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ON THE DIMINISHING PROCESS OF BÁLINT TÓTH

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Abstract. Let K and K_0 be convex bodies in $\mathbb{R}^d$, such that K contains the origin, and define the process (K_n, p_n), $n \geq 0$, as follows: let p_{n+1} be a uniform random point in K_n, and set $K_{n+1} = K_n \cap (p_{n+1} + K)$. Clearly, (K_n) is a nested sequence of convex bodies which converge to a non-empty limit object, again a convex body in $\mathbb{R}^d$. We study this process for K being a regular simplex, a cube, or a regular convex polygon with an odd number of vertices. We also derive some new results in one dimension for non-uniform distributions.

1. Introduction

The following problem was formulated by Bálint Tóth some 20 years ago with $K = K_0$ being the unit disc of the plane. Let K and K_0 be convex bodies in $\mathbb{R}^d$, such that K contains the origin, and define the process (K_n, p_n), $n \geq 0$, as follows: let p_{n+1} be a uniform random point in K_n, and set $K_{n+1} = K_n \cap (p_{n+1} + K)$. Clearly, (K_n) is a nested sequence of convex bodies which converge to a non-empty limit object, again a convex body in $\mathbb{R}^d$. What can we say about the distribution of this limit body? What can we say about the speed of the process? In Figure 1 one can see the evolution of the process up to $n = 10$ on the right, and K_{10} on the left, when $K = K_0$ is a regular heptagon. When $K = K_0$ is a disc on the plane with radius R this process was mentioned by Bavaud [3], where in Theorem 5 he proved that the limit object is almost surely a convex set with constant width R.

In [1] Ambrus, Kevei and Vígh investigated the process in one dimension, when $K = K_0 = [-1, 1]$. In this case the limit object is a random unit interval, whose center has the arcsine distribution (see Theorem 1 in [1]). So even in the simplest case the process has very interesting features. Moreover, in Theorem 2 in [1] it is shown that if r_n is the radius of the interval K_n, then $4n(r_n - 1/2)$ converges in distribution to a standard exponential random variable. The idea of the proof is to observe that $(r_n - 1/2)$ behaves as the minimum of iid random variables, and thus obtain the limit theorem via extreme value theory.

We also would like to point out the formal relationship between the diminishing process and the so-called Rényi's Parking Problem from 1958 [10]. Rényi studied the following random process: consider an interval I of length $x \gg 1$, and sequentially and randomly pack (non-overlapping) unit intervals into I. In each step we choose the center of the next unit interval uniformly from the possible space. The process stops when there is no space for placing a new unit interval. (Intuitively I is the parking lot and the unit intervals are the cars.) The first possible question is to determine the expectation $M(x)$ of the covered space. Many other variants of this

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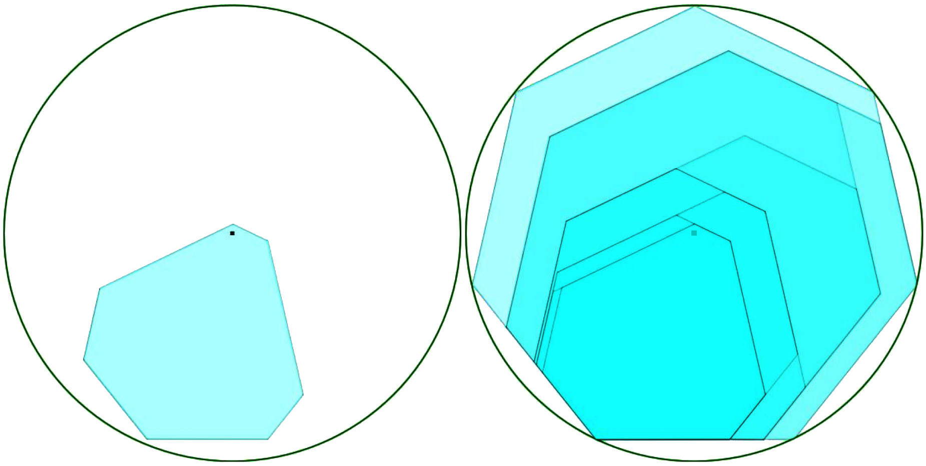

FIGURE 1. The evolution of the process for $K = K_0$ being a regular heptagon

problem have been studied for over 50 years; for an up-to-date state of the art we refer to Clay and Simányi [6]. The connection between the diminishing process and Rényi's Parking Problem can be seen easily as follows: if we choose in the definition of the diminishing process $K_0 = I$, and we drop the conditions we put on K, and define K as the complement of the closed interval of length 2 centered at 0, then we get exactly Rényi's Parking Problem.

In the present paper we analyze the diminishing process in more general cases. In Section 2 we consider the case, when instead of choosing p_{n+1} uniformly in the interval, we choose it according to a translated and scaled version of a fixed distribution F. Again, the limit object is a random unit interval. In Theorem 2.4 we determine the asymptotic behavior of the speed, while in Theorem 2.5 we show that for an appropriate choice of F the distribution of the center has the beta law. In Sections 3 and 4 we consider the case when $K = K_0$ is a cube and a regular d-dimensional simplex, respectively. The cube process can be represented as d independent interval processes, thus the results in Section 3 follow from the corresponding results in Section 2. In the case of the simplex process, the limit object is also a random regular simplex. The main result of this part is that the center of the limit simplex in barycentric coordinates has multidimensional Dirichlet law, which is a natural generalization of the beta laws to any dimension. The rate of the process is also determined. The processes considered thus far are 'self-similar' in the sense that at each step the process is a scaled and translated version of the original one.

In Sections 5, 6 and 7 we consider diminishing processes in the plane. In the case of the pentagon process even the shape of the limiting object is random. We prove that it is a pentagon with equal angles, however it is not regular a.s. This process is not 'self-similar', and its behavior is more complicated. We determine the rate of the convergence of the maximal height, but as the area of the limit object is random, limit theorem with deterministic normalization is not possible. Also the behavior of the center of mass is intractable with our methods. Finally, in

Section 7 we consider regular polygons with an odd number of vertices, i.e., $K = K_0$ is a regular polygon. Using the theory of stochastic orderings for random vectors we prove that the rate of the speed is $n^{-1/2}$. We conjecture that in the case when the number of vertices is even the speed of the process is n^{-1}. This is established in the case of the square, but in general it is open.

2. One dimension, general density

In this section we consider the process in the interval $[-1, 1]$, and the random point is chosen according to a not necessarily uniform distribution.

Fix a distribution on $[0, 1]$ with distribution function F, and in each step we choose the random point according to this distribution. That is, if the center and radius is (Z_n, r_n) the random point p_{n+1} is given by $2r_nX_{n+1}+Z_n-r_n$, where X_{n+1} is independent of Z_n, r_n, and has distribution function F. The initial condition is $(Z_0, r_0) = (0, 1)$, i.e., we start from the interval $[-1, 1]$.

Let $X, X_1, X_2, \ldots$ be iid random variables with distribution function F. It is easy to see that for $n \geq 0$

$$r_{n+1} = \begin{cases} \frac{1}{2} + r_n \min\{X_{n+1}, 1 - X_{n+1}\}, & \min\{X_{n+1}, 1 - X_{n+1}\} \leq 1 - \frac{1}{2r_n}, \\ r_n, & \text{otherwise.} \end{cases} \tag{2.1}$$

To simplify the recursions above we have to pose some assumptions on F. The following lemmas contain these assumptions. To determine the rapidness of the process we only need part (i), while for the limit distribution of the center we need both parts. In fact, in both cases we only need the 'if' part. In the following, the distribution of the random variable X is denoted by $\mathcal{L}(X)$, and given an event A the conditional distribution of X given A is $\mathcal{L}(X|A)$.

For $\alpha > 0$, $\beta > 0$ the random variable X has the beta(α, β) law, if its density is $x^{\alpha-1}(1 - x)^{\beta-1}B(\alpha, \beta)^{-1}$, $x \in (0, 1)$, where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the usual Beta function, with $\Gamma(\,\cdot\,)$ being the Gamma function.

Lemma 2.1. *Let X be a random variable with distribution function F, such that* $\mathbf{P}\{X \in [0, 1]\} = 1$.

(i) *For all $a \in [0, 1]$, for which $\mathbf{P}\{X \leq a\} > 0$, the distributional equality*

$$\mathcal{L}(X|X \leq a) = \mathcal{L}(aX)$$

holds, if and only if either X is a degenerate random variable at 0 or at 1, or $\mathcal{L}(X) = \mathrm{beta}(\delta, 1)$, for some $\delta > 0$.

(ii) *The random variables $I(X \leq 1/2)$ and $\max\{X, 1 - X\}$ are independent if and only if $F(1/2) = 0$, or $F(1/2) = 1$, or*

$$F(x) = 1 - \frac{1 - F(1/2)}{F(1/2)} F(1 - x-)$$

for all $x \in [1/2, 1]$.

Note that part (i) of the lemma is a characterization of the beta$(\delta, 1)$ law. This characterization might be known; however, we were unable to find a reference. The simple proof of Lemma 2.1 is given in the Appendix.

As an immediate consequence we obtain the following.

Lemma 2.2. *Let Y be a random variable in $[0,1]$ with continuous distribution function F. Then for any $a \in (0,1)$ the distributional equality*

$$\mathcal{L}\left(2\min\{Y,1-Y\} | 2\min\{Y,1-Y\} \le a\right) = \mathcal{L}(2a \min\{Y,1-Y\})$$

holds, and $I(Y \le 1/2)$ and $\max\{Y,1-Y\}$ are independent if and only if

$$F(x) = \begin{cases} c2^\delta x^\delta, & x \in [0,1/2], \\ 1-(1-c)2^\delta(1-x)^\delta, & x \in [1/2,1], \end{cases} \tag{2.2}$$

for some $c \in [0,1]$ and $\delta > 0$.

During the analysis of diminishing processes we frequently end up with a recursion of the following type.

Let $V, V_1, \ldots$ be a sequence of independent beta$(\delta,1)$ random variables for some $\delta > 0$, and let (a_n) be a sequence of bounded non-negative random variables, such that $a_n \downarrow a$, a.s., where $a > 0$ is deterministic. Assume that $\ell_0 = 1$, and for $n \ge 0$, for some $c > 0$

$$\ell_{n+1} = \begin{cases} \ell_n V_{n+1}, & \text{w.p. } c\frac{\ell_n^\delta}{a_n}, \\ \ell_n, & \text{w.p. } 1 - c\frac{\ell_n^\delta}{a_n}, \end{cases} \tag{2.3}$$

where $c\frac{\ell_n^\delta}{a_n} \in [0,1]$ and the abbreviation w.p. stands for 'with probability'.

To be precise this means the following throughout the paper. On our probability space $(\Omega, \mathcal{A}, \mathbf{P})$ there is a filtration $(\mathcal{F}_n)_{n\ge0}$. The filtration is usually generated by the random points p_n, i.e., $\mathcal{F}_n = \sigma(p_1,\ldots,p_n)$. The random variables a_n and ℓ_n are $\mathcal{F}_n$ measurable, and almost surely $a_n \downarrow a > 0$. Conditionally on a_n and ℓ_n let ω_{n+1} be a Bernoulli$(c\frac{\ell_n^\delta}{a_n})$ random variable and independently V_{n+1} is a beta$(\delta,1)$ random variable. Then $\ell_{n+1} = \ell_n V_{n+1}$ whenever $\omega_{n+1} = 1$, and $\ell_{n+1} = \ell_n$ otherwise. (Here and in the following section a_n is simply a function of ℓ_n. However, when dealing with the polygon process a_n is the area of K_n, and it does depend on the chosen points, and not only on ℓ_n. This is the reason for the complication.)

In the next lemma we determine the asymptotic behavior of such ℓ_n sequences. The idea of the proof is to show that ℓ_n behaves like the minimum of n iid random variables, as in the proof of Theorem 1 in [1]. The proof is deferred to the Appendix.

For $\delta > 0$, the Weibull(δ) distribution function is given by $1 - \mathrm{e}^{-x^\delta}$, for $x > 0$, and 0 otherwise.

Lemma 2.3. *Assume that ℓ_n is defined by (2.3). Then*

$$\left(\frac{c}{a}n\right)^{1/\delta} \ell_n \xrightarrow{\mathcal{D}} \mathrm{Weibull}(\delta).$$

Moreover, for any $\alpha > 0$

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{c}{a}n\right)^{\alpha/\delta} \ell_n^\alpha = \Gamma\left(1 + \frac{\alpha}{\delta}\right).$$

With the help of these lemmas we can analyze the speed of the process.

Theorem 2.4. *Assume that for the distribution of X we have*

$$\mathbf{P}\{2\min\{X, 1-X\} \le x\} = x^{\delta}, \quad x \in [0,1],$$

for some $\delta > 0$. Then as $n \to \infty$

$$4n^{1/\delta}\left(r_n - \frac{1}{2}\right) \xrightarrow{\mathcal{D}} \text{Weibull}(\delta),$$

i.e., for any $x > 0$

$$\lim_{n\to\infty} \mathbf{P}\left\{4n^{1/\delta}\left(r_n - \frac{1}{2}\right) > x\right\} = \mathrm{e}^{-x^{\delta}}.$$

Moreover, for any $\alpha > 0$

$$\lim_{n\to\infty} \mathbf{E}\, 4^{\alpha} n^{\alpha/\delta}\left(r_n - \frac{1}{2}\right)^{\alpha} = \Gamma\left(1 + \frac{\alpha}{\delta}\right).$$

Proof. Using the assumption and Lemma 2.1 (i) we see that (2.1) can be rewritten as

$$\ell_{n+1} = \begin{cases} \ell_n V_{n+1}, & \text{w.p. } (2 - r_n^{-1})^{\delta}, \\ \ell_n, & \text{w.p. } 1 - (2 - r_n^{-1})^{\delta}, \end{cases} \tag{2.4}$$

with $\ell_n = r_n - 1/2$, and $V, V_1, \dots$ are independent beta$(\delta, 1)$ random variables. Now the theorem follows from Lemma 2.3, with $a_n = r_n^{\delta} \downarrow 1/2^{\delta} = a$ and $c = 2^{\delta}$. □

To determine the limit distribution of the center consider the thinned process $(\widetilde{Z}_n, \widetilde{r}_n)$, which is obtained from the original process (Z_n, r_n) by dropping those steps when nothing changes, i.e., when $r_n = r_{n+1}$. Clearly, the limit of the center is not affected. After some calculation we obtain the recursion

$$\begin{aligned} \widetilde{Z}_{n+1} &= \widetilde{Z}_n + \frac{2\widetilde{r}_n \max\{X_{n+1}, 1 - X_{n+1}\} - 1}{2} \operatorname{sgn}(X_{n+1} - 1/2), \\ \widetilde{r}_{n+1} &= \frac{1}{2} + \widetilde{r}_n \min\{X_{n+1}, 1 - X_{n+1}\}, \end{aligned} \tag{2.5}$$

where X_{n+1} has the distribution of X conditioned on the event $\min\{X, 1-X\} < 1 - (2\widetilde{r}_n)^{-1}$.

Note that in (2.2) in Lemma 2.2 for $c = 1$ the distribution is concentrated on $[0, 1/2]$, in which case the center always moves towards $-1/2$, so the limit distribution of the center is degenerate at $-1/2$. Similarly, for $c = 0$ the limit is deterministic $1/2$. In the following theorem we exclude these cases.

Theorem 2.5. *Let us assume that for some $c \in (0,1)$ and $\delta > 0$ (2.2) holds. Then the distribution of Z is the translated beta$(\delta(1-c), \delta c)$ law, i.e., its density function is*

$$f_{\delta,c}(x) = \frac{\Gamma(\delta)}{\Gamma(\delta(1-c))\Gamma(\delta c)} (1/2 + x)^{\delta(1-c)-1} (1/2 - x)^{\delta c - 1}, \quad x \in (-1/2, 1/2).$$

In the symmetric case, when $c = 1/2$, we obtain the so-called *power semicircle laws.* For further properties of power semicircle distributions and for their role in non-commutative probability, we refer to Arizmendi and Pérez-Abreu [2].

Proof. By Lemma 2.1 and (2.5) we obtain the recursion

$$\begin{aligned} \widetilde{Z}_{n+1} &= \widetilde{Z}_n + \xi_{n+1} \widetilde{\ell}_n (1 - V_{n+1}), \\ \widetilde{\ell}_{n+1} &= \widetilde{\ell}_n V_{n+1}, \end{aligned} \tag{2.6}$$

where $\widetilde{\ell}_n = \widetilde{r}_n - 1/2$, and $\xi_1, \xi_2, \dots$ are iid Bernoulli random variables, such that $\mathbf{P}\{\xi_1 = 1\} = 1 - c = 1 - \mathbf{P}\{\xi_1 = -1\}$, and independently of $\{\xi_i\}_{i=1}^\infty$, the random variables $V_1, V_2, \dots$ are iid beta$(\delta, 1)$. The initial value is $(\widetilde{Z}_0, \widetilde{\ell}_0) = (0, 1/2)$.

Formula (2.6) implies the infinite series representation of the limit

$$Z_\infty = \frac{1}{2} \sum_{i=1}^\infty \xi_i V_1 \dots V_{i-1}(1 - V_i), \tag{2.7}$$

and thus the distributional equation perpetuity

$$Z_\infty \stackrel{\mathcal{D}}{=} \frac{1}{2}\xi_1(1 - V_1) + V_1 Z_\infty, \tag{2.8}$$

where on the right-hand side V_1, ξ_1, Z_∞ are independent.

Corollary 1.2 in Hitczenko and Letac [9] (or the proof of Theorem 3.4 in Sethuraman [11]) implies that $Z_\infty + 1/2$ has the beta$(\delta(1-c), \delta c)$ distribution. □

Note that once we have the infinite series representation (2.7) the proof can be finished using the properties of GEM(δ) (or Poisson–Dirichlet) law; see Hirth [8], or Bertoin [4, Section 2.2.5].

Distributional equations of type

$$R \stackrel{\mathcal{D}}{=} Q + MR, \quad R \text{ independent of } (Q, M),$$

where R, Q are random vectors, and M is a random variable, are called *perpetuities*. Equation (2.8) is an example. Necessary and sufficient conditions for the existence of a unique solution of one-dimensional perpetuities are given by Goldie and Maller [7]. However, in special cases (for example for $M \in [-1, 1]$) the existence of a unique solution in any dimension was known earlier; see Lemma 3.3 by Sethuraman [11]. Therefore, in (2.8) above, or in d dimension in (4.4) below, the assertion that certain distribution G satisfies the perpetuity equation is equivalent to saying that the perpetuity equation has a unique solution G.

The perpetuities (2.8) and (4.4) are interesting in their own right, because there are relatively few perpetuities when the exact solution is known. The results of Sethuraman [11] (proof of Theorem 3.4; see also Theorem 1.1 in [9]) cover those equations which appear in our investigations. For more general perpetuity equations with exact solutions we refer to the recent paper by Hitczenko and Letac [9].

3. The cube

We consider the d-dimensional cube process, where $K = K_0 = [-1, 1]^d$. Now the limiting convex body is a unit cube. Let us denote by $m_1(n), \dots, m_d(n)$ the edge lengths of the rectangular box K_n, and $(Z_1(n), \dots, Z_d(n))$ the center of K_n.

We generalize the results obtained in Section 2 into higher dimensions; to do this, we consider scaled product measures. More precisely, we pick some positive real numbers $\delta_1, \delta_2, \dots, \delta_d$, and real numbers $c_1, c_2, \dots, c_d$ such that $c_i \in (0, 1)$ for all $i = 1, \dots, d$. As in (2.2), we define for all $i = 1, \dots, d$ the distribution functions

$$F_i(x) = \begin{cases} c_i 2^{\delta_i} x^{\delta_i}, & x \in [0, 1/2], \\ 1 - (1 - c_i) 2^{\delta_i} (1 - x)^{\delta_i}, & x \in [1/2, 1]. \end{cases}$$

We introduce the joint distribution function

$$F(x_1, \dots, x_d) = \prod_{i=1}^d F_i(x_i). \tag{3.1}$$

Now, the random point p_{n+1}, $n \geq 0$, is given by

$$p_{n+1} = (m_1(n)X_1(n+1), \ldots, m_d(n)X_d(n+1)) + (Z_1(n), \ldots, Z_d(n)) - \frac{1}{2}(m_1(n), \ldots, m_d(n)),$$

where $\{\mathbf{X}(n+1) = (X_1(n+1), \ldots, X_d(n+1))\}_{n\geq 0}$ are iid random vectors with distribution function F in (3.1). The initial conditions are $(Z_1(0), \ldots, Z_d(0)) = (0, \ldots, 0)$ and $(m_1(0), \ldots, m_d(0)) = (2, \ldots, 2)$.

The following theorem readily follows using the results in Section 2.

Theorem 3.1. *For the speed of the cube process we have*

$$2\begin{pmatrix} n^{1/\delta_1}(m_1(n)-1) \\ \vdots \\ n^{1/\delta_d}(m_d(n)-1) \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} W_1 \\ \vdots \\ W_d \end{pmatrix},$$

where $W_1, \ldots, W_d$ are independent Weibull random variables with parameters δ_1, …, δ_d, respectively.

For the limit distribution of the center

$$\begin{pmatrix} Z_1(n) \\ \vdots \\ Z_d(n) \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix},$$

where $Z_1, \ldots, Z_d$ are independent, and for all $i = 1, \ldots, d$, Z_i is the translated beta$(\delta_i(1-c_i), \delta_i c_i)$ law, i.e., its density function is

$$f_{\delta_i,c_i}(x) = \frac{\Gamma(\delta_i)}{\Gamma(\delta_i(1-c_i))\Gamma(\delta_i c_i)}(1/2+x)^{\delta_i(1-c_i)-1}(1/2-x)^{\delta_i c_i-1}, \quad x \in (-1/2, 1/2).$$

We note that if $\delta_1 = \delta_2 = \ldots = \delta_d = 1$ and $c_1 = c_2 = \ldots = c_d = 1/2$, then in each step the point p_{n+1} is chosen from K_n according to the uniform distribution. In this special case for the maximum of the edge lengths $m_n = \max\{m_1(n), \ldots, m_d(n)\}$ it follows that

$$2n(m_n - 1) \xrightarrow{\mathcal{D}} W,$$

where $\mathbf{P}\{W \leq x\} = (1 - \mathrm{e}^{-x})^d$, $x \geq 0$.

4. The simplex

Now we turn to the simplex process in any dimension.

Let K be a regular d-dimensional simplex with centroid $(0, 0, \ldots, 0)$ and vertices $(\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_d)$, such that $\mathrm{e}_0 = (1, 0, \ldots, 0)$. Let us denote by $\rho_d = 1/d$ the radius of the inscribed sphere of K.

Let the initial simplex be $K_0 = \frac{2}{d+1}K$ (for reasons explained below), and for K_n given, choose a random point p_{n+1} uniformly in K_n and let $K_{n+1} = K_n \cap (p_{n+1} + K)$. Let m_n denote the height of K_n. Then K_n is a nested sequence of regular simplices and the limit object is a regular simplex with height ρ_d.

It turns out that this process can be investigated by the same methods as for $d = 1$, in the case of the segment process, in [1]. The idea is that for the simplex in any dimension the process is 'self-similar', i.e., after each step the process is a translated and scaled version of the original one.

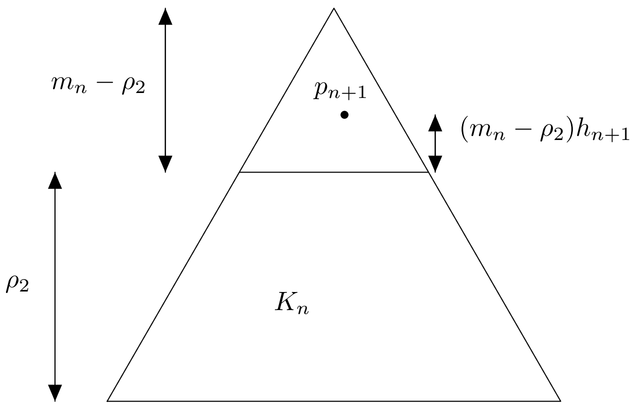

FIGURE 2. The triangle process

4.1. **The rapidness of the process.** If in the $(n+1)$th step the point p_{n+1} falls close to the center, then nothing happens, i.e., $K_{n+1} = K_n$. The 'change regions' are $d+1$ congruent, regular simplices of height $m_n - \rho_d$, and each of them sits at a vertex of K_n. Note that since the height of K_n is at most $2\rho_d$ these simplices are disjoint, so the process is simpler. This is the reason we assume $K_0 = \frac{2}{d+1}K$, since its height $m_0 = 2\rho_d$. Although, if we would start with a larger K_0, as $m_n \downarrow \rho_d$ a.s., in a random number of steps the height of K_n would be smaller than $2\rho_d$, thus the assumption $K_0 = \frac{2}{d+1}K$ has no effect on the rapidness of the process.

Theorem 4.1. *For the height process m_n*

$$\frac{(d+1)^{1/d}}{\rho_d} n^{1/d}(m_n - \rho_d) \xrightarrow{\mathcal{D}} \text{Weibull}(d).$$

Moreover, for any $\alpha > 0$

$$\lim_{n\to\infty} \mathbf{E}\frac{[(d+1)n]^{\alpha/d}}{\rho_d^\alpha}(m_n - \rho_d)^\alpha = \Gamma\left(1 + \frac{\alpha}{d}\right).$$

Proof. With disjoint change regions for the height process we have

$$m_{n+1} = \begin{cases} m_n - h_{n+1}(m_n - \rho_d), & \text{w.p. } (d+1)\left(1 - \frac{\rho_d}{m_n}\right)^d, \\ m_n, & \text{w.p. } 1 - (d+1)\left(1 - \frac{\rho_d}{m_n}\right)^d, \end{cases}$$

where $h_1, h_2, \ldots$ are independent beta$(1, d)$ random variables, which is the distribution of the distance from the base of a uniformly distributed random point in a regular simplex with height 1; see Figure 2.

Putting $\ell_n = m_n - \rho_d$, we have $\ell_n \downarrow 0$ a.s., and

$$\ell_{n+1} = \begin{cases} \ell_n(1 - h_{n+1}), & \text{w.p. } (d+1)\left(1 - \frac{\rho_d}{m_n}\right)^d, \\ \ell_n, & \text{w.p. } 1 - (d+1)\left(1 - \frac{\rho_d}{m_n}\right)^d. \end{cases} \tag{4.1}$$

The theorem follows from Lemma 2.3 with $\delta = d, c = d+1$ and $a_n = m_n^d \downarrow \rho_d^d$. $\square$

4.2. **The limit distribution of the center.** Let $\mathbf{c}_n$ denote the center of the regular simplex K_n. In this subsection we determine the limit distribution of $\mathbf{c}_n$.

As we emphasized previously the limit distribution of $n^{1/d}(m_n - \rho_d)$ is not affected if we start from any smaller regular simplex, in particular which has height $2\rho_d$. However, this is not true for the limit distribution of the center $\mathbf{c}_n$. To handle the process we have to assume that the change regions are disjoint, and so in each step the center can only move towards one of the vertices, or stay.

In order to investigate the limit distribution of the centroid, we can consider the thinned (centroid, height) process $(\widetilde{\mathbf{c}}_n, \widetilde{m}_n)$, skipping the steps when nothing happens. Put $\widetilde{\ell}_n = \widetilde{m}_n - \rho_d$.

Since the disjoint change regions have the same volume, in each step the center moves towards any of the vertices with the same probability $1/(d+1)$, according to the change region in which the chosen point falls. The size of the shift is $\frac{d}{d+1} \cdot \widetilde{\ell}_n h_{n+1}$, where $\widetilde{\ell}_n h_{n+1}$ is the distance of the chosen point from the base of the change region; see Figure 2. Thus

$$
\begin{aligned}
\widetilde{\mathbf{c}}_{n+1} &= \widetilde{\mathbf{c}}_n + \frac{d}{d+1} \widetilde{\ell}_n h_{n+1} \mathrm{e}_{\xi_{n+1}}, \\
\widetilde{\ell}_{n+1} &= \widetilde{\ell}_n (1 - h_{n+1}),
\end{aligned} \tag{4.2}
$$

where $h_1, h_2, \ldots$ are independent beta$(1, d)$ random variables, $\xi_1, \xi_2, \ldots$ are independent, uniformly distributed random variables on $\{0, 1, \ldots, d\}$, and the initial conditions are $\widetilde{\mathbf{c}}_0 = \mathbf{0}$ and $\widetilde{\ell}_0 = \rho_d$.

To obtain a more symmetric description of the center process we introduce the barycentric coordinates. The center of the limiting simplex falls in $\widehat{K} := \frac{1}{d+1} K$, i.e., in a regular simplex with height ρ_d.

Put $\widehat{\mathrm{e}}_i = \frac{1}{d+1} \mathrm{e}_i$, that is, $\widehat{\mathrm{e}}_0, \ldots, \widehat{\mathrm{e}}_d$ are the vertices of $\widehat{K}$. To parametrize the center we may use barycentric coordinates in terms of $\widehat{K}$. That is, for $\widetilde{\mathbf{c}}_n$ we have $\widetilde{\mathbf{c}}_n = \sum_{i=0}^d \lambda_n^i \widehat{\mathrm{e}}_i$, with $\sum_{i=0}^d \lambda_n^i = 1$, $\lambda_n^i \geq 0$, $i = 0, 1, \ldots, d$. It is well known that this parametrization is unique. Put $\widetilde{\Lambda}_n = (\lambda_n^0, \ldots, \lambda_n^d) \in \mathbb{R}^{d+1}$. We can rewrite (4.2) in terms of the barycentric coordinates of $\widetilde{\mathbf{c}}_n$. After some calculation we have

$$
\begin{aligned}
\widetilde{\Lambda}_{n+1} &= \widetilde{\Lambda}_n + \frac{d}{d+1} \widetilde{\ell}_n h_{n+1} \mathbf{v}_{\xi_{n+1}}, \\
\widetilde{\ell}_{n+1} &= \widetilde{\ell}_n (1 - h_{n+1}),
\end{aligned} \tag{4.3}
$$

where $\mathbf{v}_j$ is the constant -1 vector, except its jth coordinate is d. The initial values are $\widetilde{\Lambda}_0 = (1/(d+1), \ldots, 1/(d+1))$, $\widetilde{\ell}_0 = \rho_d$.

Before stating the theorem, we define the multidimensional Dirichlet distribution. Let $a_0, \ldots, a_d$ be positive numbers. The random vector $X = (X_0, \ldots, X_d)$ has Dirichlet$(a_0, \ldots, a_d)$ distribution, if $X_0 + \ldots + X_d = 1$, the components are non-negative, and $(X_1, \ldots, X_d)$ has density function

$$
\frac{\Gamma(a_0 + \ldots + a_d)}{\Gamma(a_0) \ldots \Gamma(a_d)} (1 - x_1 - \ldots - x_d)^{a_0 - 1} x_1^{a_1 - 1} \ldots x_d^{a_d - 1},
$$

on the set $\{(x_1, \ldots, x_d) : x_i \in (0, 1), i = 1, \ldots, d; \sum_{i=1}^d x_i \leq 1\}$.

Theorem 4.2. *The barycentric coordinates of the center of the limit simplex have the* Dirichlet$(d/(d+1), \ldots, d/(d+1))$ *distribution.*

Proof. Let $\widetilde{\Lambda}$ be the barycentric coordinates of the center of the limit. From (4.3) we obtain that

$$\widetilde{\Lambda} = \widetilde{\Lambda}_0 + \frac{1}{d+1} \sum_{n=0}^{\infty} (1-h_1) \dots (1-h_n) h_{n+1} \mathbf{v}_{\xi_{n+1}}.$$

Rearranging we get

$$\begin{aligned}\widetilde{\Lambda} &= h_1 \left(\frac{1}{d+1} \mathbf{v}_{\xi_1} + \widetilde{\Lambda}_0 \right) \\ &\quad + (1-h_1) \left[\widetilde{\Lambda}_0 + \frac{1}{d+1} \sum_{n=1}^{\infty} (1-h_2) \dots (1-h_n) h_{n+1} \mathbf{v}_{\xi_{n+1}} \right].\end{aligned}$$

Notice that the infinite sum in brackets is equal in distribution to $\widetilde{\Lambda}$ and it is independent of h_1 and ξ_1. Since $\frac{1}{d+1}\mathbf{v}_i + \widetilde{\Lambda}_0 = u_i$, where $(u_i)_{i=0,\dots,d}$ are the usual unit vectors in $\mathbb{R}^{d+1}$, we obtain the distributional equality

$$\widetilde{\Lambda} \stackrel{\mathcal{D}}{=} h u_\xi + (1-h)\widetilde{\Lambda}, \tag{4.4}$$

where on the right-hand side $\xi, h, \widetilde{\Lambda}$ are independent. Applying now Theorem 1.1 in [9] (or the results in the proof of Theorem 3.4 in [11]) with $Y = h \sim \mathrm{beta}(1, d)$, and $B = u_\xi \sim \sum_{i=0}^{d} \frac{1}{d+1} \delta_{u_i}$, we obtain the theorem. □

5. Regular polygons with an odd number of vertices

Let k be an odd positive integer, and assume $k \geq 5$. Let K be a regular k-gon with circumradius 1, centroid $(0,0)$, such that $(0,1)$ is a vertex and the side $v_1 v_2$ is parallel to the x-axis. We denote the vectors pointing from the origin to the vertices of K in the counterclockwise order by $v_1, \dots, v_k$. (To avoid confusion, we distinguish between points and vectors.) Put $K_0 = K$, and consider the process as before. For simplicity we usually omit k from our notation, and assume that k is fixed, odd, and clear from the circumstances.

Obviously, K_n is a polygon for each n, and since it is the intersection of translated copies of K, its sides are parallel to the sides of K. However, note that K_n is not necessarily a k-gon. For convenience, we are still going to consider K_n as a (possibly degenerated) k-gon with the following definitions. Let l_i and l_i' be two parallel support lines of K_n with equations $l_i \colon \langle x \,, v_i \rangle = \alpha_i$ and $l_i' \colon \langle x \,, v_i \rangle = \alpha_i'$, where $\alpha_i > \alpha_i'$. Now, we denote $K_n \cap l_i$ by $A_i = A_i(n)$ and we consider it as the ith vertex of K_n. Similarly, $K_n \cap l_i'$ is denoted by $s_i = s_i(n)$ and we call it the ith side of K_n. Note that with these notation some vertices might coincide and correspondingly some sides might degenerate into a point. We also introduce the ith height of K_n as $m_i(n) = \alpha_i - \alpha_i'$. We put $\mathbf{m}_n = (m_1(n), m_2(n), \dots, m_k(n))$, and $m_n = \max_i m_i(n)$.

The radius of the inscribed circle of K is denoted by $\rho_k = \cos(\pi/k)$. We also introduce the notion of change region here:

$$\mathcal{R}_i(n) = K_n \cap \{ x \mid \langle x \,, v_i \rangle \geq \alpha_i' + \rho_k \}, \quad i = 1, 2, \dots, k;$$

see Figure 5. Intuitively, the ith side moves, if we choose the next random point in $\mathcal{R}_i$. (Note that, this is not entirely true, since a degenerated side can move in other ways.) Obviously, if $p_{n+1} \notin \bigcup_1^k \mathcal{R}_i(n)$, then $K_{n+1} = K_n$.

We define

$$K_\infty = \bigcap_{n=0}^{\infty} K_n,$$

the so-called *limit object.*

Lemma 5.1. *The limit object K_∞ is a possibly degenerated, closed k-gon whose sides are parallel to the sides of K. Furthermore, the maximal height of K_∞ is exactly ρ_k almost surely.*

Proof. Since K_∞ is the intersection of closed half-planes with possible outer normals $-v_1, \ldots, -v_k$, it follows that K_∞ is a closed, possibly degenerated k-gon with sides parallel to the sides of K.

First we show that no height of K_∞ is larger than ρ_k. Suppose that $m_1(\infty) > \rho_k$; in this case $\mathcal{R}_1(\infty)$ is of positive area. Observe that no point was selected from $\mathcal{R}_1(\infty)$ by definition, which is a contradiction.

Next we prove that the maximal height of K_∞ is at least ρ_k. Clearly, it is enough to see that $m_n \geq \rho_k$ for every n. This follows from the observation that if $p_{n+1} \notin \bigcup_1^k \mathcal{R}_i(n)$, then $K_{n+1} = K_n$. $\square$

In the following lemma we show that K_n always contains a small circle of radius $1/10$. In particular this implies that the area of K_n (and thus the area of K_∞ as well) is uniformly bounded from below by $\pi/100$. To ease the notation we put $p_0 = 0$.

Lemma 5.2. *Let $k \geq 5$, and assume that*

$$K_n = \bigcap_{j=0}^{n} (K + p_j),$$

where $p_j \in \bigcap_{m=0}^{j-1}(K + p_m)$ for all j. Then K_n contains a circle of radius $1/10$.

Proof. Denote by B the unit circle centered at the origin, which is the circumcircle of K by definition. Also by definition $\rho_k B$ is the incircle of K. We consider

$$B_n = \bigcap_{j=0}^{n} (B + p_j),$$

and we observe that $K_n \subset B_n$ holds for all n.

We claim that for all $j = 0, 1, \ldots, n$, we have $p_j \in B_n$. By definition $p_j \in K_j \subset B_j$. Suppose that $p_j \notin B_n$; then there exists an index n_0 with $j < n_0 \leq n$ such that $p_j \notin (B + p_{n_0})$, and thus $p_{n_0} \notin (B + p_j)$. But by definition $p_{n_0} \in B_{n_0} \subset (B + p_j)$, a contradiction.

We obtained that B_n is the intersection of the unit circles $B + p_j$ such that all centers p_j are contained in B_n. This readily implies that the minimal width of B_n is at least one. Then Blaschke's Theorem (see [13, p. 18, Th. 2–5.]) implies that there exists x such that $B/3 + x \subseteq B_n$. Obviously for all $j \leq n$ we have that $x \in 2B/3 + p_j$, and thus $\rho_k \geq \rho_5 = \cos \pi/5 \approx 0.809 > 2/3 + 1/10$ implies that for all $j \leq n$ we have $B/10 + x \subset K + p_j$, which proves the statement. $\square$

Lemma 5.3. *There exists a $\delta_k > 0$ such that if every height of K_n is smaller than $\rho_k + \delta_k$, then the change regions $\mathcal{R}_i$ are pairwise disjoint.*

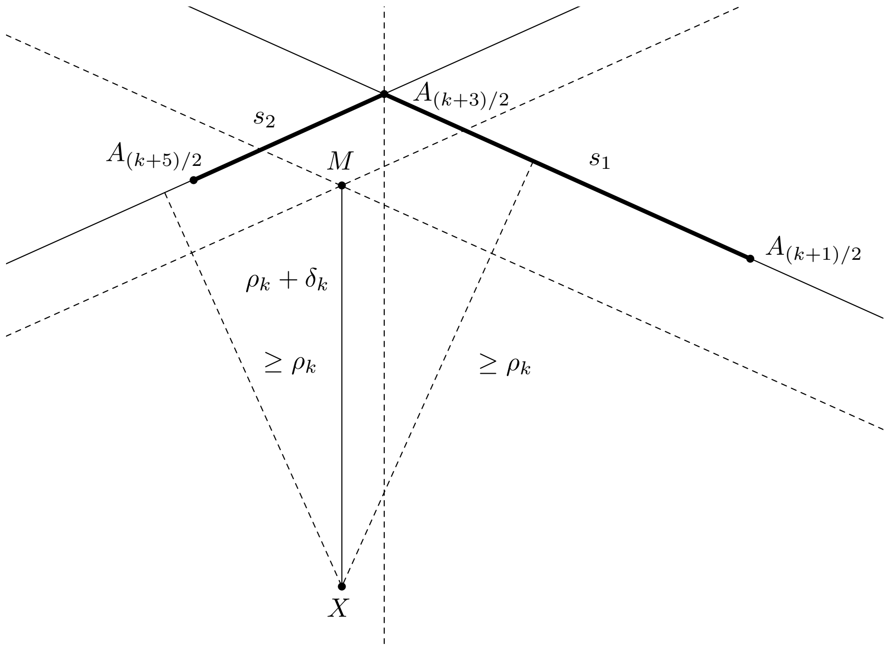

FIGURE 3. Adjacent change regions are disjoint

Proof. We show that $\mathcal{R}_i$ and $\mathcal{R}_j$ are disjoint for every $i \neq j$.

First we show that the statement is true for adjacent regions. Suppose that $X \in \mathcal{R}_1 \cap \mathcal{R}_2$ (see Figure 3).

According to Figure 3 we draw two lines parallel to l'_1 and l'_2 respectively that are at distance exactly ρ_k from the point X; these two lines meet in the point M. Obviously, there exists a $\delta_k > 0$ (depending only on k), such that $\overline{XM} = \rho_k + \delta_k$. It readily follows that $m_{(k+3)/2} \geq \rho_k + \delta_k$, a contradiction.

Next we prove that if $2 \leq m \leq (k-1)/2$, and $X \in \mathcal{R}_1 \cap \mathcal{R}_m$, then $X \in \bigcap_1^m \mathcal{R}_j$. This obviously implies the statement of the lemma. We proceed by induction on m. For $m = 2$ we are done. Now we assume that the statement is true till $m-1$, and we prove it for m.

Pick $X \in \mathcal{R}_1 \cap \mathcal{R}_m$. We may assume that $X \notin \mathcal{R}_j$ for any $j = 2, 3, \ldots, m-1$, otherwise we would be done by applying the hypothesis twice. We may also assume that we changed the coordinate system such that the slope of l'_m is positive, the slope of l'_1 is negative, and the bisectors of the line l'_1 and l'_m are vertical and horizontal; see Figure 4.

Draw the translated copy K_X of K whose center is X, the incircle of K_X is of radius ρ_k and of center X. Consider the vertices $A_{(k+1)/2+1}$ and $A_{(k+1)/2+m-1}$ of K_n, and the vertices $A'_{(k+1)/2+1}$ and $A'_{(k+1)/2+m-1}$ of K_X. From the assumptions it clearly follows that the 'horizontal distance' (the difference of the x coordinates) of $A_{(k+1)/2+1}$ and $A_{(k+1)/2+m-1}$ is larger than the horizontal distance of $A'_{(k+1)/2+1}$ and $A'_{(k+1)/2+m-1}$. But this is a contradiction, since the sides $s_1, s_2, \ldots, s_{m-1}$ form a fixed angle with the x-axis, and each of them is at most as long as the side

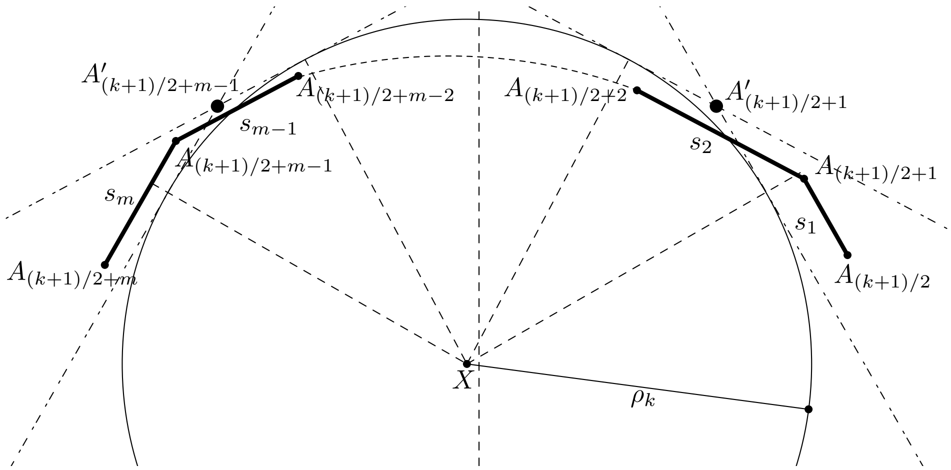

FIGURE 4. Non-adjacent change regions are disjoint

length of K, and thus the horizontal distance of $A'_{(k+1)/2+1}$ and $A'_{(k+1)/2+m-1}$ is maximal. $\square$

A configuration is called *reduced* if the change regions are disjoint. In a reduced state it is possible to follow the process. That gives the importance of the following simple corollary which readily follows from the fact that $\mathbf{m}_n$ is componentwise monotone decreasing and $m_n \downarrow \rho_k$.

Corollary 5.4. *The process a.s. reaches a reduced state in a random number of steps. After reaching a reduced state, the process always stays in a reduced state.*

6. THE PENTAGON

In this section we consider the pentagon process. This is the simplest case when not only the position, but also the shape of the limit object is random. We show that exactly one height of the limit object is ρ_5, which allows us to determine the speed of the process.

6.1. **On the limit pentagon.** First we prove that the process cannot degenerate in the following sense.

Lemma 6.1. *K_n is always a pentagon with equal inner angles.*

Proof. The key observation is that the directions of the sides of K_n are prescribed, thus the only thing we have to show is that a side cannot disappear. Suppose the opposite, and seek a contradiction. Let K_n be the first non-pentagonal state, and first assume that it is a quadrilateral and the side A_1A_5 disappears. It is easy to calculate the inner angles of K_n, three of them equal the inner angle of a regular pentagon, $3\pi/5$ (at vertices A_2, A_3 and A_4), while the fourth one is $\pi/5$ (at the vertex A_1). Also note, that the side lengths of K_n cannot exceed the side length of K. Thus K_n is contained in a deltoid (see Figure 6), where s is the side length of K. This implies that the heights m_2 and m_4 of K_n are at most $s \cdot \sin(\pi/5) = 2 \cdot \sin^2(\pi/5) \approx 0.69$. A simple argument shows that we may assume

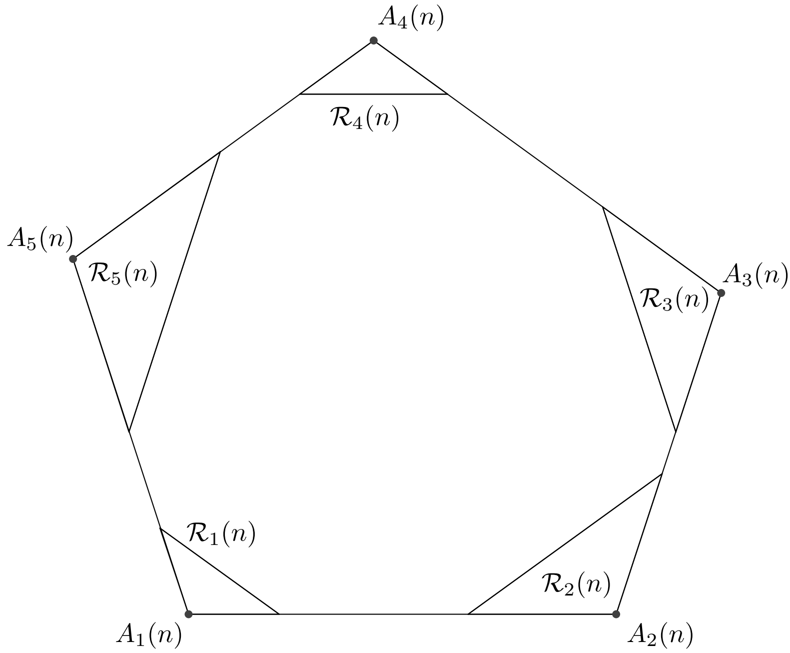

FIGURE 5. Change regions in a reduced state

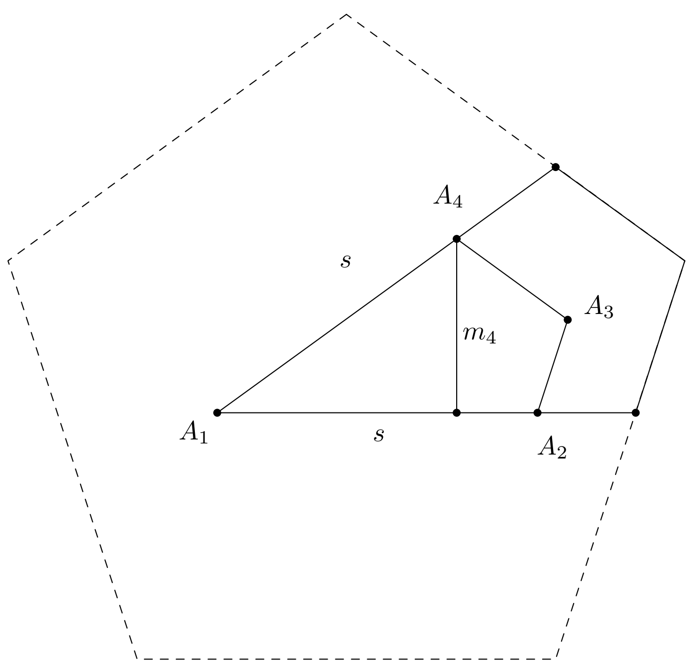

FIGURE 6. The deltoid containing K_n

that A_4 was a vertex of K_{n-1}, but A_1 and A_2 were not. This implies that the side A_1A_2 comes from K (more precisely, $A_1A_2 \subset p_n + \partial K$), and so $m_4 \geq \rho_5$. But this is not possible, since $m_4 < \rho_5$, a contradiction. A similar argument settles the case when K_n is a triangle. □

By Corollary 5.4 in a random number of steps we reach a reduced state, and so as in the simplex case we may and do assume that the process starts from a reduced state. It also follows that in a reduced state the change regions are always triangles.

Note that if the random point falls in $\mathcal{R}_1$, then beside m_1, the opposite heights m_3 and m_4 also decrease. Some calculation shows that if m_1 decreases by x, then m_3 and m_4 both decrease by $c\,x$, with

$$c = \frac{\sqrt{5}-1}{2} \tag{6.1}$$

being the reciprocal of the golden ratio. We say that m_i and m_j are *competing* heights, if $m_i > \rho_5$, $m_j > \rho_5$, and they are not adjacent.

To describe the dynamics of the process we define the following vectors: $\mathbf{v}_1 = (1,0,c,c,0)$, $\mathbf{v}_2 = (0,1,0,c,c)$, $\mathbf{v}_3 = (c,0,1,0,c)$, $\mathbf{v}_4 = (c,c,0,1,0)$, and $\mathbf{v}_5 = (0,c,c,0,1)$. With this notation, if in a reduced state in the $(n+1)$th step the random point falls in $\mathcal{R}_i(n)$, then

$$\mathbf{m}_{n+1} = \mathbf{m}_n - h_n(m_i(n) - \rho_5)\mathbf{v}_i, \tag{6.2}$$

where $h_1, h_2, \ldots$ are independent beta$(1,2)$ random variables, i.e., h is the distribution of the distance from the base of a uniformly chosen point in a triangle with height 1. That is, $h_{n+1}(m_i(n) - \rho_5)$ is the distance of p_{n+1} and the side of $\mathcal{R}_i(n)$ which is opposite to $A_i(n)$. The probability of this event is $|\mathcal{R}_i(n)|/|K_n|$, where $|\cdot|$ is the area.

Lemma 6.2. *The limit pentagon cannot have non-adjacent heights equal to* ρ_5.

Proof. Emphasizing that the process can be at any reduced state we omit the index n.

Assume that there is a state with at least two competing heights greater than ρ_5. Let, say, m_1 be the maximum height, which has a competing pair, say m_3. If the maximum height has no competing pair greater than ρ_5, then its change has no effect on the two competing heights. Thus m_1 will change eventually. So we may and do assume that m_1 is the largest height.

Case 1. $c(m_1 - \rho_5)/2 > m_3 - \rho_5$, with c defined in (6.1). Then the probability that in the next change step the uniform random point falls in $\mathcal{R}_1$ is greater than $1/5$, and given this the probability that m_1 decreases at least with $(m_1 - \rho_5)/2$ equals $\mathbf{P}\{h > 1/2\} = 1/4$. In this case m_3 decreases below ρ_5, and so the probability of this event is at least $1/20$.

Case 2. $c(m_1 - \rho_5)/2 \leq m_3 - \rho_5$. The probability that in the next change step the random point falls in $\mathcal{R}_3$ is

$$\frac{(m_3 - \rho_5)^2}{\sum_{i=1}^5 (m_i - \rho_5)_+^2} \geq \frac{(m_3 - \rho_5)^2}{5(m_1 - \rho_5)^2} \geq \frac{c^2}{20}.$$

We show that with positive probability we end up in a state corresponding to Case 1. In the next step

$$m_1' = m_1 - ch(m_3 - \rho_5),$$
$$m_3' = m_3 - h(m_3 - \rho_5).$$

We want an $h \in (0,1)$, such that $c(m_1' - \rho_5)/2 > m_3' - \rho_5$. Some calculation shows that this happens if and only if

$$h > \frac{1}{1 - \frac{c^2}{2}} \left(1 - \frac{c}{2} \frac{m_1 - \rho_5}{m_3 - \rho_5} \right),$$

where the right side is at most

$$\frac{1 - \frac{c}{2}}{1 - \frac{c^2}{2}} = \frac{3\sqrt{5} - 5}{2}.$$

The probability of this event is at least

$$\mathbf{P} \left\{ h > \frac{3\sqrt{5} - 5}{2} \right\} = \frac{(7 - 3\sqrt{5})^2}{4} \approx 0.0213.$$

So we are almost in Case 1, but it can happen that m_1' is not maximal. Notice that

$$\frac{m_1' - \rho_5}{m_1 - \rho_5} = \frac{m_1 - \rho_5 - c(m_3 - \rho_5)h}{m_1 - \rho_5} \geq 1 - c,$$

which implies that the probability of choosing in $\mathcal{R}_1$ in the next change step is at least $(1-c)^2/5$.

So we showed that starting from any state with at least two competing heights greater than ρ_5, the probability that in two change steps one of them decreases below ρ_5 is at least

$$\frac{c^2}{20} \frac{(7 - 3\sqrt{5})^2}{4} \frac{(1-c)^2}{20} \approx 2.97 \cdot 10^{-6}.$$

This proves that the process cannot have this configuration for infinite number of steps. $\square$

Lemma 6.3. *There is no non-regular pentagon with equal angles, in which the two largest heights are consecutive.*

Proof. As a first step we prove a somewhat surprising result that provides a linear relationship between any four heights of the pentagon. We assume that m_1, m_3 and m_4 are given, and we express m_2 as a linear combination of the previous three. To simplify the calculations, we place the pentagon into a new coordinate system such that A_1 is the origin and A_1A_2 agrees with the x-axis, and the whole pentagon lies in the upper half-plane. Recall that $-v_1 = (\cos(3\pi/10), \sin(3\pi/10))$, $-v_2 = (\cos(7\pi/10), \sin(7\pi/10))$, $-v_3 = (\cos(11\pi/10), \sin(11\pi/10))$, $-v_4 = (0, -1)$, $-v_5 = (\cos(-\pi/10), \sin(-\pi/10))$ are the outer normals of the sides, as we defined earlier. From the setup the equations of $l_3' = A_5A_1$ and $l_4' = A_1A_2$ readily follow: $l_3' \colon \langle -v_3, (x,y) \rangle = 0$ and $l_4' \colon y = 0$. Using the definition of m_1 we obtain $l_1' \colon \langle -v_1, (x,y) \rangle = m_1$. And again by the definition of m_3 and m_4, A_4 is on the line

$l_4\colon y = m_4$ and A_3 is on $l_3\colon \langle -v_3, (x,y)\rangle = -m_3$. We can express A_3 and A_4 by solving the system of equations:

$$\begin{aligned} A_3 &= \left(\frac{m_1 \sin\frac{11\pi}{10} + m_3 \sin\frac{3\pi}{10}}{\sin\frac{8\pi}{10}}, \frac{m_1 \cos\frac{11\pi}{10} + m_3 \cos\frac{3\pi}{10}}{-\sin\frac{8\pi}{10}} \right), \\ A_4 &= \left(\frac{m_1 - m_4 \sin\frac{3\pi}{10}}{\cos\frac{3\pi}{10}}, m_4 \right). \end{aligned}$$

Now, we can find the equation of l_2' and l_5'. After suitable simplifications, introducing the golden ratio $\lambda = (\sqrt{5}+1)/2$, we obtain

$$l_2'\colon \cos\frac{7\pi}{10}x + \sin\frac{7\pi}{10}y = -m_1 + \lambda m_4, \tag{6.3}$$

$$l_5'\colon \cos\frac{-\pi}{10}x + \sin\frac{-\pi}{10}y = -m_1 + \lambda m_3. \tag{6.4}$$

Thus $A_2 = ((-m_1 + \lambda m_3)/\cos(-\pi/10), 0)$, and to obtain m_2 we need to calculate the distance between A_2 and l_2':

$$\begin{aligned} m_2 &= \left| \cos\frac{7\pi}{10} \cdot \frac{-m_1 + \lambda m_3}{\cos(\pi/10)} + m_1 - \lambda m_4 \right| = \left| \left(\frac{1}{\lambda} + 1 \right) m_1 - m_3 - \lambda m_4 \right| \\ &= |\lambda m_1 - m_3 - \lambda m_4|. \end{aligned}$$

From (6.3) and (6.4) it readily follows that $m_3 > m_1/\lambda$ and $\lambda m_4 > m_1$, hence

$$m_2 = -\lambda m_1 + m_3 + \lambda m_4. \tag{6.5}$$

Now, suppose that m_1 and m_2 are the two largest heights. If $m_2 \neq m_3$, then we have a contradiction by (6.5). If $m_2 = m_3$, then since m_1 and m_2 are the two largest, it follows that $m_1 = m_2 = m_3 = m_4$, and hence the pentagon is regular. □

As a consequence of the previous lemmas we obtain

Theorem 6.4. *The limit pentagon has exactly one height equal to ρ_5 a.s.*

Remark. With a rather tedious case analysis one can prove that for any height of the limit pentagon $m_i \geq \rho_5 + 2 - 4c \approx 0.33688$, which is sharp.

6.2. **Rapidness of the pentagon process.** In the previous section we proved that the limit pentagon has exactly one height equal to ρ_5 a.s., i.e., after finite number of steps K_n has only one height greater than ρ_5. This observation allows us to prove some asymptotic results for the speed, however, as the area of the limit is now *random*, we cannot prove limit theorem, only upper and lower bounds.

Let t^* denote the maximum and t_* the minimum of the area of the possible limit pentagons. Note that $t_* \geq \pi/100$ by Lemma 5.2. Then we have the following.

Theorem 6.5. *For any $x > 0$*

$$\begin{aligned} \mathrm{e}^{-\frac{x^2}{t_*}} &\leq \liminf_{n\to\infty} \mathbf{P}\left\{ \sqrt{n \tan\frac{3\pi}{10}}(m_n - \rho_5) > x \right\} \\ &\leq \limsup_{n\to\infty} \mathbf{P}\left\{ \sqrt{n \tan\frac{3\pi}{10}}(m_n - \rho_5) > x \right\} \leq \mathrm{e}^{-\frac{x^2}{t^*}}. \end{aligned}$$

FIGURE 7. The evolution of $\sqrt{n}(m_n - \rho_7)$ for 1800 iterations of the heptagon process

Moreover,

$$\lim_{n\to\infty} \mathbf{E}\sqrt{n \tan \frac{3\pi}{10}}(m_n - \rho_5) = \frac{\mathbf{E}\sqrt{t}}{4\sqrt{\pi}},$$

where t denotes the area of the limit pentagon.

Proof. Put $t_n = |K_n|$. Once there is only one height greater than ρ_5 the limit pentagon is determined and so is its area $\lim_{n\to\infty} t_n = t$. The area of the only non-empty change region $|\mathcal{R}_i(n)| = (m_n - \rho_5)^2 \tan \frac{3\pi}{10}$. This means that the height process $\ell_n = m_n - \rho_5$ behaves as

$$\ell_{n+1} = \begin{cases} \ell_n(1 - h_{n+1}), & \text{w.p. } \frac{\ell_n^2}{t_n} \tan \frac{3\pi}{10}, \\ \ell_n, & \text{w.p. } 1 - \frac{\ell_n^2}{t_n} \tan \frac{3\pi}{10}, \end{cases}$$

where $h_1, h_2, \ldots$ are iid beta$(1,2)$ random variables. Since $t_n \downarrow t$ a.s., by Lemma 2.3 with $\delta = 2$, $a_n = t_n$, $c = \tan(3\pi/10)$ we obtain that given t we have for any $x > 0$

$$\mathbf{P}\left\{ \sqrt{\frac{n \tan \frac{3\pi}{10}}{t}}(m_n - \rho_5) > x \,\Big|\, t \right\} \to \mathrm{e}^{-x^2},$$

or

$$\mathbf{P}\left\{ \sqrt{n \tan \frac{3\pi}{10}}(m_n - \rho_5) > x \,\Big|\, t \right\} \to \mathrm{e}^{-\frac{x^2}{t}}.$$

The convergence of moments also holds (as in Lemma 2.3), in particular

$$\begin{aligned} \mathbf{E}\sqrt{n \tan \frac{3\pi}{10}}(m_n - \rho_5) &= \mathbf{E}\left[\mathbf{E}\left[\sqrt{n \tan \frac{3\pi}{10}}(m_n - \rho_5) \,\Big|\, t \right] \right] \\ &\to \mathbf{E} \int_0^\infty \mathrm{e}^{-\frac{x^2}{t}} \mathrm{d}x = \frac{\mathbf{E}\sqrt{t}}{4\sqrt{\pi}}, \end{aligned}$$

and the theorem is proved. □

7. Rapidness estimates

In general the polygon process is too complicated to say anything more about the limit object than Lemma 5.1. According to this lemma the maximal height of the limit object is ρ_k. Figure 7 shows the evolution of the process $\sqrt{n}(m_n - \rho_7)$, and we see that $\sqrt{n}$ is indeed the right normalization for the heptagon process too. Using stochastic majorization and minorization we are able to prove this in general.

Theorem 7.1. *Let $k \geq 5$ be an odd integer. For any $x > 0$ we have*

$$0 < \liminf_{n\to\infty} \mathbf{P}\{\sqrt{n}(m_n - \rho_k) > x\} \leq \limsup_{n\to\infty} \mathbf{P}\{\sqrt{n}(m_n - \rho_k) > x\} < 1.$$

Proof. Let $\mathbf{m}_n = (m_1(n), \ldots, m_k(n))$ be the height vector, m_n its maximum, and $A_n = \sum_{i=1}^k |\mathcal{R}_i(n)|$ the total area of the change regions. By Corollary 5.4 we may and do assume that the change regions are already disjoint. The probability of no change is the probability that the random point does not fall in $\bigcup \mathcal{R}_i(n)$, that is, $\mathbf{P}\{\mathbf{m}_{n+1} = \mathbf{m}_n\} = 1 - A_n/|K_n|$. The probability of change is $A_n/|K_n|$, in particular $|\mathcal{R}_i(n)|/|K_n|$ is the probability that we choose the point in $\mathcal{R}_i(n)$. In this case $m_i(n+1) = m_i(n) - h^i_{n+1}(m_i(n) - \rho_k)$, and all the other heights decrease at most $h^i_{n+1}(m_i(n) - \rho_k)$, where $h^i_{n+1}(m_i(n) - \rho_k)$ is the distance from the base of a uniformly chosen point in $\mathcal{R}_i(n)$, and so h^i_{n+1} is the distance from the base of a uniformly chosen point in $\mathcal{R}_i(n)(m_i(n) - \rho_k)^{-1}$, i.e., we scale the change region to have height 1. So we have that in case of change $\mathbf{m}_{n+1} \geq \mathbf{m}_n - h^i_{n+1}(m_n - \rho_k)\mathbf{1}$, where $\mathbf{1}$ stands for the constant 1 vector, and so $m_{n+1} \geq m_n - h^i_{n+1}(m_n - \rho_k)$.

We want to construct simple processes, serving as lower and upper bound for m_n. In order to do so we recall same basic properties of stochastic ordering. For random variables X and Y we say that X is *stochastically larger* than Y ($Y \leq_{\mathrm{st}} X$) if $\mathbf{P}\{X \leq x\} \leq \mathbf{P}\{Y \leq x\}$ for any $x \in \mathbb{R}$. This is equivalent to the condition $\mathbf{E}f(X) \geq \mathbf{E}f(Y)$ for any increasing function f. For random vectors the definition is somewhat trickier. In $\mathbb{R}^k$ a set U is an *upper set* if for $\mathbf{x}_1 \in U$, $\mathbf{x}_2 \geq \mathbf{x}_1$ imply $\mathbf{x}_2 \in U$. For k-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ we have $\mathbf{Y} \leq_{\mathrm{st}} \mathbf{X}$ if $\mathbf{P}\{\mathbf{X} \in U\} \geq \mathbf{P}\{\mathbf{Y} \in U\}$ for any upper set U. This is equivalent to the condition $\mathbf{E}f(\mathbf{X}) \geq \mathbf{E}f(\mathbf{Y})$ for any $f : \mathbb{R}^k \to \mathbb{R}$ that is increasing in each argument. We refer to Shaked and Shanthikumar [12] (Chapter 1.A and Chapters 6.A and 6.B).

The first step is to obtain a stochastic majorant and minorant for h^i_n for any type of scaled change regions. Let us fix such a region, and let t_x be the area of those points in the region, which are farther than $1 - x$ from the base. If h is the distance of the random point from the base, then $\mathbf{P}\{h > 1 - x\} = t_x/t_1$. The angle of the upper vertex is at most $\frac{k-2}{k}\pi$, and the corresponding angle bisector is orthogonal to the base, so for all $x \in [0, 1]$

$$t_x \leq \frac{1}{2} x\, 2x \tan \frac{(k-2)\pi}{2k} = x^2 \tan \frac{(k-2)\pi}{2k}.$$

By Lemma 5.2 a disc of radius 1/10 is contained in K_n, which together with convexity imply that the angle of the upper vertex is at least $2 \arcsin \frac{1}{20}$. Therefore

$$t_x \geq x^2 \tan\left(\arcsin \frac{1}{20}\right) =: x^2 \delta_1.$$

Summarizing, we have

$$x^2 \delta_1 \leq \mathbf{P}\{h > 1 - x\} = \frac{t_x}{t_1} \leq x^2 c_1,$$

where $c_1 = \tan \frac{(k-2)\pi}{2k} \gg 1$. Note that δ_1 in the lower bound does not depend on k. For $x \geq 0$ put

$$H^*(x) = \min\{x^2 c_1, 1\}, \tag{7.1}$$

$$H_*(x) = \begin{cases} x^2 \delta_1, & x \in [0, 1), \\ 1, & x \geq 1, \end{cases} \tag{7.2}$$

for the distribution functions of the stochastic minorant and majorant of $1 - h$.

The previous reasoning also shows that

$$\delta_1 (m_i(n) - \rho_k)_+^2 \leq |\mathcal{R}_i(n)| \leq c_1 (m_i(n) - \rho_k)_+^2,$$

and so

$$\delta_1(m_n - \rho_k)^2 \le A_n \le kc_1(m_n - \rho_k)^2. \tag{7.3}$$

By the trivial bound and by Lemma 5.2 we have the following upper and lower bounds for the area:

$$\pi/100 \le |K_n| \le |K| \le \pi. \tag{7.4}$$

The lower bound. Using (7.3) and (7.4) the change probability can be estimated as

$$\frac{A_n}{|K_n|} \le \frac{100kc_1}{\pi}(m_n - \rho_k)^2 =: c_2(m_n - \rho_k)^2.$$

Let us define the process

$$m'_{n+1} = \begin{cases} m'_n - (m'_n - \rho_k)h_{n+1}, & \text{w.p. } c_2(m'_n - \rho_k)^2, \\ m'_n, & \text{w.p. } 1 - c_2(m'_n - \rho_k)^2, \end{cases} \tag{7.5}$$

where $h_1, h_2, \ldots,$ iid, and $1 - h_1$ has distribution function H^* in (7.1). We claim that

$$\mathbf{P}\{m_{n+1} \le x | m_n = y\} \le \mathbf{P}\{m'_{n+1} \le x | m'_n = y\}. \tag{7.6}$$

Indeed, m'_n decreases with higher probability, and if it decreases, the decrement is greater. Putting $\ell'_n = m'_n - \rho_k$

$$\ell'_{n+1} = \begin{cases} \ell'_n(1 - h_{n+1}), & \text{w.p. } c_2(\ell'_n)^2, \\ \ell'_n, & \text{w.p. } 1 - c_2(\ell'_n)^2. \end{cases}$$

We can write $\ell'_{n+1} = \min\{\ell'_n, U_{n+1}\}$, with U_{n+1} independent of ℓ'_n and having distribution function

$$\mathbf{P}\{U_{n+1} \le x\} = \begin{cases} c_1c_2x^2, & x < \ell'_n/\sqrt{c_1}, \\ c_2\,(\ell'_n)^2, & x \in [\ell'_n/\sqrt{c_1}, \ell'_n), \\ 1, & x \ge \ell'_n. \end{cases}$$

If V has distribution function

$$\widetilde{H}(x) = \min\left\{c_1c_2x^2, 1\right\}, \tag{7.7}$$

and it is independent of ℓ'_n, then $\min\{\ell'_n, U_{n+1}\} \ge_{\mathrm{st}} \min\{\ell'_n, V\}$ for any n and ℓ'_n. For $V_1, V_2, \ldots$ iid with distribution function $\widetilde{H}$, put $\underline{V}_n = \min\{V_1, \ldots, V_n\}$. We obtained that for all n

$$\mathbf{P}\{\ell'_{n+1} \le x | \ell'_n = y\} \le \mathbf{P}\{\underline{V}_{n+1} \le x | \underline{V}_n = y\},$$

and combining this with (7.6) we deduce

$$\mathbf{P}\{\ell_{n+1} \le x | \ell_n = y\} \le \mathbf{P}\{\underline{V}_{n+1} \le x | \underline{V}_n = y\}, \tag{7.8}$$

where $\ell_n = m_n - \rho_k$. We claim that these inequalities imply the unconditional inequality.

The latter process can be written as (we assume that the process starts from a sufficiently small state)

$$\underline{V}_{n+1} = \begin{cases} \underline{V}_n k_{n+1}, & \text{w.p. } c_1c_2\underline{V}_n^2, \\ \underline{V}_n, & \text{w.p. } 1 - c_1c_2\underline{V}_n^2, \end{cases}$$

where $k_1, k_2, \dots$ are iid beta$(2,1)$ random variables. Short calculation gives that

$$\mathbf{P}\left\{\underline{V}_{n+1} \le x | \underline{V}_n = y\right\} = \begin{cases} 1, & x \ge y, \\ c_1 c_2 x^2, & x < y, \end{cases}$$

which is decreasing in y for any fix x.

Let us assume that $\ell_0 = \underline{V}_0$, and it is sufficiently small. The law of total probability and (7.8) imply $\mathbf{P}\{\ell_1 \le x\} \le \mathbf{P}\{\underline{V}_1 \le x\}$. Assume that for any $x > 0$, $\mathbf{P}\{\ell_n \le x\} \le \mathbf{P}\{\underline{V}_n \le x\}$ for some $n \ge 1$. Then

$$\begin{aligned} \mathbf{P}\{\ell_{n+1} \le x\} &= \int \mathbf{P}\{\ell_{n+1} \le x | \ell_n = y\} \, \mathrm{d}\mathbf{P}\{\ell_n \le y\} \\ &\le \int \mathbf{P}\{\underline{V}_{n+1} \le x | \underline{V}_n = y\} \, \mathrm{d}\mathbf{P}\{\ell_n \le y\} \\ &\le \int \mathbf{P}\{\underline{V}_{n+1} \le x | \underline{V}_n = y\} \, \mathrm{d}\mathbf{P}\{\underline{V}_n \le y\} \\ &= \mathbf{P}\{\underline{V}_{n+1} \le x\}, \end{aligned}$$

where we used the law of total probability, (7.8), the induction hypothesis, the monotonicity of the conditional probabilities, and that for two distribution functions F, G, such that $F(x) \le G(x)$, and for a monotone decreasing function f we have $\int f \mathrm{d}F \le \int f \mathrm{d}G$ ([12], Chapter 1.A). So we proved that $\underline{V}_n \le_{\mathrm{st}} \ell_n$ for every n.

For the asymptotic behavior of $\underline{V}_n$ we have

$$\mathbf{P}\left\{\sqrt{c_1 c_2}\sqrt{n}\underline{V}_n > x\right\} \to \mathrm{e}^{-x^2},$$

and since $\underline{V}_n \le_{\mathrm{st}} \ell_n$, we obtain

$$\liminf_{n\to\infty} \mathbf{P}\left\{\sqrt{c_1 c_2}\sqrt{n}(m_n - \rho_k) > x\right\} \ge \mathrm{e}^{-x^2}.$$

In particular we have

$$\begin{aligned} \mathbf{E}\left[\sqrt{c_1 c_2}\sqrt{n}(m_n - \rho_k)\right] &= \int_0^\infty \mathbf{P}\left\{\sqrt{c_1 c_2}\sqrt{n}(m_n - \rho_k) > x\right\} \mathrm{d}x \\ &\ge \int_0^\infty \mathbf{P}\left\{\sqrt{c_1 c_2}\sqrt{n}\underline{V}_n > x\right\} \mathrm{d}x \\ &\to \int_0^\infty \mathrm{e}^{-x^2} \mathrm{d}x, \end{aligned}$$

where at the last convergence we used the uniform integrability of $\sqrt{n}\underline{V}_n$.

The upper bound. Now we turn to the construction of the upper bound process. If the random point falls in the change region $\mathcal{R}_i(n)$, then we have $m_i(n+1) = m_i(n) - h^i_{n+1}(m_i(n) - \rho_k)$, and the other heights may change or may not. In any case $\mathbf{m}_{n+1} \le \mathbf{m}_n - \mathbf{e}_i h^i_{n+1}(m_i(n) - \rho_k)$, where $\mathbf{e}_i$ is the ith standard, k-dimensional unit vector. The probability of this event is $|\mathcal{R}_i(n)|/|K_n|$ for which by (7.3) and (7.4)

$$\frac{|\mathcal{R}_i(n)|}{|K_n|} \ge \frac{\delta_1}{\pi}(m_i(n) - \rho_k)_+^2 =: c_3(m_i(n) - \rho_k)_+^2.$$

Instead of h^i we put the stochastically smaller h, for which $1 - h$ has distribution function H_* defined in (7.2). Note that for this h we have $\mathbf{P}\{h = 0\} = 1 - \delta_1$.

We define the k-dimensional process $\widehat{\mathbf{m}}_n$ as follows. Let $i \in \{1, 2, \dots, k\}$ such that $n + 1 \equiv i \pmod{k}$. Then define

$$\widehat{\mathbf{m}}_{n+1} = \begin{cases} \widehat{\mathbf{m}}_n - \mathbf{e}_i h_{n+1} \left(\widehat{m}_i(n) - \rho_k\right), & \text{w.p. } c_3(m_i(n) - \rho_k)_+^2, \\ \widehat{\mathbf{m}}_n, & \text{w.p. } 1 - c_3(m_i(n) - \rho_k)_+^2, \end{cases} \tag{7.9}$$

where $h_1, h_2, \dots$ are iid and $1 - h_1$ has distribution function H_* in (7.2), that is, in each step at most one component decreases, and component i can decrease only in steps $\ell k + i$, $\ell \in \mathbb{N}$. From the construction it is clear that for each $\mathbf{y} \in \mathbb{R}^k$, and for each upper set U

$$\mathbf{P}\{\mathbf{m}_{n+1} \in U | \mathbf{m}_n = \mathbf{y}\} \le \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U | \widehat{\mathbf{m}}_n = \mathbf{y}\}. \tag{7.10}$$

Now we show that $\mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U | \widehat{\mathbf{m}}_n = \mathbf{y}\}$ is a monotone increasing function of $\mathbf{y}$ for any fixed upper set U. To do so, let $n + 1 \equiv i \pmod{k}$, and define $u_i(\mathbf{y}) = \inf\{u : (y_1, \dots, y_{i-1}, u, y_{i+1}, \dots, y_k) \in U\}$. We may assume that $\mathbf{y} \in U$, $\widehat{m}_i(n) > \rho_k$ and $u_i(\mathbf{y}) > \rho_k$, otherwise the statement is obvious. Recall that in one step only coordinate i can change, and so by (7.9) we have

$$\begin{aligned} \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U | \widehat{\mathbf{m}}_n = \mathbf{y}\} &= 1 - \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \notin U | \widehat{\mathbf{m}}_n = \mathbf{y}\} \\ &= 1 - \mathbf{P}\{\widehat{m}_i(n+1) < u_i(\mathbf{y}) | \widehat{\mathbf{m}}_n = \mathbf{y}\} \\ &= 1 - c_3 (y_i - \rho_k)^2 \, \mathbf{P}\left\{1 - h < \frac{u_i(\mathbf{y}) - \rho_k}{y_i - \rho_k}\right\} \\ &= 1 - c_3 \delta_1 (u_i(\mathbf{y}) - \rho_k)_+^2. \end{aligned}$$

By the properties of the upper set we have that $\mathbf{y} \le \mathbf{y}'$ implies $u_i(\mathbf{y}) \ge u_i(\mathbf{y}')$ and so the conditional probability is monotone increasing. As in the case of the lower estimation this allows us to prove the majorization $\mathbf{m}_n \le_{\mathrm{st}} \widehat{\mathbf{m}}_n$ as follows: If $\mathbf{m}_0 = \widehat{\mathbf{m}}_0$ in distribution, then we have the majorization for $n = 1$, and if it is true for some $n \ge 1$, then for any upper set U

$$\begin{aligned} \mathbf{P}\{\mathbf{m}_{n+1} \in U\} &= \int \mathbf{P}\{\mathbf{m}_{n+1} \in U | \mathbf{m}_n = \mathbf{y}\} \, \mathrm{d}\mathbf{P}\{\mathbf{m}_n \le \mathbf{y}\} \\ &\le \int \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U | \widehat{\mathbf{m}}_n = \mathbf{y}\} \, \mathrm{d}\mathbf{P}\{\mathbf{m}_n \le \mathbf{y}\} \\ &\le \int \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U | \widehat{\mathbf{m}}_n = \mathbf{y}\} \, \mathrm{d}\mathbf{P}\{\widehat{\mathbf{m}}_n \le \mathbf{y}\} \\ &= \mathbf{P}\{\widehat{\mathbf{m}}_{n+1} \in U\}, \end{aligned}$$

where we used the law of total probability, (7.10), the induction hypothesis, the monotonicity of the conditional probabilities, and that for two distribution functions F, G, such that $F(\mathbf{x}) \ge G(\mathbf{x})$, and for a monotone increasing function f we have $\int f \mathrm{d}F \le \int f \mathrm{d}G$ ([12, Chapter 6.B]).

Putting

$$\overline{H}(x) = \min\{\delta_1 c_3 x^2, 1\}, \tag{7.11}$$

as before we see that

$$\widehat{m}_i(n) - \rho_k = \min\{W_{i,j} : j = 1, 2, \dots; k(j-1) + i \le n\},$$

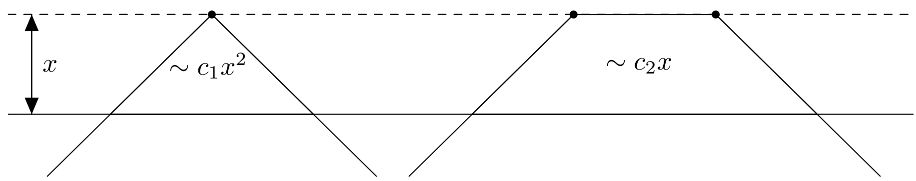

FIGURE 8. Change regions of different shapes

where $\{W_{i,j} : i = 1, 2, \ldots, k; j \in \mathbb{N}\}$ are iid random variables with distribution function $\overline{H}$. We have

$$\begin{aligned}
&\mathbf{P}\left\{\sqrt{\delta_1 c_3 n} \max_{1\le i\le k} \min\{W_{i,j} : j = 1, 2, \ldots; \, k(j-1)+i \le n\} \le x\right\} \\
&= \prod_{i=1}^{k} \left[1 - \mathbf{P}\left\{\sqrt{\delta_1 c_3 n} \min\{W_{i,j} : j = 1, 2, \ldots; \, k(j-1)+i \le n\} > x\right\}\right] \\
&\to \left(1 - \mathrm{e}^{-\frac{x^2}{k}}\right)^k .
\end{aligned}$$

This, together with the stochastic majorization $\mathbf{m}_n \le_{\mathrm{st}} \widehat{\mathbf{m}}_n$ implies that

$$\limsup_{n\to\infty} \mathbf{P}\left\{\sqrt{\delta_1 c_3 n}(m_n - \rho_k) > x\right\} \le 1 - \left(1 - \mathrm{e}^{-\frac{x^2}{k}}\right)^k .$$

In particular we have

$$\begin{aligned}
\mathbf{E}\left[\sqrt{\delta_1 c_3 n}(m_n - \rho_k)\right] &= \int_0^\infty \mathbf{P}\left\{\sqrt{\delta_1 c_3 n}(m_n - \rho_k) > y\right\} \mathrm{d}y \\
&\le \int_0^\infty \mathbf{P}\left\{\sqrt{\delta_1 c_3 n}(\widehat{m}_n - \rho_k) > y\right\} \mathrm{d}y \\
&\to \int_0^\infty \left[1 - \left(1 - \mathrm{e}^{-\frac{y^2}{k}}\right)^k\right] \mathrm{d}y.
\end{aligned}$$

□

8. CONCLUDING REMARKS

The major difference between regular polygons with an odd and an even number of vertices hides in the fact that while in the odd case the change regions are always triangles, in the even case change regions might be trapezoids or (in the degenerated case) triangles, hence their area might be of different order (see Figure 8). We conjecture that in the latter case the 'typical' change regions are trapezoids, which would imply that the speed of the process is $1/n$. (Compare with Theorem 7.1, where we obtained $1/\sqrt{n}$ for the speed in the odd case.) This conjecture is well supported by numerical experiments. We conclude the paper with the results of some computer simulations; see Figure 9 and Figure 10. It is transparent that $\sqrt{n}$ and n are the right normalizations, respectively.

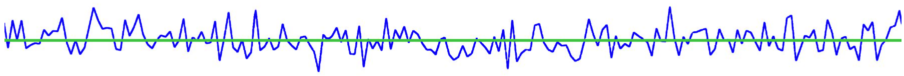

FIGURE 9. $\sqrt{100}(m_{100} - \rho_7)$ for 200 outcomes of the heptagon process (the minimal value is 0.215, the maximal value is 1.078)

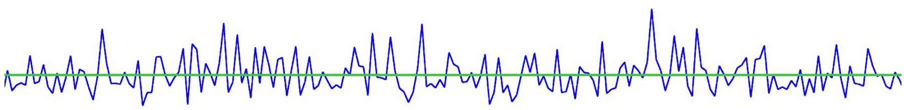

FIGURE 10. $100(m_{100} - \rho_8)$ for 200 outcomes of the octagon process (the minimal value is 0.236, the maximal value is 5.381)

APPENDIX

Proof of Lemma 2.1. To prove part (i) note that the distributional equality means

$$F(x) = F(a)F(x/a),$$

for all $0 \le x \le a$. The monotonicity of F easily implies that the solution has the stated form for some $\delta > 0$. The 'if' part follows by simple calculation.

We turn to part (ii). For any $x \in [1/2, 1]$

$$\mathbf{P}\{I(X \le 1/2) = 0,\ \max\{X, 1-X\} > x\} = \mathbf{P}\{X > x\} = 1 - F(x),$$

and

$$\begin{aligned}&\mathbf{P}\{I(X \le 1/2) = 0\}\,\mathbf{P}\{\max\{X, 1-X\} > x\} \\ &= (1 - F(1/2))\,(1 - F(x) + F(1-x-))\,.\end{aligned}$$

Solving the equation for F we obtain the statement. □

Proof of Lemma 2.3. After some calculation one obtains that given ℓ_n and a_n, $\ell_{n+1} \stackrel{\mathcal{D}}{=} \min\{\ell_n, Y\}$, where Y is a non-negative random variable, such that Y^δ is uniformly distributed on $[0, a_n/c]$.

For any $\varepsilon \ge 0$ let $U^{(\varepsilon)}, U_1^{(\varepsilon)}, U_2^{(\varepsilon)}, \ldots$ be iid non-negative random variables, such that

$$\mathbf{P}\{U^{(\varepsilon)} \le x\} = x^\delta \frac{c}{a+\varepsilon}, \quad x \in \left[0, [(a+\varepsilon)/c]^{1/\delta}\right],$$

that is, $(U^{(\varepsilon)})^\delta \sim \text{Uniform}[0, (a+\varepsilon)/c]$. Put

$$M_n^{(\varepsilon)} = \min\{U_1^{(\varepsilon)}, U_2^{(\varepsilon)}, \ldots, U_n^{(\varepsilon)}\}.$$

Since a_n is decreasing, $Y \ge_{\text{st}} U^{(0)}$, therefore

$$\ell_n \ge_{\text{st}} \min\{U_1^{(0)}, U_2^{(0)}, \ldots, U_n^{(0)}\} = M_n^{(0)}.$$

As

$$\mathbf{P}\left\{\left(\frac{cn}{a}\right)^{1/\delta} M_n^{(0)} > x\right\} \to \mathrm{e}^{-x^\delta},$$

we have

$$\liminf_{n\to\infty} \mathbf{P}\left\{\left(\frac{cn}{a}\right)^{1/\delta} \ell_n > x\right\} \ge \lim_{n\to\infty} \mathbf{P}\left\{\left(\frac{cn}{a}\right)^{1/\delta} M_n^{(0)} > x\right\} = \mathrm{e}^{-x^\delta}.$$

To prove the reverse inequality, let us fix $\varepsilon > 0$, $\beta > 0$. Given that $a_n \le a + \varepsilon$ we have $Y \le_{\mathrm{st}} U^{(\varepsilon)}$, and thus given that $a_{\lfloor \beta n \rfloor} < a + \varepsilon$ we have

$$\ell_n \le_{\mathrm{st}} \min \left\{ U_1^{(\varepsilon)}, U_2^{(\varepsilon)}, \dots, U_{\lfloor (1-\beta) n \rfloor}^{(\varepsilon)} \right\} =: M_{\lfloor (1-\beta) n \rfloor}^{(\varepsilon)}.$$

By the law of total probability

$$\begin{aligned}
&\mathbf{P}\left\{ \left(\frac{cn}{a} \right)^{1/\delta} \ell_n > x \right\} \\
&= \mathbf{P}\left\{ \left(\frac{cn}{a} \right)^{1/\delta} \ell_n > x \middle| a_{\lfloor \beta n \rfloor} < a + \varepsilon \right\} \cdot \mathbf{P}\left\{ a_{\lfloor \beta n \rfloor} < a + \varepsilon \right\} \\
&\quad + \mathbf{P}\left\{ \left(\frac{cn}{a} \right)^{1/\delta} \ell_n > x \middle| a_{\lfloor \beta n \rfloor} \ge a + \varepsilon \right\} \cdot \mathbf{P}\left\{ a_{\lfloor \beta n \rfloor} \ge a + \varepsilon \right\} \\
&\le \mathbf{P}\left\{ \left(\frac{cn}{a} \right)^{1/\delta} M_{\lfloor (1-\beta) n \rfloor}^{(\varepsilon)} > x \right\} + \mathbf{P}\left\{ a_{\lfloor \beta n \rfloor} \ge a + \varepsilon \right\}.
\end{aligned}$$

By the assumption $a_n \downarrow a > 0$ a.s., so the second term goes to 0, while from extreme value theory (see e.g. [5], p. 192) we have

$$\lim_{n\to\infty} \mathbf{P}\left\{ \left(\frac{c}{a} n \right)^{1/\delta} M_{\lfloor (1-\beta) n \rfloor}^{(\varepsilon)} > x \right\} = \mathrm{e}^{-x^\delta \frac{a}{a+\varepsilon}(1-\beta)},$$

that is, by the stochastic dominance

$$\limsup_{n\to\infty} \mathbf{P}\left\{ \left(\frac{c}{a} n \right)^{1/\delta} \ell_n > x \right\} \le \mathrm{e}^{-x^\delta \frac{a}{a+\varepsilon}(1-\beta)}.$$

Since $\varepsilon > 0$ and $\beta > 0$ are as small as we want, we obtain

$$\limsup_{n\to\infty} \mathbf{P}\left\{ \left(\frac{c}{a} n \right)^{1/\delta} \ell_n > x \right\} \le \mathrm{e}^{-x^\delta},$$

and the convergence in distribution is proved.

Once we have the distributional convergence, to prove the moment convergence it is enough to show that $\{n^{\alpha/\delta} \ell_n^\alpha\}$ is uniformly integrable (see e.g. [5], Theorem 25.12). Since a_n is bounded, for some $\eta > 0$ we have $a_n \le a + \eta$ a.s. for all $n \ge 1$, and thus $\ell_n \le_{\mathrm{st}} M_n^{(\eta)}$. Therefore

$$\mathbf{P}\{n^{1/\delta} \ell_n > x\} \le \mathbf{P}\{n^{1/\delta} M_n^{(\eta)} > x\} = \left(1 - \frac{x^\delta}{n} \frac{c}{a+\eta} \right)^n \le \mathrm{e}^{-x^\delta \frac{c}{a+\eta}},$$

and the uniform integrability follows. □

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