightarrow \text{grmod } A^{[r]}$ defined by

$$Q(M) = \begin{pmatrix} M^{(r)} \\ M(-1)^{(r)} \\ \vdots \\ M(-r+1)^{(r)} \end{pmatrix}$$

where the right action of $A^{[r]}$ on $Q(M)$ is given by $(m_i)(a_{ij}) = (\sum_k m_k a_{ik})$ (cf. [11, Remark 4.9]). Since

$$e(A^{[r]})e = \begin{pmatrix} A^{(r)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cong A^{(r)}$$

as graded algebras and

$$Q(M)e = \begin{pmatrix} M^{(r)} \\ M(-1)^{(r)} \\ \vdots \\ M(-r+1)^{(r)} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} M^{(r)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cong M^{(r)}$$

in $\text{grmod } A^{(r)}$, $(-)^{(r)} : \text{tails } A \rightarrow \text{tails } A^{(r)}$ is an equivalence functor if and only if the composition of functors

$$(-)e : \text{tails } A^{[r]} \xrightarrow{Q^{-1}} \text{tails } A \xrightarrow{(-)^{(r)}} \text{tails } A^{(r)}$$

is an equivalence functor. $\square$

Let $A = k\langle x_1, \dots, x_n \rangle / I$ be a right noetherian connected graded algebra of depth $A \geq 2$ satisfying χ_1 and $G = \langle \text{diag}(\xi^{\deg x_1}, \dots, \xi^{\deg x_n}) \rangle \leq \text{GL}(n, k)$ a cyclic subgroup generated by a diagonal matrix where $\xi \in k^\times$ is a primitive r -th root of unity. By [13, Theorem 4.4 (1)], $A^{[r]} \cong A * G$ as ungraded algebras, and in this isomorphism, it is shown that the idempotent $e \in A^{[r]}$ above is sent to $\frac{1}{|G|} \sum 1 * g \in A * G$. Since $e(A^{[r]})e \cong A^{(r)} \cong A^G$ as ungraded algebras, we define the notion of ampleness of a group action motivated by the above result.

Definition 2.11. Let A be a right noetherian graded algebra, $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e = \frac{1}{|G|} \sum 1 * g \in A * G$. We say that G is ample for A if

$$(-)e : \text{tails } A * G \rightarrow \text{tails } A^G$$

is an equivalence functor.

Note that, in the above setting, $A * G$ is right noetherian. Moreover, it follows from [12, Corollary 1.12] that A^G is also right noetherian.

Lemma 2.12. *Let A be a right noetherian graded algebra of finite global dimension. If $G \leq \operatorname{GrAut} A$ is a finite ample subgroup such that $\operatorname{char} k$ does not divide $|G|$, then A^G is a graded isolated singularity.*

Proof. Since A has finite global dimension,

$$\operatorname{gldim}(\operatorname{tails} A^G) = \operatorname{gldim}(\operatorname{tails} A * G) \leq \operatorname{gldim} A * G = \operatorname{gldim} A < \infty$$

by [10, Theorem 7.5.6], so A^G is a graded isolated singularity. $\square$

Theorem 2.13. *Let A be a noetherian graded algebra, $G \leq \operatorname{GrAut} A$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$, and $e := \frac{1}{|G|} \sum 1 * g \in A * G$. If $A * G/(e)$ is finite dimensional, then G is ample for A . In fact,*

$$- \otimes_{A * G} (A * G)e : \operatorname{tails} A * G \rightleftarrows \operatorname{tails} A^G : - \otimes_{A^G} e(A * G)$$

are equivalence functors quasi-inverse to each other. Moreover, if A is a locally finite $\mathbb{N}$ -graded algebra satisfying χ_1 , then

$$\underline{\operatorname{Hom}}_{A * G}(A, -) : \operatorname{tails} A * G \rightleftarrows \operatorname{tails} A^G : \underline{\operatorname{Hom}}_{A^G}(A, -)$$

are equivalence functors quasi-inverse to each other.

Proof. Since A is noetherian, both $A * G$ and A^G are noetherian. Moreover, $(A * G)e \cong A$ is finitely generated as a graded right A^G -module and $e(A * G) \cong A$ is finitely generated as a graded left A^G -module by Lemma 1.4 and [12, Corollary 5.9], so

$$- \otimes_{A * G} (A * G)e : \operatorname{tails} A * G \rightleftarrows \operatorname{tails} A^G : - \otimes_{A^G} e(A * G)$$

are equivalence functors quasi-inverse to each other by Lemma 2.7.

Suppose that A is a locally finite $\mathbb{N}$ -graded algebra satisfying χ_1 . Recall that we identify $A = (A * G)e$ in $\operatorname{grmod} A^G$, and $A = e(A * G)$ in $\operatorname{grmod} A * G$. Since A satisfies χ_1 and $A * G$ is finitely generated as a graded left and right A -module, $A * G$ also satisfies χ_1 by [2, Theorem 8.3 (1)] (cf. [2, Proposition 3.1 (3), Corollary 3.6 (1), Proposition 3.11 (2)]), so

$$\underline{\operatorname{Hom}}_{A * G}(A, -) : \operatorname{tails} A * G \rightleftarrows \operatorname{tails} A^G : \underline{\operatorname{Hom}}_{A^G}(A, -)$$

are equivalence functors quasi-inverse to each other by Lemma 2.9. $\square$

3. GRADED SKEW BIMODULE CALABI-YAU ALGEBRAS

Graded bimodule Calabi-Yau algebras are an important class of algebras studied in representation theory of algebras. In this section, we will prove that a graded skew bimodule Calabi-Yau algebra is an AS-regular algebra over R defined in [11], which is one of the generalizations of an AS-regular algebra.

3.1. Conventions on bimodule structures. In this subsection, we will fix some conventions on bimodule structures. Although results in this subsection are known to the experts, it is sensitive to check the bimodule structures in each isomorphism in the proof of Theorem 3.5, so we will give proofs in some detail.

For a graded algebra A , we define the enveloping algebra of A by $A^e := A^o \otimes_k A$. The graded A - A bimodule X can be viewed as a graded right A^e -module by $x(a \otimes b) := axb$. By this right action, we have a commutative diagram

$$\begin{array}{ccc} A^e \times A^e & \xrightarrow{\text{product}} & A^e \\ \parallel & & \parallel \\ (A \otimes_k A) \times A^e & \xrightarrow{\text{right action}} & A \otimes_k A, \end{array}$$

so we can identify A^e with $A \otimes_k A$ as a graded right A^e -module. The map $\varphi : A^e \rightarrow (A^e)^o$; $a \otimes b \rightarrow b \otimes a$ is an isomorphism of graded algebras so the graded A - A bimodule X can be viewed as a graded left A^e -module via φ , that is, $(a \otimes b)x := x\varphi(a \otimes b) = x(b \otimes a) = bxa$. For $X, Y \in \text{GrMod } A^e$, we define the graded A^e - A^e bimodule structure on $X \otimes_k Y$ by

$$(a \otimes b)(x \otimes y)(c \otimes d) := cxa \otimes byd.$$

For an abelian category $\mathcal{C}$, we denote by $\mathcal{D}(\mathcal{C})$ the derived category of $\mathcal{C}$, and by $\mathcal{D}^b(\mathcal{C})$ the bounded derived category of $\mathcal{C}$.

Lemma 3.1. *Let A be a graded algebra. For $X, Y, Z \in \text{GrMod } A^e$,*

- (1) $Z \otimes_{A^e}^{\mathbf{L}} (X \otimes_k Y) \cong X \otimes_A^{\mathbf{L}} Z \otimes_A^{\mathbf{L}} Y$,
- (2) $(X \otimes_k Y) \otimes_{A^e}^{\mathbf{L}} Z \cong Y \otimes_A^{\mathbf{L}} Z \otimes_A^{\mathbf{L}} X$

in $\mathcal{D}(\text{GrMod } A^e)$.

Proof. (1) It is easy to check that the map

$$\phi : Z \otimes_{A^e} (X \otimes_k Y) \rightarrow X \otimes_A Z \otimes_A Y$$

defined by $\phi(z \otimes x \otimes y) = x \otimes z \otimes y$ is a well-defined homomorphism in $\text{GrMod } A^e$. If $Z = A^e$, then ϕ induces an isomorphism of graded vector spaces

$$\begin{array}{ccccccc} A^e \otimes_{A^e} (X \otimes_k Y) & \cong & X \otimes_k Y & \cong & X \otimes_A A \otimes_k A \otimes_A Y & \cong & X \otimes_A A^e \otimes_A Y \\ (a \otimes b) \otimes (x \otimes y) & \rightarrow & xa \otimes by & \leftarrow & (x \otimes a) \otimes (b \otimes y) & \leftarrow & x \otimes (a \otimes b) \otimes y, \end{array}$$

so ϕ is an isomorphism in $\text{GrMod } A^e$. Since tensor products commute with arbitrary direct sum and degree shift, if Z is free in $\text{GrMod } A^e$, then ϕ is an isomorphism in $\text{GrMod } A^e$. If F is a free resolution of Z in $\text{GrMod } A^e$, then F is simultaneously a free resolution of Z in $\text{GrMod } A$ and $\text{GrMod } A^o$, so

$$Z \otimes_{A^e}^{\mathbf{L}} (X \otimes_k Y) \cong F \otimes_{A^e} (X \otimes_k Y) \cong X \otimes_A F \otimes_A Y \cong X \otimes_A^{\mathbf{L}} Z \otimes_A^{\mathbf{L}} Y$$

in $\mathcal{D}(\text{GrMod } A^e)$.

(2) The proof is similar to (1). □

For $X, Y \in \text{GrMod } A^e$, we define the graded A^e - A^e bimodule structure on $\underline{\text{Hom}}_k(X, Y)$ by

$$((a \otimes b)\varphi(c \otimes d))(x) = b\varphi(axc)d.$$

Lemma 3.2. *Let A be a graded algebra. For $X, Y \in \text{GrMod } A^e$,*

$$\text{R}\underline{\text{Hom}}_A(X \otimes_k A, Y) \cong \underline{\text{Hom}}_A(X \otimes_k A, Y) \cong \underline{\text{Hom}}_k(X, Y) \cong DX \otimes_k Y$$

in $\mathcal{D}(\text{GrMod}(A^e)^e)$ (in the derived category of graded A^e - A^e bimodules).

Proof. Viewing $X \otimes_k A, Y$ as graded A -modules,

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_A(X \otimes_k A, Y) &\cong \mathrm{R}\underline{\mathrm{Hom}}_A(X \otimes_k^{\mathbf{L}} A, Y) \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_k(X, \mathrm{R}\underline{\mathrm{Hom}}_A(A, Y)) \\ &\cong \underline{\mathrm{Hom}}_k(X, Y) \end{aligned}$$

in $\mathcal{D}(\mathrm{GrMod} k)$. It follows that $\mathrm{R}\underline{\mathrm{Hom}}_A(X \otimes_k A, Y)$ is concentrated in complex degree 0, so $\mathrm{R}\underline{\mathrm{Hom}}_A(X \otimes_k A, Y) \cong \underline{\mathrm{Hom}}_A(X \otimes_k A, Y)$ in $\mathcal{D}(\mathrm{GrMod}(A^e)^e)$. It is easy to check that the map

$$\alpha : \underline{\mathrm{Hom}}_A(X \otimes_k A, Y) \rightarrow \underline{\mathrm{Hom}}_k(X, \underline{\mathrm{Hom}}_A(A, Y)) \rightarrow \underline{\mathrm{Hom}}_k(X, Y)$$

defined by $\alpha(\phi)(x) = \phi(x \otimes 1)$ is an isomorphism as graded A^e - A^e -bimodules. On the other hand, it is easy to check that the map

$$\beta : DX \otimes_k Y = \underline{\mathrm{Hom}}_k(X, k) \otimes_k Y \rightarrow \underline{\mathrm{Hom}}_k(X, k \otimes_k Y) \rightarrow \underline{\mathrm{Hom}}_k(X, Y)$$

defined by $\beta(\psi \otimes y)(x) = \psi(x)y$ is an isomorphism of graded A^e - A^e bimodules. $\square$

3.2. Graded skew bimodule Calabi-Yau algebras. AS-regular algebras are an important class of algebras studied in noncommutative algebraic geometry, while Calabi-Yau algebras are an important class of algebras studied in representation theory. In this subsection, after extending the notions of these two classes of algebras, we will see some relationships between them.

For a graded algebra A , we denote by $\mathrm{perf}^{\mathbb{Z}} A$ the full subcategory of $\mathcal{D}^b(\mathrm{grmod} A)$ consisting of complexes quasi-isomorphic to perfect complexes, that is, complexes of finite length whose terms are finitely generated projectives. Extending the notion of a Calabi-Yau algebra, the following class of algebras has been defined and studied in several papers (see [15]).

Definition 3.3. A graded algebra A is called a graded skew bimodule Calabi-Yau of dimension d and of Gorenstein parameter ℓ if $A \in \mathrm{perf}^{\mathbb{Z}} A^e$ and $\mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, A^e) \cong {}_{\nu}A(\ell)[-d]$ in $\mathcal{D}(\mathrm{GrMod} A^e)$ for some $\nu \in \mathrm{GrAut} A$ called the Nakayama automorphism of A .

There are a few generalizations of the notion of AS-regularity in the literature. The following generalization is used in this paper.

Definition 3.4 ([11, Definition 3.1]). A locally finite $\mathbb{N}$ -graded algebra S with $R = S_0$, is called AS-regular over R of dimension d and Gorenstein parameter ℓ if

- (1) $\mathrm{gldim} R < \infty$,
- (2) $\mathrm{gldim} S = d < \infty$, and
- (3) $\mathrm{R}\underline{\mathrm{Hom}}_S(R, S) \cong {}_{\sigma}DR(\ell)[-d]$ in $\mathcal{D}(\mathrm{GrMod} R^e)$ for some $\sigma \in \mathrm{Aut}_k R$.

AS-regularity as defined in Definition 1.2 is the same as AS-regularity over k as defined above. The above two classes of algebras are closely related. In fact, they are equivalent if A is connected graded by [15, Lemma 1.2]. We will show one implication for a nonconnected case.

Theorem 3.5. *Let A be a locally finite $\mathbb{N}$ -graded skew bimodule Calabi-Yau algebra with $R = A_0$ of dimension d and of Gorenstein parameter ℓ . If $\mathrm{gldim} R < \infty$ and $\mathrm{gldim} A = d$, then A is an AS-regular algebra over R of dimension d and of Gorenstein parameter ℓ .*

Proof. It is enough to check condition (3) in the above definition. Since A is a graded skew bimodule Calabi-Yau algebra of dimension d and of Gorenstein parameter ℓ , we see that $\mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, A^e) \cong {}_\nu A(\ell)[-d]$ in $\mathcal{D}(\mathrm{GrMod} A^e)$ where $\nu \in \mathrm{GrAut} A$ is the Nakayama automorphism of A . Since

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_A(R, A) &\cong \mathrm{R}\underline{\mathrm{Hom}}_A(R \otimes_A^{\mathbf{L}} A \otimes_A^{\mathbf{L}} A, A) \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_A(A \otimes_{A^e}^{\mathbf{L}} (R \otimes_k A), A) && [\text{Lemma 3.1 (1)}] \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, \mathrm{R}\underline{\mathrm{Hom}}_A(R \otimes_k A, A)) \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, DR \otimes_k A) && [\text{Lemma 3.2}] \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, (DR \otimes_k A) \otimes_{A^e}^{\mathbf{L}} A^e) \\ &\cong (DR \otimes_k A) \otimes_{A^e}^{\mathbf{L}} \mathrm{R}\underline{\mathrm{Hom}}_{A^e}(A, A^e) && [A \in \mathrm{perf}^{\mathbb{Z}} A^e] \\ &\cong (DR \otimes_k A) \otimes_{A^e}^{\mathbf{L}} {}_\nu A(\ell)[-d] \\ &\cong A \otimes_A^{\mathbf{L}} {}_\nu A \otimes_A^{\mathbf{L}} DR(\ell)[-d] && [\text{Lemma 3.1 (2)}] \\ &\cong {}_\nu DR(\ell)[-d], \end{aligned}$$

in $\mathcal{D}(\mathrm{GrMod} A^e)$ (in $\mathcal{D}^b(\mathrm{GrMod} R^e)$), A is an AS-regular algebra over R of dimension d and of Gorenstein parameter ℓ . $\square$

Corollary 3.6. *If S is a noetherian AS-regular algebra over k of dimension d and of Gorenstein parameter ℓ , and $G \leq \mathrm{GrAut} S$ is a finite subgroup such that $\mathrm{char} k$ does not divide $|G|$, then $S * G$ is a noetherian AS-regular algebra over kG of dimension d and of Gorenstein parameter ℓ .*

Proof. Since S is locally finite $\mathbb{N}$ -graded and G is finite, $S * G$ is locally finite $\mathbb{N}$ -graded. Since $\mathrm{char} k$ does not divide $|G|$, we see that $(S * G)_0 = kG$ is semi-simple, that is, $\mathrm{gldim} kG = 0$. Since S is connected graded AS-regular and kG is semi-simple, $S * G$ is graded skew bimodule Calabi-Yau of dimension d and of Gorenstein parameter ℓ by (the proof of) [15, Theorem 4.1]. Since S is noetherian, $S * G$ is also noetherian. By [10, Theorem 7.5.6], $\mathrm{gldim} S * G = \mathrm{gldim} S = d$. By Theorem 3.5, $S * G$ is an AS-regular algebra over kG of dimension d and of Gorenstein parameter ℓ . $\square$

Theorem 3.7. *Let S be a noetherian AS-regular algebra over k of dimension $d \geq 2$, and $G \leq \mathrm{GrAut} S$ a finite ample subgroup. If $\mathrm{char} k$ does not divide $|G|$, then the map*

$$S * G \rightarrow \underline{\mathrm{End}}_{S^G}(S); \quad s * g \mapsto [t \mapsto sg(t)]$$

is an isomorphism of graded algebras.

Proof. Since $S \in \mathrm{grmod} S^G$, it follows that $\mathrm{depth}_{S^G} S = \mathrm{depth}_S S = d \geq 2$. Since S^G is a noetherian connected graded algebra,

$$\underline{\mathrm{End}}_{S^G}(S) = B(\mathrm{grmod} S^G, S, (1)) \cong B(\mathrm{tails} S^G, \pi S, (1))$$

as graded algebras by [13, Lemma 3.3]. Since $S * G$ is a noetherian AS-regular algebra over kG of dimension $d \geq 2$ by Corollary 3.6,

$$B(\mathrm{tails}(S * G), \pi(S * G), (1)) \cong S * G$$

as graded algebras by [11, Proposition 4.4] (cf. [11, Remark 4.5]). By Lemma 1.4,

$$S * G \otimes_{S * G} (S * G)e \cong (S * G)e \cong S$$

in $\mathrm{grmod} S^G$. Since G is ample for S , the equivalence functor

$$- \otimes_{S * G} (S * G)e : \mathrm{tails}(S * G) \rightarrow \mathrm{tails}(S^G)$$

induces an isomorphism of algebraic triples

$$(\text{tails}(S * G), \pi(S * G), (1)) \rightarrow (\text{tails}(S^G), \pi S, (1)),$$

hence

$$S * G \cong B(\text{tails}(S * G), \pi(S * G), (1)) \cong B(\text{tails}(S^G), \pi S, (1)) \cong \underline{\text{End}}_{S^G}(S)$$

as graded algebras. It is easy to see that the composition of the above isomorphisms $\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S)$ sends an element $s * g \in S * G$ to the right multiplication of $(s * g)$ on S via the identification $(\star) S \rightarrow (S * G)e$; $t \mapsto (t * 1)e$ in Lemma 1.4, so

$$\begin{aligned} \Phi(s * g)(t) &=^{(*)} (s * g)(t * 1)e = (sg(t) * g) \left(\frac{1}{|G|} \sum_{h \in G} 1 * h \right) \\ &= \frac{1}{|G|} \sum_{h \in G} sg(t) * gh = \frac{1}{|G|} \sum_{h \in G} sg(t) * h \\ &= (sg(t) * 1) \left(\frac{1}{|G|} \sum_{h \in G} 1 * h \right) = (sg(t) * 1)e \\ &=^{(*)} sg(t). \end{aligned}$$

□

Lemma 3.8. *Let A be a noetherian connected graded algebra. For $M, N \in \text{grmod } A$, there exists an exact sequence*

$$\begin{array}{ccccccc} N \otimes_A \underline{\text{Hom}}_A(M, A) & \xrightarrow{\rho_M^N} & \underline{\text{Hom}}_A(M, N) & \rightarrow & \underline{\text{Hom}}_A(M, N)/P(M, N) & \rightarrow & 0 \\ n \otimes \phi & \mapsto & [m \mapsto n\phi(m)] & & & & \end{array}$$

where $P(M, N)$ is a graded subgroup of $\underline{\text{Hom}}_A(M, N)$ consisting of morphisms factoring through a finitely generated projective graded A -module (i.e. free).

Proof. The proof is analogous to the proof in the ungraded commutative case, as given in [20, Lemma 3.8]. □

Theorem 3.9. *Let A be a noetherian graded algebra of finite global dimension. Assume that there exists an idempotent $e \in A$ such that eAe is a noetherian AS-Gorenstein algebra over k of dimension $d \geq 2$ and $Ae \in \text{CM}^{\mathbb{Z}}(eAe)$. If eAe is a graded isolated singularity, and the graded algebra homomorphism*

$$\Phi : A \rightarrow \underline{\text{End}}_{eAe}(Ae); \quad a \mapsto [xe \mapsto axe]$$

is an isomorphism, then $A/(e)$ is finite dimensional over k .

Proof. Let $\psi : Ae \otimes_{eAe} eA \rightarrow A$ be a homomorphism defined by $\psi(ae \otimes eb) = aeb$. Since the composition of the following homomorphisms:

$$\begin{array}{ll} A & \cong \underline{\text{Hom}}_A(A, A) & a & \mapsto [x \mapsto ax] \\ & \xrightarrow{\underline{\text{Hom}}(\psi, A)} \underline{\text{Hom}}_A(Ae \otimes_{eAe} eA, A) & & \mapsto [xe \otimes ey \mapsto axey] \\ & \cong \underline{\text{Hom}}_{eAe}(Ae, \underline{\text{Hom}}_A(eA, A)) & & \mapsto [xe \mapsto [ey \mapsto axey]] \\ & \cong \underline{\text{Hom}}_{eAe}(Ae, Ae) = \underline{\text{End}}_{eAe}(Ae) & & \mapsto [xe \mapsto axe] \end{array}$$

is equal to Φ , we see that $\underline{\text{Hom}}(\psi, A)$ is an isomorphism by our assumption. Let $\epsilon : A \rightarrow eA$ be a split epimorphism defined by $\epsilon(a) = ea$. Then the commutative

diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_A(A, A) & \xrightarrow[\cong]{\underline{\mathrm{Hom}}(\psi, A)} & \underline{\mathrm{Hom}}_A(Ae \otimes_{eAe} eA, A) \\ \underline{\mathrm{Hom}}(A, \epsilon) \downarrow & & \downarrow \underline{\mathrm{Hom}}(Ae \otimes_{eAe} eA, \epsilon) \\ \underline{\mathrm{Hom}}_A(A, eA) & \xrightarrow{\underline{\mathrm{Hom}}(\psi, eA)} & \underline{\mathrm{Hom}}_A(Ae \otimes_{eAe} eA, eA) \end{array}$$

implies that $\underline{\mathrm{Hom}}(\psi, eA)$ is surjective. Let Ψ be the composition of the following homomorphisms:

$$\begin{array}{ll} eA \cong \underline{\mathrm{Hom}}_A(A, eA) & eb \mapsto [x \mapsto ebx] \\ \xrightarrow{\underline{\mathrm{Hom}}(\psi, eA)} \underline{\mathrm{Hom}}_A(Ae \otimes_{eAe} eA, eA) & \mapsto [xe \otimes ey \mapsto ebxey] \\ \cong \underline{\mathrm{Hom}}_{eAe}(Ae, \underline{\mathrm{Hom}}_A(eA, eA)) & \mapsto [xe \mapsto [ey \mapsto ebxey]] \\ \cong \underline{\mathrm{Hom}}_{eAe}(Ae, eAe) & \mapsto [xe \mapsto ebxe]. \end{array}$$

Since $\underline{\mathrm{Hom}}(\psi, eA)$ is surjective, Ψ is surjective, so

$$\Phi' := Ae \otimes_{eAe} \Psi : Ae \otimes_{eAe} eA \rightarrow Ae \otimes_{eAe} \underline{\mathrm{Hom}}_{eAe}(Ae, eAe)$$

is surjective. By Lemma 3.8, we have a commutative diagram

$$\begin{array}{ccccccc} Ae \otimes_{eAe} eA & \xrightarrow{\psi} & A & \xrightarrow{\quad} & A/(e) & \longrightarrow & 0 \\ \downarrow \Phi' & & \cong \downarrow \Phi & & \downarrow & & \\ Ae \otimes_{eAe} \underline{\mathrm{Hom}}_{eAe}(Ae, eAe) & \xrightarrow{\rho_{Ae}^{Ae}} & \underline{\mathrm{End}}_{eAe}(Ae) & \longrightarrow & \frac{\underline{\mathrm{End}}_{eAe}(Ae)}{P(Ae, Ae)} & \longrightarrow & 0 \end{array}$$

with exact rows. By the five lemma, the induced map

$$A/(e) \rightarrow \underline{\mathrm{End}}_{eAe}(Ae)/P(Ae, Ae)$$

is injective. Since eAe is a noetherian AS-Gorenstein graded isolated singularity of dimension $d \geq 2$, we have

$$\underline{\mathrm{End}}_{eAe}(Ae)/P(Ae, Ae) \cong \underline{\mathrm{End}}_{\mathrm{CM}^{\mathbb{Z}} eAe}(Ae) \cong D \underline{\mathrm{Ext}}_{eAe}^1(Ae, \tau Ae)$$

by [19, Corollary 4.6] where $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$ is the stable category of $\mathrm{CM}^{\mathbb{Z}}(A)$ and τ is the Auslander-Reiten translation on $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$. By [19, Lemma 5.7], $\underline{\mathrm{Ext}}_{eAe}^1(Ae, \tau Ae)$ is finite dimensional over k , so $A/(e)$ is finite dimensional over k . $\square$

Theorem 3.10. *Let S be a noetherian AS-regular algebra over k of dimension $d \geq 2$ and $G \leq \mathrm{HSL}(S)$ a finite subgroup such that $\mathrm{char} k$ does not divide $|G|$. Then the following are equivalent:*

- (1) G is ample for S .
- (2) S^G is a graded isolated singularity, and

$$\Phi : S * G \rightarrow \underline{\mathrm{End}}_{S^G}(S); \quad s * g \mapsto [t \mapsto sg(t)]$$

is an isomorphism of graded algebras.

- (3) $S * G/(e)$ is finite dimensional over k where $e = \frac{1}{|G|} \sum_{g \in G} 1 * g \in S * G$.

Proof. (1) $\Rightarrow$ (2): This follows from Lemma 2.12 and Theorem 3.7.

(2) $\Rightarrow$ (3): Since $S * G$ is a noetherian graded algebra of finite global dimension, $e(S * G)e \cong S^G$ is a noetherian AS-Gorenstein algebra over k of dimension $d \geq 2$ by Lemma 1.4 and Theorem 1.3, and $(S * G)e \cong S \in \mathrm{CM}^{\mathbb{Z}}(S^G)$ by Lemma 1.4, this follows from Theorem 3.9.

- (3) $\Rightarrow$ (1): This follows from Theorem 2.13. $\square$

Corollary 3.11. *Let $S = k[x_1, \dots, x_d]$ be a commutative polynomial algebra such that $\text{char } k = 0$, $\deg x_i = 1$ and $d \geq 2$, and let $G \leq \text{SL}(d, k)$ be a finite subgroup. Then the following are equivalent:*

- (1) G is ample.
- (2) S^G is an isolated singularity.
- (3) $S * G / (e)$ is finite dimensional over k where $e = \frac{1}{|G|} \sum_{g \in G} 1 * g \in S * G$.
- (4) G acts freely on $\mathbb{A}^n \setminus \{0\}$.

Proof. Since $G \leq \text{SL}(d, k)$, it follows that G is small, so

$$\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S); \quad s * g \mapsto [t \mapsto sg(t)]$$

is an isomorphism of graded algebras by [5, Theorem 3.2]. Since $\text{SL}(d, k) = \text{HSL}(S)$, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 3.10. The equivalence (2) $\Leftrightarrow$ (4) follows from [6, 8.2]. $\square$

3.3. Beilinson algebras. The following is a noncommutative generalization of a Beilinson algebra.

Definition 3.12 ([11, Definition 4.7]). The Beilinson algebra of an AS-regular algebra over R of Gorenstein parameter ℓ is defined by

$$\nabla S := \text{End}_S \left(\bigoplus_{i=0}^{\ell-1} S(i) \right) = \begin{pmatrix} S_0 & S_1 & \cdots & S_{\ell-1} \\ 0 & S_0 & \cdots & S_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_0 \end{pmatrix}$$

with the multiplication $(a_{ij})(b_{ij}) = \left(\sum_{k=0}^{\ell-1} a_{kj} b_{ik} \right)$. (We impose this multiplication formula so that $\nabla S = (S^{[\ell]})_0$.)

Note that if a group G acts on an AS-regular algebra S , then G naturally acts on ∇S by $g((a_{ij})) := (g(a_{ij}))$. It is easy to check the following result.

Lemma 3.13. *Let S be a noetherian AS-regular algebra over k of Gorenstein parameter ℓ and $G \leq \text{GrAut } S$ a finite subgroup. If $\text{char } k$ does not divide $|G|$, then $(\nabla S) * G \cong \nabla(S * G)$ as algebras.*

Theorem 3.14. *Let S be a noetherian AS-regular algebra over k of dimension $d \geq 2$ and of Gorenstein parameter ℓ , and $G \leq \text{GrAut } S$ a finite ample subgroup. If $\text{char } k$ does not divide $|G|$, then*

$$\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod}(\nabla S) * G) \cong \mathcal{D}^b \left(\text{mod } \text{End}_{S^G} \left(\bigoplus_{i=0}^{\ell-1} S(i) \right) \right).$$

Moreover, $\bigoplus_{i=0}^{\ell-1} \pi S(i)$ is a tilting object in $\mathcal{D}^b(\text{tails } S^G)$.

Proof. Since G is ample and $S * G$ is a noetherian AS-regular algebra over kG of dimension $d \geq 2$ and of Gorenstein parameter ℓ by Corollary 3.6,

$$\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{tails}(S * G)) \cong \mathcal{D}^b(\text{mod } \nabla(S * G)) \cong \mathcal{D}^b(\text{mod}(\nabla S) * G)$$

by [11, Theorem 4.14] and Lemma 3.13. Since the equivalence functor

$$(-)e : \text{tails}(S * G) \rightarrow \text{tails } S^G$$

sends $\pi(S * G)(i)$ to $\pi(S * G)e(i) \cong \pi S(i)$ for $i \in \mathbb{Z}$ by Lemma 1.4, and we have $\text{depth}_{S^G}(\bigoplus_{i=0}^{\ell-1} S(i)) = d \geq 2$,

$$\begin{aligned} \nabla(S * G) &\cong \text{End}_{\text{tails}(S * G)} \left(\bigoplus_{i=0}^{\ell-1} \pi(S * G)(i) \right) \\ &\cong \text{End}_{\text{tails } S^G} \left(\bigoplus_{i=0}^{\ell-1} \pi S(i) \right) \\ &\cong \text{End}_{S^G} \left(\bigoplus_{i=0}^{\ell-1} S(i) \right) \end{aligned}$$

as algebras by [11, Lemma 4.11] and [13, Lemma 2.9]. Since $\bigoplus_{i=0}^{\ell-1} \pi(S * G)(i)$ is a tilting object in $\mathcal{D}^b(\text{tails } S * G)$ by [11, Proposition 4.3, Proposition 4.4], $\bigoplus_{i=0}^{\ell-1} \pi S(i)$ is a tilting object in $\mathcal{D}^b(\text{tails } S^G)$. $\square$

Let $S = k[x, y]$ be a polynomial algebra and $G \leq \text{SL}(2, k)$ a finite subgroup. The classical McKay correspondence claims that

$$\mathcal{D}^b(\widetilde{\text{Spec } S^G}) \cong \mathcal{D}^b(\text{mod } S * G)$$

where $\widetilde{\text{Spec } S^G}$ is the minimal resolution of $\text{Spec } S^G$. On the other hand, if S is a noetherian AS-regular algebra over k of dimension $d \geq 2$ and $G \leq \text{GrAut } S$ is a finite ample subgroup, then S^G is a graded isolated singularity by Lemma 2.12. This shows that $\text{Proj } S^G$ is smooth so that $\text{Proj } S^G = \text{Proj } S^G$. In this case,

$$\mathcal{D}^b(\widetilde{\text{Proj } S^G}) = \mathcal{D}^b(\text{Proj } S^G) = \mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } \nabla S * G)$$

by Theorem 3.14, which is similar to McKay correspondence (see [13] for more details).

It is interesting to compare with the following theorem.

Theorem 3.15. *Let S be a noetherian AS-regular algebra over k of dimension $d \geq 2$, and $G \leq \text{HSL}(S)$ a finite ample subgroup. If $\text{char } k$ does not divide $|G|$, then S is a $(d - 1)$ -cluster tilting object in $\text{CM}^{\mathbb{Z}}(S^G)$.*

Proof. Since G is ample for S , we see that S^G is a graded isolated singularity by Lemma 2.12. By Theorem 3.7, $\text{End}_{S^G}(S) \cong S * G$ is a finitely generated graded free module over S , so S is a $(d - 1)$ -cluster tilting object in $\text{CM}^{\mathbb{Z}}(S^G)$ by [19, Theorem 5.9]. $\square$

4. NOETHERIAN AS-REGULAR ALGEBRAS OF DIMENSION 2

Let S be a noetherian AS-regular algebra of dimension 2 and $G \leq \text{HSL}(S)$ a finite subgroup. The purpose of this last section is to find a quiver $Q_{S,G}$ such that $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } kQ_{S,G})$. Throughout this section, we assume that k is an algebraically closed field of characteristic 0.

4.1. Classification of group actions. In this subsection, we will show that, for a noetherian AS-regular algebra S of dimension 2, any finite subgroup $G \leq \text{HSL}(S)$ is ample unless $S \cong k\langle x, y \rangle / (xy \pm yx)$.

Lemma 4.1. *Let S be a noetherian AS-regular algebra and $\sigma \in \text{GrAut } S$.*

- (1) *If $x \in S$ is a regular normal element such that $\sigma(x) = ax$ for some $a \in k^\times$, then $\text{hdet}_S \sigma = a(\text{hdet}_A \sigma)$ where $A = S/(x)$.*
- (2) *If S is Koszul and $\sigma|_{S_1} = (c_{ij})$, then $\sigma^t|_{S_1^!} = (c_{ji})$ defines an automorphism $\sigma^t \in \text{GrAut } S^!$, and $\sigma^t(u) = (\text{hdet } \sigma)u$ for any $u \in S_d^!$.*

Proof. (1) is [7, Proposition 2.4]. (2) is a special case of [8, Lemma 4.6]. □

It is known that if S is a noetherian AS-regular algebra of dimension 2, then $S \cong k\langle x, y \rangle / (f)$ for some homogeneous element $f \in k\langle x, y \rangle$ where $\deg x = m, \deg y = n \in \mathbb{N}^+$. If $\gcd(m, n) = r \neq 1$, then $S^{(r)} \cong k\langle x, y \rangle / (f)$ with $\deg x = m/r, \deg y = n/r$ so that $\gcd(\deg x, \deg y) = 1$, and $\text{tails } S^G \cong (\text{tails } (S^{(r)})^G)^{\times r}$ for any $G \leq \text{GrAut } S = \text{GrAut } S^{(r)}$, so it is enough to study the case that $\gcd(\deg x, \deg y) = 1$ in understanding the category $\text{tails } S^G$.

Lemma 4.2. *Let $S = k\langle x, y \rangle / (f)$ be a noetherian AS-regular algebra of dimension 2 such that $\gcd(\deg x, \deg y) = 1$. The following is a list of all possible f (up to isomorphism of graded algebras), $\sigma \in \text{GrAut } S$ and corresponding $\text{hdet } \sigma$:*

$(\deg x, \deg y)$	f	σ	$\text{hdet } \sigma$
$(1, 1)$	$xy - yx$	$x \mapsto ax + by$ $y \mapsto cx + dy$	$ad - bc$
$(1, 1)$	$xy + yx$	$x \mapsto ax$ $y \mapsto dy$	ad
$(1, 1)$	$xy + yx$	$x \mapsto by$ $y \mapsto cx$	bc
$(1, 1)$	$xy - \alpha yx, \alpha \neq 0, \pm 1$	$x \mapsto ax$ $y \mapsto dy$	ad
$(1, q), q \geq 2$	$xy - yx$	$x \mapsto ax$ $y \mapsto cx^q + dy$	ad
$(1, q), q \geq 2$	$xy - \alpha yx, \alpha \neq 0, 1$	$x \mapsto ax$ $y \mapsto dy$	ad
$(p, q), p, q \geq 2$	$xy - \alpha yx, \alpha \neq 0$	$x \mapsto ax$ $y \mapsto dy$	ad
$(1, q), q \geq 1$	$xy - yx - x^{q+1}$	$x \mapsto ax$ $y \mapsto cx^q + a^q y$	a^{q+1}

Proof. Since S is isomorphic to a graded Ore extension of the polynomial algebra in one variable by [17, Proposition 3.3], the list of f follows from a calculation. In each case, it is easy to find all graded algebra automorphisms of S . If $\deg x = \deg y = 1$, then S is Koszul, so we can use Lemma 4.1 (2) to compute their homological determinants. Otherwise, $\sigma(x) = ax$ for some $a \in k^\times$, so we can use Lemma 4.1 (1) to compute their homological determinants. □

Lemma 4.3. *Let $S = k\langle x, y \rangle / (f)$ be a noetherian AS-regular algebra of dimension 2 such that $\gcd(\deg x, \deg y) = 1$. If $S \not\cong k\langle x, y \rangle / (xy \pm yx)$, then every finite*

subgroup $G \leq \text{HSL}(S)$ is of the form

$$G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$$

where ξ is a primitive r -th root of unity with $r = |G|$.

Proof. For $\sigma \in G$, since $\text{char } k = 0$ and σ is of finite order, it is easy to see that $\sigma = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda, \mu \in k^\times$ are roots of unity by Lemma 4.2. Since $\text{hdet } \sigma = \lambda\mu = 1$ by Lemma 4.2, we may write $\sigma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. If $r := \max\{|\lambda| \mid \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in G\} \in \mathbb{N}^+$, then it is easy to see that $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$ where ξ is a primitive r -th root of unity. $\square$

By the above lemma, we focus on studying S^G when $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$.

Lemma 4.4. *Let $S = k\langle x, y \rangle / (xy - yx - x^{q+1})$ where $\deg x = 1, \deg y = q \in \mathbb{N}^+$. For any $1 \leq r \leq q+1$ such that $\gcd(q, r) = 1$, $(\mathcal{S}, (r))$ is ample for tails S .*

Proof. For any $1 \leq m \leq q+1$, since $x^m \in S$ is a normal element, $S/x^m S \cong S/(x^m) \cong k[x, y]/(x^m)$, so, for any $n \geq 1$,

$$\dim_k S/(x^m S + y^n S) = \dim_k k[x, y]/(x^m, y^n) < \infty.$$

For any $1 \leq j \leq q+1$, there exists $i \in \mathbb{N}^+$ such that $1 \leq ir - j \leq r \leq q+1$. Since $\gcd(q, r) = 1$, there exist $m, n \in \mathbb{N}^+$ such that $nr - mq = 1$, so that $\deg(y^{mj}) = mqj = nrj - j$. Define a homomorphism

$$\phi_j : S(-ir) \oplus S(-nrj) \xrightarrow{(x^{ir-j}, y^{mj})} S(-j).$$

Since $1 \leq ir - j \leq q+1$ and $mj \geq 1$, we have that $\text{Coker } \phi_j = (S/x^{ir-j}S + y^{mj}S)(-j)$ is finite dimensional by the above argument. Since the Gorenstein parameter of S is $q+1$, it follows that $\{S(-j)\}_{1 \leq j \leq q+1}$ generates tails S , so $(\mathcal{S}, (r))$ is ample for tails S by [13, Theorem 3.5] (cf. [13, Lemma 2.7]). $\square$

We expect that, for a noetherian AS-regular algebra S of dimension 2, any finite subgroup $G \leq \text{HSL}(S)$ is ample for S . The following theorem shows that this is the case unless $S \cong k\langle x, y \rangle / (xy \pm yx)$.

Theorem 4.5. *Let $S = k\langle x, y \rangle / (f)$ be a noetherian AS-regular algebra of dimension 2 such that $\gcd(\deg x, \deg y) = 1$, and $G \leq \text{HSL}(S)$ a finite subgroup. If $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$ where $\xi \in k^\times$ is a primitive r -th root of unity with $r = |G|$, then*

- (1) *the map $\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S)$ defined by $\Phi(s * g) = [t \mapsto sg(t)]$ is an isomorphism of graded algebras, and*
- (2) *G is ample for S .*

Proof. (1) Since Φ preserves grading by definition, it is enough to show that Φ is an isomorphism of ungraded algebras.

Suppose that $S = k\langle x, y \rangle / (xy - \alpha yx)$, $\alpha \neq 0$. Since $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{r-1} \end{pmatrix} \right\rangle$, by regrading S as $\deg x = 1, \deg y = r - 1$, we see that $(\mathcal{S}, (r))$ becomes ample for tails S by [13, Corollary 3.6], so the map $\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S)$ is an isomorphism of algebras by [19, Theorem 5.11].

Suppose that $S = k\langle x, y \rangle / (xy - yx - x^{q+1})$ with $\deg x = 1, \deg y = q \geq 1$. By Lemma 4.2, $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix} \right\rangle$ such that $r \mid q+1$. Since $\gcd(q, r) = 1$, $(\mathcal{S}, (r))$ is ample

for tails S by Lemma 4.4, so the map $\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S)$ is an isomorphism of graded algebras by [19, Theorem 5.11].

(2) Since S^G is a noetherian AS-Gorenstein graded isolated singularity by [19, Corollary 5.3], and Φ is an isomorphism of graded algebras by (1), G is ample for S by Theorem 3.10. $\square$

Remark 4.6. If $S \not\cong k\langle x, y \rangle / (xy \pm yx)$, then any finite subgroup $G \leq \text{HSL}(S)$ is of the form in the above theorem by Lemma 4.3, so it is always ample. If $S \cong k[x, y]$ with $\deg x = \deg y = 1$ and $G \leq \text{HSL}(S) = \text{SL}(2, k)$ is a finite subgroup, then G is small, so the map $\Phi : S * G \rightarrow \underline{\text{End}}_{S^G}(S)$ is an isomorphism of graded algebras by [5, Theorem 3.2], and G is ample by Corollary 3.11. For $S \cong k\langle x, y \rangle / (xy + yx)$ with $\deg x = \deg y = 1$, finite subgroups of $\text{HSL}(S)$ were classified in [3], but we do not know if they are all ample.

4.2. Quiver representations. If S is a noetherian AS-regular algebra of dimension 2, then $\nabla S \cong kQ_S$ for some quiver Q_S by [13, Theorem 5.4], so if $G \leq \text{HSL}(S)$ is a finite subgroup, then $(\nabla S) * G \cong kQ_S * G$ is Morita equivalent to a path algebra $kQ_{S,G}$ for some quiver $Q_{S,G}$. If $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$, then G is ample, so $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } kQ_{S,G})$ by Theorem 4.5. The purpose of this subsection is to compute $Q_{S,G}$ for each pair (S, G) . (We will see that, in this case, $kQ_S * G$ is isomorphic to $kQ_{S,G}$.)

We fix notation used throughout this subsection. Let $S = k\langle x, y \rangle / (f)$ be a noetherian AS-regular algebra of dimension 2 and of Gorenstein parameter $\ell = \deg x + \deg y$. Then the quiver Q_S is given as follows. The set of vertices is $\{0, 1, \dots, \ell - 1\}$. For any vertex i such that $i + \deg x < \ell$, we draw an arrow $i \xrightarrow{x} i + \deg x$. For any vertex i such that $i + \deg y < \ell$, we draw an arrow $i \xrightarrow{y} i + \deg y$. In addition, we assume that $\gcd(\deg x, \deg y) = 1$. Then the underlying graph of the quiver Q_S is the extended Dynkin diagram of type $\tilde{A}_{\ell-1}$. Let $G \leq \text{HSL}(S)$ be a finite subgroup. Assume that $G = \langle g \rangle$ where

$$g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

and ξ is a primitive r -th root of unity with $r = |G|$. If $f \neq xy \pm yx$, then G is always of this form by Lemma 4.3. Put $\Lambda = (\nabla S) * G \cong kQ_S * G$.

Lemma 4.7. *The following hold:*

- (1) *Let $\{e_0, \dots, e_{\ell-1}\}$ be a complete set of primitive orthogonal idempotents of kQ_S , and let $\rho_i = \frac{1}{r} \sum_{p=0}^{r-1} \xi^{ip} g^p \in kG$ for $i = 0, \dots, r-1$. Then*

$$\{e_i^j := e_i * \rho_j \mid 0 \leq i \leq \ell - 1, 0 \leq j \leq r - 1\}$$

is a complete set of primitive orthogonal idempotents of Λ .

- (2) *Λ is basic.*

Proof. (1) This follows from a straightforward calculation.

- (2) It is enough to show $\Lambda / \text{rad } \Lambda = k \times \dots \times k$ as algebras:

$$\begin{aligned} \Lambda / \text{rad } \Lambda &\cong (kQ_S / \text{rad } kQ_S) * G \\ &\cong (ke_0 \times \dots \times ke_{\ell-1}) * G \\ &\cong ke_0 * G \times \dots \times ke_{\ell-1} * G \end{aligned}$$

where the first isomorphism is by [14, Section 1.2] and the last isomorphism is by the fact that G acts on e_i trivially. Moreover, we have an isomorphism

$$\begin{aligned} ke_i * G &\cong e_i^0(ke_i * G)e_i^0 \times \cdots \times e_i^{r-1}(ke_i * G)e_i^{r-1}; \\ e_i * g^s &\mapsto (e_i^0(e_i * g^s)e_i^0, \dots, e_i^{r-1}(e_i * g^s)e_i^{r-1}). \end{aligned}$$

Since $e_i^j(ke_i * G)e_i^j$ is 1-dimensional over k , we obtain the desired assertion. $\square$

By the above lemma and [14, Theorem 1.3(c)], $\Lambda = kQ_S * G$ is basic and hereditary, so there exists a quiver $Q_{S,G}$ such that $\Lambda \cong kQ_{S,G}$. We now compute the quiver $Q_{S,G}$.

Theorem 4.8. *The quiver $Q_{S,G}$ is given as follows. The set of vertices is $\{(i, j) \mid 0 \leq i \leq \ell-1, 0 \leq j \leq r-1\}$. For any arrow $i \xrightarrow{x} i'$ ($= i + \deg x$) in Q_S and any $j \in \mathbb{Z}/r\mathbb{Z}$, we draw an arrow $(i, j-1) \rightarrow (i', j)$. For any arrow $i \xrightarrow{y} i'$ ($= i + \deg y$) in Q_S and any $j \in \mathbb{Z}/r\mathbb{Z}$, we draw an arrow $(i, j+1) \rightarrow (i', j)$.*

Proof. This result can be obtained by applying arguments in [14, Section 2.3] to our setting. $\square$

Example 4.9. We consider the case $\deg x = 1, \deg y = 1$ and $|G| = 3$ (see [5]). In this case, $\ell = 2, r = 3$. The quiver Q_S is given as follows:

$$0 \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} 1.$$

Then the quiver $Q_{S,G}$ is given as follows:

Example 4.10. We consider the case $\deg x = 1, \deg y = 3$ and $|G| = 2$. In this case, $\ell = 4, r = 2$. The quiver Q_S is given as follows:

$$0 \xrightarrow{x} 1 \xrightarrow{x} 2 \xrightarrow{x} 3.$$

y

Then the quiver $Q_{S,G}$ is given as follows:

$(0,0) \quad (1,0) \quad (2,0) \quad (3,0)$
 $(0,1) \quad (1,1) \quad (2,1) \quad (3,1)$
 $= Q_S \sqcup Q_S.$

Example 4.11. We consider the case $\deg x = 3, \deg y = 5$ and $|G| = 4$. In this case, $\ell = 8, r = 4$. The quiver Q_S is given as follows:

By appropriate renumbering of vertices, it is equal to

$$0 \xrightarrow{x} 1 \xrightarrow{x} 2 \xleftarrow{y} 3 \xrightarrow{x} 4 \xrightarrow{x} 5 \xleftarrow{y} 6 \xrightarrow{x} 7.$$

y

Then the quiver $Q_{S,G}$ is given as follows:

As illustrated in the above examples, the quiver Q_S can be written as

$$0 \xrightarrow{x} \cdots \xrightarrow{x} \ell - 1.$$

y

For a quiver Q_S and a positive integer i , we define the i -covering quiver Q_S^i as follows:

This quiver is uniquely determined by Q_S and i .

By observation, we see that the quiver of $Q_{S,G}$ is given by

$$\underbrace{Q_S^{\frac{m}{\ell}} \sqcup \cdots \sqcup Q_S^{\frac{m}{\ell}}}_{n \text{ times}}$$

where $m = \text{lcm}(\ell, r)$, $n = \text{gcd}(\ell, r)$.

Example 4.12. We consider the case $\deg x = 1$, $\deg y = 3$ and $|G| = 6$. In this case, $\ell = 4$, $r = 6$. The quiver Q_S is given as follows:

$$0 \xrightarrow{x} 1 \xrightarrow{x} 2 \xrightarrow{x} 3.$$

y

Note that $m = \text{lcm}(\ell, r) = 12, n = \text{gcd}(\ell, r) = 2$. Then the quiver $Q_{S,G}$ is given as follows:

For positive integers i, j , we define the quiver $Q_{(i,j)}$ by

A BGP-reflection of a quiver is a new quiver obtained by inverting all arrows incident to a vertex which is a sink or a source. Let Q and Q' be two quivers without oriented cycles. Then the path algebras kQ and kQ' are derived equivalent if and only if Q' can be obtained from Q by a sequence of BGP-reflections (see [4, Chapter I, 5.7]).

As a consequence, we obtain the following result, saying that tails S^G is determined by the triple $(\deg x, \deg y, |G|)$ up to derived equivalence.

Theorem 4.13. *We have the following equivalences of triangulated categories:*

$$\begin{aligned} \mathcal{D}^b(\text{tails } S^G) &\cong \mathcal{D}^b(\text{mod } kQ_{S,G}) \\ &\cong \underbrace{\mathcal{D}^b(\text{mod } kQ_{(\frac{m}{\ell} \deg x, \frac{m}{\ell} \deg y)}) \times \cdots \times \mathcal{D}^b(\text{mod } kQ_{(\frac{m}{\ell} \deg x, \frac{m}{\ell} \deg y)})}_{n \text{ times}}. \end{aligned}$$

Proof. Since Q_S has $\deg y$ arrows labeled as “ x ” and $\deg x$ arrows labeled as “ y ”, the $\frac{m}{\ell}$ -covering quiver $Q_S^{\frac{m}{\ell}}$ has $\frac{m}{\ell} \deg y$ arrows labeled as “ x ” and $\frac{m}{\ell} \deg x$ arrows labeled as “ y ”. So by a sequence of BGP-reflection from $Q_S^{\frac{m}{\ell}}$, we have the quiver $Q_{(\frac{m}{\ell} \deg x, \frac{m}{\ell} \deg y)}$. Hence it follows that

$$\begin{aligned} \mathcal{D}^b(\text{mod } kQ_{S,G}) &\cong \underbrace{\mathcal{D}^b(\text{mod } kQ_S^{\frac{m}{\ell}}) \times \cdots \times \mathcal{D}^b(\text{mod } kQ_S^{\frac{m}{\ell}})}_{n \text{ times}} \\ &\cong \underbrace{\mathcal{D}^b(\text{mod } kQ_{(\frac{m}{\ell} \deg x, \frac{m}{\ell} \deg y)}) \times \cdots \times \mathcal{D}^b(\text{mod } kQ_{(\frac{m}{\ell} \deg x, \frac{m}{\ell} \deg y)})}_{n \text{ times}}. \quad \square \end{aligned}$$

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