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LEVEL 14 AND 15 ANALOGUES OF RAMANUJAN'S ELLIPTIC FUNCTIONS TO ALTERNATIVE BASES

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ABSTRACT. We briefly review Ramanujan's theories of elliptic functions to alternative bases, describe their analogues for levels 5 and 7, and develop new theories for levels 14 and 15. This gives rise to a rich interplay between theta functions, eta-products and Eisenstein series. Transformation formulas of degrees five and seven for hypergeometric functions are obtained, and the paper ends with some series for $1/\pi$ similar to ones found by Ramanujan.

1. Introduction

One of the fundamental functions studied by Ramanujan in his paper "*Modular equations and approximations to* π", [27], is

$$f(\ell) = \frac{\ell P(q^\ell) - P(q)}{\ell - 1}, \tag{1.1}$$

where $\ell \geq 2$ is an integer, called the level, $|q| < 1$, and

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1 - q^j}.$$

The functions $f(\ell)$ have rich properties. For example, $f(4)$ is the generating function for the number of representations of an integer as a sum of four squares, that is (e.g., see [14, (3.13)]),

$$f(4) = \left(\sum_{j=-\infty}^{\infty} q^{j^2} \right)^4. \tag{1.2}$$

Another interesting property is

$$f(4) = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4 \, f(4)} \right)^{2j}, \tag{1.3}$$

where η_m is defined for any positive integer m by

$$\eta_m = q^{m/24} \prod_{j=1}^{\infty} (1 - q^{mj}).$$

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By (1.2) and [7, (3.1.6)–(3.1.8), (4.2.1)–(4.2.3) and Theorem 5.7 (a)(i)], the identity (1.3) is equivalent to

$$\left(\frac{2K}{\pi}\right)^2 = {}_3F_2\left(\begin{matrix}\frac12, \frac12, \frac12\\ 1, 1\end{matrix}; 4k^2k'^2\right), \tag{1.4}$$

where k, k' and K are, respectively, the modulus, complementary modulus, and complete elliptic integral of the first kind, from Jacobi's theory of elliptic functions. In [27], Ramanujan showed how (1.4) (equivalently, (1.3)) can be used to produce some remarkable series that converge to $1/\pi$, for example,

$$\frac{1}{\pi} = \frac{1}{16}\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{(42n+5)}{2^{12n}}. \tag{1.5}$$

Ramanujan indicated that similar results hold for levels 1, 2 and 3. Almost no details of the theories for levels 1, 2 and 3 are given in Ramanujan's paper [27]. There are some details in his notebooks [28], however, and they have been completely analyzed by Berndt, Bhargava and Garvan [6]. A different analysis, taking the function $f(\ell)$ as the starting point, has been given in [14].

Collectively, the theories for levels 1, 2 and 3 are known as "Ramanujan's theories of elliptic functions to alternative bases". The level 3 analogues of (1.2) and (1.3) are

$$f(3) = \left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2}\right)^2 \tag{1.6}$$

and

$$f(3) = \sum_{j=0}^{\infty}\binom{2j}{j}^2\binom{3j}{j}\left(\frac{\eta_1^2\eta_3^2}{f(3)}\right)^{3j}, \tag{1.7}$$

and the level 2 analogues are

$$f(2) = \frac12\left(\left(\sum_{j=-\infty}^{\infty} q^{j^2/2}\right)^4 + \left(\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2/2}\right)^4\right) \tag{1.8}$$

and

$$f(2) = \sum_{j=0}^{\infty}\binom{2j}{j}^2\binom{4j}{2j}\left(\frac{\eta_1^2\eta_2^2}{f(2)}\right)^{4j}. \tag{1.9}$$

In connection with the level 2 theory, Ramanujan gave the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{j=0}^{\infty}\binom{2j}{j}^2\binom{4j}{2j}\frac{(1103+26390j)}{396^{4j}}. \tag{1.10}$$

It converges sufficiently fast—each term adds about 8 decimal digits of accuracy—so that it was used by R. W. Gosper in 1985 to compute the value of π to 17,526,100 decimal places, then a world record. More information about the formulas (1.5) and (1.10), including some details of how they may be derived from (1.3) and (1.9), respectively, has been given in the survey articles [2] and [9]. Further series will be described in Section 7.

For level 1, the appropriate function $f(1)$ is not given by (1.1) but instead by

$$(f(1))^2 = 1 + 240\sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j}. \tag{1.11}$$

The analogue of (1.3) is

$$f(1) = \sum_{j=0}^{\infty} \binom{2j}{j}\binom{3j}{j}\binom{6j}{3j}\left(\frac{\eta_1^4}{f(1)}\right)^{6j}. \tag{1.12}$$

The identities (1.6)–(1.12) can be found in [6] or [14], or they can be proved by putting together identities in those references and applying Clausen's identity in the form given by [12, (20) with $c=0$].

Analogous theories are now also known for levels $5 \le \ell \le 13$. One of the main differences in the theories for levels $\ell \ge 5$ is that the coefficients in the analogues of (1.3), (1.7), (1.9) and (1.12) are no longer given by products of binomial coefficients, but instead by recurrence relations that involve three or more terms. For details and further references, see [12] for levels $5 \le \ell \le 9$, $\ell \ne 7$; [17] for level 7; and [16] for level 10. The results for levels 11, 12 and 13 are given in [21], [19] and [20], respectively. We shall briefly discuss the analogues of the theories for levels 5 and 7, as these are particularly relevant to the present work.

The analogue of (1.3) for level 5 is given by

$$f(5) = \sum_{j=0}^{\infty} \binom{2j}{j}\left\{\sum_{k=0}^{j}\binom{j}{k}^2\binom{j+k}{k}\right\}\left(\frac{\eta_1^2\eta_5^2}{f(5)}\right)^{2j}. \tag{1.13}$$

If b_j denotes the sum in braces, that is,

$$b_j = \sum_{k=0}^{j}\binom{j}{k}^2\binom{j+k}{k},$$

then the b_j do not have a simple closed form, but they satisfy a three-term recurrence relation given by

$$(j+1)^2 b_{j+1} = (11j^2+11j+3)b_j + j^2 b_{j-1}. \tag{1.14}$$

The sequence $\{b_j\}$ was studied by R. Apéry and discussed by van der Poorten [33]. The level 5 analogue of (1.2) involves the theta series of a lattice; see, e.g., [30, A028887]. There is also a close connection with the Rogers-Ramanujan continued fraction $r(q)$ defined by

$$r = r(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}.$$

For example, the power series variable in the identity (1.13) is related to the Rogers-Ramanujan continued fraction by

$$\left(\frac{\eta_1^2\eta_5^2}{f(5)}\right)^2 = \frac{r^5(1-11r^5-r^{10})}{(1+r^{10})^2}.$$

The level 7 analogue of (1.2) is given by

$$f(7)=\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}q^{j^2+jk+2k^2}\right)^2, \tag{1.15}$$

while the analogue of (1.3) is

$$f(7)=\sum_{j=0}^{\infty}\left\{\sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{k}^2\binom{2j-k}{j}\binom{2j-2k}{j}\right\}\left(\frac{\eta_1^2\eta_7^2}{f(7)}\right)^{3j/2}. \tag{1.16}$$

If the coefficients in braces are denoted by c_j, then the following three-term recurrence relation holds:

$$(j+1)^3c_{j+1}=(2j+1)(13j^2+13j+4)c_j+3j(9j^2-1)c_{j-1}. \tag{1.17}$$

The sequence $\{c_j\}$ has been studied in [17], [22], [23] and [32].

The goal of this work is to systematically develop the theories for levels 14 and 15. The theories for these two levels are strikingly similar, and this is largely due to the seemingly trivial observation

$$\sum_{d|14}d=\sum_{d|15}d=24.$$

The theory for level 14 is related to the theories for levels 2 and 7, and the theory for level 15 has connections with the theories for levels 3 and 5. The reader who wishes to skim ahead to see the level 14 and 15 analogues of (1.3) may refer to Theorems 4.3 and 6.3. It may be mentioned that the analogues of the recurrence relations (1.14) and (1.17) for levels 14 and 15 are four-term recurrence relations.

This work is organized as follows. Section 2 contains some background information on modular forms. Sections 3 and 4 contain the results for level 14, and Sections 5 and 6 contain the analogous results for level 15. The basic interrelationships between theta functions, Eisenstein series and eta-products are analyzed, and analogues of (1.3) are established. Finally, as is customary in this subject, some series for $1/\pi$ similar to ones given by Ramanujan are presented in Section 7.

2. Background on modular forms

Let τ be any complex number with positive imaginary part and let $q=\exp(2\pi i\tau)$. The Dedekind eta-function is defined by

$$\eta(\tau)=q^{1/24}\prod_{j=1}^{\infty}(1-q^j).$$

Let $P(q)$ denote Ramanujan's Eisenstein series of weight 2 defined by

$$P(q)=1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}.$$

For any positive integer ℓ, let η_ℓ and P_ℓ be defined by

$$\eta_\ell=\eta(\ell\tau)\quad\text{and}\quad P_\ell=P(q^\ell).$$

An eta-product is a function of the form

$$f(\tau)=\prod_{\delta|\ell}\left(\eta(\delta\tau)\right)^{r_\delta}, \tag{2.1}$$

where ℓ is a positive integer, the product is taken over the positive divisors of ℓ, and the r_δ are integers.

Let $M_k(\Gamma_0(\ell))$ be the space of modular forms of weight k with trivial multiplier system for the modular subgroup $\Gamma_0(\ell)$; see, e.g., [25, Chapter 1] for the definitions. When k is an even integer there is a simple test that can be used to determine if an eta-product is in $M_k(\Gamma_0(\ell))$:

Lemma 2.1. *Let ℓ be a positive integer and consider the eta-product $f(\tau)$ defined by* (2.1)*. Let*

$$k = \frac{1}{2}\sum_{\delta|\ell} r_\delta \quad \textit{and} \quad s = \prod_{\delta|\ell} \delta^{|r_\delta|}.$$

Suppose that

(1) *k is an even integer;*
(2) *s is the square of an integer;*
(3) $\displaystyle\sum_{\delta|\ell} \delta\, r_\delta \equiv 0 \pmod{24}$*;*
(4) $\displaystyle\sum_{\delta|\ell} \frac{\ell}{\delta} r_\delta \equiv 0 \pmod{24}$*;*
(5) $\displaystyle\sum_{\delta|\ell} \gcd(d,\delta)^2\, \frac{r_\delta}{\delta} \geq 0$ *for all $d|\ell$.*

Then $f \in M_k(\Gamma_0(\ell))$.

Proof. This is immediate from [25, Thms. 1.64, 1.65]. The main ideas of the proof are given in [24, Theorem 1]. □

We will also need the following result about Eisenstein series of weight 2.

Lemma 2.2. *Let $\ell \geq 2$ be an integer. Then, $\ell P(q^\ell) - P(q) \in M_2(\Gamma_0(\ell))$.*

Proof. See [29, pp. 177–178]. □

3. Level 14: Basic properties

3.1. **Eta-products and Eisenstein series.** Let u, v, w, x, z and f be defined by

$$u = \left(\frac{\eta_7\eta_{14}}{\eta_1\eta_2}\right)^4, \quad v = \left(\frac{\eta_2\eta_{14}}{\eta_1\eta_7}\right)^3, \quad w = \left(\frac{\eta_1\eta_{14}}{\eta_2\eta_7}\right)^4, \tag{3.1}$$

$$x = \frac{v}{(1+v)(1+8v)} = \frac{w}{(1+w)^2}, \tag{3.2}$$

$$z = \eta_1\eta_2\eta_7\eta_{14} \quad \text{and} \quad f = \frac{z}{x}. \tag{3.3}$$

We will not study f again until Section 4; it has been defined here so that all of the main definitions are in the same place. The two different expressions for x will be shown to be equivalent in Theorem 3.3, below. We shall encounter the functions zv, zv^{-1}, zw and zw^{-1} frequently. Their explicit representations as eta-products are given by

$$zv = \frac{\eta_2^4\eta_{14}^4}{\eta_1^2\eta_7^2}, \quad \frac{z}{v} = \frac{\eta_1^4\eta_7^4}{\eta_2^2\eta_{14}^2}, \quad zw = \frac{\eta_1^5\eta_{14}^5}{\eta_2^3\eta_7^3} \quad \text{and} \quad \frac{z}{w} = \frac{\eta_2^5\eta_7^5}{\eta_1^3\eta_{14}^3}.$$

The eta-products

$$zu = \frac{\eta_7^5\eta_{14}^5}{\eta_1^3\eta_2^3} \quad \text{and} \quad \frac{z}{u} = \frac{\eta_1^5\eta_2^5}{\eta_7^3\eta_{14}^3}$$

will not be featured in our work; see (3.7), below.

Theorem 3.1. *The following results hold:*

The dimension of the space of modular forms of weight 2 *for the modular subgroup* $\Gamma_0(14)$ *is given by*

$$\dim M_2(\Gamma_0(14)) = 4. \tag{3.4}$$

If c_1, c_2, c_7 *and* c_{14} *are any constants that satisfy*

$$14c_1 + 7c_2 + 2c_7 + c_{14} = 0,$$

then

$$c_1P_1 + c_2P_2 + c_7P_7 + c_{14}P_{14} \in M_2(\Gamma_0(14)). \tag{3.5}$$

Furthermore,

$$z,\ zv,\ zv^{-1},\ zw,\ zw^{-1} \in M_2(\Gamma_0(14)) \tag{3.6}$$

and

$$zu,\ zu^{-1} \notin M_2(\Gamma_0(14)). \tag{3.7}$$

Proof. The dimension formula (3.4) follows from [31, Prop. 6.1]. The result (3.5) follows from Lemma 2.2 and the trivial property that

$$M_k(\Gamma_0(\ell)) \subseteq M_k(\Gamma_0(m)) \quad \text{if} \quad \ell|m.$$

The results in (3.6) are immediate consequences of Lemma 2.1.

It remains to prove (3.7). The q-expansions can be used to show that any four of z, zv, zv^{-1}, zw, zw^{-1} are linearly independent. By the dimension formula (3.4), the set $\{zv,\ zv^{-1},\ zw,\ zw^{-1}\}$ is a basis for $M_2(\Gamma_0(14))$. The q-expansions can be used to show that neither zu nor zu^{-1} are linear combinations of zv, zv^{-1}, zw and zw^{-1}. It follows that $zu,\ zu^{-1} \notin M_2(\Gamma_0(14))$. □

The functions zu and zu^{-1} satisfy all of the conditions in Lemma 2.1 except for (5); that is, they are not holomorphic at all of the cusps. It turns out that the weight 6 modular forms z^3u and z^3u^{-1} are holomorphic at the cusps. This will be used in Theorem 3.4 to establish an algebraic equation that relates u to v and w.

It is well known (e.g., [31, p. 83]) that

$$M_k(\Gamma_0(\ell)) = E_k(\Gamma_0(\ell)) \oplus S_k(\Gamma_0(\ell)), \tag{3.8}$$

where $E_k(\Gamma_0(\ell))$ and $S_k(\Gamma_0(\ell))$ are the subspaces of Eisenstein series and cusp forms, respectively, of weight k for $\Gamma_0(\ell)$. From the dimension formulas in [31, p. 93] we find that $\dim E_2(\Gamma_0(14)) = 3$ and $\dim S_2(\Gamma_0(14)) = 1$. In fact,

$$E_2(\Gamma_0(14)) = \{\, c_1P_1 + c_2P_2 + c_7P_7 + c_{14}P_{14} \mid 14c_1 + 7c_2 + 2c_7 + c_{14} = 0\}$$

and

$$S_2(\Gamma_0(14)) = \mathbb{C}z.$$

The next result gives a representation of each of zv, zv^{-1}, zw and zw^{-1} as the sum of an Eisenstein series and a cusp form.

Theorem 3.2. *The following identities hold:*

$$\begin{aligned} zv &= \frac{1}{72}\left(-P_1+P_2+7P_7-7P_{14}\right)-\frac{1}{3}z, \\ \frac{z}{v} &= \frac{1}{18}\left(P_1-4P_2-7P_7+28P_{14}\right)-\frac{8}{3}z, \\ zw &= \frac{1}{144}\left(5P_1-26P_2+91P_7-70P_{14}\right)+\frac{5}{6}z, \\ \frac{z}{w} &= \frac{1}{144}\left(-13P_1+10P_2-35P_7+182P_{14}\right)+\frac{5}{6}z. \end{aligned}$$

Proof. By (3.5) and (3.6) we have

$$z,\ 2P_2-P_1,\ 7P_7-P_1,\ 14P_{14}-P_1 \in M_2(\Gamma_0(14)).$$

It is easy to check that the four functions are linearly independent, so by (3.4) they form a basis for $M_2(\Gamma_0(14))$. The claimed results are just explicit representations for various functions in terms of this basis. □

The next result gives an algebraic relation between v and w.

Theorem 3.3. *The following identity holds:*

$$\frac{v}{(1+v)(1+8v)} = \frac{w}{(1+w)^2}.$$

Proof. By (3.4) and (3.6) we have that z, zv, zv^{-1}, zw, zw^{-1} are linearly dependent; in fact, from Theorem 3.2 we have

$$7z+8zv+\frac{z}{v}-zw-\frac{z}{w}=0.$$

The claimed identity may be obtained by dividing both sides by z and rearranging. □

Theorem 3.3 is equivalent to the modular equation given by Ramanujan in his second notebook [28, Ch. 19, Entry 19, (ix)] but with $-q$ in place of q. Other proofs of this identity have been given by Berndt [3, pp. 314–324], Chan and Lang [13, (3.6)] and Ramanathan [26].

The next result gives an algebraic relation that connects u to x, and hence relates u to v and w.

Theorem 3.4. *The following identity holds:*

$$\frac{u}{(1+49u)^2} = \frac{x^3}{(1-11x)^2}.$$

Proof. Consider the set

$$H = \left\{z^3u, z^3u^{-1}\right\} \cup \left\{z^3w^j \,\middle|\, -3\le j\le 3\right\} \cup \left\{z^3v,\ z^3v^2,\ z^3v^3,\ z^3vw\right\}.$$

By Lemma 2.1 we may deduce that each of the 13 functions in H are in $M_6(\Gamma_0(14))$. By [31, pp. 92–93] the dimension of $M_6(\Gamma_0(14))$ is 12, so H is a linearly dependent set. On equating coefficients of q^n for $-3\le n\le 9$ we deduce that

$$\begin{aligned} &z^3\left(2401u+\frac{1}{u}\right) \\ &= z^3\left(w^3+\frac{1}{w^3}\right)-16z^3\left(w^2+\frac{1}{w^2}\right)+48z^3\left(w+\frac{1}{w}\right)+32z^3, \end{aligned}$$

with the coefficients of z^3v, z^3v^2, z^3v^3 and z^3vw in the linear relation being zero. On dividing by z^3, applying (3.2) and simplifying, we obtain the required identity. □

3.2. **Theta functions.** Ramanujan's theta functions φ and ψ and the septic theta function σ are defined by

$$\varphi = \varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \qquad \psi = \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}$$

and

$$\sigma = \sigma(q) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+2k^2}.$$

The septic theta function is related to Ramanujan's theta functions by:

Theorem 3.5.

$$\sigma(q) = \varphi(-q)\varphi(-q^7) + 4q\psi(q)\psi(q^7) \tag{3.9}$$

and

$$\sigma(q^2) = \varphi(-q)\varphi(-q^7) + 2q\psi(q)\psi(q^7). \tag{3.10}$$

Proof. By a special case of a theorem of Dirichlet (e.g., see [11, Ex. 1]), we have

$$\sigma(q) = 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j}.$$

Ramanujan [3, p. 302], [28, Ch. 19, Entry 17] gave the identities

$$\varphi(-q)\varphi(-q^7) = 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{(-q)^j}{1-q^j}$$

and

$$q\psi(q)\psi(q^7) = \sum_{\substack{j>0 \\ j \text{ odd}}} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j}.$$

Hence,

$$\begin{aligned}\varphi(-q)\varphi(-q^7) + 4q\psi(q)\psi(q^7) &= 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{(-q)^j}{1-q^j} + 4\sum_{\substack{j>0 \\ j \text{ odd}}} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j} \\ &= 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j} = \sigma(q)\end{aligned}$$

and

$$\begin{aligned}\varphi(-q)\varphi(-q^7) + 2q\psi(q)\psi(q^7) &= 1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{(-q)^j}{1-q^j} + 2\sum_{\substack{j>0 \\ j \text{ odd}}} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j} \\ &= 1 + 2\sum_{\substack{j>0 \\ j \text{ even}}} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j} = \sigma(q^2),\end{aligned}$$

where the last step holds since

$$\left(\frac{2j}{7}\right)=\left(\frac{2}{7}\right)\left(\frac{j}{7}\right)=\left(\frac{j}{7}\right).$$

□

By Jacobi's triple product identity, Ramanujan's theta functions have the following representations as infinite products (e.g., see [3, pp. 36, 37]):

$$\varphi(-q)=\frac{\eta_1^2}{\eta_2} \quad \text{and} \quad q^{1/8}\psi(q)=\frac{\eta_2^2}{\eta_1}. \tag{3.11}$$

The septic theta function does not have a simple representation as an infinite product. However, certain linear combinations of $\sigma(q)$ and $\sigma(q^2)$ may be expressed as simple infinite products:

Theorem 3.6. *The following identities hold:*

$$\sigma(q)-\sigma(q^2)=2\frac{\eta_2^2\eta_{14}^2}{\eta_1\eta_7} \quad \textit{and} \quad 2\sigma(q^2)-\sigma(q)=\frac{\eta_1^2\eta_7^2}{\eta_2\eta_{14}}.$$

Proof. These are immediate from Theorem 3.5 and (3.11). □

The next result gives two relations between level 7 and level 14 modular functions.

Theorem 3.7. *Let s be the level 7 modular function defined by $s=\eta_7^4/\eta_1^4$ and let $v=(\eta_2\eta_{14}/\eta_1\eta_7)^3$ be the level 14 modular function defined in (3.1). Then*

$$\frac{s}{1+13s+49s^2}=\frac{v}{(1+4v)^3}$$

and

$$\left.\frac{s}{1+13s+49s^2}\right|_{q\to q^2}=\frac{v^2}{(1+2v)^3}.$$

Proof. Ramanujan found that $\sigma^3(q)$ may be expressed as a sum of three infinite products, viz.,

$$\sigma^3(q)=\frac{\eta_1^7}{\eta_7}+13\eta_1^3\eta_7^3+49\frac{\eta_7^7}{\eta_1}. \tag{3.12}$$

See [1, p. 404, Entry 18.2.14] and [3, p. 467, Entry 5(i)], or [11, (13)], for proofs and references. If we cube the identities (3.9) and (3.10) and make use of (3.11) and (3.12), we find that

$$\frac{\eta_1^7}{\eta_7}+13\eta_1^3\eta_7^3+49\frac{\eta_7^7}{\eta_1}=\left(\frac{\eta_1^2\eta_7^2}{\eta_2\eta_{14}}+4\frac{\eta_2^2\eta_{14}^2}{\eta_1\eta_7}\right)^3$$

and

$$\frac{\eta_2^7}{\eta_{14}}+13\eta_2^3\eta_{14}^3+49\frac{\eta_{14}^7}{\eta_2}=\left(\frac{\eta_1^2\eta_7^2}{\eta_2\eta_{14}}+2\frac{\eta_2^2\eta_{14}^2}{\eta_1\eta_7}\right)^3.$$

These may be rearranged to give the desired results. □

The next result expresses the squares and product of septic theta functions as linear combinations of Eisenstein series and the cusp form.

Theorem 3.8. *The following identities hold:*

$$\sigma^2(q) = \frac{1}{6}\left(7P_7 - P_1\right) = \frac{z}{v}(1+4v)^2, \tag{3.13}$$

$$\sigma^2(q^2) = \frac{1}{6}\left(7P_{14} - P_2\right) = \frac{z}{v}(1+2v)^2, \tag{3.14}$$

$$\sigma(q)\sigma(q^2) = \frac{1}{18}\left(14P_{14} + 7P_7 - 2P_2 - P_1\right) + \frac{2}{3}z, \tag{3.15}$$

$$\varphi^2(-q)\varphi^2(-q^7) = \frac{z}{v} = \frac{1}{18}\left(28P_{14} - 7P_7 - 4P_2 + P_1\right) - \frac{8}{3}z \tag{3.16}$$

and

$$q^2\psi^2(q)\psi^2(q^7) = zv = \frac{1}{72}\left(-7P_{14} + 7P_7 + P_2 - P_1\right) - \frac{1}{3}z. \tag{3.17}$$

Proof. The first equality in (3.13) was known to Ramanujan; see [11, Ex. 3] for references. The second equality in (3.13) follows directly from Theorem 3.2.

The first equality in (3.14) is the trivial consequence of replacing q with q^2 in the first equality on (3.13). The second equality in (3.14) follows immediately from Theorem 3.2.

By elementary algebra and Theorem 3.6 we have

$$\begin{aligned}\sigma(q)\sigma(q^2) &= \frac{1}{3}\sigma^2(q) + \frac{2}{3}\sigma^2(q^2) + \frac{1}{3}\left(\sigma(q) - \sigma(q^2)\right)\left(2\sigma(q^2) - \sigma(q)\right) \\ &= \frac{1}{3}\sigma^2(q) + \frac{2}{3}\sigma^2(q^2) + \frac{2}{3}\eta_1\eta_2\eta_7\eta_{14}.\end{aligned}$$

Therefore, (3.15) follows from (3.13) and (3.14) and the definition of z. Alternatively, by (3.13) and (3.14) we have

$$\sigma(q)\sigma(q^2) = \frac{z}{v}(1+2v)(1+4v),$$

and (3.15) follows from this by applying Theorem 3.2.

The first equalities in (3.16) and (3.17) are immediate consequences of (3.1) and (3.11); the second equalities in (3.16) and (3.17) are from Theorem 3.2. $\square$

The next result links the level 2 theta functions $\varphi(-q)$ and $\psi(q)$ to the level 14 functions z and x. It is an analogue of Theorem 3.8.

Theorem 3.9. *The following identities hold:*

$$\sqrt{\varphi^8(-q) + 64q\psi^8(q)} = 2P_2 - P_1 = \frac{z}{x}\left(4\sqrt{1-4x} - 3\sqrt{1-18x+49x^2}\right) \tag{3.18}$$

and

$$\sqrt{\varphi^8(-q^7) + 64q\psi^8(q^7)} = 2P_{14} - P_7 = \frac{z}{7x}\left(4\sqrt{1-4x} + 3\sqrt{1-18x+49x^2}\right). \tag{3.19}$$

Proof. The first equality in (3.18) (and hence, the first equality in (3.19)) is well known; for example, see [14, (4.7)]. It remains to prove the second equality in each of (3.18) and (3.19). By Theorem 3.2 we may deduce that

$$2P_2 - P_1 = 4z\left(\frac{1}{w} - w\right) - 3z\left(\frac{1}{v} - 8v\right) \tag{3.20}$$

and

$$14P_{14} - 7P_7 = 4z\left(\frac{1}{w} - w\right) + 3z\left(\frac{1}{v} - 8v\right). \tag{3.21}$$

From (3.2) it follows that

$$\frac{1}{w} - w = \frac{\sqrt{1-4x}}{x} \qquad \text{and} \qquad \frac{1}{v} - 8v = \frac{\sqrt{1-18x+49x^2}}{x}. \tag{3.22}$$

By using (3.22) in (3.20) and (3.21) we obtain the required results. □

4. Level 14: Differential equations

In this section we will find a third order linear differential equation for f with respect to x. We begin by computing some derivatives.

Lemma 4.1. *Let v, w, x and z be defined by* (3.1)–(3.3)*. The following differentiation formulas hold:*

$$q\frac{d}{dq}\log v = \frac{z}{v}\sqrt{(1+v)(1+8v)(1+5v+8v^2)}, \tag{4.1}$$

$$q\frac{d}{dq}\log w = \frac{z}{w}\sqrt{1-14w+19w^2-14w^3+w^4} \tag{4.2}$$

and

$$q\frac{d}{dq}\log x = \frac{z}{x}\sqrt{(1-4x)(1-18x+49x^2)}. \tag{4.3}$$

Proof. By direct calculations using the definitions of v and w we find that

$$q\frac{d}{dq}\log v = \frac{1}{8}\left(14P_{14} - 7P_7 + 2P_2 - P_1\right)$$

and

$$q\frac{d}{dq}\log w = \frac{1}{6}\left(14P_{14} - 7P_7 - 2P_2 + P_1\right).$$

Hence, by applying Theorem 3.2, we deduce

$$q\frac{d}{dq}\log v - \left(\frac{z}{v} + 7z + 8zv\right) = -2zw, \tag{4.4}$$

$$q\frac{d}{dq}\log v + \left(\frac{z}{v} + 7z + 8zv\right) = \frac{2z}{w} \tag{4.5}$$

and

$$q\frac{d}{dq}\log w - \left(\frac{z}{w} - 7z + zw\right) = -16zv, \tag{4.6}$$

$$q\frac{d}{dq}\log w + \left(\frac{z}{w} - 7z + zw\right) = \frac{2z}{v}. \tag{4.7}$$

The identity (4.1) may be obtained by multiplying (4.4) and (4.5) and simplifying. Similarly, multiplying (4.6) and (4.7) and simplifying gives (4.2). The identity (4.3) can be deduced from (4.2) by making the change of variable $x = w/(1+w)^2$. □

We are now ready to derive a third order linear differential equation for f with respect to x.

Theorem 4.2. *Let x and f be as defined by* (3.2) *and* (3.3)*. Then*

$$x^2(1-4x)(1-18x+49x^2)\frac{d^3f}{dx^3}+3x(1-33x+242x^2-490x^3)\frac{d^2f}{dx^2}$$
$$+(1-76x+867x^2-2352x^3)\frac{df}{dx}-(5-141x+588x^2)f=0. \tag{4.8}$$

Proof. Let t, g and h be defined by

$$t=\frac{s}{1+13s+49s^2},\quad g=\frac{1}{6}\left(7P_7-P_1\right)\quad\text{and}\quad h=\frac{z}{v}, \tag{4.9}$$

where $s=\eta_7^4/\eta_1^4$. It is known (e.g., see [17]) that g satisfies the following third order differential equation with respect to t:

$$t^2(1+t)(1-27t)\frac{d^3g}{dt^3}+3t(1-39t-54t^2)\frac{d^2g}{dt^2}$$
$$+(1-86t-186t^2)\frac{dg}{dt}=4(1+6t)g. \tag{4.10}$$

We change variables from (t,g) to (v,h). By Theorem 3.7 we have

$$t=\frac{v}{(1+4v)^3}. \tag{4.11}$$

By the definitions of g and h in (4.9) and Theorem 3.2 we may deduce that

$$g=\frac{1}{6}\left(7P_7-P_1\right)=\frac{z}{v}+8z+16zv=\frac{(1+4v)^2}{v}z=(1+4v)^2h. \tag{4.12}$$

By applying the change of variables (4.11) and (4.12) to (4.10) we deduce that

$$v^2(1+v)(1+8v)(1+5v+8v^2)\frac{d^3h}{dv^3}$$
$$+3v(1+21v+122v^2+280v^3+192v^4)\frac{d^2h}{dv^2}$$
$$+(1+50v+454v^2+1408v^3+1216v^4)\frac{dh}{dv}$$
$$+4(1+22v+108v^2+128v^3)h=0. \tag{4.13}$$

Now we make another change of variables from (v,h) to (x,f). By (3.2), (3.3) and (4.9) we have

$$f=\frac{z}{x}=\frac{hv}{x}=h(1+v)(1+8v)\quad\text{and}\quad x=\frac{v}{(1+v)(1+8v)}. \tag{4.14}$$

By applying this change of variables to (4.13) we obtain the required differential equation for f with respect to x. □

Let $\{a_n\}$ and $\{c_n\}$ be the sequences defined by the recurrence relations

$$(n+1)^3a_{n+1}=(2n+1)(11n^2+11n+5)a_n$$
$$-n(121n^2+20)a_{n-1}+98n(n-1)(2n-1)a_{n-2},\quad n\ge 0, \tag{4.15}$$

and

$$(n+1)^3c_{n+1}=(2n+1)(13n^2+13n+4)c_n+3n(9n^2-1)c_{n-1},\quad n\ge 0, \tag{4.16}$$

and initial conditions $a_0 = 1$, $c_0 = 1$. Let the generating functions of $\{a_n\}$ and $\{c_n\}$ be $\lambda(y)$ and $\omega(y)$, respectively, that is,

$$\lambda(y) = \sum_{n=0}^{\infty} a_n y^n \qquad \text{and} \qquad \omega(y) = \sum_{n=0}^{\infty} c_n y^n. \tag{4.17}$$

Theorem 4.3. *Let x and f be defined by* (3.2) *and* (3.3)*; equivalently, let*

$$f = \frac{(\eta_1^3\eta_7^3 + \eta_2^3\eta_{14}^3)(\eta_1^3\eta_7^3 + 8\eta_2^3\eta_{14}^3)}{\eta_1^2\eta_2^2\eta_7^2\eta_{14}^2} = \frac{(\eta_2^4\eta_7^4 + \eta_1^4\eta_{14}^4)^2}{\eta_1^3\eta_2^3\eta_7^3\eta_{14}^3}$$

and

$$x = \frac{\eta_1\eta_2\eta_7\eta_{14}}{f}.$$

Suppose t and g are as in (4.9)*. Then*

$$f = \lambda(x) \qquad \text{and} \qquad g = \omega(t),$$

that is,

$$f = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g = \sum_{n=0}^{\infty} c_n t^n. \tag{4.18}$$

Proof. This is immediate from the differential equations (4.8) and (4.10) and the properties that $f = g = 1$ and $x = t = 0$ when $q = 0$. □

It is known (e.g., see [17] or [22]) that

$$c_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j}^2 \binom{2n-j}{n} \binom{2n-2j}{n}. \tag{4.19}$$

The sequence $\{a_n\}$ was first studied in [22]. To the best of our knowledge, a similar formula for a_n as a single sum of terms involving binomial coefficients that is analogous to (4.19) has not yet been given.

The functions λ and ω are interrelated by

Theorem 4.4. *The following identities hold in a neighborhood of $y = 0$:*

$$\begin{aligned} \frac{1}{(1+y)(1+8y)}\,\lambda\left(\frac{y}{(1+y)(1+8y)}\right) &= \frac{1}{(1+4y)^2}\,\omega\left(\frac{y}{(1+4y)^3}\right) \\ &= \frac{1}{(1+2y)^2}\,\omega\left(\frac{y^2}{(1+2y)^3}\right). \end{aligned}$$

Proof. By (4.11), (4.12) and (4.14) we have

$$x = \frac{v}{(1+v)(1+8v)}, \qquad t = \frac{v}{(1+4v)^3} \tag{4.20}$$

and

$$\frac{f}{(1+v)(1+8v)} = h = \frac{g}{(1+4v)^2}. \tag{4.21}$$

By substituting the series expansions (4.18) into (4.21) and using (4.20) we obtain

$$\frac{1}{(1+v)(1+8v)} \sum_{n=0}^{\infty} a_n \left(\frac{v}{(1+v)(1+8v)}\right)^n = \frac{1}{(1+4v)^2} \sum_{n=0}^{\infty} c_n \left(\frac{v}{(1+4v)^3}\right)^n.$$

Replacing v with y gives the first equality.

For the second equality, by Theorem 3.8 we have

$$\frac{1}{(1+4v)^2}\times\frac{1}{6}\left(7P_7-P_1\right)=\frac{1}{(1+2v)^2}\times\frac{1}{6}\left(7P_{14}-P_2\right).$$

By applying (4.9) and (4.18), this becomes

$$\frac{1}{(1+4v)^2}\sum_{n=0}^{\infty}c_n\left(t(q)\right)^n=\frac{1}{(1+2v)^2}\sum_{n=0}^{\infty}c_n\left(t(q^2)\right)^n. \tag{4.22}$$

By (4.9) and Theorem 3.7 we have

$$t(q)=\frac{s}{1+13s+49s^2}=\frac{v}{(1+4v)^3} \tag{4.23}$$

and

$$t(q^2)=\left.\frac{s}{1+13s+49s^2}\right|_{q\to q^2}=\frac{v^2}{(1+2v)^3}. \tag{4.24}$$

Using (4.23) and (4.24) in (4.22) and replacing v with y, we complete the proof. □

The first equality in Theorem 4.4 is due to Guillera and Zudilin [22]. The second identity in Theorem 4.4 is a quadratic transformation formula for the level 7 function ω.

The next result expresses the function λ in terms of the hypergeometric function. It also gives a seventh degree transformation formula for the level 2 hypergeometric function.

Theorem 4.5. *Suppose that x, v and w are related, in a neighborhood of $x=0$, by*

$$x=\frac{v}{(1+v)(1+8v)}=\frac{w}{(1+w)^2}.$$

Let λ be the function defined by (4.17)*. Then,*

$$\begin{aligned} x\lambda(x)&=\frac{vw}{4v(1-w^2)-3w(1-8v^2)}\,{}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix};\frac{256v^4w^3}{(w^3+64v^4)^2}\right)\\ &=\frac{7vw}{4v(1-w^2)+3w(1-8v^2)}\,{}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix};\frac{256v^4w^3}{(1+64v^4w^3)^2}\right).\end{aligned}$$

Proof. By (3.18) and (3.19) we have

$$\begin{aligned} z&=\frac{x}{4\sqrt{1-4x}-3\sqrt{1-18x+49x^2}}\times\left(2P_2-P_1\right)\\ &=\frac{7x}{4\sqrt{1-4x}+3\sqrt{1-18x+49x^2}}\times\left(2P_{14}-P_7\right).\end{aligned} \tag{4.25}$$

The remainder of the proof consists of expressing each term in (4.25) in terms of v and w.

By (3.3), (4.17) and (4.18) we have

$$z=xf=x\sum_{n=0}^{\infty}a_nx^n=x\lambda(x). \tag{4.26}$$

By (3.22) it follows that

$$\frac{x}{4\sqrt{1-4x}-3\sqrt{1-18x+49x^2}}=\frac{vw}{4v(1-w^2)-3w(1-8v^2)} \tag{4.27}$$

and

$$\frac{7x}{4\sqrt{1-4x}+3\sqrt{1-18x+49x^2}}=\frac{7vw}{4v(1-w^2)+3w(1-8v^2)}. \tag{4.28}$$

By (1.9) we have

$$2P_2-P_1=\sum_{j=0}^{\infty}\binom{2j}{j}^2\binom{4j}{2j}\left(\frac{\eta_1^2\eta_2^2}{2P_2-P_1}\right)^{4j}={}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix};\ \frac{256\eta_1^8\eta_2^8}{(2P_2-P_1)^4}\right). \tag{4.29}$$

Applying (3.18), (3.11) and then (3.1) gives

$$\frac{256\eta_1^8\eta_2^8}{(2P_2-P_1)^4}=\frac{256\eta_1^8\eta_2^8}{(\varphi^4(-q)+16q\psi^4(q))^2}=\frac{256\eta_1^8\eta_2^8}{(\eta_1^{16}/\eta_2^8+64\eta_2^{16}/\eta_1^8)^2}=\frac{256v^4w^3}{(w^3+64v^4)^2}. \tag{4.30}$$

Substituting (4.30) into (4.29) gives

$$2P_2-P_1={}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix};\ \frac{256v^4w^3}{(w^3+64v^4)^2}\right), \tag{4.31}$$

and a similar procedure can be used to give

$$2P_{14}-P_7={}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix};\ \frac{256v^4w^3}{(1+64v^4w^3)^2}\right). \tag{4.32}$$

The proof may be completed by substituting (4.26), (4.27), (4.28), (4.31) and (4.32) into (4.25). ☐

5. Level 15: Basic properties

The theory for level 15 parallels the theory for level 14 in many ways. To emphasize the analogy, we use capital letters (e.g., V, Z, etc.) for the level 15 analogues of the corresponding level 14 quantities v, z, etc.

5.1. **Eta-products and Eisenstein series.** Let U, V, W, X, Z and F be defined by

$$U=\left(\frac{\eta_5\eta_{15}}{\eta_1\eta_3}\right)^3,\quad V=\left(\frac{\eta_3\eta_{15}}{\eta_1\eta_5}\right)^2,\quad W=\left(\frac{\eta_1\eta_{15}}{\eta_3\eta_5}\right)^3, \tag{5.1}$$

$$X=\frac{V}{(1+3V)^2}=\frac{W}{1+W-W^2}, \tag{5.2}$$

$$Z=\eta_1\eta_3\eta_5\eta_{15}\quad\text{and}\quad F=\frac{Z}{X}. \tag{5.3}$$

The two different expressions for X will be shown to be equivalent in Theorem 5.3 below. Eta-products for ZV, ZV^{-1}, ZW and ZW^{-1} are given by

$$ZV=\frac{\eta_3^3\eta_{15}^3}{\eta_1\eta_5},\quad \frac{Z}{V}=\frac{\eta_1^3\eta_5^3}{\eta_3\eta_{15}},\quad ZW=\frac{\eta_1^4\eta_{15}^4}{\eta_3^2\eta_5^2},\quad\text{and}\quad \frac{Z}{W}=\frac{\eta_3^4\eta_5^4}{\eta_1^2\eta_{15}^2}. \tag{5.4}$$

Theorem 5.1. *The following results hold:*

The dimension of the space of modular forms of weight 2 for the modular subgroup $\Gamma_0(15)$ is given by

$$\dim M_2(\Gamma_0(15))=4. \tag{5.5}$$

If c_1, c_3, c_5 and c_{15} are any constants that satisfy

$$15c_1+5c_3+3c_5+c_{15}=0,$$

then

$$c_1P_1 + c_3P_3 + c_5P_5 + c_{15}P_{15} \in M_2(\Gamma_0(15)). \tag{5.6}$$

Furthermore,

$$Z,\ ZV,\ ZV^{-1},\ ZW,\ ZW^{-1} \in M_2(\Gamma_0(15)) \tag{5.7}$$

and

$$ZU,\ ZU^{-1} \notin M_2(\Gamma_0(15)). \tag{5.8}$$

Proof. The proof is almost identical to the proof of Theorem 3.1, so we omit the details. □

Analogous to the level 14 case, we have the subspace decomposition

$$M_2(\Gamma_0(15)) = E_2(\Gamma_0(15)) \oplus S_2(\Gamma_0(15)),$$

where

$$E_2(\Gamma_0(15)) = \{\, c_1P_1 + c_3P_3 + c_5P_5 + c_{15}P_{15} \mid 15c_1 + 5c_3 + 3c_5 + c_{15} = 0\}$$

and

$$S_2(\Gamma_0(15)) = \mathbb{C}Z.$$

By comparing coefficients in the q-expansions, we readily obtain the following representations of ZV, ZV^{-1}, ZW and ZW^{-1} as sums of an Eisenstein series and a cusp form.

Theorem 5.2. *The following identities hold:*

$$\begin{aligned}
ZV &= \frac{1}{96}\left(-P_1 + P_3 + 5P_5 - 5P_{15}\right) - \frac{1}{4}Z,\\
\frac{Z}{V} &= \frac{1}{32}\left(P_1 - 9P_3 - 5P_5 + 45P_{15}\right) - \frac{9}{4}Z,\\
ZW &= \frac{1}{96}\left(-P_1 + 21P_3 - 35P_5 + 15P_{15}\right) - \frac{1}{4}Z,\\
\frac{Z}{W} &= \frac{1}{96}\left(-7P_1 + 3P_3 - 5P_5 + 105P_{15}\right) + \frac{1}{4}Z.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.2. □

The next result gives an algebraic relation between V and W.

Theorem 5.3. *The following identity holds:*

$$\frac{V}{(1+3V)^2} = \frac{W}{1+W-W^2}. \tag{5.9}$$

Proof. By Theorem 5.2 it follows that

$$5Z + 9ZV + \frac{Z}{V} + ZW - \frac{Z}{W} = 0,$$

and this implies the result. □

The identity (5.9) is equivalent to one given by Ramanujan at the bottom of one of the pages in his second notebook [28, 1st ed., p. 324; 2nd ed., p. 333]. Proofs have been given by Berndt [4, p. 221, Entry 62] and Chan and Lang [13, (3.10)]. The proof given above is different from those proofs.

The result in the next theorem is simpler than the level 14 analogue in Theorem 3.4 because the modular forms in the proof have weight 4 instead of weight 6.

Theorem 5.4. *The following identity holds:*

$$\frac{U}{1-125U^2} = \frac{V^2}{(1+V+9V^2)(1+3V)(1-3V)}.$$

Proof. By the dimension formulas in [31, Prop. 6.1], the dimension of $M_4(\Gamma_0(15))$ is 8. By Lemma 2.1 we have

$$Z^2V^j \in M_4(\Gamma_0(15)) \quad \text{for} \quad -2 \le j \le 2$$

and

$$Z^2U,\ Z^2U^{-1},\ Z^2VW,\ Z^2V^{-1}W^{-1} \in M_4(\Gamma_0(15)).$$

It follows that a non-trivial linear relation holds among the 9 functions, and by comparing coefficients we deduce that

$$Z^2\left(\frac{1}{U} - 125U\right) = Z^2\left(\frac{1}{V^2} - 81V^2\right) + Z^2\left(\frac{1}{V} - 9V\right).$$

The required identity follows by rearrangement. □

5.2. **Theta functions.** The Borweins' theta functions a, b and c and the level 15 theta functions σ_A and σ_B are defined by

$$a = a(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2},$$

$$b = b(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2}\omega^{j-k}, \quad \omega = \exp(2\pi i/3),$$

$$c = c(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{(j+\frac{1}{3})^2+(j+\frac{1}{3})(k+\frac{1}{3})+(k+\frac{1}{3})^2},$$

$$\sigma_A = \sigma_A(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+4k^2}$$

and

$$\sigma_B = \sigma_B(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{2j^2+jk+2k^2}.$$

One of the fundamental properties of cubic functions is the identity [8]

$$a^3(q) = b^3(q) + c^3(q). \tag{5.10}$$

Analogous to (3.11), the functions b and c have simple representations as infinite products given by [5, p. 109]:

$$b(q) = \frac{\eta_1^3}{\eta_3}, \qquad c(q) = 3\frac{\eta_3^3}{\eta_1}. \tag{5.11}$$

Comparing these infinite products with definitions (5.1) and (5.3), it follows immediately that
(5.12)

$$b(q)b(q^5) = \frac{Z}{V}, \quad c(q)c(q^5) = 9ZV, \quad b(q)c(q^5) = 3ZW^{2/3}, \quad b(q^5)c(q) = \frac{3Z}{W^{2/3}}$$

and

$$b(q)b(q^5)c(q)c(q^5) = 9Z^2. \tag{5.13}$$

Although a does not have a simple representation as an infinite product, the identities (5.10) and (5.11) imply that its cube may be expressed as a sum of two infinite products; this is an analogue of (3.12).

The next result was known to Ramanujan [5, p. 124, Th. 7.6], [28, p. 259].

Theorem 5.5. *The following identity holds:*

$$a(q)a(q^5) = b(q)b(q^5) + c(q)c(q^5) + 3\sqrt{b(q)b(q^5)c(q)c(q^5)}.$$

Proof. By (5.10), expanding, and applying (5.12) we have

$$\begin{aligned} a^3(q)a^3(q^5) &= \left(b^3(q) + c^3(q)\right)\left(b^3(q^5) + c^3(q^5)\right) \\ &= Z^3\left(\frac{1}{V^3} + 729V^3 + \frac{27}{W^2} + 27W^2\right). \end{aligned}$$

Now use Theorem 5.3 to eliminate the terms that involve W, to get

$$\begin{aligned} a^3(q)a^3(q^5) &= Z^3\left(\frac{1}{V^3} + 729V^3 + 27\left(\frac{1}{V} + 5 + 9V\right)^2 + 54\right) \\ &= \left(\frac{Z}{V} + 9ZV + 9Z\right)^3. \end{aligned}$$

By taking the cube roots and applying (5.12) we complete the proof. □

An analogue of Theorem 3.6 is given by

Theorem 5.6. *The following identities hold:*

$$\sigma_A(q) - \sigma_B(q) = 2\frac{\eta_1^2\eta_{15}^2}{\eta_3\eta_5} \quad \textit{and} \quad \sigma_A(q) + \sigma_B(q) = 2\frac{\eta_3^2\eta_5^2}{\eta_1\eta_{15}}.$$

Proof. See [15, Theorem 2.3]. □

The next result expresses various products and squares of theta functions as sums of Eisenstein series and a cusp form.

Theorem 5.7. *The following identities hold:*

$$
\begin{aligned}
a(q)a(q^5) &= \frac{1}{16}\left(-P_1 - 3P_3 + 5P_5 + 15P_{15}\right) + \frac{9}{2}Z,\\
b(q)b(q^5) &= \frac{1}{32}\left(P_1 - 9P_3 - 5P_5 + 45P_{15}\right) - \frac{9}{4}Z,\\
c(q)c(q^5) &= \frac{3}{32}\left(-P_1 + P_3 + 5P_5 - 5P_{15}\right) - \frac{9}{4}Z,\\
a^2(q) &= \frac{1}{2}\left(3P(q^3) - P(q)\right),\\
\sigma_A^2(q) &= \frac{1}{12}\left(-P_1 + 3P_3 - 5P_5 + 15P_{15}\right) + 2Z,\\
\sigma_B^2(q) &= \frac{1}{12}\left(-P_1 + 3P_3 - 5P_5 + 15P_{15}\right) - 2Z,\\
\sigma_A(q)\sigma_B(q) &= \frac{1}{16}\left(-P_1 - 3P_3 + 5P_5 + 15P_{15}\right) + \frac{1}{2}Z.
\end{aligned}
$$

Proof. The results for $b(q)b(q^5)$ and $c(q)c(q^5)$ are immediate consequences of Theorem 5.2 and (5.12); the result for $a(q)a(q^5)$ then follows by applying Theorem 5.5 and (5.13). The result for $a^2(q)$ is well known; e.g., see [11, Ex. 2].

It remains to prove the identities that involve σ_A and σ_B. By Theorem 5.6 and (5.4) we have

$$\sigma_A(q) = \frac{\eta_3^2\eta_5^2}{\eta_1\eta_{15}} + \frac{\eta_1^2\eta_{15}^2}{\eta_3\eta_5} = \sqrt{\frac{Z}{W}} + \sqrt{ZW} \tag{5.14}$$

and

$$\sigma_B(q) = \frac{\eta_3^2\eta_5^2}{\eta_1\eta_{15}} - \frac{\eta_1^2\eta_{15}^2}{\eta_3\eta_5} = \sqrt{\frac{Z}{W}} - \sqrt{ZW}. \tag{5.15}$$

The claimed results for σ_A^2, σ_B^2 and $\sigma_A\sigma_B$ now follow by squaring or multiplying, and then applying Theorem 5.2. ☐

Numerous identities can be obtained by making use of the linear relations among the right-hand sides of the results in Theorem 5.7; we only mention the example

$$a^2(q) + 5a^2(q^5) = 3\sigma_A^2(q) + 3\sigma_B^2(q).$$

The next result gives two relations between level 5 and level 15 modular functions. It is an analogue of Theorem 3.7.

Theorem 5.8. *Let S be the level 5 modular function defined by $S = \eta_5^6/\eta_1^6$ and let $V = (\eta_3\eta_{15}/\eta_1\eta_5)^2$ be the level 15 modular function defined in (5.1). Then*

$$\frac{S}{1+22S+125S^2} = \frac{V}{(1+9V+27V^2)^2}$$

and

$$\left.\frac{S}{1+22S+125S^2}\right|_{q\to q^3} = \frac{V^3}{(1+3V+3V^2)^2}.$$

Proof. By Theorem 5.2 we have

$$\frac{1}{4}\left(5P_5 - P_1\right) = Z\left(\frac{1}{V} + 9 + 27V\right) \tag{5.16}$$

and it is well known (e.g., [18, (2.8)]) that

$$\frac{1}{4}(5P_5 - P_1) = \zeta\sqrt{1+22S+125S^2}, \tag{5.17}$$

where $\zeta = \eta_1^5/\eta_5$. By combining (5.16) and (5.17) and squaring, we get

$$Z^2\left(\frac{1}{V}+9+27V\right)^2 = \zeta^2(1+22S+125S^2).$$

Since

$$\frac{Z^2}{\zeta^2} = \frac{(\eta_1\eta_3\eta_5\eta_{15})^2}{(\eta_1^5/\eta_5)^2} = \frac{\eta_3^2\eta_{15}^2}{\eta_1^2\eta_5^2} \times \frac{\eta_5^6}{\eta_1^6} = VS,$$

this completes the proof of the first result.

The second result may be proved in a similar way, starting with

$$\frac{1}{4}(5P_{15} - P_3) = Z\left(\frac{1}{V}+3+3V\right). \tag{5.18}$$

□

The next theorem is an analogue of Theorem 3.9. The results may be compared with (5.16) and (5.18).

Theorem 5.9. *The following identities that link the cubic theta function $a(q)$ to the level 15 functions X and Z hold:*

$$a^2(q) = \frac{1}{2}(3P_3 - P_1) = \frac{Z}{X}\left(3\sqrt{1-2X+5X^2} - 2\sqrt{1-12X}\right) \tag{5.19}$$

and

$$a^2(q^5) = \frac{1}{2}(3P_{15} - P_5) = \frac{Z}{5X}\left(3\sqrt{1-2X+5X^2} + 2\sqrt{1-12X}\right). \tag{5.20}$$

Proof. The first equality in (5.19) (and hence, the first equality in (5.20)) is well known; for example, see [14, (3.22)]. It remains to prove the second equality in each of (5.19) and (5.20). By Theorem 5.2 we may deduce that

$$\frac{1}{2}(3P_3 - P_1) = 3Z\left(\frac{1}{W}+W\right) - 2Z\left(\frac{1}{V}-9V\right) \tag{5.21}$$

and

$$\frac{1}{2}(3P_{15} - P_5) = \frac{3Z}{5}\left(\frac{1}{W}+W\right) + \frac{2Z}{5}\left(\frac{1}{V}-9V\right). \tag{5.22}$$

From (5.2) it follows that

$$\frac{1}{V} - 9V = \frac{\sqrt{1-12X}}{X} \qquad \text{and} \qquad \frac{1}{W}+W = \frac{\sqrt{1-2X+5X^2}}{X}. \tag{5.23}$$

By using (5.23) in (5.21) and (5.22) we obtain the required results. □

6. Level 15: Differential equations

In this section we will find a third order linear differential equation for F with respect to X. We begin by calculating some derivatives.

Lemma 6.1. *Let V, W, X and Z be defined by (5.1)–(5.3). The following differentiation formulas hold:*

$$q\frac{d}{dq}\log V = \frac{Z}{V}\sqrt{1+10V+47V^2+90V^3+81V^4}, \tag{6.1}$$

$$q\frac{d}{dq}\log W = \frac{Z}{W}\sqrt{(1+W-W^2)(1-11W-W^2)} \tag{6.2}$$

and

$$q\frac{d}{dq}\log X = \frac{Z}{X}\sqrt{(1-12X)(1-2X+5X^2)}. \tag{6.3}$$

Proof. The proof is similar to the proof of Lemma 4.1. □

Theorem 6.2. *The following differential equation holds:*

$$\begin{aligned} &X^2(1-12X)(1-2X+5X^2)\frac{d^3F}{dX^3} \\ &+3X(1-21X+58X^2-150X^3)\frac{d^2F}{dX^2} \\ &+(1-48X+207X^2-720X^3)\frac{dF}{dX} \\ &-3(1-11X+60X^2)F=0. \end{aligned} \tag{6.4}$$

Proof. Let T, G and H be defined by

$$T=\frac{S}{1+22S+125S^2}, \quad G=\frac{1}{4}(5P_5-P_1) \quad \text{and} \quad H=\frac{Z}{V}, \tag{6.5}$$

where $S=\eta_5^6/\eta_1^6$. It is known (e.g., see [17, (23), (24), (30)]) that G satisfies the following third order differential equation with respect to T:

$$\begin{aligned} &T^2(1-44T-16T^2)\frac{d^3G}{dT^3}+3T(1-66T-32T^2)\frac{d^2G}{dT^2} \\ &+(1-144T-108T^2)\frac{dG}{dT}=6(1+2T)G. \end{aligned} \tag{6.6}$$

We change variables from (T,G) to (V,H). By Theorem 5.8 we have

$$T=\frac{V}{(1+9V+27V^2)^2}. \tag{6.7}$$

By (5.16) and (6.5) we find that

$$G=\frac{1}{4}(5P_5-P_1)=Z\left(\frac{1}{V}+9+27V\right)=H(1+9V+27V^2). \tag{6.8}$$

Applying the change of variables (6.7) and (6.8) to (6.6) we find that

$$\begin{aligned}(6.9)\qquad & V^2(1+10V+47V^2+90V^3+81V^4)\frac{d^3H}{dV^3} \\ & + 3V(1+15V+94V^2+225V^3+243V^4)\frac{d^2H}{dV^2} \\ & + (1+36V+351V^2+1134V^3+1539V^4)\frac{dH}{dV} \\ & + 3(1+23V+117V^2+216V^3)H = 0.\end{aligned}$$

Now we make another change of variables from (V, H) to (X, F). By (5.2), (5.3) and (6.5) we have

$$(6.10)\qquad F = \frac{Z}{X} = \frac{HV}{X} = H(1+3V)^2 \quad \text{and} \quad X = \frac{V}{(1+3V)^2}.$$

By applying this change of variables to (6.9) we obtain the required differential equation for F with respect to X. □

Let $\{A_n\}$ and $\{C_n\}$ be the sequences defined by the recurrence relations

$$\begin{aligned}(6.11)\quad (n+1)^3A_{n+1} &= (2n+1)(7n^2+7n+3)A_n \\ &\quad - n(29n^2+4)A_{n-1} + 30n(n-1)(2n-1)A_{n-2}, \quad n \ge 0,\end{aligned}$$

and

(6.12)

$$(n+1)^3C_{n+1} = 2(2n+1)(11n^2+11n+3)C_n + 4n(4n^2-1)C_{n-1}, \quad n \ge 0,$$

and initial conditions $A_0 = 1$, $C_0 = 1$. Let the generating functions be $\Lambda(y)$ and $\Omega(y)$, respectively; that is,

$$(6.13)\qquad \Lambda(y) = \sum_{n=0}^{\infty} A_n y^n \qquad \text{and} \qquad \Omega(y) = \sum_{n=0}^{\infty} C_n y^n.$$

Theorem 6.3. *Let X and F be defined by* (5.2) *and* (5.3)*; equivalently, let*

$$F = \frac{(\eta_1^2\eta_5^2 + 3\eta_3^2\eta_{15}^2)^2}{\eta_1\eta_3\eta_5\eta_{15}} = \frac{\eta_3^6\eta_5^6 + \eta_1^3\eta_3^3\eta_5^3\eta_{15}^3 - \eta_1^6\eta_{15}^6}{\eta_1^2\eta_3^2\eta_5^2\eta_{15}^2}$$

and

$$X = \frac{\eta_1\eta_3\eta_5\eta_{15}}{F}.$$

Suppose T and G are as in (6.5)*. Then*

$$F = \Lambda(X) \qquad \text{and} \qquad G = \Omega(T),$$

that is,

$$(6.14)\qquad F = \sum_{n=0}^{\infty} A_n X^n \qquad \text{and} \qquad G = \sum_{n=0}^{\infty} C_n T^n.$$

Proof. This is immediate from the differential equations (6.4) and (6.6) and the properties $F = G = 1$ and $X = T = 0$ when $q = 0$. □

It is known (e.g., see [17, (1), (7)]) that

$$C_n = \binom{2n}{n} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}.$$

A similar formula for A_n as a single sum of terms involving binomial coefficients has not yet been given.

The functions Λ and Ω are interrelated by

Theorem 6.4. *The following identities hold in a neighborhood of $y = 0$:*

$$\begin{aligned} \frac{1}{(1+3y)^2}\,\Lambda\left(\frac{y}{(1+3y)^2}\right) &= \frac{1}{1+9y+27y^2}\,\Omega\left(\frac{y}{(1+9y+27y^2)^2}\right) \\ &= \frac{1}{1+3y+3y^2}\,\Omega\left(\frac{y^3}{(1+3y+3y^2)^2}\right). \end{aligned}$$

Proof. The proof is similar to the proof to Theorem 4.4, so we omit the details. □

The second equality in Theorem 6.4 gives a cubic transformation for the level 5 function $\Omega(y)$.

The next result expresses the function Λ in terms of the hypergeometric function. It also gives a quintic transformation for the level 3 hypergeometric function.

Theorem 6.5. *Suppose that X, V and W are related, in a neighborhood of $X = 0$, by*

$$X = \frac{V}{(1+3V)^2} = \frac{W}{1+W-W^2}.$$

Then

$$\begin{aligned} X\Lambda(X) &= \frac{VW}{3V(1+W^2)-2W(1-9V^2)}\; {}_3F_2\left(\begin{matrix} \frac13, \frac12, \frac23 \\ 1, 1 \end{matrix}; \frac{108V^3W^2}{(W^2+27V^3)^2}\right) \\ &= \frac{5VW}{3V(1+W^2)+2W(1-9V^2)}\; {}_3F_2\left(\begin{matrix} \frac13, \frac12, \frac23 \\ 1, 1 \end{matrix}; \frac{108V^3W^2}{(1+27V^3W^2)^2}\right). \end{aligned}$$

Proof. By (5.19) and (5.20) we have

$$\begin{aligned} Z &= \frac{X}{3\sqrt{1-2X+5X^2}-2\sqrt{1-12X}} \times \frac{3P_3-P_1}{2} \\ &= \frac{5X}{3\sqrt{1-2X+5X^2}+2\sqrt{1-12X}} \times \frac{3P_{15}-P_5}{2}. \end{aligned} \tag{6.15}$$

The remainder of the proof consists of expressing each term in (6.15) in terms of V and W.

By (5.3), (6.13) and (6.14) we have

$$Z = XF = X\sum_{n=0}^{\infty} A_n X^n = X\Lambda(X). \tag{6.16}$$

By (5.23) it follows that

$$\frac{X}{3\sqrt{1-2X+5X^2}-2\sqrt{1-12X}} = \frac{VW}{3V(1+W^2)-2W(1-9V^2)} \tag{6.17}$$

and

$$\frac{5X}{3\sqrt{1-2X+5X^2}+2\sqrt{1-12X}} = \frac{5VW}{3V(1+W^2)+2W(1-9V^2)}. \tag{6.18}$$

By (1.6) and (1.7) we have

$$\frac{3P_3 - P_1}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left(\frac{\eta_1^2\eta_3^2}{a^2(q)}\right)^{3j} = {}_3F_2\left(\begin{matrix}\frac13,\ \frac12,\ \frac23\\ 1,\ 1\end{matrix};\ \frac{108\eta_1^6\eta_3^6}{a^6(q)}\right). \tag{6.19}$$

Next, applying (5.10), (5.11) and then (5.1), gives

$$\frac{108\eta_1^6\eta_3^6}{a^6(q)} = \frac{108\eta_1^6\eta_3^6}{(b^3(q)+c^3(q))^2} = \frac{108\eta_1^6\eta_3^6}{(\eta_1^9/\eta_3^3 + 27\eta_3^9/\eta_1^3)^2} = \frac{108V^3W^2}{(W^2+27V^3)^2}. \tag{6.20}$$

Substituting (6.20) into (6.19) gives

$$\frac{3P_3 - P_1}{2} = {}_3F_2\left(\begin{matrix}\frac13,\ \frac12,\ \frac23\\ 1,\ 1\end{matrix};\ \frac{108V^3W^2}{(W^2+27V^3)^2}\right), \tag{6.21}$$

and a similar procedure can be used to give

$$\frac{3P_{15} - P_5}{2} = {}_3F_2\left(\begin{matrix}\frac13,\ \frac12,\ \frac23\\ 1,\ 1\end{matrix};\ \frac{108V^3W^2}{(1+27V^3W^2)^2}\right). \tag{6.22}$$

The proof may be completed by substituting (6.16), (6.17), (6.18), (6.21) and (6.22) into (6.15). □

7. Ramanujan-type series for $1/\pi$

Ramanujan's paper [27] contains several interesting series for $1/\pi$, of which (1.5) and (1.10) are representative examples. All of Ramanujan's series may be classified according to two parameters, the level ℓ and the degree N; e.g., see [12]. The examples (1.5) and (1.10) correspond to the instances $(\ell, N) = (4, 7)$ and $(\ell, N) = (2, 29)$, respectively. The series in Ramanujan's paper correspond to levels $\ell \in \{1, 2, 3, 4\}$. Ramanujan's series have been studied by many authors, and analogous series are known for $5 \le \ell \le 13$.

In this section we will present analogues of (1.5) and (1.10) for levels 14 and 15. We begin with the results for level 14.

Theorem 7.1. *Let x and f be defined by* (3.2) *and* (3.3) *and let a_n satisfy the four-term recurrence relation* (4.15) *and initial condition $a_0 = 1$. For any integer $N \ge 2$ let x_N and λ_N be defined by*

$$x_N = x\left(\exp(-2\pi\sqrt{N/14})\right)$$

and

$$\lambda_N = \frac{x(q)}{2N}\,\frac{d}{dx}\,\frac{f(q)}{f(q^N)}\bigg|_{q=\exp(-2\pi/\sqrt{14N})}.$$

Then

$$\frac{1}{2\pi} = \sqrt{\frac{N}{14}}\sqrt{(1-4x_N)(1-18x_N+49x_N^2)}\sum_{n=0}^{\infty} a_n(n+\lambda_N)x_N^n.$$

A similar result holds with $-\exp(-\pi\sqrt{N/7})$ in place of $\exp(-2\pi\sqrt{N/14})$:

Theorem 7.2. *Let x and f be defined by* (3.2) *and* (3.3) *and let a_n satisfy the four term recurrence relation* (4.15) *and initial condition $a_0 = 1$. For any integer $N \ge 7$ let x_N and λ_N be defined by*

$$x_N = x\left(-\exp(-\pi\sqrt{N/7})\right)$$

TABLE 1. Data for Theorems 7.1 and 7.2

q	N	x_N	λ_N
$\exp\left(-2\pi\sqrt{N/14}\right)$	2	$\frac{1}{18}$	$\frac{1}{7}$
	3	$\frac{1}{25}$	$\frac{8}{45}$
	5	$\frac{1}{49}$	$\frac{11}{60}$
$-\exp\left(-\pi\sqrt{N/7}\right)$	3	$-\frac{1}{3}$	series does not converge
	19	$-\frac{1}{171}$	$\frac{73}{340}$

TABLE 2. Data for Theorems 7.3 and 7.4

q	N	X_N	λ_N
$\exp\left(-2\pi\sqrt{N/15}\right)$	2	$\frac{1}{15}$	$\frac{1}{4}$
	4	$\frac{1}{30}$	$\frac{3}{13}$
$-\exp\left(-\pi\sqrt{N/15}\right)$	5	$-\frac{1}{3}$	series does not converge
	13	$-\frac{1}{15}$	$\frac{11}{26}$
	29	$-\frac{1}{75}$	$\frac{251}{986}$
	37	$-\frac{1}{135}$	$\frac{113}{518}$
	53	$-\frac{1}{363}$	$\frac{2327}{13250}$

and

$$\lambda_N = \frac{x(q)}{2N}\,\frac{d}{dx}\,\frac{f(q)}{f(q^N)}\bigg|_{q=-\exp(-\pi/\sqrt{7N})}.$$

Then

$$\frac{1}{\pi} = \sqrt{\frac{N}{7}}\sqrt{(1-4x_N)(1-18x_N+49x_N^2)}\,\sum_{n=0}^{\infty} a_n(n+\lambda_N)x_N^n.$$

The condition $N \geq 7$ in Theorem 7.2 is to ensure convergence, for which we require $|x_N| < 1/(9+4\sqrt{2})$. Examples of values of x_N and λ_N that appear to be rational are given in Table 1.

Here are the corresponding results for level 15.

Theorem 7.3. *Let X and F be defined by* (5.2) *and* (5.3) *and let A_n satisfy the four-term recurrence relation* (6.11) *and initial condition $A_0 = 1$. For any integer $N \geq 2$ let X_N and λ_N be defined by*

$$X_N = X\left(\exp(-2\pi\sqrt{N/15})\right)$$

and

$$\lambda_N = \frac{X(q)}{2N}\frac{d}{dX}\frac{F(q)}{F(q^N)}\bigg|_{q=\exp(-2\pi/\sqrt{15N})}.$$

Then

$$\frac{1}{2\pi} = \sqrt{\frac{N}{15}}\sqrt{(1-12X_N)(1-2X_N+5X_N^2)}\sum_{n=0}^{\infty}A_n(n+\lambda_N)X_N^n.$$

A similar result holds with $-\exp(-\pi\sqrt{N/15})$ in place of $\exp(-2\pi\sqrt{N/15})$:

Theorem 7.4. *Let X and F be defined by* (5.2) *and* (5.3) *and let A_n satisfy the four-term recurrence relation* (6.11) *and initial condition $A_0 = 1$. For any integer $N \geq 12$ let X_N and λ_N be defined by*

$$X_N = X\left(-\exp(-\pi\sqrt{N/15})\right)$$

and

$$\lambda_N = \frac{X(q)}{2N}\frac{d}{dX}\frac{F(q)}{F(q^N)}\bigg|_{q=-\exp(-\pi/\sqrt{15N})}.$$

Then

$$\frac{1}{2\pi} = \sqrt{\frac{N}{15}}\sqrt{(1-12X_N)(1-2X_N+5X_N^2)}\sum_{n=0}^{\infty}A_n(n+\lambda_N)X_N^n.$$

The condition $N \geq 12$ in Theorem 7.4 has been determined numerically. It is to ensure convergence, for which we require $|X_N| < 1/12$. Examples of values of X_N and λ_N that appear to be rational are given in Table 2.

Proof of Theorems 7.1–7.4. All four theorems follow directly from [10, Theorem 2.1]. □

The data in Tables 1 and 2 has been obtained numerically. The parameter values can be proved rigorously, in principle, using the procedure of Chan et al. [10, pp. 408–409]; or see [12, pp. 370–371] for another example. A different method for determining the values of x_N and λ_N has been proposed in [22].

References

[1] George E. Andrews and Bruce C. Berndt, *Ramanujan's lost notebook. Part I*, Springer, New York, 2005. MR2135178 (2005m:11001)

[2] Nayandeep Deka Baruah, Bruce C. Berndt, and Heng Huat Chan, *Ramanujan's series for $1/\pi$: a survey*, Amer. Math. Monthly **116** (2009), no. 7, 567–587, DOI 10.4169/193009709X458555. MR2549375

[3] Bruce C. Berndt, *Ramanujan's notebooks. Part III*, Springer-Verlag, New York, 1991. MR1117903 (92j:01069)

[4] Bruce C. Berndt, *Ramanujan's notebooks. Part IV*, Springer-Verlag, New York, 1994. MR1261634 (95e:11028)

[5] Bruce C. Berndt, *Ramanujan's notebooks. Part V*, Springer-Verlag, New York, 1998. MR1486573 (99f:11024)

[6] Bruce C. Berndt, S. Bhargava, and Frank G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4163–4244, DOI 10.2307/2155035. MR1311903 (97h:33034)

[7] Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM.* A study in analytic number theory and computational complexity. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1987. MR877728 (89a:11134)

[8] J. M. Borwein and P. B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 691–701, DOI 10.2307/2001551. MR1010408 (91e:33012)

[9] J. M. Borwein, P. B. Borwein, and D. H. Bailey, *Ramanujan, modular equations, and approximations to pi, or how to compute one billion digits of pi*, Amer. Math. Monthly **96** (1989), no. 3, 201–219, DOI 10.2307/2325206. MR991866 (90d:11143)

[10] Heng Huat Chan, Song Heng Chan, and Zhiguo Liu, *Domb's numbers and Ramanujan-Sato type series for* $1/\pi$, Adv. Math. **186** (2004), no. 2, 396–410, DOI 10.1016/j.aim.2003.07.012. MR2073912 (2005e:11170)

[11] Heng Huat Chan and Shaun Cooper, *Powers of theta functions*, Pacific J. Math. **235** (2008), no. 1, 1–14, DOI 10.2140/pjm.2008.235.1. MR2379766 (2009a:11095)

[12] Heng Huat Chan and Shaun Cooper, *Rational analogues of Ramanujan's series for* $1/\pi$, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 2, 361–383, DOI 10.1017/S0305004112000254. MR2981931

[13] Heng Huat Chan and Mong Lung Lang, *Ramanujan's modular equations and Atkin-Lehner involutions*, Israel J. Math. **103** (1998), 1–16, DOI 10.1007/BF02762264. MR1613532 (99f:11053)

[14] Shaun Cooper, *Inversion formulas for elliptic functions*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 2, 461–483, DOI 10.1112/plms/pdp007. MR2533672 (2010g:11068)

[15] Shaun Cooper, *Theta function identities for level 15*, Ramanujan rediscovered, Ramanujan Math. Soc. Lect. Notes Ser., vol. 14, Ramanujan Math. Soc., Mysore, 2010, pp. 79–86. MR2856958 (2012i:11039)

[16] Shaun Cooper, *Level 10 analogues of Ramanujan's series for* $1/\pi$, J. Ramanujan Math. Soc. **27** (2012), no. 1, 59–76. MR2933486

[17] Shaun Cooper, *Sporadic sequences, modular forms and new series for* $1/\pi$, Ramanujan J. **29** (2012), no. 1-3, 163–183, DOI 10.1007/s11139-011-9357-3. MR2994096

[18] Shaun Cooper and Pee Choon Toh, *Quintic and septic Eisenstein series*, Ramanujan J. **19** (2009), no. 2, 163–181, DOI 10.1007/s11139-008-9123-3. MR2511669 (2010d:11103)

[19] S. Cooper and D. Ye, *The Level 12 Analogue of Ramanujan's function* k, to appear in J. Aust. Math. Soc.

[20] Shaun Cooper and Dongxi Ye, *The Rogers-Ramanujan continued fraction and its level 13 analogue*, J. Approx. Theory **193** (2015), 99–127, DOI 10.1016/j.jat.2014.01.008. MR3324566

[21] Shaun Cooper, Jinqi Ge, and Dongxi Ye, *Hypergeometric transformation formulas of degrees 3, 7, 11 and 23*, J. Math. Anal. Appl. **421** (2015), no. 2, 1358–1376, DOI 10.1016/j.jmaa.2014.07.061. MR3258323

[22] Jesús Guillera and Wadim Zudilin, *Ramanujan-type formulae for* $1/\pi$*: the art of translation*, The legacy of Srinivasa Ramanujan, Ramanujan Math. Soc. Lect. Notes Ser., vol. 20, Ramanujan Math. Soc., Mysore, 2013, pp. 181–195. MR3221311

[23] Michael D. Hirschhorn, *A sequence considered by Shaun Cooper*, New Zealand J. Math. **43** (2013), 37–42. MR3095664

[24] Taira Honda and Isao Miyawaki, *Zeta-functions of elliptic curves of* 2*-power conductor*, J. Math. Soc. Japan **26** (1974), 362–373. MR0360594 (50 #13042)

[25] Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and* q*-series*, CBMS Regional Conference Series in Mathematics, vol. 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 2004. MR2020489 (2005c:11053)

[26] K. G. Ramanathan, *Ramanujan's modular equations*, Acta Arith. **53** (1990), no. 4, 403–420. MR1067771 (91h:33021)

[27] S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math., **45** (1914), 350–372.

[28] Srinivasa Ramanujan, *Notebooks. Vols. 1, 2*, Tata Institute of Fundamental Research, Bombay, 1957. MR0099904 (20 #6340)

[29] Bruno Schoeneberg, *Elliptic modular functions: an introduction.* Translated from the German by J. R. Smart and E. A. Schwandt. Die Grundlehren der mathematischen Wissenschaften, Band 203, Springer-Verlag, New York-Heidelberg, 1974. MR0412107 (54 #236)

[30] N. J. A. Sloane, *The on-line encyclopedia of integer sequences,* published electronically at http://oeis.org, 2015.

[31] William Stein, *Modular forms, a computational approach.* With an appendix by Paul E. Gunnells. Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007. MR2289048 (2008d:11037)

[32] Zhiwei Sun, *Some new series for* $1/\pi$ *and related congruences* (English, with English and Chinese summaries), Nanjing Daxue Xuebao Shuxue Bannian Kan **31** (2014), no. 2, 150–164. MR3362545

[33] Alfred van der Poorten, *A proof that Euler missed...Apéry's proof of the irrationality of* $\zeta(3)$*.* An informal report. Math. Intelligencer **1** (1978/79), no. 4, 195–203, DOI 10.1007/BF03028234. MR547748 (80i:10054)

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