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REFINED ESTIMATES FOR SIMPLE BLOW-UPS OF THE SCALAR CURVATURE EQUATION ON S^n

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Abstract. In their work on a sharp compactness theorem for the Yamabe problem, Khuri, Marques and Schoen [J. Differential Geom. 81 (2009), 143–196] apply a refined blow - up analysis (what we call 'second order blow - up argument' in this article) to obtain highly accurate approximate solutions for the Yamabe equation. As for the conformal scalar curvature equation on S^n with $n \ge 4$, we examine the second order blow - up argument and obtain a refined estimate for a blow - up sequence near a simple blow - up point. The estimate involves the local effect from the Taylor expansion of the scalar curvature function, the global effect from other blow - up points, and the balance formula as expressed in the Pohozaev identity in an essential way.

1. Introduction

In this article, we expound local and global contributions to a refined 'second order' estimate for simple blow - ups (or *simple isolated blow - ups* as known in some literature) of the prescribed scalar curvature equation

$$\Delta_1 u - \tilde c_n \, n(n-1)\, u + (\tilde c_n \, \mathcal{K})\, u^{\frac{n+2}{n-2}} = 0 \qquad \text{on} \quad S^n. \tag{1.1}$$

Here $\mathcal{K}$, fixed once it is given, is assumed to be smooth enough (say, in $C^{\,n+4}(S^n)$), Δ_1 is the Laplacian on S^n with the standard metric g_1, and $\tilde c_n = \frac{n-2}{4\,(n-1)}$ $(n \ge 3)$. Via the stereographic *projection* $\dot{\mathcal{P}} : S^n \setminus \{\mathbf{N}\} \longrightarrow \mathbb{R}^n$, which sends the north pole $\mathbf{N} \in S^n$ to infinity, equation (1.1) can be expressed in the simple form

$$\Delta_o v + (\tilde c_n K)\, v^{\frac{n+2}{n-2}} = 0\,, \tag{1.2}$$

(1.3)

$$v(y) := u(\dot{\mathcal{P}}^{-1}(y)) \cdot \left(\frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}} \ \text{and} \ K(y) := \mathcal{K}(\dot{\mathcal{P}}^{-1}(y)) \qquad \text{for} \quad y \in \mathbb{R}^n .$$

In (1.2), Δ_o is the Laplacian on $\mathbb{R}^n$ with the standard Euclidean metric g_o. Considered as a 'dual' to the Yamabe equation, the study on the non - compact behavior (or blow - up phenomenon) of equation (1.1) is found to be rich and vibrant. See for example [7], [8], [15], [20], [21] and the references therein.

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1a. Simple blow-up. When $(\tilde{c}_n K)$ is equal to a constant, say, $n(n-2)$, equation (1.2) has a family of solutions:

$$\mathbf{A}_{\epsilon_i, \zeta_i}(y) = \left(\frac{\epsilon_i}{\epsilon_i^2 + |y - \zeta_i|^2} \right)^{\frac{n-2}{2}} . \tag{1.4}$$

For non-constant K, a sequence of positive solutions $\{v_i\}$ of (1.2) which blows up at 0 is shown to be 'close' to a sequence found in (1.4). Precisely,

(1.5)

$$\left| v_i(y) - \mathbf{A}_{\epsilon_i, \zeta_i}(y) \right| \le \varepsilon_i \cdot \epsilon_i^{-\frac{n-2}{2}} \qquad \text{for} \quad |y - \zeta_i| \le \epsilon_i R_i \ \text{ and } \ i \gg 1,$$

with parameters $\epsilon_i \to 0$, $|\zeta_i| \to 0$ and $R_i \to \infty$ specific to $\{v_i\}$ (cf. (2.21) in §2d). Here (via a rescaling), we assume throughout this article that

$$(\tilde{c}_n K)(0) = n(n-2). \tag{1.6}$$

Estimate (1.5) is rather weak; its accuracy in general deteriorates when $i \to \infty$. Moreover, (1.5) is valid (generally) in a sequence of shrinking balls $B_{\zeta_i}(\epsilon_i R_i)$. (The order of shrinkage $O(\epsilon_i)$ makes space for the bubbles described in (1.4) to be stacked up (developed vertically; cf. the Delaunay solution [13]) or be put in juxtaposition (developed horizontally). See [15] for a classification of blow-ups for equation (1.1).)

One can characterize simple blow-ups in a geometric manner when the bound in (1.5) can be stabilized in terms of scale and accuracy, namely,

(1.7)

$$\frac{1}{C} \cdot \mathbf{A}_{\epsilon_i, \zeta_i}(y) \le v_i(y) \le C \cdot \mathbf{A}_{\epsilon_i, \zeta_i}(y) \quad \text{for} \quad \text{all } |y - \zeta_i| \le \rho_o \ \text{ and } \ i \gg 1.$$

Here ρ_o and C are fixed positive numbers (see Proposition 2.24 for the precise statement; cf. also the notion of quasi-isometry). Simple blow-ups are by far the most common non-compact behavior we encounter in equation (1.1). In [16], [17], [18], blow-up sequences with a fixed non-constant $\mathcal{K}$ (may not be symmetric) are constructed using the Lyapunov-Schmidt reduction method (see also [25]).

1b. Description of the main result. In this article, we identify three factors affecting the fixed scale behavior of simple blow-ups.

(I) *The local behavior of K in terms of the Taylor expansion*

$$(\tilde{c}_n K)(y) = n(n-2) + [-\mathbf{P}_\ell(y)] + R_{\ell+1}(y) \qquad \text{for} \quad y \in B_o(\rho_o). \tag{1.8}$$

Here $\mathbf{P}_\ell$ is a *homogeneous polynomial* of degree $\ell \in \mathbb{N}$, and $R_{\ell+1}$ is the remainder in the Taylor expansion. (See (2.29) and (3.7) for the sign convention we use on $\mathbf{P}_\ell$.) We know that if 0 is a blow-up point for equation (1.2), then $\ell \ge 2$ (that is, $\bigtriangledown K(0) = \vec{0}$; see Theorem 5.1 in [15] for the precise statement; cf. also [7]). Hence

$$\begin{aligned} &\text{number of critical points of } \mathcal{K} \text{ is finite} \\ &\Longrightarrow \text{equation (1.1) has at most finite number of blow-up points.} \end{aligned} \tag{1.9}$$

The leading polynomial term $\mathbf{P}_\ell$ comes into the picture when we find the difference between v_i and the standard solutions given in (1.4). See (3.11). The second order blow-up argument allows us to discern the central information enveloped in $\mathbf{P}_\ell$. We discuss this point more in §1c and §1d.

(II) *'Flexibility' of the simple blow-up as measured by* $|\xi_i| = O(\lambda_i^{\alpha})$, where

$$v_i(\xi_i) = \max \left\{ v_i(y) \mid y \in \overline{B_o(\rho_o)} \right\} \quad \text{for} \quad i \gg 1 \tag{1.10}$$

and

$$\lambda_i := [v_i(\xi_i)]^{-\frac{2}{n-2}}, \qquad \xi_i \to 0 \quad (\text{the blow-up point}). \tag{1.11}$$

Here ρ_o is a small fixed number (its size is related to other blow-up points). v_i could have other maximal points near ξ_i, but their distances to ξ_i are at most $o(\lambda_i)$ for $i \gg 1$. Refer to §2g. The position parameter ξ_i appears in the expression for the difference $[v_i - A_{\lambda_i, \xi_i}]$; see (3.11). Thanks to the work of Chen and Lin [7], [8], one can impose conditions, including the following main ones (see §2g for the full details):

$$\| \nabla \mathbf{P}_\ell(y) \| \ge C |y|^{\ell - 1} \qquad \text{for} \quad y \in B_o(\rho_o)$$

and

$$\int_{\mathbb{R}^n} \nabla \mathbf{P}_\ell(y + \mathcal{X}) [V_1(y)]^{\frac{2n}{n-2}} \, dy \ne \vec{0} \qquad \text{for all } \mathcal{X} \in \mathbb{R}^n \setminus \{0\},$$

resulting in

$$|\xi_i| = o(\lambda_i) \quad \text{modulo a subsequence} \quad \big(\text{that is}, \ \lambda_i^{-1} \cdot \xi_i \to 0\big). \tag{1.12}$$

(III) *Interaction with other blow-ups.* This is expressed by a global harmonic function (or Green's function)

$$\sum_{j=0}^{k} \frac{A_j}{|y - \hat{Y}_j|^{n-2}} \qquad \text{for} \quad y \in \mathbb{R}^n \setminus \{\hat{Y}_o = 0, \hat{Y}_1, \cdots, \hat{Y}_k\}, \tag{1.13}$$

'effective' outside a neighborhood containing all the blow-up points $\{\hat{Y}_j\}_{j=0}^{k}$. See §2e. In (1.13), A_j are positive numbers. A major challenge here is to match the information expressed in (1.13) (the 'collapsed region') with the one in (1.5) (the 'blow-up' region). See §2d.1 and §6a for a fuller discussion.

Main Theorem 1.14. *For $n \ge 4$, let $u_i \in C^{n+4}(S^n)$ be a sequence of positive solutions of equation* (1.1)*, with $\mathcal{K} \in C^{n+4}(S^n)$, and let v_i and K be associated to u_i and $\mathcal{K}$ via* (1.3)*, respectively. Assume that $\{u_i\}$ has a finite number of blow-up points: one of them is at the south pole, but none at the north pole. Take the following conditions* (1.15)–(1.19) *into account:*

(1.15) *0 is a simple blow-up point for $\{v_i\}$.*

(1.16) *$\mathcal{K} > 0$ in S^n, and K is given by the Taylor expansion in* (1.8) *in $B_o(\rho_o)$.*

(1.17) $(2 \le)\ \ell \le n - 2$.

(1.18) *The parameters λ_i and ξ_i corresponding to the simple blow-up point at 0 (via* (1.10) *and* (1.11)*, respectively) satisfy* (1.12)*, that is, $|\xi_i| = o(\lambda_i)$.*

(1.19) *When $\ell = n - 2$ is even and there is more than one blow-up point or when ℓ is odd, we require that $\Delta_o^{(h_\ell)} \mathbf{P}_\ell(y) \equiv 0$. Here h_ℓ is the biggest integer less than or equal to $\ell/2$.*

Then we can determine a polynomial Γ *(constructible from* $\mathbf{P}_\ell$ *via a fixed procedure) so that the following estimate holds (modulo a subsequence):*

(1.20)

$$\left| \, v_i\,(y) \;-\; \mathbf{A}_{\lambda_i\,,\,\xi_i}\,(y) \;-\; \left[\,\lambda_i^{\ell+1} \times \Gamma\,(\mathcal{Y})\,\right]\cdot\left[\,\mathbf{A}_{\lambda_i\,,\,\xi_i}\,(y)\,\right]^{\frac{n}{n\,-\,2}} \;-\; O_H\!\left(\lambda_i^{\frac{n\,-\,2}{2}}\right)\right|$$

$$= o\!\left(\lambda_i^{\ell \,-\, \frac{n-2}{2}}\right) \quad \text{for} \;\; y \in B_o\,(\rho_1) \quad (\,\rho_1 \le \rho_o \;\; \text{is fixed}\,)\,, \;\; \text{where } \mathcal{Y} = \frac{y \,-\, \xi_i}{\lambda_i}\,.$$

Here the term $O_H\!\left(\lambda_i^{\frac{n\,-\,2}{2}}\right)$ *is defined via the global harmonic term* (1.13)*, and its precise expression is found in* (6.58)*.* (*The precise construction of* Γ *is given in Proposition* 4.49.)

1c. Necessity of the condition $\Delta_o^{(h_\ell)}\,\mathbf{P}_\ell\,(y) \equiv 0$**.** At first sight the condition

$$\text{“}\; \Delta_o^{(h_\ell)}\,\mathbf{P}_\ell\,(y) \;=\; \underbrace{\Delta_o\,(\cdots[\,\Delta_o\,(\Delta_o\,\mathbf{P}_\ell)\,]\cdots)}_{\leftarrow\;\; h_\ell \;\;\rightarrow}(y) \;\equiv\; 0 \tag{1.21}$$

for all $y \in \mathbb{R}^n$” appears to be technical. Closer examination reveals that it is an integrated part of the discussion. In fact, under the conditions in Main Theorem 1.14, when ℓ is even, we obtain $\Delta_o^{(h_\ell)}\,\mathbf{P}_\ell\,(y) \equiv 0$ with the help of the Pohozaev identity (see Proposition 6.1). The vanishing of $\Delta_o^{(h_\ell)}\,\mathbf{P}_\ell$ allows us to construct the polynomial Γ in Main Theorem 1.14 via a reduction method, which we begin to expound.

1d. Key features of the proof. In [12], Khuri, Marques and Schoen introduce refined blow-up estimates for the Yamabe equation. The method is based on a second order approximation coupled with a second order blow-up argument. We apply these methods to the scalar curvature equation (1.1) and highlight the following differences.

The second order inhomogeneous equation is given by

$$\Delta_o\,\Phi + n\,(n+2)\,A_1^{\frac{4}{n\,-\,2}}\cdot\Phi = \mathbf{P}_\ell\cdot A_1^{\frac{n+2}{n\,-\,2}} \qquad \text{in } \mathbb{R}^n. \tag{1.22}$$

Here $A_1 = A_{1\,,\,0}$ as given in (1.4). We observe that the linear operator appeared on the left-hand side of (1.22); it is used extensively in the Lyapunov-Schmidt reduction method (see for example [2], [4], [5], [16], [17], [18]). In [12], a solution of (1.22) is found by a linear algebra method. The method does not disclose the precise form of the solution, which is desirable when we construct sharper estimates for simple blow-ups. In this manuscript, we introduce a *reduction method* which explores the recursive relations in equation (1.22) (expounded in §4). The condition $\Delta_o^{(h_\ell)}\,\mathbf{P}_\ell \equiv 0$ comes into the picture when we terminate the recursive process. As a consequence, we can determine in a step-by-step manner the exact form of the solution Φ. Although the detail is shown in §4, we indicate here that we know precisely what Γ is in (1.23) once $\mathbf{P}_\ell$ is given.

Another unique feature here is that the global influence from other blow-up points has to be taken into account when we estimate the accuracy of $O\,(\lambda_i^{n\,-\,2})$ (see §5). To do so, we have to extend the information given by the harmonic function in (1.13) to the whole neighborhood of the blow-up point at 0 in a manner so that the second blow-up argument still works. See (6.7), §5b and §6.

1e. Applications: Limitation on 'flexibility' of simple blow-up, and locations of the blow-up points. Consider the parameters λ_i and ξ_i corresponding to the simple blow-up point at 0 via (1.10) and (1.11). Suppose that

$$\xi_i = \lambda_i^{\alpha} \cdot \vec{X} \quad \text{for a fixed vector } \vec{X} \text{ and a fixed number } \alpha > 0, \tag{1.23}$$

where $1 < \alpha < 2$. Assume also that 0 is the only simple blow-up point and $\ell = n - 2$. Then we have

$$\mathbf{P}_{\ell}(\vec{X}) = 0 \qquad (\text{here } \vec{X} \text{ is considered as a point in } \mathbb{R}^n). \tag{1.24}$$

See Theorem A. 6. 62 in the e-Appendix[1] for the precise statement and full layers of information available, as well as the conditions for (1.24) to hold. In case of multiple simple blow-up points with Taylor expansions at each blow-up point given as in (1.8), where uniformly $\ell = n - 2$, similar limitations exist, and they involve the *locations* of the simple blow-up points. See §A.7 in the e-Appendix for the exact formulas.

The information should be helpful when one seeks examples and investigates situations with multiple simple blow-up points (cf. [19] on using the interaction of two close bubbles to find solutions of equation (1.1) for certain functions $\mathcal{K}$.) As a footnote, only recently a blow-up sequence with a *single* simple blow-up point was constructed for a fixed and non-identically constant K [16], [17], [18]; cf. also [14], and [4], [5] for the Yamabe equation.

1f. General conditions, assumptions and conventions. To keep the notation clean, and without losing sight of the technical details, we assume that

(1.25)
u_i and $\mathcal{K}$ are in $C^{n+4}(S^n)$, and u_i is a positive solution of (1.1).

(1.26)
"v_i and K descend from u and $\mathcal{K}$ via (1.3). Moreover, $\mathcal{K} > 0$ in S^n,

and $(\tilde{c}_n K)(0) = n(n-2)$."

The degree of smoothness assumed on u and $\mathcal{K}$ can be reduced according to the content (especially in §2).

$\bullet_1$ Throughout this work, the dimension $n \geq 3$, except when otherwise specifically mentioned, and $\tilde{c}_n = (n-2)/[4(n-1)]$. We observe the practice on using C, possibly with sub-indices, to denote various positive constants which may be rendered *differently* from line to line according to the contents, *while we use $\bar{c}$ and $\bar{C}$, possibly with sub-index, to denote a* fixed *positive constant which always keeps the same value as first defined*.

$\bullet_2$ Denote by $B_y(r)$ the open ball in $(\mathbb{R}^n, g_o)$ with center at y and radius $r > 0$. Likewise, let $\mathcal{B}_x(\rho)$ be the open ball in (S^n, g_1) with center at $x \in S^n$ and radius $\rho \in (0, \pi]$. We also use the standard notation $\langle\ ,\ \rangle$ to denote the inner product in $(\mathbb{R}^n, g_o)$.

$\bullet_3$ Given a sequence of positive numbers $\{\lambda_i\}$, and a positive number m, we say that a sequence of numbers $\{\gamma_i\}$ satisfies

$$\gamma_i = O_{\lambda_i}(m) \iff |\gamma_i| \leq C\lambda_i^m \qquad \text{for} \quad i \gg 1. \tag{1.27}$$

[1] e-Appendix is available at `https://arxiv.org/pdf/1707.02401.pdf` (pp. 44–83) and from `https://doi.org/10.1090/tran/6983` (Supplementary appendix).

Likewise,

$$\gamma_i \;=\; o_{\lambda_i}(m) \iff |\gamma_i| \;\le\; c_i\,\lambda_i^m \quad \text{for} \quad i \gg 1, \ \text{where } c_i \ge 0 \text{ and } c_i \to 0$$

as $i \to \infty$. The notation helps to highlight the order and manage longer expressions inside the brackets.

$\bullet_4$ A statement involving a sequence is said to hold "*modulo a subsequence*" if we can select a subsequence (from the original sequence in the statement) so that the statement is valid for this subsequence. As a rule, we assume that the statement is true for the original sequence so that the notation remains clean.

2. Simple blow-up

2a. Simple blow - up and its analytic definition. Intuitively, simple blow - up develops precisely one bubble in a neighborhood. Its analytic definition is given by R. Schoen in [24]. See also [12] and [20]. Via a rotation, we assume without loss of generality that the blow - up point is at the south pole. Let $\{v_i\}$ be given as in (1.3). For a simple blow - up point, there exists a sequence $\{\xi_{m_i}\} \to 0$ such that

(2.1)
$$\text{for each } i \gg 1,\ \xi_{m_i} \text{ is a } \textit{local} \text{ maximum of } v_i\,, \text{ with } \lim_{i\to\infty} v_i(\xi_{m_i}) = \infty\,,$$

and the rescaled average

$$r \longmapsto r^{\frac{n-2}{2}} \cdot \left[\frac{\int_{\partial B_{\xi_{m_i}}(r)} v_i\, dS}{\int_{\partial B_{\xi_{m_i}}(r)} 1\, dS} \right] \tag{2.2}$$

has precisely one critical point in $(0,\ \rho_o)$. Here $\rho_o > 0$ is fixed (independent on v_i for $i \gg 1$).

2b. Proportionality of simple blow - ups. The following estimate is essentially taken from Proposition 2.3 in [20]. We present it in the setting of this article.

Proportionality Proposition 2.3. *Under the standard conditions* (1.6), (1.25) *and* (1.26), *let* 0 *be a simple blow - up point for* $\{v_i\}$, *and the sequence* $\xi_{m_i} \to 0$ *carries the meaning as in* (2.1) *and* (2.2). *Then there exist positive constants* $\bar C_1$ *and* $\bar\rho_o$ *such that*

(2.4)
$$v_i(y) \;\le\; \frac{\bar C_1}{v_i(\xi_{m_i})} \cdot \frac{1}{|\,y - \xi_i|^{n-2}} \qquad \textit{for} \quad 0 < |\,y - \xi_i| \;\le\; \bar\rho_o \quad \textit{and for all } i \gg 1\,.$$

In addition, there is a number $\bar\rho_1 \in (0,\ \bar\rho_o)$ *such that* (*modulo a subsequence*)

$$[\,v_i(\xi_{m_i})] \cdot v_i(y) \;\to\; \frac{1}{|\,y|^{n-2}} \;+\; h(y) \qquad \textit{in} \quad C^2_{\rm loc}(\,B_o(\bar\rho_1) \setminus \{0\}\,)\,, \tag{2.5}$$

where h *is a harmonic function in* $B_o(\bar\rho_1)$. (*Recall that* $(\tilde c_n\,K)(0) \;=\; n\,(n-2)$.)

2c. Harmonic expression of the collapsed part. Consider a blow - up sequence of positive solutions $\{u_i\}$ of equation (1.1). Consider the situations where

(2.6)
"the number of blow - up points is finite, say at $\beta_o \;=\; \mathbf{S}\,,\ \cdots,\ \beta_k \in S^n \setminus \{\mathbf{N}\}$,

and at least one of them is a simple blow - up point (say, β_o)."

Take a point

$$x_c \notin \{\beta_o, \cdots, \beta_k, \mathbf{N}\}. \tag{2.7}$$

Under the general conditions (1.25), (1.26), and also (2.6), a subsequence of

$$\left\{ \frac{u_i}{u_i(x_c)} \right\} \tag{2.8}$$

converges to a positive C^2-function $\mathcal{H}$ defined on $S^n \setminus \{\beta_1, \cdots, \beta_k\}$. See [15]. With the stereographic projection $\dot{\mathcal{P}}$ onto $\mathbb{R}^n$, which sends $\mathbf{N}$ to infinity, $\mathcal{H}$ can be expressed as (cf. the transformation in (1.3))

$$H(y) := [\mathcal{H} \circ \dot{\mathcal{P}}^{-1}(y)] \cdot \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}}, \tag{2.9}$$

$$H(y) = \sum_{j=0}^{k} \frac{A_j}{|y - \hat{Y}_j|^{n-2}} \qquad \text{for} \quad y \in \mathbb{R}^n \setminus \{\hat{Y}_o, \cdots, \hat{Y}_k\}. \tag{2.10}$$

Here

$$\hat{Y}_j := \mathcal{P}(\beta_j) \qquad \text{for} \qquad 0 \le j \le k, \tag{2.11}$$

and A_j are positive numbers. Refer to §4 in [15]. The convergence can be quantified in the following manner. Given a sequence of positive numbers $\varepsilon_j \downarrow 0$ and a sequence of compact sets $\{\mathcal{C}_j\}$ such that

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots, \qquad \bigcup_{j=0}^{\infty} \mathcal{C}_j = \mathbb{R}^n \setminus \{\hat{Y}_1, \cdots, \hat{Y}_k\}, \tag{2.12}$$

there exists a sequence of natural numbers $N_i \uparrow \infty$ so that

$$\left| v_i(y) - [u_i(x_c)] \cdot H(y) \right| \le \varepsilon_j \cdot [u_i(x_c)] \qquad \text{for} \quad i \ge N_j \quad \text{and } y \in \mathcal{C}_j. \tag{2.13}$$

We point out that when $i \to \infty$,

(2.14) "right-hand side of (2.13)" $= \varepsilon_j \cdot [u_i(x_c)] \to 0$,

(2.15) the domain in which (2.13) holds is $\mathcal{C}_j$ "$\to$" $\mathbb{R}^n \setminus \{\hat{Y}_1, \cdots, \hat{Y}_k\}$;

cf. (2.22) and (2.23) in §2d.

2c.1. Change of the base point. We observe that, in (2.8), one can replace the base point $u_i(x_c)$ by a sequence of numbers $\{\gamma_i\}$ so that

$$C^{-1} \cdot \gamma_i \le u_i(x_c) \le C\gamma_i \qquad \text{for} \quad i \gg 1. \tag{2.16}$$

A subsequence of $\{\gamma_i^{-1} \cdot u_i\}$ converges to a positive C^2-function $\tilde{\mathcal{H}}$ defined on $S^n \setminus \{\beta_o, \cdots, \beta_k\}$. With the stereographic projection $\dot{\mathcal{P}}$ onto $\mathbb{R}^n$, $\tilde{\mathcal{H}}$ can be expressed as in (2.10) and (2.11), with a scaling factor $\lim\limits_{1 \to \infty} \gamma_i^{-1} \cdot u_i(x_c)$ inserted.

2d. Renormalization and first order approximation. Let 0 be a simple blow-up point for the sequence of positive solutions $\{v_i\}$ of equation (1.2). With the notation in (2.1) and (2.2), define

(2.17)
$$\mathcal{V}_i(\mathcal{Y}) := \frac{v_i(\xi_{m_i} + \lambda_{m_i} \cdot \mathcal{Y})}{v_i(\xi_{m_i})} \qquad \text{for} \quad \mathcal{Y} \in \mathbb{R}^n \quad \text{with} \quad \lambda_{m_i} \cdot \mathcal{Y} \in B_o(\rho_o),$$

where $\lambda_{m_i} := [v_i(\xi_{m_i}]^{-\frac{2}{n-2}}$. Here $\mathcal{V}_i$ satisfies the equation (extendable to $\mathbb{R}^n$)

$$\Delta_o \mathcal{V}_i + [(\tilde{c}_n K)(\xi_{m_i} + \lambda_{m_i} \cdot \mathcal{Y})] \mathcal{V}_i^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad B_o(\lambda_{m_i}^{-1} \cdot \rho_o). \tag{2.18}$$

Assuming (1.6), under the conditions (1.25) and (1.26), we invoke Proposition 2.1 in [20] (p. 333) to conclude that, modulo a subsequence, $\{\mathcal{V}_i\}$ converges to $A_1 = A_{1,0}$ as given in (1.4);[2] cf. [6] and [10]. The convergence happens in the C^1-sense, uniformly in compact subsets in $\mathbb{R}^n$ (for the variable $\mathcal{Y}$). This translates into a weak approximation of v_i, which can be described in the following manner. Given sequences of positive numbers $\{\varepsilon_i\}$ and $\{R_i\}$ with $\varepsilon_i \downarrow 0$ and $R_i \uparrow \infty$, via the Cantor diagonal argument on subsequences, we have

$$|\mathcal{V}_i(\mathcal{Y}) - A_1(\mathcal{Y})| \le \varepsilon_i \tag{2.19}$$

for all $\mathcal{Y} \in B_o(R_i)$ and $i \gg 1$ (modulo a subsequence). Moreover, by choosing R_i to be *smaller* if necessary, we can take it that

$$\varepsilon_i \cdot R_i^{2(n-1)} \to 0 \quad \text{and} \quad \lambda_i \cdot R_i \to 0 \quad \text{as} \quad i \to \infty. \tag{2.20}$$

(See also §3a in [15] and the proof of Proposition A.6.34 in the e-Appendix for the application of (2.20).) Via the change of variables

$$y = \xi_{m_i} + \lambda_{m_i} \mathcal{Y}; \qquad \mathcal{Y} \in B_o(R_i) \iff y \in B_{\xi_{m_i}}(\lambda_{m_i} \cdot R_i),$$

(2.17), (2.19) and (1.4) yield

(2.21)
$$|v_i(y) - \mathbf{A}_{\lambda_{m_i}, \xi_{m_i}}(y)| \le \frac{\varepsilon_i}{\lambda_{m_i}^{\frac{n-2}{2}}} \qquad \text{for} \quad |y - \xi_{m_i}| \le \lambda_{m_i} \cdot R_i \quad \text{and} \quad i \gg 1;$$

cf. (1.5). However, we do not know, a priori, how small we can take ε_i (relative to λ_i) and how large we can choose R_i (relative to λ_i^{-1}). In particular, the following scenario can occur:

$$\text{“right-hand side of (2.21)”} = \varepsilon_i \cdot \lambda_{m_i}^{-\frac{n-2}{2}} \to \infty, \tag{2.22}$$

$$\text{the radius of the ball in which (2.21) holds} = \lambda_{m_i} \cdot R_i \to 0. \tag{2.23}$$

Our goal is to introduce bubble estimates that are accurate up to $O(\lambda_{m_i}^\tau)$ for $\tau > 0$ (as big as possible) and to “stabilize” the domain in which the estimates hold.

---

[2] The job is made easier as we explain in §2f; for simple blow-up, we can take ξ_i to be a global maximum point of v_i in $\overline{B_o(\rho_o)}$.

2d.1. Joint between shrinking bubble estimate and the expanding global harmonic term. As mentioned, there are diametric contrasts between bubble estimate (2.21) and the global harmonic estimate in (2.13). Adding to the list, observe that $\Delta_o H = 0$, whereas

$$\Delta_o \mathbf{A}_{\lambda_{m_i}, \xi_{m_i}} = -n(n-2)[\mathbf{A}_{\lambda_{m_i}, \xi_{m_i}}]^{\frac{n+2}{n-2}} \quad (< 0).$$

These two estimates do not immediately link to each other. We demonstrate their intricate relation when we present estimates that are accurate up to order $O_{\lambda_{m_i}}(n-2)$.

2e. An equivalent geometric expression. Before we proceed to a closer relation between $\mathcal{V}_i$ and A_1, we examine a simpler estimate here. Not only is the estimate useful in later discussion, it is interesting in its own right. As for the proof, we present it in §A.1 in the e-Appendix.

Proposition 2.24. *Under the standard conditions in* (1.6), (1.25) *and* (1.26) *modulo a subsequence,* 0 *is a simple blow-up point for* $\{v_i\}$ *if and only if there exists a sequence* $\zeta_i \in \mathbb{R}^n$, *with*

$$\zeta_i \to 0 \quad \textit{and} \quad \epsilon_i := \frac{1}{[v_i(\zeta_i)]^{\frac{2}{n-2}}} \to 0, \qquad \textit{so that} \tag{2.25}$$

$$\frac{1}{C} \cdot \mathbf{A}_{\epsilon_i, \zeta_i}(y) \le v_i(y) \le C \cdot \mathbf{A}_{\epsilon_i, \zeta_i}(y) \qquad \textit{for all} \quad |y - \zeta_i| \le \rho_1. \tag{2.26}$$

Here $C \ge 1$ *and* ρ_1 *are positive constants independent on* i.

2f. Shifting to the maximal point. Let $\xi_i \in B_o(\rho_o)$ be given in (1.10). We can take $\xi_{m_i} = \xi_i$ in (2.1) and (2.2). Moreover, suppose that there exists another sequence of points $\{\tilde{\xi}_i\}$ which also satisfies (2.1) and (2.2) in the definition of simple blow-up points. We have (modulo a subsequence)

$$|\tilde{\xi}_i - \xi_i| = o(\lambda_i) \qquad (\lambda_i \text{ as in } (1.11)). \tag{2.27}$$

The proofs of the above statements, which require only standard techniques, can be found in §A.2 in the e-Appendix.

2g. Non-degenerate conditions and $o(\lambda_i)$ restriction on flexibility. Non-vanishing derivatives at the blow-up point tend to post restriction on the blow-up flexibility. One good example can be found in [8], which we highlight here, using the setting of the present article. Via Taylor's expansion,

$$(\tilde{c}_n K)(y) = n(n-2) + [-\mathbf{P}_\ell(y)] + R_{\ell+1}(y) \qquad \text{for} \quad y \in B_o(\rho). \tag{2.28}$$

Here we use multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and

$$\mathbf{P}_\ell(y) = \sum_{|\alpha| = \ell} \left[D^{(\ell)}_\alpha(-\tilde{c}_n K)\Big|_{y=0} \cdot \frac{y^\alpha}{\alpha!} \right], \tag{2.29}$$

$$R_{\ell+1}(y) = O\left(\max_{B_o(\rho_o + \varepsilon')} |\nabla^{(\ell+1)} K| \times |y|^{\ell+1} \right). \tag{2.30}$$

(The negative sign is introduced for later matching; see (3.7).) One can verify that

$$\frac{|R_{\ell+1}(y)|}{|y|^\ell} \to 0 \quad \text{and} \quad \frac{\| \nabla R_{\ell+1}(y)\|}{|y|^{\ell-1}} \to 0 \quad \text{as } |y| \to 0. \tag{2.31}$$

A more demanding condition is the lower bound

(2.32)

$$C^{-1}\,|\,y|^{\,\ell\,-\,1} \le \sqrt{\left(\frac{\partial\,\mathbf{P}_\ell\,(y)}{\partial y_{|_1}}\right)^2 + \,\cdots\, + \left(\frac{\partial\,\mathbf{P}_\ell\,(y)}{\partial y_{|_n}}\right)^2} \quad (\le\; C\,|\,y|^{\,\ell\,-\,1})$$

for $y \,\in\, B_o\,(\rho)$; cf. the example below. From (2.32), we have
(2.33)
$0 < c\,(\varepsilon) \,\le\, \|\,\bigtriangledown K\,(y)\| \,\le\, C \qquad \text{for} \quad \varepsilon \,\le\, |\,y| \,\le\, \rho\,, \quad \text{where } \rho \text{ is small enough}.$

The following is a direct application of Lemma 3.6 in [8] after checking (1.2), (1.6), and (3.2), and the conditions stated at the beginning of §3 in [8] (in particular, $\alpha_i \,\le\, n\,-\,2$, p. 127 in [8]) , also verifying the conditions stated in Lemma 3.4 and Lemma 3.6 in [8], and taking $p_i \,\equiv\, \frac{n+2}{n\,-\,2}$.

Proposition 2.34. *Granted the general conditions in* (1.6) , (1.25) *and* (1.26), *suppose that* 0 *is a simple blow-up point for* $\{v_i\}$. *Assume also* (2.28) *and* (2.32) *for* $2 \,\le\, \ell \,\le\, n\,-\,2$. *If*

$$\int_{\mathbb{R}^n} \bigtriangledown \mathbf{P}_\ell\,(y + \mathcal{X})\,[\,A_1\,(y)]^{\frac{2n}{n\,-\,2}}\;dy \;\neq\; \vec{0} \qquad \textit{for all}\;\; \mathcal{X} \,\in\, \mathbb{R}^n \setminus \{0\}\,, \tag{2.35}$$

then, modulo a subsequence, we have $|\,\xi_i| = o\,(\lambda_i\,)$. (*Recall that* $A_1 \,=\, A_{1\,,\,0}$ *is given in* (1.4), ξ_i *fulfills* (1.10), λ_i *is given in* (1.11), *and* $\mathbf{P}_\ell$ *in* (2.29).)

2g.1. Examples on K with local expansions fulfilling the conditions $\Delta_o^{h_\ell}\,\mathbf{P}_\ell \,\equiv\, 0$, (2.32) and (2.35). Recall that h_ℓ is defined as the largest integer that is less than or equal to $\ell/2$. Consider n and $\ell \,\ge\, 2$, both even numbers , and

$$(\tilde{c}_n\,K)\,(y) \;=\; n\,(n\,-\,2) \,+\, \left[\,\left(y^\ell_{|_1} \,-\, y^\ell_{|_2}\right) \,+\, \cdots \,+\, \left(y^\ell_{|_{n\,-\,1}} \,-\, y^\ell_{|_n}\right)\right] \tag{2.36}$$

for $y \,\in\, B_o\,(\rho_o)$. Using Hölder's inequality, one can verify (2.32) . Moreover,

$$\begin{aligned}
(y_{|_1} + \mathcal{X}_1)^{\ell\,-\,1} \;=\; & y_1^{\ell-1} + C(\ell\,-\,1\,,\,2)\cdot y_1^{\ell-\,1\,-\,2}\,\mathcal{X}_1^2 \\
& + \cdots + C(\ell\,-\,1\,,\,\ell-2)\cdot y_1^{\ell-1\,-\,2}\,\mathcal{X}_1^{\ell\,-\,2} \\
& + \mathcal{X}_1\cdot\big[\,C(\ell\,-\,1\,,\,2)\cdot y_1^{\ell-1\,-\,1} \\
& \qquad\qquad + \cdots + C(\ell\,-\,1\,,\,\ell-3)\cdot y_1^2\cdot \mathcal{X}_1^{\ell\,-\,4} + \mathcal{X}_1^{\ell\,-\,2}\,\big]\,.
\end{aligned}$$

Here $\mathcal{X} = (\mathcal{X}_1\,,\;\cdots\;\mathcal{X}_n)$, and

$$C(j\,,\;k) \;=\; \frac{j\,!}{(j\,-\,k)\,!\;k\,!} \qquad (j \;\ge\; k)$$

is the binomial coefficient. Note that all the powers in $\mathcal{X}_1$ inside the brackets are even numbers. As

$$\int_{\mathbb{R}^n} y^{2j+1}_{|_1}\;[\,A_1\,(y)]^{\frac{2n}{n\,-\,2}}\,dy \;=\; 0\,,$$

we have

$$\int_{\mathbb{R}^n} (y_{|_1} + \mathcal{X}_1)^{\ell\,-\,1}\;[\,A_1\,(y)]^{\frac{2n}{n\,-\,2}}\,dy \;=\; 0 \qquad \Longleftrightarrow \qquad \mathcal{X}_1 \;=\; 0\,.$$

It follows that (2.35) is fulfilled with the form in (2.36) . In addition, observe that

$$\Delta_o^{h_\ell}\;\left[\,\left(y^\ell_{|_1} \,-\, y^\ell_{|_2}\right) \,+\, \cdots \,+\, \left(y^\ell_{|_{n\,-\,1}} \,-\, y^\ell_{|_n}\right)\right] \;=\; 0 \qquad (\ell \;\text{ being }\;\text{ even})\,.$$

One can generalize equality (2.36) by introducing positive multipliers onto each $(y^\ell_{|_{2j-1}} - y^\ell_{|_{2j}})$.

3. Difference between the normalization $\mathcal{V}_i$ and A_1

After shifting from ξ_{m_i} to ξ_i as described in §2f, for the sake of simplicity, we continue to use the notation

$$\mathcal{V}_i(\mathcal{Y}) := \frac{v_i(\xi_i + \lambda_i \mathcal{Y})}{M_i} \qquad \text{for} \quad \mathcal{Y} \in \mathbb{R}^n. \tag{3.1}$$

Here

$$M_i := v_i(\xi_i) \quad \text{and} \quad \lambda_i = M_i^{-\frac{2}{n-2}}, \quad \xi_i \text{ is given in } (1.10); \tag{3.2}$$

cf. (2.17) and §2f. As $A_1 = A_{1,\,0}$ satisfies the equation

$$\Delta_o A_1 + n(n-2) A_1^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{3.3}$$

together with equation (2.18), which holds after the changes $\xi_{m_i} \to \xi_i$ and $\lambda_{m_i} \to \lambda_i$, it can be seen that

$$\begin{aligned} &\Delta_o(\mathcal{V}_i - A_1)(\mathcal{Y}) \\ &= n(n-2)\Big\{[A_1(\mathcal{Y})]^{\frac{n+2}{n-2}} - [\mathcal{V}_i(\mathcal{Y})]^{\frac{n+2}{n-2}}\Big\} \\ &\quad + [n(n-2) - \tilde{c}_n K(\lambda_i \mathcal{Y} + \xi_i)]\,[A_1(\mathcal{Y})]^{\frac{n+2}{n-2}} \\ &\quad + [n(n-2) - \tilde{c}_n K(\lambda_i \mathcal{Y} + \xi_i)]\Big\{[\mathcal{V}_i(\mathcal{Y})]^{\frac{n+2}{n-2}} - [A_1(\mathcal{Y})]^{\frac{n+2}{n-2}}\Big\} \end{aligned} \tag{3.4}$$

for $\mathcal{Y} \in \mathbb{R}^n$.

3a. Linear approximation to $\left(A_1^{\frac{n+2}{n-2}} - \mathcal{V}_i^{\frac{n+2}{n-2}}\right)$ in case of simple blow-up. It follows from Proposition 2.24 that (see §A.10 in the e-Appendix for details)

$$\begin{aligned} [A_1(\mathcal{Y})]^{\frac{n+2}{n-2}} - [\mathcal{V}_1(\mathcal{Y})]^{\frac{n+2}{n-2}} &= \left(\frac{n+2}{n-2}\right)[A_1(\mathcal{Y})]^{\frac{4}{n-2}} \cdot [A_1(\mathcal{Y}) - \mathcal{V}_i(\mathcal{Y})] \\ &\quad + O(1)\,[A_1(\mathcal{Y}) - \mathcal{V}_i(\mathcal{Y})]^2 \cdot [A_1(\mathcal{Y})]^{\frac{4}{n-2}-1} \end{aligned} \tag{3.5}$$

for $|\mathcal{Y}| \le \rho_o \lambda_i^{-1}$.

3b. Taylor expansion of $(\tilde{c}_n K)$ above 0. Since we know that $\bigtriangledown K(0) = 0$, and by (1.6), $(\tilde{c}_n K)(0) = n(n-2)$, we assume that all the derivatives of K vanish at 0 up to (and equal to) order $\ell - 1$. Here $\ell \ge 2$ is an integer. Using multi-index $\alpha = (\alpha_1\,,\ \cdots,\ \alpha_n)$ and the Taylor expansion of $(\tilde{c}_n\,K)$, we obtain

$$
\begin{aligned}
(3.6) \qquad n\,(n\,-\,2)\,-\,\tilde{c}_n\,K(\lambda_i\,\mathcal{Y} + \xi_i)\ &=\ \sum_{|\,\alpha| = \ell} D_\alpha^{(\ell)}\,(-\,\tilde{c}_n K)\Big|_0 \cdot \frac{(\,\lambda_i\,\mathcal{Y} + \xi_i)^{\,\alpha}}{\alpha\,!} \\
&\quad + O\,(1)\left[\max_{|\,\lambda_i\,\mathcal{Y} + \xi_i| \,\le\, \rho_o^+} \|\ \bigtriangledown^{(\ell+1)}\ K\|\right] \cdot |\,\lambda_i\,\mathcal{Y} + \xi_i|^{\,\ell+1} \\
&= \lambda_i^\ell \cdot \mathbf{P}_\ell\,(\mathcal{Y})\ +\ \sum_{k=1}^{\ell} O\left(|\,\xi_i|^{\,k} \cdot (\lambda_i|\,\mathcal{Y}|\,)^{\,\ell\,-\,k}\right)\ +\ \mathcal{R}_3\,(\mathcal{Y})
\end{aligned}
$$

for $\lambda_i \cdot |\,\mathcal{Y}|\ \le\ \rho_o\,,\quad i \gg 1$. In the above ρ_o^+ is slightly bigger than ρ_o. Moreover,

$$
(3.7) \qquad \mathbf{P}_\ell\,(\mathcal{Y})\ =\ \sum_{|\,\alpha\,| = \ell}\left[D_\alpha^{(\ell\,)}\,(-\,\tilde{c}_n\,K)\Big|_0 \cdot \frac{\mathcal{Y}^{\,\alpha}}{\alpha\,!}\right],
$$

$$
(3.8) \qquad \mathcal{R}_3\,(\mathcal{Y})\ =\ O\left(\max_{|\,\lambda_i\,\mathcal{Y} + \xi_i| \,\le\, \rho_o^+} \|\ \bigtriangledown^{(\ell+1)}\ K\| \cdot |\,\lambda_i\,\mathcal{Y} + \xi_i|^{\,\ell+1}\right).
$$

3c. The mixed term. Consider the last term in (3.4). Using Taylor expansion as in (3.6) and the inequality

$$
(3.9) \qquad a > b > 0 \quad \text{and} \quad p \ge 1 \quad \Longrightarrow \quad a^p\,-\,b^p\ \le\ \frac{1}{p} \cdot (a\,-\,b) \cdot a^{\,p\,-\,1}\,,
$$

we obtain

$$
\begin{aligned}
(3.10) \qquad &[\,n\,(n\,-\,2)\,-\,\tilde{c}_n\,K(\lambda_i\,\mathcal{Y} + \xi_i)\,]\left\{[\,\mathcal{V}_i\,(\mathcal{Y})]^{\frac{n+2}{n\,-\,2}}\,-\,[\,A_1\,(\mathcal{Y})]^{\frac{n+2}{n\,-\,2}}\right\} \\
&= O\left(\max_{|\,\lambda_i\,\mathcal{Y} + \xi_i| \,\le\, \rho_o^+)} \|\ \bigtriangledown^{(\ell)}\ K\,\| \cdot |\,\lambda_i\,\mathcal{Y} + \xi_i|^{\,\ell}\right) \\
&\quad \times \left[\ O\,(1)\,|\,\mathcal{V}_i\,-\,V| \times \max\left\{\ [\,\mathcal{V}_i\,(\mathcal{Y})]^{\frac{4}{n\,-\,2}}\ ,\ [\,A_1\,(\mathcal{Y})]^{\frac{4}{n\,-\,2}}\right\}\ \right]
\end{aligned}
$$

for $|\,\mathcal{Y}|\ \le\ \lambda_i^{-1}\,\rho_o\,.$

3d. Isolating the key terms and the remainder. It follows from (3.4), (3.5), (3.6) and (3.10) that

(3.11)

$$
\Delta_o\,[\,\mathcal{V}_i\,-\,A_1\,]\,(\mathcal{Y}) + n\,(n+2)\,A_1^{\frac{4}{n\,-\,2}}\,[\,\mathcal{V}_i\,-\,A_1\,]\,(\mathcal{Y}) = \lambda_i^\ell \cdot \mathbf{P}_\ell\,(\mathcal{Y}) \cdot A_1^{\frac{n+2}{n\,-\,2}}\ +\ \mathbf{RM}\,(\mathcal{Y})
$$

$$
\text{for} \quad |\,\mathcal{Y}|\ \le\ \lambda_i^{-1}\,\rho_o \qquad \left(\mathcal{Y}\ =\ \frac{y\,-\,\xi_i}{\lambda_i}\right).
$$

Here (refer to §6b and §6c)

$$\text{(3.12)}\qquad \mathbf{RM} \;=\; \mathbf{RM}_1 + \mathbf{RM}_1 + \mathbf{RM}_3 + \mathbf{RM}_4\,,$$

$$\mathbf{RM}_1(\mathcal{Y}) \quad \text{“}=\text{”} \quad \sum_{k=1}^{\ell} O\left(|\xi_i|^k \cdot (\lambda_i|\mathcal{Y}|)^{\ell-k}\right),$$

$$\mathbf{RM}_2(\mathcal{Y}) \quad \text{“}=\text{”} \quad O\left(\max_{|\lambda_i y+\xi_i|\le\rho_o^+} \|\bigtriangledown^{(\ell+1)} K\| \times |\lambda_i\mathcal{Y}+\xi_i|^{\ell+1}\right),$$

$$\mathbf{RM}_3(\mathcal{Y}) \quad \text{“}=\text{”} \quad O(1)\left\{[A_1(\mathcal{Y})]^{\frac{4}{n-2}-1}\cdot[\mathcal{V}_i(\mathcal{Y})-A_1(\mathcal{Y})]^2\right\},$$

$$\mathbf{RM}_4(\mathcal{Y}) \quad \text{“}=\text{”} \quad O\left(\max_{|\lambda_i\mathcal{Y}+\xi_i|\le\rho_o^+} \|\bigtriangledown^{(\ell)} K\| \times |\lambda_i\mathcal{Y}+\xi_i|^{\ell}\right)$$

$$\times\left[O(1)|\mathcal{V}_i(\mathcal{Y})-A_1(\mathcal{Y})|\times\max\left\{[\mathcal{V}_i(\mathcal{Y})]^{\frac{4}{n-2}},\quad [A_1(\mathcal{Y})]^{\frac{4}{n-2}}\right\}\right]$$

for $|\mathcal{Y}| \le \lambda_i^{-1}\rho_o$. (We use “ = ” to tell us that the right-hand side is the order of the term.)

4. Cancellation of the $O(\lambda_i^\ell)$ term in (3.11)

We first ignore the order λ_i^ℓ in equation (3.11) and consider the linear inhomogeneous equation

$$\text{(4.1)}\qquad \Delta_o\Pi + n(n+2)A_1^{\frac{4}{n-2}}\cdot\Pi = \mathcal{P}_\ell\cdot A_1^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^n$$

(with unknown Π). Here $\mathcal{P}_\ell$ is a *homogeneous* polynomial defined on $\mathbb{R}^n$ of degree $\ell\ge 1$, and $A_1 = A_{1,\,0} = \left(\frac{1}{1+|\mathcal{Y}|^2}\right)^{\frac{n-2}{2}}$. As in [12], potential solutions Π can be expressed in the following form:

$$\text{(4.2)}\qquad \Pi(\mathcal{Y}) = \frac{\Gamma(\mathcal{Y})}{(1+\mathcal{R}^2)^{\frac{n}{2}}} \qquad \text{for}\quad \mathcal{Y}\in\mathbb{R}^n \qquad (\mathcal{R}=|\mathcal{Y}|).$$

Putting (4.2) into (4.1), we obtain

$$\text{(4.3)}\qquad (1+\mathcal{R}^2)\cdot[\Delta_o\Gamma] - 2n[\mathcal{Y}\cdot\bigtriangledown\Gamma] + 2n\Gamma = \mathcal{P}_\ell \qquad \text{in}\quad \mathbb{R}^n.$$

Our goal is to find polynomial solutions Γ of equation (4.3) and to keep the degree of Γ as close to ℓ as possible (this is for the second order blow-up argument) and to make explicit the dependence on $\mathcal{P}_\ell$. We first look at some selected examples.

Example 4.4. When $\mathcal{P}_\ell\equiv 0$. We can take

$$\text{(4.5)}\qquad \Gamma_1(\mathcal{Y}) := \sum_{j=1}^{n} c_j\,\mathcal{Y}_{|j} \qquad \text{or} \qquad \Gamma_2(\mathcal{Y}) := \mathcal{R}^2 - 1.$$

Here c_j are any constants, and $\mathcal{Y} = (\mathcal{Y}_{|1},\,\cdots,\,\mathcal{Y}_{|n})\in\mathbb{R}^n$. Interestingly, up to linear combinations of Γ_1 and Γ_2, these are the only possible solutions when $\mathcal{P}_\ell\equiv 0$ and when we restrict Γ to be a polynomial of degree less than n; cf. Theorem 4.16 and Proposition 4.21.

Example 4.6. When $\ell \geq 2$ and $\Delta_o \mathcal{P}_\ell = 0$. In this case we simply take $\Gamma_3 = c\,\mathcal{P}_\ell$:

$$(4.7)\qquad (1+\mathcal{R}^2)\,\Delta_o\,\Gamma_3 \,-\, 2n\,[\,(\mathcal{Y}\cdot\nabla\,\Gamma_3) \,-\, \Gamma_3] \;=\; -\,2\,n\,(\ell\,-\,1)\,(c\,\mathcal{P}_\ell) = \mathcal{P}_\ell$$

$$\Longrightarrow \quad c \;=\; -\,\frac{1}{2\,n\,(\ell\,-\,1)} \quad \Longrightarrow \quad \Gamma_3 \;=\; -\,\frac{\mathcal{P}_\ell}{2\,n\,(\ell\,-\,1)}\;.$$

Example 4.8. When $\ell = 1$. As the left-hand side of (4.3) is linear, we may assume that $\mathcal{P}_\ell\,(\mathcal{Y}) = \mathcal{Y}_{|_1}$. Consider

$$(4.9)\qquad \Gamma_4\,(\mathcal{Y}) \;:=\; a\,\mathcal{R}^2\,\mathcal{Y}_{|_1} \;+\; b\,\mathcal{R}^4\,\mathcal{Y}_{|_1} \qquad (\text{maximum degree } = 5)\,.$$

Direct calculation shows that when $n = 4$, $a = \frac{1}{2n+4}$, and $b = \frac{2n-4}{2n+4}\cdot\frac{1}{4n+16}$, then Γ_4 is a solution of (4.3) with $\mathcal{P}_\ell\,(\mathcal{Y}) = \mathcal{Y}_{|_1}$. The example demonstrates that, in general, $\ell = 1$ makes it harder to solve equation (4.3) (cf. an existence result for equation (1.1) obtained by Aubin in [1]).

4a. Solving (4.1) via the linear method ($\ell < n$). As in [12] (p. 152), we introduce the collection (h_ℓ is the biggest integer less than or equal to $\ell/2$)

$$(4.10)\qquad \mathcal{F}\,(\,\mathcal{P}_\ell\,) \;:=\; \Big\{\text{linear combinations of } (\mathcal{R}^2)^j\,\Delta_o^{(k)}\,\mathcal{P}_\ell\,, \qquad 0 \leq j \leq k\,,\; k = 0,\,1,\,\cdots,\,h_\ell\,\Big\}\,,$$

where $\mathcal{R} = |\,\mathcal{Y}|$. Note that $\Delta_o^{(k)}\,\mathcal{P}_\ell \equiv 0$ for $k \geq h_\ell + 1$. Comparing to the one introduced in [12], (4.10) has the index j limited more strictly from above. Assuming that $\Delta_o^{(j)}\,\mathcal{P}_\ell \not\equiv 0$ for $0 \leq j \leq h_\ell$, $\mathcal{F}\,(\,\mathcal{P}_\ell\,)$ is a vector space with dimension, in general, equal to $\frac{1}{2}\,(h_\ell + 1)\,(h_\ell + 2)$ ($= O\,(n^2)$ when ℓ is close to n; an exceptional case is when $\mathcal{P}_\ell\,(\mathcal{Y}) = \mathcal{R}^\ell$, where ℓ is an even number). We list some simple properties concerning the operator in (4.3) and $\mathcal{F}\,(\,\mathcal{P}_\ell\,)$. All these can be checked readily (see §A.3 in the e-Appendix for a proof).

Lemma 4.11. $\mathcal{P}_\ell \in \mathcal{F}\,(\,\mathcal{P}_\ell\,)$. *Moreover,*

$$(4.12)\qquad \text{if } \ell \text{ is odd and } \Delta_o^{(h_\ell)}\,\mathcal{P}_\ell \not\equiv 0 \;\Longrightarrow\; \sum_j c_j\,\mathcal{Y}_{|_j} \,\in\, \mathcal{F}\,(\,\mathcal{P}_\ell\,)\,;$$

$$(4.13)\qquad \text{if } \ell \text{ is even and } \Delta_o^{(h_\ell)}\,\mathcal{P}_\ell \neq 0 \;\Longrightarrow\; (\mathcal{R}^2 \,-\, 1) \,\in\, \mathcal{F}\,(\,\mathcal{P}_\ell\,)\,.$$

Here at least one of the coefficients is $c_j \neq 0$.

$$(4.14)\qquad \text{The degree of each term in } \mathcal{F}\,(\,\mathcal{P}_\ell\,) \text{ is at most } \ell\,.$$

$$(4.15)\qquad (1+\mathcal{R}^2)\,\Delta_o\,\bullet \,-\, 2n(\,\mathcal{Y}\cdot\nabla\,\bullet) + 2n\,\bullet \;:\; \mathcal{F}\,(\,\mathcal{P}_\ell\,) \to \mathcal{F}\,(\,\mathcal{P}_\ell\,) \text{ is linear}.$$

In principle, one can express the linear operator in (4.15) of Lemma 4.11 into a matrix by using the basis of $\mathcal{F}\,(\mathcal{P}_\ell)$ as shown in (4.10) and determine whether there is a solution or not. However, for genuine cases and when ℓ is close to n, the matrix is of the size (number of entries) on the order of $O\,(n^4)$. In [12], the authors observe that one can make use of the following Liouville-type theorem (shown in [8]; see also [3]) to demonstrate that a solution exists.

Theorem 4.16. *Suppose ψ is a smooth solution of the equation*

$$(4.17)\qquad (1+\mathcal{R}^2)\cdot[\,\Delta_o\,\psi\,] \,-\, 2n\,[\,\mathcal{Y}\cdot\nabla\,\psi\,] \,+\, 2n\,\psi \;=\; 0 \qquad \text{in} \quad \mathbb{R}^n$$

with

$$\lim_{r\to\infty} \frac{\psi(\mathcal{Y})}{\mathcal{R}^n} = 0 \qquad (\mathcal{R} = |\mathcal{Y}|)\,. \tag{4.18}$$

Then

$$\psi(\mathcal{Y}) = c_o(\mathcal{R}^2 - 1) + \sum_{j=1}^{n} c_j\, \mathcal{Y}_{|_j} \qquad \textit{for} \quad \mathcal{Y} \in \mathbb{R}^n\,. \tag{4.19}$$

Moreover,

$$\psi(0) = 0 \quad \textit{and} \quad \bigtriangledown \psi(0) = 0 \implies \phi \equiv 0 \qquad \textit{in} \ \ \mathbb{R}^n. \tag{4.20}$$

We now describe the linear method used in [12] to find a polynomial solution to equation (4.3). The following result begins to reveal that the condition $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell \equiv 0$ is tightly knitted together with the refined estimate we seek.

Proposition 4.21. *Assume that $\mathcal{P}_\ell$ is a homogeneous polynomial of degree ℓ, with $2 \le \ell < n$. The linear operator which appears in* (4.15) *of Lemma* 4.11 *is a bijection if and only if $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell \equiv 0$.*

Proof. For the sufficient part, the proof goes on in an essential manner as in the proof of Proposition 4.1 in [12], using Lemma 4.11 and Theorem 4.16 to show that the linear operator is an injection, and hence a bijection. As for the necessary part, it follows from (4.12) and (4.13) of Lemma 4.11, and Example 4.4 (the kernel contains a non-identically zero element in $\mathcal{F}(\mathcal{P}_\ell)$). □

4b. The reduction method. Concerning the solution we find via Proposition 4.21, besides being in $\mathcal{F}(\mathcal{P}_\ell)$ (in particular, we have property (4.14) in Lemma 4.11), there is little we know about the solution itself. When we come to the bubble estimates, it is natural to ask for the precise form of the solution Γ. In this section we introduce a constructive method which allows us to determine each coefficient in Γ. We present the precise result.

Lemma 4.22. *Let $\mathcal{P}_\ell$ be a homogeneous polynomial of degree $\ell \ge 2$ (defined on $\mathbb{R}^n$). When n is even, assume also that $\ell < n+2$ (no such condition when n is odd). Define a polynomial $\mathcal{G}$ via*

$$\mathcal{G}(\mathcal{Y}) = \sum_{\substack{k\,\le\, h_\ell - 1\\ 0\,\le\, j\,\le\, k}} C_k^j \cdot (\mathcal{R}^2)^j\, [\,\Delta_o^{(k)}\, \mathcal{P}_\ell(\mathcal{Y})\,] \qquad (\mathcal{R} = |\mathcal{Y}|)\,, \tag{4.23}$$

where the coefficients C_k^j can be determined by using (4.48). *Then $\mathcal{G}$ satisfies*

$$\begin{aligned}(1+\mathcal{R}^2)\,\Delta_o\,\mathcal{G} - 2n\,(\mathcal{Y}\cdot\bigtriangledown\mathcal{G}) + 2n\,\mathcal{G} &= \mathcal{P}_\ell + [\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] \\ &\cdot \left\{ a_{h_\ell}\cdot(\mathcal{R}^2)^{h_\ell} + a_{h_\ell - 1}\cdot(\mathcal{R}^2)^{h_\ell - 1} + \cdots + a_1\cdot(\mathcal{R}^2)^1 + a_o \right\}.\end{aligned} \tag{4.24}$$

Here the numbers a_k can be found by using (4.45).

The precise definitions of C_k^j and a_k are obtained on the way toward the proof of Lemma 4.22. The key property is that they depend only on n, ℓ, j and k, and are formed by an algebraic iteration process, which we start to describe.

4b.1. First step in the proof of Lemma 4.22 : the recurrent and reduction to powers of $(\mathcal{R}^2)$. Consider first the situation where

$$\Delta_o^{(k)}\,\mathcal{P}_\ell \not\equiv 0 \qquad \text{for} \qquad 1 \le k \le h_\ell - 1\,. \tag{4.25}$$

As in Example 4.4, we take

$$[\,C_o^o \cdot \mathcal{P}_\ell\,]\,, \qquad \text{where } C_o^o = \frac{1}{2n\,(1-\ell)}\,, \tag{4.26}$$

and obtain

(4.27)

$$(1+\mathcal{R}^2)\,\Delta_o\,[\,C_o^o \cdot \mathcal{P}_\ell\,] + 2n\,[1-(\mathcal{Y}\cdot\bigtriangledown)]\,[\,C_o^o \cdot \mathcal{P}_\ell\,] = \mathcal{P}_\ell + C_o^o \cdot \big[\,(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell + \Delta_o\,\mathcal{P}_\ell\,\big]\,.$$

That is, we obtain $\mathcal{P}_\ell$ in the right-hand side, but "pay the price" by introducing $[\,(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell\,]$ and $[\,\Delta_o\,\mathcal{P}_\ell\,]$. Observe that the degree of $\Delta_o\,\mathcal{P}_\ell$ is lowered to $\ell-2$, while the degree of $[\,(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell\,]$ is still equal to ℓ, but its structure appears simpler in the sense that 2 of the degree is taken over by $(\mathcal{R}^2)$.

If $\ell = 2$ or 3, we are done. Assume that $\ell \ge 4$. To proceed, we simplify the notation and highlight the change in order and introduce

$$\mathbf{R} = (\mathcal{R}^2) \implies \mathbf{R}^j = (\mathcal{R}^2)^j\,, \qquad \mathbf{D} = [\,\Delta_o\,\mathcal{P}_\ell\,] \longrightarrow \mathbf{D}_k := [\,\Delta_o^{(k)}\,\mathcal{P}_\ell\,]\,. \tag{4.28}$$

Using this notation, we have (cf. §A.3 in the e-Appendix)

(4.29)

$$\begin{aligned}
&(1+\mathbf{R})\cdot\Delta_o\,[\mathbf{R}^j\ \mathbf{D}_k] \qquad \Big(= (1+\mathcal{R}^2)\,\Delta_o\,[\,(\mathcal{R}^2)^j\cdot\Delta_o^{(k)}\,\mathcal{P}_\ell\,]\Big)\\
&= A_{\ell,j,k}\cdot(\mathbf{R}^j\ \mathbf{D}_k) + \mathbf{R}^{j+1}\,\mathbf{D}_{k+1} \quad (\text{degree} = \ell + 2\,(j-k) \ \text{ on both terms})\\
&\qquad + A_{\ell,j,k}\cdot(\mathbf{R}^{j-1}\,\mathbf{D}_k) + \mathbf{R}^j\,\mathbf{D}_{k+1}\\
&\qquad\qquad (\uparrow \ \ \text{degree} = \ell + 2\,(j-k) - 2 \ \text{ on both terms})\,.
\end{aligned}$$

Here

$$A_{\ell,j,k} = (2j)\cdot(2j+n-2+2\,\ell-4k)\,.$$

We realize that the process produces one same term $[\mathbf{R}^j\ \mathbf{D}_k]$ (times a constant), one same order term, plus two lower order terms. We illustrate the procedure when the linear operator (the right-hand side of (4.3)) acts on $[\,\mathbf{R}^j\ \mathbf{D}_k\,]$ via the following diagram ($j \ge 1$):

$$\begin{array}{ccccc}
 & & (\times\,A_{\ell,j,k} \ \to) & \mathbf{R}^{j-1}\,\mathbf{D}_k & \\
 & & \nearrow & & \\
(\times\,\{\,A_{\ell,j,k} - 2n\,[\,\ell+2\,(j-k)-1\,]\,\}) \hookrightarrow & \mathbf{R}^j\ \mathbf{D}_k & \longrightarrow & \mathbf{R}^j\,\mathbf{D}_{k+1} & (\leftarrow \ \times 1)\\
 & \downarrow\ (\leftarrow\ \times 1) & & & \\
 & \mathbf{R}^{j+1}\,\mathbf{D}_{k+1} & & & \\
 & & & & \\
\text{Degree of the term } (\le)\ : & \ell + 2\,(j-k) & & \ell + 2\,(j-k) - 2 &
\end{array}$$

FIGURE 4.30. The four terms, their degrees, and the multipliers.

We represent schematically part of the reduction procedure in the diagram in Figure 4.31, showing the terms produced (indicated by the arrows, including itself) when the term is acted upon by the operator $(1+\mathbf{R})\cdot\Delta_o$.

$$
\begin{array}{cccccccccc}
 & \mathcal{P}_\ell & & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{D}_2 \;\cdots & \longrightarrow & [\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] \\
 & \downarrow & & \nearrow & \downarrow & \nearrow & & & \\
(1^{\text{st}}) & \mathbf{R}\;\mathbf{D} & & \longrightarrow & \mathbf{R}\,\mathbf{D}_2 & \longrightarrow & \mathbf{R}\,\mathbf{D}_3 \;\cdots & \longrightarrow & (\mathcal{R}^2)\,[\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] \\
 & \downarrow & & \nearrow & \downarrow & \nearrow & & & \\
(2^{\text{nd}}) & \mathbf{R}^2\;\mathbf{D}_2 & & \longrightarrow & \mathbf{R}^2\,\mathbf{D}_3 & \longrightarrow & \cdots & \longrightarrow & (\mathcal{R}^2)^2\,[\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] \\
 & \downarrow\;\vdots & & & \downarrow & & \cdots & & \\
 & \mathbf{R}^{h_\ell-1}\,\mathbf{D}_{h_\ell-1} & & \rightarrow & \mathbf{R}^{h_\ell-1}\,\mathbf{D}_{h_\ell} & = & (\mathcal{R}^2)^{h_\ell-1}\,[\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] & & \\
 & \downarrow & & & & & & & \\
(h_\ell^{\text{th}}) & \mathbf{R}^{h_\ell}\;\mathbf{D}_{h_\ell} & & & & & & & \\
 & \| & & & & & & & \\
 & (\mathcal{R}^2)^{h_\ell}\;[\,\Delta_o^{(h_\ell)}\,\mathcal{P}_\ell\,] & & & & & & & \\
 & & & & & & & & \\
\text{Degree } (\le) & \ell & & & \ell-2 & \cdots & & & 1\;/\;0 \\
 & & & & & & & & \ell \text{ is odd/even}
\end{array}
$$

FIGURE 4.31. Showing the cancellation order (top $\rightarrow$ down) on the first column.

Back to the case when $\ell = 4$. In the second step, we seek to eliminate the term

$$C_o^o\cdot[\,(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell\,] \;=\; C_o^o\cdot\mathbf{R}\,\mathbf{D}\,,$$

which appears in (4.27). From (4.29) and (4.27) we take

$$C_1^1 \;=\; \frac{-\,C_o^o}{A_{\ell,1,1} - 2n\,(\ell-1)} \;=\; \frac{-\,C_o^o}{2\,(n-2)\,(2-\ell)} \qquad (\text{here } \ell \ge 4)\,. \tag{4.32}$$

It follows that

$$\begin{aligned}(1+\mathcal{R}^2)\,\Delta_o\{\,[\,C_o^o\cdot\mathcal{P}_\ell\,]+[\,C_1^1\cdot(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell\,]\,\}\\ +\,2\,n\,[1-(\mathcal{Y}\cdot\bigtriangledown)]\,\{\,[\,C_o^o\cdot\mathcal{P}_\ell\,]+[\,C_1^1\cdot(\mathcal{R}^2)\,\Delta_o\,\mathcal{P}_\ell\,]\,\}\end{aligned} \tag{4.33}$$

$$= \mathcal{P}_\ell \;+\; [\,C_o^o + A_{\ell,1,1}\cdot C_1^1\,]\cdot[\,\Delta_o\,\mathcal{P}_\ell\,] + C_1^1\cdot[\,(\mathcal{R}^2)^2\,\Delta_o^2\,\mathcal{P}_\ell + (\mathcal{R}^2)\,\Delta_o^2\,\mathcal{P}_\ell\,]\,.$$

Inductively, we find (refer to (4.23)) that

$$C_j^j \;=\; \frac{-\,C_{j-1}^{j-1}}{A_{\ell,j,j} - 2n\,(\ell-1)} \;=\; \frac{-\,C_{j-1}^{j-1}}{(2j)\,[\,n-2+2\,(\ell-j)\,] - 2n\,(\ell-1)}\,, \tag{4.34}$$

where $1 \le j \le h_\ell$. This enables us to cancel the terms in the first column in Figure 4.31, ending with the term

$$C^{h_\ell - 1}_{h_\ell - 1} \cdot [\,(\mathcal{R}^2)^{h_\ell}\, \Delta_o^{(h_\ell)}\, \mathcal{P}_\ell\,]\,, \tag{4.35}$$

which is present on the right-hand side of equation (4.24).

Next, we proceed to cancel the terms in the second column (Figure 4.31), starting from top toward the bottom. Gradually we move right to the next column, always proceeding from top to bottom. We summarize the cancellation in the following two cases.

**Cancellation of terms in the top row in Figure* 4.31. From (4.33), and also from Figure 4.31, we have the term

$$[\,C^o_o + C^1_1 \cdot A_{\ell,1,1}\,] \cdot \mathbf{D}$$

to be canceled. This is done by adding the term

$$-\,\frac{C^o_o + C^1_1 \cdot A_{\ell,1,1}}{-\,2n\,[\,(\ell - 2) - 1])} \cdot \mathbf{D} \implies C^o_1 = -\,\frac{C^o_o + C^1_1 \cdot A_{\ell,1,1}}{-\,2n\,[\,(\ell - 2) - 1]\,)}\,.$$

With the help of the information depicted in Figure 4.30 and via induction, we have

$$C^o_k = -\,\frac{C^o_{k-1} + C^1_k \cdot A_{\ell,1,k}}{-\,2n\,[\,(\ell - 2k) - 1]} \qquad \text{for} \qquad 1 \le k \le h_\ell - 1\,. \tag{4.36}$$

The numbers C^1_2, $C^1_3, \cdots, C^1_{h_\ell - 1}$ are obtained below; see Remark 4.40. Note that

$$\ell - 2k \ge \ell - 2\,(h_\ell - 1) \ge 2 \implies (\ell - 2k) - 1 \ne 0 \qquad (\text{recall} \quad (4.25))\,.$$

This enables us to cancel the terms in the first row, ending with

$$C^o_{h_\ell - 1} \cdot \Delta_o^{(h_\ell)}\, \mathcal{P}_\ell\,,$$

which appears in the right-hand side of (4.24).

**Cancellation of the* '*inside*' *terms*. Finally, consider any 'inside' term $\mathbf{R}^j\,\mathbf{D}_k$. We observe that $k > j$ ($k = j$ appears in the first column only).

$$\begin{array}{ccc}
 & C^{j-1}_{k-1}\, \mathbf{R}^{j-1}\, \mathbf{D}_{k-1} & (j \ge 1) \\
 & (\times 1 \;\; \to)\ \downarrow & \\
C^j_{k-1}\, \mathbf{R}^j\, \mathbf{D}_{k-1}\ (\times 1) \ \to & \mathbf{R}^j\, \mathbf{D}_k & \\
 & \nearrow\ (\leftarrow\ \times A_{\ell,\,j+1,\,k}) & \\
C^{j+1}_k\, \mathbf{R}^{j+1}\, \mathbf{D}_k & &
\end{array}$$

FIGURE 4.37. The three terms which give rise to an inside term (with multipliers).

Via induction and the discussion in (4.34) and (4.36), we may assume that the coefficients C^{j-1}_{k-1}, C^j_{k-1} and C^{j+1}_k are determined. The term $\mathbf{R}^j\,\mathbf{D}_k$ makes its presence on the right-hand side given by

$$[\; C^{j-1}_{k-1} + C^j_{k-1} + C^{j+1}_k \cdot A_{\ell,\,j+1,\,k}\;] \cdot \mathbf{R}^j\, \mathbf{D}_k\,. \tag{4.38}$$

To cancel it, we introduce the term

$$-\,\frac{C^{j-1}_{k-1} \;+\; C^{j}_{k-1} \;+\; C^{j+1}_{k}\cdot A_{\ell\,,\,j+1\,,\;k}}{A_{\ell\,,\,j\,,\,k} \;-\; 2n\,[\,(\ell \;-\; 2k+2j) \;-\; 1]}\cdot \mathbf{R}^{j}\,\mathbf{D}_{k}$$

to the *left*-hand side,

$$(4.39)\qquad \cdots \;\Longrightarrow\; C^{j}_{k} \;=\; -\,\frac{C^{j-1}_{k-1} \;+\; C^{j}_{k-1} \;+\; C^{j+1}_{k}\cdot A_{\ell\,,\,j+1\,,\;k}}{A_{\ell\,,\,j\,,\;k} \;-\; 2n\,[\,\ell+2\,(j \;-\; k) \;-\; 1]}\;.$$

Remark 4.40. Concerning the usage of $C^1_2\,,\ C^1_3\,,\ \cdots,\ C^1_{h_\ell-1}$ in (4.36), we remark that, based on (4.39), in order to determine C^1_2, we need only $C^o_1\,,\ C^1_1$ and $C^2_2\,,$ which are known via (4.34) and (4.36). Afterward, we can determine C^{j-1}_{j} for $3\ \le\ j\ \le\ h_\ell$ (the other coefficient in the second column in Figure 4.31). $C^2_3\,,$ together with C^o_2 and C^1_2, helps to determine C^1_3, and so on.

4b.2. Non-zero characters for $(\mathcal{R}^{2})^{j}\,\Delta^{(k)}_{o} \;=\; (\mathcal{R}^{2})^{j}\,\Delta^{(j+\sqcup)}_{o}$ ($j\ \ge\ 1$ and $\sqcup\ \ge\ 0$). In order to finish the proof for Lemma 4.22, we are required to show that the denominators in (4.34) and (4.39) are non-zero under the conditions on ℓ as stated in Lemma 4.22. Note that

$$\text{Degree}\,\{\,(\mathcal{R}^{2})^{j}\cdot[\,\Delta^{(j+\sqcup)}_{o}\,\mathcal{P}_\ell\,]\,\} \;=\; \ell+2\,[\,j-(j+\sqcup)\,] = \ell-2\sqcup \;\le\; \ell \;\;(\sqcup\ \ge\ 0)\,.$$

Moreover, as the process stops when $j+\sqcup = h_\ell$, we need only consider the situation where $j+\sqcup\ \le\ h_\ell-1$. It follows that

$$(4.41)\quad k-j \;=\; \sqcup \ge 0 \;\text{ and }\; k \;=\; j+\sqcup \;\le\; h_\ell-1 \;\Longrightarrow\; j \;\le\; h_\ell-1\,;$$

$$j \;\ge\; 1 \;\Longrightarrow\; \sqcup \;\le\; h_\ell-2\,.$$

We investigate the *characteristic equation*, which is given by the denominator in (4.39) (note that the denominator in (4.34) corresponds to $k = j$):

$$\begin{aligned}
(4.42)\qquad & A_{\ell\,,\,j\,,\,k} \;-\; 2n\,[\,\ell+2\,(j-k)-1] \;=\; 0\\
\Longleftrightarrow\qquad & (2j)\,[\,2j+(n-2)+2\,(\ell-2k)\,] - 2n\,[\,\ell-2\,(k-j)-1] \;=\; 0\\
\Longleftrightarrow\qquad & (2j)^{2} - (2j)[\,(n-2)+2\,\ell-4\sqcup\,]+n\,(2\ell-4\sqcup-2\,) \;=\; 0\\
\Longleftrightarrow\qquad & [\,(2j)-n]\cdot[\,(2j)-(\,2\,\ell-4\sqcup-2)\,] \;=\; 0\\
\cdots\;\Longleftrightarrow\qquad & j \;=\; \frac{n}{2} \quad\text{or}\quad j \;=\; (\ell-1)-2\sqcup\,.
\end{aligned}$$

* *When n is even.* Here $n/2$ is an integer, and (4.41) requires us to post the restriction

$$(4.43)\qquad j \;\le\; h_\ell-1 \;<\; \frac{n}{2} \;\Longleftrightarrow\; h_\ell-1 \;<\; \frac{n}{2} \;\Longleftrightarrow\; 2\cdot h_\ell \;<\; n+2\,.$$

That is, when ℓ is even, we require

$$(4.44)\qquad \ell \;<\; n+2\,.$$

Similarly, when ℓ is odd, we need

$$2\cdot\frac{\ell-1}{2} \;<\; n+2 \;\Longleftrightarrow\; \ell \;<\; n+3 \;\Longleftrightarrow\; \ell \;<\; n+2\,,$$

as n is even $\Longrightarrow n+3$ is odd, and ℓ is also odd in this case. For the second root in (4.42), since $k \le h_\ell - 1$, we have

$$j+\sqcup < \frac{\ell}{2} \Longrightarrow j+\sqcup \le \frac{\ell}{2} - 1 \Longrightarrow \sqcup \le \frac{\ell}{2} - 2 \quad (j \ge 1) \Longrightarrow j+2\sqcup \le \ell - 3 .$$

Thus the solution $j = (\ell - 1) - 2\sqcup \iff j+2\sqcup = \ell - 1$ is too big to happen.

* *When n is odd.* In this case, $n/2$ is not an integer. We need only consider the second root in (4.42). As we want the term $(\mathcal{R}^2)$ to be present, and $(\Delta_o^{(j+\sqcup)}\, \mathcal{P}_\ell)$ is not yet reduced to first order, we have

$$j+\sqcup < \frac{\ell-1}{2} \Longrightarrow j+\sqcup \le \frac{\ell-1}{2} - 1 \Longrightarrow \sqcup \le \frac{\ell-1}{2} - 2 \quad (j \ge 1)$$

$$\Longrightarrow j+2\sqcup \le \ell - 3 \iff j < (\ell - 3) - 2\sqcup .$$

Once again, the solution $j = (\ell - 1) - 2\sqcup$ is too big. This completes the proof that the denominators in (4.34) and (4.39) are non-zero under the conditions of Lemma 4.22.

* *The residue*. As the 'pure' $(\mathcal{R}^2)$, $(\mathcal{R}^2)^2$, $\cdots$, $(\mathcal{R}^2)^{h_\ell}$ terms are obtained as the by-products of the last cancellations in each column (see Figure 4.31), except the last coefficient a_o, all the others are a combination of the horizontal arrow and the downward arrow (refer to Figure 4.31). Hence (together with Figure 4.30; cf. also (4.35) and (4.23)) we have

$$a_{h_\ell} = C^{h_\ell - 1}_{h_\ell - 1}, \; a_{h_\ell - 1} = \left[C^{h_\ell - 1}_{h_\ell - 1} + C^{h_\ell - 2}_{h_\ell - 1} \right], \cdots, \tag{4.45}$$
$$a_1 = \left[C^1_{h_\ell - 1} + C^o_{h_\ell - 1} \right] \text{ and } \quad a_o = C^o_{h_\ell - 1} .$$

The argument is completed under condition (4.25). Finally, suppose that

$$\Delta_o^{(k_o)}\, \mathcal{P}_\ell \equiv 0 \qquad \text{for an integer } k_o \in [\,1,\ h_\ell - 1\,]. \tag{4.46}$$

The process described in Figure 4.31 ends earlier. In this case

$$\mathcal{G} = \sum_{0 \le j \le k \le k_o - 1} C^j_k \cdot (\mathcal{R}^2)^j \,[\, \Delta_o^{(k)}\, \mathcal{P}_\ell \,], \tag{4.47}$$

where the coefficients C^j_k are the same as the above. Moreover, in this case (that is, with (4.46)) $\mathcal{G}$ satisfies

$$(1+\mathcal{R}^2)\,\Delta_o \mathcal{G} - 2n\,(\mathcal{Y}\cdot \bigtriangledown \mathcal{G}) + 2n\,\mathcal{G} = \mathcal{P}_\ell .$$

This completes the proof of Lemma 4.22. $\square$

We summarize the coefficient in the following:

$$\uparrow \qquad C^o_o = \frac{-1}{2n\,(\ell - 1)}, \ \cdots, \ C^j_j = \frac{-C^{j-1}_{j-1}}{A_{\ell,j,j} - 2n\,(\ell - 1)}, \cdots,$$

$$C^o_1 = -\frac{C^o_o + C^1_1 \cdot A_{\ell,1,1}}{-2n\,[\,(\ell - 2) - 1]}, \ \cdots, \ C^o_k = -\frac{C^o_{k-1} + C^1_k \cdot A_{\ell,1,k}}{-2n\,[\,(\ell - 2k) - 1]}, \ \cdots, \tag{4.48}$$

$$\downarrow \qquad C^j_k = -\frac{C^{j-1}_{k-1} + C^j_{k-1} + C^{j+1}_k \cdot A_{\ell,j+1,k}}{A_{\ell,j,k} - 2n\,\{\,[\,\ell + 2\,(j - k)] - 1\}}$$

for $1 \le j < k \le h_\ell - 1$.

Proposition 4.49. *Let $\mathcal{P}_\ell$ be a homogeneous polynomial of degree ℓ defined on $\mathbb{R}^n$. Assume that*

(i) *when ℓ is even: $2 \le \ell < n+2$ and $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell = 0$ ($\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell$ is degree zero);*

(ii) *when ℓ is odd: $2 \le \ell$ and $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell \equiv 0$ (here $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell$ is degree one). Then equation (4.3) has a polynomial solution $\mathcal{G}_o$ given by*

$$\mathcal{G}_o = \sum_{0 \le j \le k \le h_\ell - 1} C_k^j \cdot (\mathcal{R}^2)^j \,[\, \Delta_o^{(k)}\, \mathcal{P}_\ell \,]\,. \tag{4.50}$$

The coefficients C_k^j are presented in (4.48). In particular, the constant and the linear terms are not present in the solution, and the degree of each term in $\mathcal{G}$ is at most ℓ.

Refer to §A.4 in the e-Appendix for the case when $\Delta_o^{(h_\ell)}\, \mathcal{P}_\ell \not\equiv 0$.

5. Mezzo-scale effect of the global harmonic term

In this section we show that for estimates of v_i with accuracy of order $O_{\lambda_i}(n-2)$ or better, the contribution from other blow-up points has to be taken into account. We continue to assume the following:

(5.1) "*the standard conditions* (1.6), (1.25) *and* (1.26), *plus* (2.16), *with the notation in* (1.10) *and* (1.11)".

5a. Rescaled harmonic part. From Proposition 2.24, we can find small positive numbers c_o and c_1 such that for $i \gg 1$,

$$C^{-1} \cdot \lambda_i^{\frac{n-2}{2}} \le v_i\,(y) \le C\, \lambda_i^{\frac{n-2}{2}} \qquad \text{for} \quad i \gg 1 \quad \text{and} \quad c_o \le |y| \le c_1\,.$$

Together with the Harnack inequality (cf. Theorem 8.20 and Corollary 8.21 in [11], p. 199), and the discussion in §2c.1, a subsequence of

$$\{ M_i^{-1} \cdot u_i \} = \left\{ \lambda_i^{-\frac{n-2}{2}} \cdot u_i \right\} \qquad (M_i \text{ is given in (3.2)}) \tag{5.2}$$

converges to a positive C^2-function $\mathcal{H}_\lambda$ in $S^n \setminus \{\beta_o, \cdots, \beta_k\}$. (The convergence is uniform in every compact set in $S^n \setminus \{\beta_o, \cdots, \beta_k\}$.) With the stereographic projection $\dot{\mathcal{P}}$ onto $\mathbb{R}^n$, $\mathcal{H}_\lambda$ can be expressed (cf. (1.3) and (2.9)) as

$$H_\lambda\,(y) := [\, \mathcal{H}_\lambda \circ \dot{\mathcal{P}}^{-1}(y)] \cdot \left(\frac{2}{1 + |\,y|^2} \right)^{\frac{n-2}{2}}\,, \tag{5.3}$$

$$H_\lambda\,(y) = \sum_{j=0}^{k} \frac{\mathcal{A}_l}{|\,y - \hat{Y}_l|^{\,n-2}} \tag{5.4}$$

for $y \in \mathbb{R}^n \setminus \{\, \hat{Y}_o = 0, \cdots, \hat{Y}_k\}$. Here $\hat{Y}_j := \mathcal{P}\,(\beta_j)$, $0 \le j \le k$, and $\mathcal{A}_j$ are positive constants (a constant times A_j which appears in (2.10)). From the Proportionality Proposition 2.3, (5.3) and §2f, together with (1.10) and (1.11), we obtain

$$[\, v_i\,(\xi_i)] \cdot v_i\,(y) = \lambda_i^{-\frac{n-2}{2}} \cdot v_i\,(y) \longrightarrow \frac{1}{|\,y|^{\,n-2}} + h\,(y) \quad \text{in } C^2_{\text{loc}}(B_o\,(\bar{\rho}_1) \setminus \{0\})\,.$$

Recall that we assume (without loss of generality) $(\tilde{c}_n\,K)\,(0) \;=\; n\,(n\;-\;2)$. Hence we know that

$$\mathcal{A}_o \;=\; 1\,. \tag{5.5}$$

(5.6) Define $$\mathrm{H}_{\lambda_{\ge 1}}\,(y) \;:=\; \sum_{j\,=\,1}^{k} \frac{\mathcal{A}_j}{|\,y\;-\hat{Y}_j|^{\,n\,-\,2}} \qquad \text{for} \quad y \,\in\, \mathbb{R}^n\setminus\{\,\hat{Y}_1\,,\,\cdots,\,\hat{Y}_k\}\,.$$

Note that $\mathrm{H}_{\lambda_{\ge 1}}\,(y) \;=\; H_\lambda\,(y) - |\,y|^{-\,(n\,-\,2)}$ is well-defined and smooth on a neighborhood of 0.

5b. Estimating $|\;\mathcal{V}_i\,(\mathcal{Y})\;-\;A_1\,(\mathcal{Y})\,|$ on the mezzo-scale $C_o\;\le\;\lambda_i\,|\,\mathcal{Y}|\;\le\;C_1$. We start with the convergence occurring in (5.2):

(5.7)

$$\frac{u_i\,(x)}{\lambda_i^{\frac{n\,-\,2}{2}}} \;\to\; \mathcal{H}_\lambda\,(x) \quad \text{for } x\,\in\,S^n\Big\backslash\left[\,\bigcup_{j\,=\,0}^{k}\,\mathcal{B}_{\beta_l}\,(\rho)\right]\,, \quad \rho>0 \;\;\text{small} \;\;\text{but} \;\;\text{fixed}$$

$$\Longrightarrow\quad \left|\,\frac{u_i\,(x)}{\lambda_i^{\frac{n\,-\,2}{2}}} \;-\; \mathcal{H}_\lambda\,(x)\right| \;\le\; \varepsilon \quad \text{for all} \quad i\,\gg\,1 \;\;\text{and}\;\; x\,\in\,S^n\Big\backslash\left[\,\bigcup_{j\,=\,0}^{k}\,\mathcal{B}_{\beta_l}\,(\rho)\right]$$

$$\Longrightarrow\quad \left|\,\frac{u_i\,(x)}{\lambda_i^{\frac{n\,-\,2}{2}}}\cdot\left(\frac{2}{1+|\,y|^{\,2}}\right)^{\frac{n\,-\,2}{2}} \;-\; \mathcal{H}_\lambda\,(x)\cdot\left(\frac{2}{1+|\,y|^{\,2}}\right)^{\frac{n\,-\,2}{2}}\right| \;\le\; \varepsilon\cdot\left(\frac{2}{1+|\,y|^{\,2}}\right)^{\frac{n\,-\,2}{2}}$$

$$\text{for} \quad \text{all} \quad i\,\gg\,1 \;\;\text{and}\;\; x\,\in\,S^n\Big\backslash\left[\,\mathbf{N}\,\cup\,\bigcup_{j\,=\,0}^{k}\,\mathcal{B}_{\beta_l}\,(\rho)\right]; \quad y=\dot{\mathcal{P}}\,(x)$$

$$\Longrightarrow\quad \left|\,\frac{v_i\,(y)}{\lambda_i^{\frac{n\,-\,2}{2}}} \;-\; H_\lambda\,(y)\right| \;\le\; \frac{C\,\varepsilon}{(1+r)^n}$$

$$\text{for} \quad \text{all} \quad i\,\gg\,1 \;\;\text{and}\;\; y\,\in\,\mathbb{R}^n\Big\backslash\left[\,\bigcup_{l\,=\,0}^{k}\,B_{\hat{Y}_l}\;(r_o)\right]$$

$$\Longrightarrow\quad \left|\,\frac{v_i\,(y)}{\lambda_i^{\frac{n\,-\,2}{2}}} \;-\; \frac{1}{|\,y|^{\,n\,-\,2}} \;-\; \mathrm{H}_{\lambda_{\ge 1}}\;(y)\right| \;\le\; \frac{C\,\varepsilon}{(1+r)^n} \qquad \text{(using (5.5) and (5.6))}$$

$$\cdots\qquad \Longrightarrow\quad \left|\,v_i\,(y) \;-\; \frac{\lambda_i^{\frac{n\,-\,2}{2}}}{|y|^{\,n\,-\,2}} \;-\; [\,\mathrm{H}_{\lambda_{\ge 1}}\;(y)\,]\cdot\lambda_i^{\frac{n\,-\,2}{2}}\right| \;\le\; \frac{C\,\varepsilon\cdot\lambda_i^{\frac{n\,-\,2}{2}}}{(1+r)^n}$$

for all $i\,\gg\,1$ and $c_o\;\le\;|\,y|\;\le\;c_1$. As usual, $r\;=\;|\,y|$. Note that we may take

$$c_1 \;\le\; \frac{1}{2}\cdot\min_{1\,\le\,j\,\le\,k}|\,\hat{Y}_j|\,. \tag{5.8}$$

In (5.7), we replace $y \to \xi_i + \lambda_i\,\mathcal{Y}$. For $i \gg 1$ and $c_o \le |\,\xi_i + \lambda_i\,\mathcal{Y}| \le c_1$:

(5.9)

$$\left| v_i\,(\xi_i+\lambda_i\,\mathcal{Y}) - \frac{\lambda_i^{\frac{n-2}{2}}}{|\,\xi_i+\lambda_i\,\mathcal{Y}|^{\,n-2}} - [\,\mathrm{H}_{\lambda_{\ge 1}}\,(\xi_i+\lambda_i\,\mathcal{Y})\,]\cdot\lambda_i^{\frac{n-2}{2}} \right| \le \frac{C\,\varepsilon\cdot\lambda_i^{\frac{n-2}{2}}}{(1+|\,\xi_i+\lambda_i\,\mathcal{Y}|\,)^{\,n}}$$

$$\Longrightarrow \left| \frac{v_i\,(\xi_i+\lambda_i\,\mathcal{Y})}{v_i\,(\xi_i)} - \frac{\lambda_i^{n-2}}{|\,\xi_i+\lambda_i\,\mathcal{Y}|^{\,n-2}} - [\,\mathrm{H}_{\lambda_{\ge 1}}\,(\xi_i+\lambda_i\,\mathcal{Y})\,]\cdot\lambda_i^{n-2} \right| \le C\,\varepsilon\cdot\lambda_i^{n-2}$$

$$\Longrightarrow \left| \mathcal{V}_i\,(\mathcal{Y}) - \frac{1}{|\,[\lambda_i^{-1}\cdot\xi_i\,]+\mathcal{Y}|^{\,n-2}} - \lambda_i^{n-2}\cdot[\,\mathrm{H}_{\lambda_{\ge 1}}\,(\xi_i+\lambda_i\,\mathcal{Y})\,] \right| \le C\,\varepsilon\cdot\lambda_i^{n-2}\,.$$

Next, we seek to show that the second term in the last inequality above is "close" to A_1 in the mezzo-range, under the condition that $|\,\lambda_i^{-1}\cdot\xi_i\,| = O\,(1)$. We first note that

$$y = \xi_i + \lambda_i\,\mathcal{Y}\,, \qquad \text{where} \quad c_o \le |\,y| \le c_1$$

$$\Longrightarrow c_o \le |\,\xi_i+\lambda_i\,\mathcal{Y}| \le c_1 \Longrightarrow c_o\cdot\lambda_i^{-1} \le |\,(\lambda_i^{-1}\cdot\xi_i\,)+\mathcal{Y}| \le c_1\cdot\lambda_i^{-1}$$

$$(\text{assuming } \lambda_i^{-1}\cdot|\,\xi_i| = O\,(1))$$

$$\cdots \Longrightarrow [\,c_o - o\,(1)\,]\cdot\lambda_i^{-1} \le |\mathcal{Y}| \le [\,c_1+o\,(1)\,]\cdot\lambda_i^{-1}\,, \tag{5.10}$$

where $o\,(1)\to 0$ as $i\to\infty$. Thus again, for $i \gg 1$ and $c_o \le |\,\xi_i+\lambda_i\,\mathcal{Y}| \le c_1$, we continue with

$$\left| A_1\,(\mathcal{Y}) - \frac{1}{|\,[\lambda_i^{-1}\cdot\xi_i\,]+\mathcal{Y}|^{\,n-2}} \right| \tag{5.11}$$

$$= \left| \left(\frac{1}{1+|\,\mathcal{Y}|^2}\right)^{\frac{n-2}{2}} - \left(\frac{1}{|\mathcal{Y}|^2+2\,\mathcal{Y}\cdot[\,\lambda_i^{-1}\cdot\xi_i\,]+O\,(1)}\right)^{\frac{n-2}{2}} \right|$$

$$\le \frac{2}{n-2}\cdot\left| \frac{1}{1+|\,\mathcal{Y}|^2} - \frac{1}{|\mathcal{Y}|^2+2\,\mathcal{Y}\cdot[\,\lambda_i^{-1}\cdot\xi_i\,]+O\,(1)} \right|$$

$$\times\ \max\left\{ \left(\frac{1}{1+|\,\mathcal{Y}|^2}\right)^{\frac{n-2}{2}-1}, \left(\frac{1}{|\mathcal{Y}|^2+2\langle\,\mathcal{Y}\,,[\,\lambda_i^{-1}\cdot\xi_i]\,\rangle+O\,(1)}\right)^{\frac{n-2}{2}-1} \right\}$$

$$\le C'\cdot\left| \frac{2\,\langle\,\mathcal{Y}\,,[\,\lambda_i^{-1}\cdot\xi_i]\,\rangle+O\,(1)-1}{[\,1+|\,\mathcal{Y}|^2]\cdot[\,|\,\mathcal{Y}|^2+2\,\langle\,\mathcal{Y}\,,[\,\lambda_i^{-1}\cdot\xi_i]\,\rangle+O\,(1)]} \right|\cdot\left(\frac{1}{|\,\mathcal{Y}|^2}\right)^{\frac{n-2}{2}-1}$$

$$\le\; C'\cdot\left[\frac{1}{|\,\mathcal{Y}|^{4}} \;+\; \frac{O\,(1)}{|\,\mathcal{Y}|^{4}} \;+\; \frac{|\,\mathcal{Y}|\cdot|\,\lambda_i^{-1}\cdot\xi_i\,|}{|\,\mathcal{Y}|^{4}}\right]\cdot\left(\frac{1}{|\,\mathcal{Y}|^{2}}\right)^{\frac{n-2}{2}-1}$$

$$\le\; C'\cdot\left[\frac{1}{|\,\mathcal{Y}|^{4}} \;+\; \frac{O\,(1)}{|\,\mathcal{Y}|^{4}} \;+\; \frac{O\,(1)}{|\,\mathcal{Y}|^{3}}\right]\cdot\left(\frac{1}{|\,\mathcal{Y}|^{2}}\right)^{\frac{n-2}{2}-1}$$

$$\le\; C'\cdot\frac{O\,(1)}{|\,\mathcal{Y}|^{3}} \;*\; \left(\frac{1}{|\,\mathcal{Y}|^{2}}\right)^{\frac{n-2}{2}-1}$$

$$\le\; C'\cdot\frac{O\,(1)}{|\,\mathcal{Y}|}\cdot\left(\frac{1}{|\,\mathcal{Y}|^{2}}\right)^{\frac{n-2}{2}} \;\le\; O\,(1)\cdot\lambda_i^{n-1}\,.$$

In the above, we apply the inequality

$$a > b > 0 \;\text{ and }\; p \ge 1 \;\Longrightarrow\; a^p - b^p \;\le\; p^{-1}\cdot(a-b)\cdot a^{p-1}\,.$$

Note that when $j \ne 0$, from (5.8) and (5.10), we have

$$\frac{A}{|\,(\xi_i+\lambda_i\,\mathcal{Y}) - \hat{Y}_j\,|^{n-2}} = \frac{A}{|\,(\lambda_i\,\mathcal{Y} - \hat{Y}_j\,) + \xi_i\,|^{n-2}} \tag{5.12}$$

$$= \frac{A}{|\,\lambda_i\,\mathcal{Y} - \hat{Y}_j\,|^{n-2}} \times \frac{1}{\left(1 + \frac{\xi_i}{|\,\lambda_i\,\mathcal{Y} - \hat{Y}_j|}\right)^{n-2}}$$

$$= \frac{A}{|\,\lambda_i\,\mathcal{Y} - \hat{Y}_j\,|^{n-2}}\cdot[\,1 + O\,(|\,\xi_i\,|\,)\,]$$

for $[\,c_o - o\,(1)]\cdot\lambda_i^{-1} \le |\,\mathcal{Y}| \le [\,c_1 + o\,(1)]\cdot\lambda_i^{-1}$. Thus if we install the terms

$$\mathrm{H}_{\ge 1}\,(\mathcal{Y}) \;:=\; \sum_{j\ge 1}\frac{\mathcal{A}_j}{|\,\lambda_i\,\mathcal{Y} - \hat{Y}_j\,|^{n-2}} \tag{5.13}$$

($\mathrm{H}_{\ge 1}\,(\mathcal{Y})$ is smooth and harmonic in $B_o\,(c_1\cdot\lambda_i^{-1}\,))$ and apply the triangle inequality, then (5.8), (5.11) and (5.12) furnish us with the following mezzo-scale estimate.

Lemma 5.14. *For $n \ge 3$, under the conditions and notation in* (5.1) *for* $\{u_i\}$ $\{v_i\}$, λ_i *and* ξ_i, *assume also that* $\lambda_i^{-1}\cdot|\,\xi_i| = O\,(1)$. *For any* $\varepsilon > 0$, *we have*

$$|\,\mathcal{V}_i\,(\mathcal{Y}) - A_1\,(\mathcal{Y}) - \lambda_i^{n-2}\cdot H_{\ge 1}\,(\mathcal{Y})\,| \;\le\; C\,\varepsilon\cdot\lambda_i^{n-2} \;+\; O\,(1)\cdot\lambda_i^{n-1} \tag{5.15}$$

for all $i \gg 1$ *and* $\bar{c}_1\cdot\lambda_i^{-1} \;\le\; |\,\mathcal{Y}| \;\le\; \bar{c}_2\cdot\lambda_i^{-1}$. *Here* $\bar{c}_1 > 0$ *can be taken to be any small (but fixed) constant as long as* $\bar{c}_1 < \bar{c}_2$. (*In* (5.15), *the order in the right-hand side is* $O_{\lambda_i}(n-2)$).

6. Second order blow-up argument and the proof of Main Theorem 1.14

In this section, *we take up all the assumptions stated in Main Theorem* 1.14. To begin with, we observe the following.

Proposition 6.1. *For $n \ge 4$, under the general conditions listed in* (5.1), *we also take the following conditions* (i)–(iv) *into account.*

(i) 0 *is a simple blow-up point for* $\{v_i\}$.

(ii) *K is given by* (1.8) *in* $B_o(\rho_o)$, *where* $2 \le \ell < n-2$.

(iii) *The parameters* λ_i *and* ξ_i *corresponding to the simple blow-up point at* 0 (*via* (1.10) *and* (1.11)) *satisfy* (1.12), *that is*, $|\xi_i| = o(\lambda_i)$.

(iv) ℓ *is even*.

Then $\Delta_o^{(h_\ell)} \mathbf{P}_\ell(y) = 0$. ($\mathbf{P}_\ell$ *is found in* (1.8). *When ℓ is even,* $h_\ell = \ell/2$, *and* $\Delta_o^{(h_\ell)} \mathbf{P}_\ell$ *is a number*.) *The same conclusion also holds when* $\ell = n-2$ *with an additional assumption that* 0 *is the only blow-up point* (*ℓ is still required to be even*).

The key to the proof is to combine the change of center formula (see (A.6.33) in the e-Appendix) with the condition $|\xi_i| = o(\lambda_i)$. Other arguments actually proceed in similar fashion as those found in [15] and [20]. For the benefit of the reader, we present the estimates in §A.6.d in the e-Appendix.

Together with condition (1.19) in the Main Theorem, Proposition 4.49 and Proposition 6.1, we can secure a solution $\Pi_{\mathbf{p}}$ of the equation

$$\Delta_o \Pi_{\mathbf{p}} + n(n+2) A_1^{\frac{4}{n-2}} \cdot \Pi_{\mathbf{p}} = \mathbf{P}_\ell \cdot A_1^{\frac{n+2}{n-2}} \qquad \text{in} \quad \mathbb{R}^n .$$

Moreover,

$$\Pi_{\mathbf{p}}(\mathcal{Y}) = \frac{\Gamma_{\mathbf{p}}(\mathcal{Y})}{(1+\mathcal{R}^2)^{\frac{n}{2}}} \qquad \text{for} \quad \mathcal{Y} \in \mathbb{R}^n, \quad \text{where } \mathcal{R} = |\mathcal{Y}|. \tag{6.2}$$

Thanks to Proposition 4.49, the precise form of $\Gamma_{\mathbf{p}}$ is known once $\mathbf{P}_\ell$ is given. It follows from (3.11) that

$$\Delta_o(\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_{\mathbf{p}}) + n(n+2) A_1^{\frac{4}{n-2}} (\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_{\mathbf{p}}) = \mathbf{RM} \tag{6.3}$$

in $\mathbb{R}^n$, where the 'remainder' $\mathbf{RM}$ is given in (3.12); cf. (3.1)–(3.3).

6a. Inclusion of the harmonic term via interpolation. Consider

$$\mathcal{D}_i^{\Pi}(\mathcal{Y}) := [\mathcal{V}_i(\mathcal{Y}) - A_1(\mathcal{Y}) - \lambda_i^\ell \cdot \Pi_{\mathbf{p}}(\mathcal{Y}) - \lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}(\mathcal{Y})] + [\lambda_i^{n-2} \cdot h_o] \cdot \left[1 - \frac{\tilde{\mathcal{R}}(\mathcal{Y})}{c\,\lambda_i^{-1}}\right] \tag{6.4}$$

in the region $|\mathcal{Y}| \le c\,\lambda_i^{-1}$. Here $c \le c_1$ is a positive constant to be fixed (cf. (5.8)), $\mathrm{H}_{\ge 1}$ is given in (5.13), and

$$h_o := \mathrm{H}_{\ge 1}(0) \qquad (\text{that is, setting } \mathcal{Y} = 0 \text{ in } (5.13)). \tag{6.5}$$

In addition, $\tilde{\mathcal{R}} \in C^\infty(\mathbb{R}^n)$ is a non-negative function which satisfies

$$\tilde{\mathcal{R}}(\mathcal{Y}) = |\mathcal{Y}| \quad \text{for} \quad |\mathcal{Y}| \ge 1, \; \tilde{\mathcal{R}}(0) = 0, \; \nabla_{\mathcal{Y}} \tilde{\mathcal{R}}(0) = \vec{0}, \; \text{and } |\Delta_o \tilde{\mathcal{R}}| \le C \text{ in } \mathbb{R}^n. \tag{6.6}$$

6a.1. Joint between bubble estimate and the global harmonic term. From (1.4), (1.6) and (2.17), we know that $\mathcal{V}_i(0) = A_1(0) = 1$ for all i. Here, we group together the terms in (6.4) which link to the harmonic function $\mathrm{H}_{\ge 1}$ and note that

$$\lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}(\mathcal{Y}) + [\lambda_i^{n-2} \cdot h_o] \cdot \left(1 - \frac{\tilde{\mathcal{R}}(\mathcal{Y})}{c\,\lambda_i^{-1}}\right) = 0 \qquad \text{if } \mathcal{Y} = 0, \tag{6.7}$$

$$\lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}(\mathcal{Y}) + [\lambda_i^{n-2} \cdot h_o] \cdot \left(1 - \frac{\tilde{\mathcal{R}}(\mathcal{Y})}{c\,\lambda_i^{-1}}\right) = \lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}(\mathcal{Y})$$

if $|\mathcal{Y}| = c\,\lambda_i^{-1}$. That is, via $\tilde{\mathcal{R}}$, the bubble estimate and the global harmonic term are joint. See the comments in §2d.1, (2.14), (2.15), (2.22) and (2.23).

6b. Ingredients for the method to work. [12] provides the framework for what we call the "second level blow-up argument" (see also [8]). It is an exquisite method which goes to the root of the blow-up phenomenon. Here we summarize the key steps and apply it to our situation.

6b.1. First order vanishing property. From (1.6), (3.1), (3.2), Proposition 4.49 (from there we know that $\Gamma_{\mathbf{p}}$ contains no constant term), and (6.7), we obtain

$$\mathcal{D}_i^{\Pi}(0) \;=\; 0 \qquad \text{for} \qquad i \;\gg\; 1\,. \tag{6.8}$$

Referring to (5.13) (observe the presence of λ_i in the right-hand side below),

(6.9)
$$\frac{\partial}{\partial \mathcal{Y}_{|_1}}\left[\frac{\mathcal{A}_l}{|\,\lambda_i\,\mathcal{Y} \,-\, \hat{Y}_j\,|^{\,n-2}}\right]\Bigg|_{\mathcal{Y}=0} = \;(n\,-\,2)\,\mathcal{A}_l\cdot\frac{\lambda_i\,\hat{Y}_{j\,|_1}}{|\,\hat{Y}_j\,|^{\,n}} \qquad [\,\hat{Y}_j = (\hat{Y}_{j\,|_1}\,,\,\cdots,\,\hat{Y}_{j\,|_n}\,)\,]\,.$$

From the definition of $A_1 \,=\, A_{1\,,\,0}$ (cf. (1.4)), (1.6), (3.1), (3.2), Proposition 4.49 (from there we recognize that $\Gamma_{\mathbf{p}}$ contains no first order terms), (6.5), (6.6) and (6.9), we obtain

$$\|\,\bigtriangledown_{\mathcal{Y}}\,\mathcal{D}_i^{\Pi}(0)\,\| \;=\; 0\,(\lambda_i^{\,n\,-\,1}\,) \qquad\quad \text{for} \qquad i \;\gg\; 1\,. \tag{6.10}$$

6b.2. The maximum in $B_o(c\,\lambda_i^{-1})$. Because $\Delta_o\,[\,\mathrm{H}_{\ge 1} \,-\, h_o\,] = 0$, we have

$$\Delta_o\,\mathcal{D}_i^{\Pi} \;+\; n\,(n+2)\,A_1^{\frac{4}{n\,-\,2}}\cdot\mathcal{D}_i^{\Pi} \;=\; \mathbf{R.H.S}_{\,i} \qquad \text{in} \quad B_o\,(c\,\lambda_i^{-1}\,)\,, \tag{6.11}$$

where (by using (6.3) and (6.6)),

(6.12)
$$\mathbf{R.H.S}_{\,i} \;:=\; \mathbf{RM} \;-\; \lambda_i^{n\,-\,2}\left\{\frac{h_o}{c\,\lambda_i^{-1}}\cdot\Delta_o\,\tilde{\mathcal{R}} \;+\; n\,(n+2)\cdot\frac{h_o}{c\,\lambda_i^{-1}}\cdot A_1^{\frac{4}{n\,-\,2}}\cdot\tilde{\mathcal{R}}\right.$$
$$\left. +\; n\,(n+2)\,A_1^{\frac{4}{n\,-\,2}}\cdot[\,\mathrm{H}_{\ge 1} \,-\, h_o\,]\right\}.$$

From (2.26) in Proposition 2.24, Proposition 4.49 and the expression for $\Pi_{\mathbf{p}}$ (that is, (6.2)), we have

$$\ell \;\le\; n\,-\,2 \;\Longrightarrow\; \Lambda_{\,i} := \max_{|\,\mathcal{Y}|\le c\,\lambda_i^{-1}}|\,\mathcal{D}_i^{\Pi}\,(\mathcal{Y})| \;<\; \infty \qquad \text{for} \quad i \gg 1\,. \tag{6.13}$$

We assert that

$$\Lambda_i = o_{\lambda_i}\,(\ell)\,. \tag{6.14}$$

The assertion is equivalent to

$$\Lambda_i \;=\; o\,(1)\,\lambda_i^{\ell} \;\Longleftrightarrow\; \frac{\lambda_i^{\ell}}{\Lambda_i} \;=\; \frac{1}{o\,(1)} \;\Longleftrightarrow\; \frac{\lambda_i^{\ell}}{\Lambda_i} \;\to\; \infty\,. \tag{6.15}$$

Suppose that this is not the case. Then (modulo a subsequence)

$$\frac{\lambda_i^{\ell}}{\Lambda_i} \;\le\; C \quad \text{for all} \;\; i \;\ge\; 1 \qquad \Longleftrightarrow \qquad \frac{1}{\Lambda_i} \;\le\; \frac{C}{\lambda_i^{\ell}} \quad \text{for all} \;\; i \;\ge\; 1\,. \tag{6.16}$$

In what follows, we seek to derive a contradiction to (6.16).

6b.3. Renormalization. Consider the function

$$\mathbf{W}_i := \frac{\mathcal{D}_i^{\Pi}}{\Lambda_i} \qquad \text{defined in} \qquad \overline{B_o\,(c\,\lambda_i^{-1})}\,. \tag{6.17}$$

From (6.11), we have

$$\Delta_o\,\mathbf{W}_i \;+\; n\,(n+2)\,A_1^{\frac{4}{n-2}} \cdot \mathbf{W}_i \;=\; \Lambda_i^{-1} \cdot \mathbf{R.H.S}_{\,i} \qquad \text{in} \quad B_o\,(\,c\,\lambda_i^{-1}\,)\,. \tag{6.18}$$

6b.4. Order of magnitude of the remainder. The key properties we want to show about $\mathbf{R.H.S}_{\,i}$ are the following (under the condition in the Main Theorem):

$$\text{Given any } \mathcal{R}_o \;>\; 0\,, \;\; \Lambda_i^{-1} \cdot |\,\mathbf{R.H.S}_{\,i}\,| \;\to\; 0 \;\; \text{uniformly in} \;\; B_o\,(\mathcal{R}_o)\,. \tag{6.19}$$

$$\frac{|\,\mathbf{R.H.S}_{\,i}\,(\mathcal{Y})|}{\Lambda_i} \;\le\; \frac{C}{(1+\mathcal{R})^4} \;+\; O\,(\lambda_i) \cdot \chi_{B_o\,(1)} \quad \text{for} \quad \mathcal{R} \;\le\; c\,\lambda_i^{-1}. \tag{6.20}$$

Here $\mathcal{R} \;=\; |\,\mathcal{Y}|$ and $\chi_{B_o\,(1)}$ is the characteristic function of the unit ball. These are demonstrated in §6c.

6b.5. Vanishing on the whole. From (6.8), (6.9), (6.10), (6.16) and (6.17), we know that

$$|\,\mathbf{W}_i| \;\le\; 1 \qquad \text{in} \quad B_o\,(c\,\lambda_i^{-1})\,, \tag{6.21}$$

$$\mathbf{W}_i\,(0) \;=\; 0 \qquad \text{for} \quad i \gg 1\,, \quad \text{and} \quad \triangledown\,\mathbf{W}_i\,(0) \;=\; O\,(\lambda_i) \;\to\; \vec{0} \;\; \text{as} \;\, i \to \infty\,. \tag{6.22}$$

It follows from (6.11), (6.18), (6.19), (6.21) and standard elliptic theory that (modulo a subsequence)

$$\mathbf{W}_i \longrightarrow \mathbf{W} \qquad \text{uniformly in every compact subset of } \mathbb{R}^n. \tag{6.23}$$

Here $\mathbf{W}$ is a C^2-function satisfying

$$\Delta_o\,\mathbf{W} \;+\; n\,(n+2)\,A_1^{\frac{4}{n-2}} \cdot \mathbf{W} \;=\; 0 \qquad \text{in} \quad \mathbb{R}^n. \tag{6.24}$$

In addition, from (6.22),

$$\mathbf{W}\,(0) \;=\; 0 \qquad \text{and} \qquad \triangledown\,\mathbf{W}\,(0) \;=\; \vec{0}\,. \tag{6.25}$$

Moreover, in §6b.7, we show that

$$|\,\mathbf{W}\,(\mathcal{Y})| \to 0 \qquad \text{when} \quad |\,\mathcal{Y}| \;\to\; \infty\,. \tag{6.26}$$

A standard bootstrap argument shows that $\mathbf{W}$ is smooth in $\mathbb{R}^n$. It follows from the Liouville-type theorem for (6.24) (see Lemma 2.4 in [8]; cf. also [3]) that

$$\mathbf{W} \;\equiv\; 0 \quad \text{in} \quad \mathbb{R}^n \quad \Longrightarrow \quad \mathbf{W}_i \;\to\; 0 \;\; \text{uniformly in} \;\; B_o\,(\mathcal{R}_o) \subset \mathbb{R}^n. \tag{6.27}$$

Here $\mathcal{R}_o$ can be any given positive number. On the other hand, by the definition of Λ_i, there is a point

$$\mathcal{Y}_{\mu_i} \;\in\; \overline{B_o\,(c\,\lambda_i^{-1})} \qquad \text{such that} \quad \mathbf{W}_i\,(\mathcal{Y}_{\mu_i}) \;=\; 1\,. \tag{6.28}$$

We produce a contradiction to (6.27) by showing that we can find a positive number $\mathcal{R}_o$ such that

$$|\,\mathcal{Y}_{\mu_i}| \;\le\; \mathcal{R}_o \qquad \text{for all} \quad i \gg 1 \qquad \left(\Longleftrightarrow \;\; \max_{B_o\,(\mathcal{R}_o)} \mathbf{W}_i \;=\; 1 \right). \tag{6.29}$$

6b.6. Smallness of $|\,\mathbf{W}_i\,|$ near the boundary $\partial B_o\,(\,c\,\lambda_i^{-1}\,)$. Given (5.15) in Lemma 5.14, we turn our attention to $\lambda_i^{\ell}\cdot\Pi_{\mathbf{p}}$ in (6.4). Based on Proposition 4.49 and condition (6.2), we have

(6.30) the degree of the polynomial $\Gamma_{\mathbf{p}}$ in (6.2) is at most $n-2$.

It follows that

$$
\begin{aligned}
(6.31)\qquad |\,\lambda_i^{\ell}\cdot\Pi_{\mathbf{p}}(\mathcal{Y})| \;&=\; \lambda_i^{\ell}\cdot\frac{|\,\Gamma_{\mathbf{p}}(\mathcal{Y})|}{(1+\mathcal{R}^{\,2})^{\frac{n}{2}}}\\
&\le\; C\cdot\frac{\lambda_i^{\ell}\,(1+\mathcal{R})^{\,n-2}}{(1+\mathcal{R})^{n}} \;\le\; \frac{C_1\,\lambda_i^{\ell}}{(1+\mathcal{R})^2} \;\le\; C_2\,\lambda_i^{\ell+2}\\
&\qquad\text{for}\quad \mathcal{R}\,=\,|\,\mathcal{Y}|\,\ge\,(1-\delta)\cdot c\,\lambda_i^{-1}
\end{aligned}
$$

$$
(6.32)\qquad \cdots\;\Longrightarrow\;\Lambda_i^{-1}\cdot|\,\lambda_i^{\ell}\cdot\Pi\,(\mathcal{Y})|\;=\;O\,(\lambda_i^2)\quad(\text{via }\;(6.16))
$$

for $\mathcal{R}\;\ge\;(1-\delta)\cdot c\,\lambda_i^{-1}$. Together with (6.4), (6.6), Lemma 5.14 and (6.31), we have

$$
(6.33)\qquad |\,\mathbf{W}_i\,(\mathcal{Y})|\,=\,O\,(\varepsilon)\;+O\,(\lambda_i^2)\qquad\text{for}\quad|\,\mathcal{Y}|\,=\,c\,\lambda_i^{-1}\quad\text{and}\quad i\gg 1.
$$

Moreover, for $(1-\delta)\cdot[\,c\,\lambda_i^{-1}\,]\,\le\,|\,\mathcal{Y}|\;\le\;[\,c\,\lambda_i^{-1}\,]$, we have

$$
(6.34)\quad [\,\lambda_i^{n-2}\cdot h_o\,]\cdot\left|\,1-\frac{\tilde{\mathcal{R}}\,(\mathcal{Y})}{c\,\lambda_i^{-1}}\,\right| \;=\;[\,\lambda_i^{n-2}\cdot h_o\,]\cdot\left|\,1-\frac{|\,\mathcal{Y}|}{c\,\lambda_i^{-1}}\,\right|\;\le\;[\,\delta\cdot h_o]\cdot\lambda_i^{n-2}\,.
$$

Thus if we choose $\delta\,>\,0$ to be small [relative to the constant C in (6.16) and h_o only], and combine (6.32) with (6.33), (6.34) and Lemma 5.14, we obtain
(6.35)

$$
|\,\mathbf{W}_i\,(\mathcal{Y})|\;\le\;\frac{1}{2}\qquad\text{for}\quad(1-\delta)\cdot[\,c\,\lambda_i^{-1}\,]\;\le\;|\,\mathcal{Y}|\;\le\;[\,c\,\lambda_i^{-1}\,]\quad\text{and}\quad i\gg 1\,.
$$

6b.7. The decay to 0 – proof of (6.26). To demonstrate (6.26), we show that for any positive number $\tilde{\varepsilon}$ (small and given), we can find a positive number $\mathcal{R}_{\tilde{\varepsilon}}$ and a natural number $I_{\tilde{\varepsilon}}$ such that

$$
(6.36)\quad |\,\mathbf{W}_i\,(\mathcal{Y})|\;\le\;\tilde{\varepsilon}\qquad\text{for}\quad\text{all }i\ge I_{\tilde{\varepsilon}}\;\text{ and }\;\mathcal{R}_{\tilde{\varepsilon}}\;\le\;|\,\mathcal{Y}|\;\le\;(1-\delta)\cdot[\,c\,\lambda_i^{-1}\,]\,.
$$

Here $\delta\,\in\,(0,\;1)$ is fixed, and the integer $I_{\tilde{\varepsilon}}$ depends on $\tilde{\varepsilon}$ only. In particular, $(1-\delta)\cdot[\,c\,\lambda_i^{-1}\,]\;\to\;\infty$ as $i\to\infty$. Once we have (6.36), together with the uniform convergence of $\mathbf{W}$ to $\mathbf{W}_i$ on any given compact subset of $\mathbb{R}^n$, we have

$$
(6.37)\qquad |\,\mathbf{W}\,(\mathcal{Y})|\;\le\;\tilde{\varepsilon}\qquad\text{for}\quad\text{all}\quad|\,\mathcal{Y}|\;\ge\;\mathcal{R}_{\tilde{\varepsilon}}\quad\Longrightarrow\quad(6.26)\,.
$$

To demonstrate the proof for (6.36), via (6. 33), we already know that $|\,\mathbf{W}_i|$ is 'small' along the boundary $\partial B_o\,(\,c\,\lambda_i^{-1}\,)$. The value in $B_o\,(\,c\,\lambda_i^{-1}\,)$ is governed by the equation describing $\Delta_o\,\mathbf{W}_i$ (that is, (6.18)), and the Green representation formula, which, in the present situation, is given by

(6.38)
$$\mathbf{W}_i\,(\mathcal{Y}_{out}) = \int_{B_o(\,c\,\lambda_i^{-1}\,)} G_i\,(\mathcal{Y},\ \mathcal{Y}_{out})\Big\{-\,n\,(n+2)[\,A_1\,(\mathcal{Y})]^{\frac{4}{n\,-\,2}}\cdot\mathbf{W}_i\,(\mathcal{Y}) + \ \Lambda_i^{-1}\cdot\mathbf{R.H.S}_{\,i}\,(\mathcal{Y})\Big\}\,d\,\mathcal{Y} + \int_{\partial B_o(\,c\,\lambda_i^{-1}\,)}[\,\mathbf{n}\cdot\bigtriangledown_{\mathcal{Y}}\,G_i(\mathcal{Y},\ \mathcal{Y}_{out})]\,\mathbf{W}_i\,(\mathcal{Y})\ dS_{\mathcal{Y}}$$

for $\mathcal{Y}_{out}\,\in\,B_o\,(\,c\,\lambda_i^{-1}\,)$. Here G_i is the Green function for Δ_o in $B_o\,(\,c\,\lambda_i^{-1}\,)$ with the Dirichlet boundary condition. See, for example, [22]. Note that

$$G_i\,(\mathcal{Y}\,,\ \mathcal{Y}'\,)\ \approx\ -\,\frac{1}{(n\,-\,2)\,\|\,S^{n\,-\,1}\|}\cdot\frac{1}{|\,\mathcal{Y}\,-\,\mathcal{Y}'|^{\,n\,-\,2}}$$

when $\mathcal{Y}$ is close to $\mathcal{Y}'$. The sign is the negative of the one used in [12]. Using (1.4) and (6.21), we obtain

$$(6.39)\quad \left|\,[\,A_1\,(\mathcal{Y})]^{\frac{4}{n\,-\,2}}\cdot\mathbf{W}_i\,(\mathcal{Y})\right|\ \le\ \left(\frac{1}{1+\mathcal{R}^2}\right)^2\ \le\ \frac{C}{1+\mathcal{R}^4}\qquad\text{for}\quad \mathcal{Y}\in\mathbb{R}^n\,.$$

Consider points $\mathcal{Y}_{out}$ so that

$$|\,\mathcal{Y}_{out}|\ \le\ (1\,-\,\delta)\cdot[\,c\,\lambda_i^{-1}]\,.$$

Via proportional property on the Green function (see §A.5 in the e-Appendix (cf. also p. 157 in [12]), we have

(6.40)
$$|\,G_i\,(\mathcal{Y},\ \mathcal{Y}_{out})|\le\left[C_1+\frac{C_2}{\delta^{\,n\,-\,2}}\right]\cdot\frac{1}{|\,y\,-\mathcal{Y}_{out}|^{\,n\,-\,2}}\qquad\text{for}\quad \mathcal{Y}\in B_o\,(c\,\lambda_i^{-1}\,)\setminus\{\mathcal{Y}_{out}\}\,,$$

$$(6.41)\qquad |\,\mathbf{n}\cdot\bigtriangledown G_i\,(\mathcal{Y},\ \mathcal{Y}_{out})|\ \le\ \frac{C_3}{\delta^{\,n}}\cdot\lambda_i^{n\,-\,1}\qquad\text{for}\quad \mathcal{Y}\,\in\,\partial B_o\,(c\,\lambda_i^{-1}\,)\,,$$

where $|\,\mathcal{Y}_{out}|\ \le\ (1\,-\,\delta)\cdot[\,c\,\lambda_i^{-1}\,]$. Here $C_1\,,\ C_2$ and C_3 are positive constant independent on i and δ. It follows from (6.33) and (6.41) that

$$(6.42)\qquad \left|\,\int_{\partial B_o(\,c\,\lambda_i^{-1})}[\,\mathbf{n}\cdot\bigtriangledown_{\mathcal{Y}}\,G_i(\mathcal{Y},\ \mathcal{Y}_{out})\,]\,\mathbf{W}_i\,(\mathcal{Y})\ dS_{\mathcal{Y}}\right|$$
$$\le\ \frac{C_2\cdot\varepsilon}{\delta^{\,n}}\cdot\lambda_i^{n\,-\,1}\cdot\int_{\partial B_o(\,c\,\lambda_i^{-1})}dS_{\mathcal{Y}}$$
$$\le\ \frac{C_2\cdot\varepsilon}{\delta^{\,n}}\cdot\lambda_i^{n\,-\,1}\cdot\left[\|\,S^{n\,-\,1}\|\cdot(c\,\lambda_i)^{n\,-\,1}\right]\ \le\ \frac{C\cdot\varepsilon}{\delta^n}\,.$$

Here we consider that C is independent on i. Putting the information into (6.38), together with (6.20) and (6.40), we obtain

(6.43)

$$
\begin{aligned}
|\,\mathbf{W}_i\,(\mathcal{Y}_{out})\,| \;&\le\; \left[C_1 + \frac{C_2}{\delta^{\,n-2}} \right] \cdot \int_{B_o(c\,\lambda_i^{-1})} \left(\frac{1}{|\,\mathcal{Y} - \mathcal{Y}_{out}|^{\,n-2}} \cdot \frac{1}{(1+|\,\mathcal{Y}\,|)^{\,4}} \right) dy \\
&\quad + \int_{B_o\,(1)} O\,(\lambda_i) + \frac{C\cdot\varepsilon}{\delta^n} \qquad (\text{from the harmonic term}) \\
&\le\; C_\delta \cdot \left[\frac{1}{(1+|\,\mathcal{Y}_{out}|\,)} + O\,(\lambda_i) + O\,(\varepsilon) \right]
\end{aligned}
$$

for $|\,\mathcal{Y}_{out}| \le (1-\delta)\,[\,c\,\lambda_i^{-1}\,]$. Refer to [8] for the estimation of the first integral in (6.43). Thus we can find $R_{\tilde\varepsilon} > 0$ and $I_{\tilde\varepsilon}$ such that for all $i \ge I_{\tilde\varepsilon}$, we have

$$
(6.44) \qquad |\,\mathbf{W}_i\,(\mathcal{Y})| \;\le\; \tilde\varepsilon \qquad \text{for} \qquad R_{\tilde\varepsilon} \;\le\; |\mathcal{Y}| \;\le\; (1-\delta)\,[\,c\,\lambda_i^{-1}\,]\,.
$$

6b.8. Further restriction on the location of the maximum − proof of (6.29). In view of (6.35), we actually have

$$
\mathcal{Y}_{\mu_i} \;\in\; \overline{B_o\,\left((1-\delta)\,[\,c\,\lambda_i^{-1}]\right)} \qquad\qquad (\text{cf. } (6.28))\,.
$$

Here $\delta > 0$ is chosen close to 0 (as explained in §6b.6). Arguing as in (6.43), we arrive at a similar conclusion:

$$
(6.45) \qquad 1 \;=\; |\,\mathbf{W}_i\,(\mathcal{Y}_{\mu_i})| \;\le\; C\left[\frac{1}{1+|\,\mathcal{Y}_{\mu_i}|} + O\,(\lambda_i) + O\,(\varepsilon) \right].
$$

It follows that there is a fixed positive number R_o such that

$$
|\,\mathcal{Y}_{\mu_i}| \;\le\; R_o \qquad \text{for} \quad i \gg 1\,.
$$

Hence we establish (6.29) and obtain a contradiction to (6.27). Thus (6.16) must be wrong. That is, (6.14) holds.

6c. Terms in the remainder R.H.S.$_i$ in (6.12). Here we verify (6.19) and (6.20). Recall that **RM** is decomposed into four components as expressed in (3.12).

6c.1. First term. Under the condition $|\,\xi_i| = o_{\lambda_i}(1)$, we have

(6.46)

$$
\begin{aligned}
&\Lambda_i^{-1} \cdot \left[\, |\,\xi_i|^{\,k} \cdot (\lambda_i\,|\,\mathcal{Y}|\,)^{\ell-k} \right] \times [\,A_1\,(\mathcal{Y})]^{\frac{n+2}{n-2}} \quad (1 \le k \le \ell;\;\; |\xi_i| = o\,(1)\cdot\lambda_i\,) \\
&\le \lambda_i^{-\ell} \cdot o\,(1) \cdot \lambda_i^{k} \cdot \lambda_i^{\ell-k} \times \mathcal{R}^{\,\ell-k} \cdot \left(\frac{1}{1+\mathcal{R}^{\,2}} \right)^{\frac{n+2}{2}} \qquad (\text{using } (6.16)\,;\;\; \mathcal{R} = |\,\mathcal{Y}|) \\
&\le o\,(1) \cdot (1+\mathcal{R})^{\,\ell-k} \cdot \left(\frac{1}{1+\mathcal{R}} \right)^{n+2} \;\le\; \frac{o\,(1)}{(1+\mathcal{R})^{\,5}} \qquad \text{for} \quad \mathcal{R} \;\le\; c\,\lambda_i^{-1} \\
&\to 0 \qquad \text{uniformly in} \quad B_o\,(\mathcal{R}_o) \qquad (\,i \gg 1\,;\;\; \ell \le n-2\,, \quad k \ge 1)\,.
\end{aligned}
$$

6c.2. Second term.

(6.47)

$$\Lambda_i^{-1} \cdot \left(\max_{B_o(\lambda_i \mathcal{Y} + \xi_i)} \| \bigtriangledown^{(\ell+1)} K \| \right) \cdot | \lambda_i \mathcal{Y} + \xi_i |^{\ell+1} \times [A_1(\mathcal{Y})]^{\frac{n+2}{n-2}}$$

$$\le C \lambda_i^{-\ell} \left| \lambda_i \left(\mathcal{Y} + \frac{\xi_i}{\lambda_i} \right) \right|^{\ell+1} \times \frac{1}{(1+\mathcal{R})^{n+2}} \le C\, \lambda_i \cdot (1+\mathcal{R})^{\ell+1} \cdot \frac{1}{(1+\mathcal{R})^{n+2}}$$

$$\text{(via (6.16); } |\xi_i| = o(\lambda_i), \ \ \ell \le n)$$

$$\le \frac{C \lambda_i}{(1+\mathcal{R})^3} \le \frac{C_1}{(1+\mathcal{R})^4} \qquad \text{for} \quad \mathcal{R} \le c \lambda_i^{-1} \ \ [\Longrightarrow \ \ \lambda_i (1+\mathcal{R}) \le C_2] \,.$$

6c.3. Third term. Using $a^2 = (a-b)^2 + 2b(a-b) + b^2$, we obtain

(6.48)

$$\Lambda_i^{-1} \cdot A_1^{\frac{4}{n-2}-1} (\mathcal{V}_i - A_1)^2 \qquad \text{(take } a = (\mathcal{V}_i - A_1), \ \ b = \lambda_i^\ell \cdot \Pi_{\mathbf{p}})$$

$$\le A_1^{\frac{4}{n-2}-1} \Lambda_i^{-1} \{ [\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_{\mathbf{p}}]^2 + 2 (\lambda_i^\ell \Pi_{\mathbf{p}}) \cdot | \mathcal{V}_i - A_1 - \lambda_i^\ell \Pi_{\mathbf{p}} | + (\lambda_i^\ell \cdot \Pi_{\mathbf{p}})^2 \}$$

$$\le \left(\frac{1}{1+\mathcal{R}^2} \right)^{\frac{6-n}{2}} \{ | (\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_{\mathbf{p}} | \cdot | \mathbf{W}_i + O_{\lambda_i}((n-2)-\ell) |$$

$$+ 2 | \lambda_i^\ell \cdot \Pi_{\mathbf{p}} | \cdot | \mathbf{W}_i + O_{\lambda_i}((n-2)-\ell) | + (\Lambda_i^{-1} \lambda_i^\ell) \cdot \lambda_i^\ell \cdot (\Pi_{\mathbf{p}})^2 \}$$

$$\text{(using (6.4), (6.16) and (6.17))}$$

$$\le \frac{C}{(1+\mathcal{R})^{6-n}} \cdot \{ | (\mathcal{V}_i(\mathcal{Y}) - A_1(\mathcal{Y}) - \lambda_i^\ell \cdot \Pi_{\mathbf{p}} | + \lambda_i^\ell | \Pi_{\mathbf{p}}(\mathcal{Y}) | + \lambda_i^\ell \cdot | \Pi_{\mathbf{p}}(\mathcal{Y}) |^2 \}$$

$$\text{(as } | \mathbf{W}_i + O_{\lambda_i}((n-2)-\ell) | \le | \mathbf{W}_i | + | O_{\lambda_i}((n-2)-\ell) | \le 1 + C \text{ when } \ell \le (n-2))$$

$$\longrightarrow 0 \ \text{ in } B_o(R_o) \text{ uniformly (from (2.59) : } | \mathcal{V}_i - A_1 | \to 0 \text{ in } B_o(R_o))$$

$$\le \frac{C_1}{(1+\mathcal{R})^{6-n}} \cdot \{ | \mathcal{V}_i(\mathcal{Y}) - A_1(\mathcal{Y}) | + \lambda_i^\ell | \Pi_{\mathbf{p}}(\mathcal{Y}) | + \lambda_i^\ell \cdot | \Pi_{\mathbf{p}}(\mathcal{Y}) |^2 \}$$

$$\le \frac{C_2}{(1+\mathcal{R})^{6-n}} \cdot \left\{ \frac{1}{(1+\mathcal{R})^{n-2}} + \frac{\lambda_i^\ell \cdot (1+\mathcal{R})^\ell}{(1+\mathcal{R})^n} + \frac{\lambda_i^\ell \cdot (1+\mathcal{R})^{2\ell}}{(1+\mathcal{R})^{2n}} \right\}$$

$$\text{(via (3.7) and Proposition 4.49)}$$

$$\le \frac{C}{(1+\mathcal{R})^4} \qquad \text{for} \quad \mathcal{R} \le c \lambda_i^{-1} \ \ (\Longrightarrow \ \ \lambda_i (1+\mathcal{R}) \le C_2) \,.$$

Here $\ell \le n-2$.

6c.4. Fourth term. From (3.12), we obtain

(6.49)

$$\begin{aligned}
&\mathbf{RM}_4\,(\mathcal{Y})\\
&= \Lambda_i^{-1}\cdot\Big\{\, O\Big(\max_{|\lambda_i\,\mathcal{Y}+\xi_i|\,\le\,\rho_o)}\|\,\bigtriangledown^{(\ell)}\ K\,\|\times|\,\lambda_i\,\mathcal{Y}+\xi_i|^{\,\ell}\Big)\Big\}\\
&\quad\times\Big[\,O\,(1)\,|\,\mathcal{V}_i\,-\,A_1\,|\times\max\Big\{\,\mathcal{V}_i^{\frac{4}{n-2}}\,,\ \ A_1^{\frac{4}{n-2}}\Big\}\Big]\\
&\le\ C\,\Lambda_i^{-1}\cdot|\,\lambda_i\,\mathcal{Y}+\xi_i|^{\,\ell}\cdot\Big[\,A_1^{\frac{4}{n-2}}\,|\,\mathcal{V}_i\,-\,A_1\,|\,\Big]\qquad\qquad(\text{cf.}\ \ (\text{A.1.3}))\\
&\quad\ \text{for}\quad|\,\mathcal{Y}|\ \le\ c\cdot\lambda_i^{-1}\qquad(\text{using Propositions 2.3 and 2.24})]\\
&\le\ C_1\,\Lambda_i^{-1}\,\lambda_i^{\ell}\cdot(\,1+\mathcal{R}\,)^{\ell}\Big[A_1^{\frac{4}{n-2}}\cdot\big(|\,\mathcal{V}_i\,-\,A_1\,-\,\lambda_i^{\ell}\,\Pi_{\mathbf{p}}\,|+\lambda_i^{\ell}\,|\,\Pi_{\mathbf{p}}\,|\,\big)\Big]\\
&\le\ C_2\,\lambda_i^{\ell}\cdot(\,1+\mathcal{R}\,)^{\,\ell}\ \times\ \frac{1}{(1+\mathcal{R})^{\,4}}\times|\,\mathbf{W}_i\ +\ O_{\lambda_i}((n\,-\,2)\,-\,\ell)|\\
&\quad+\ C_3\,(\Lambda_i^{-1}\,\lambda_i^{\ell}\,)\cdot(\,1+\mathcal{R}\,)^{\ell}\cdot\frac{1}{(1+\mathcal{R})^{\,4}}\cdot\frac{\lambda_i^{\ell}\,(1+\mathcal{R})^{\ell}}{(1+\mathcal{R})^{n}}\\
&\le\ \frac{C_4\,\lambda_i^{\ell}\cdot(1+\mathcal{R})^{\ell}}{(1+\mathcal{R})^{4}}\cdot\Big[\,1+\frac{1}{(1+\mathcal{R})^{n-\ell}}\Big]\ \longrightarrow\ 0\quad\text{uniformly}\quad\text{in}\quad B_o\,(R_o)\\
&\le\ \frac{C_5}{(1+\mathcal{R})^{\,4}}\qquad\quad\text{for}\qquad\mathcal{R}\,=\,|\,\mathcal{Y}|\,\le\,c\,\lambda_i^{-1}\qquad(\text{when}\ \ \ell\,\le\,n\,-\,2)\,.
\end{aligned}$$

6c.5. The inserted harmonic term. The last couple of terms to be considered in $\Lambda_i^{-1}\cdot\mathbf{R.H.S.}_{\cdot i}$ (cf. (6.12), (6.19) and (6.20)) are

(6.50)

$$\frac{1}{\Lambda_i}\cdot\left|\,\lambda_i^{n-2}\cdot\frac{h_o}{c\,\lambda_i^{-1}}\cdot\Delta_{\mathcal{Y}}\,\tilde{\mathcal{R}}\,(\mathcal{Y})\right|\ \le\ C\,\lambda_i\cdot\chi_{B_o\,(1)}\qquad(\text{via}\ \ (6.16)\ \ \text{and}\ \ \ell\,\le\,n-2)\,,$$

(6.51)

$$\begin{aligned}
&\frac{\lambda_i^{n-2}}{\Lambda_i}\cdot n\,(n+2)\,\frac{h_o}{c\,\lambda_i^{-1}}\cdot[\,A_1\,(\mathcal{Y})]^{\,\frac{4}{n-2}}\cdot\tilde{\mathcal{R}}\,(\mathcal{Y})\\
&\le C\,\lambda_i\cdot\chi_{B_o\,(1)}\ +\ C_1\cdot\lambda_i\cdot\frac{\mathcal{R}}{(1+\mathcal{R})^4}\ \ \longrightarrow\ 0\ \ \text{uniformly}\quad\text{in}\ \ B_o\,(R_o)\\
&\le C\,\lambda_i\cdot\chi_{B_o\,(1)}\ +\ C_2\cdot\frac{1}{(1+\mathcal{R})^4}\qquad\qquad\text{for}\quad\mathcal{R}=|\,\mathcal{Y}|\,\le\,c\,\lambda_i^{-1}\,,
\end{aligned}$$

(6.52)

$$\frac{\lambda_i^{n-2}}{\Lambda_i} \cdot n\,(n+2)\,[\,A_1\,(\mathcal{Y})]^{\frac{4}{n-2}} \cdot |\,\mathrm{H}_{\ge 1}\,(\mathcal{Y}) \;-\; h_o\,|$$

$$\le C \cdot \left(\frac{1}{1+\mathcal{R}^2}\right)^4 \times \sum_{j\ge 1} \left|\,\left[\,\frac{1}{|\,\lambda_i\,\mathcal{Y} \,-\, \hat{Y}_j\,|^{\,n-2}} \;-\; \frac{1}{|\,\hat{Y}_j\,|^{\,n-2}}\right]\,\right| \times \mathcal{A}_j$$

$\longrightarrow$ 0 uniformly for $\mathcal{Y} \in B_o\,(R_o)$

(recall (5.13) and (6.5), and observe also that $\lambda_i\,\mathcal{Y} \to 0$ for $|\mathcal{Y}| \;\le\; R_o$)

$$\le \frac{C}{(1+\mathcal{R})^4} \qquad\qquad \text{for} \quad \mathcal{R} \le c\,\lambda_i^{-1}\,.$$

Thus we estimate each term in the remainder and confirm the right orders in (6.19) and (6.20). Combining the above discussion, we obtain the following.

Theorem 6.53. *Under the conditions in Main Theorem* 1.14*, we have*
(6.54)

$$|\,\mathcal{D}_i^{\Pi}\,(\mathcal{Y})| \;=\; o_{\lambda_i}(\ell) \quad \textit{for} \quad |\mathcal{Y}| \;\le\; c\,\lambda_i^{-1} \quad \textit{and} \quad i \gg 1 \quad (\textit{modulo a subsequence}\,)\,.$$

6d. Proof of Main Theorem 1.14 – zooming out to the original scale. As in the transformation (2.17) (see also §2f), we note that, from the definitions of $\mathcal{V}_i$, A_1, and (6.2), estimate (6.54) in Theorem 6.53 leads to

$$(6.55)\qquad \left|\,\frac{v_i\,(\,\xi_i + \lambda_i\,\mathcal{Y})}{v_i\,(\xi_i)} \;-\; \left(\frac{1}{1+|\,\mathcal{Y}|^{\,2}}\right)^{\frac{n-2}{2}} \;-\; \lambda_i^{\ell} \cdot \frac{\Gamma_{\mathbf{p}}\,(\mathcal{Y})}{(1+|\,\mathcal{Y}|^{\,2})^{\frac{n}{2}}}\right.$$

$$\left. -\; \lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}\,(\mathcal{Y}) \;+\; \lambda_i^{n-2} \cdot h_o \cdot \left(1 \;-\; \frac{\tilde{\mathcal{R}}}{c\,\lambda_i^{-1}}\right)\,\right| \;=\; o\,(\lambda_i^{\ell}\,)$$

for $|\,\mathcal{Y}| \,\le\, c\,\lambda_i^{-1}$. Via the transformation $y = \lambda_i\,\mathcal{Y} \,+\, \xi_i$ and the definition $M_i :=$ $v_i\,(\xi_i)$, (6.55) is rewritten as

(6.56)

$$\left|\,v_i\,(y) \;-\; M_i \cdot \left(\frac{1}{1+\lambda_i^{-2}\,|\,y \,-\, \xi_i|^{\,2}}\right)^{\frac{n-2}{2}} \;-\; M_i \cdot \lambda_i^{\ell} \cdot \frac{\Gamma_{\mathbf{p}}\,(\lambda_i^{-1}\,(y \,-\, \xi_i))}{[\,1+\lambda_i^{-2}\,|\,y \,-\, \xi_i|^{\,2}\,]^{\frac{n}{2}}}\right.$$

$$\left. -\; M_i \cdot \lambda_i^{n-2} \cdot \mathrm{H}_{\ge 1}\,(\lambda_i^{-1}\,(y \,-\, \xi_i)) \;+\; M_i \cdot \lambda_i^{n-2} \cdot h_o \cdot \left(1 \;-\; \frac{\tilde{\mathcal{R}}\,(\lambda_i^{-1}\,(y \,-\, \xi_i))}{c\,\lambda_i^{-1}}\right)\,\right|$$

$$=\; M_i \cdot o\,(\lambda_i^{\ell}\,)$$

for $|\,y| \;=\; |\,\lambda_i\,\mathcal{Y} + \xi_i| \;\le\; c \,-\, o\,(1) \;\; (i \gg 1)$. Recall that $M_i \;=\; \lambda_i^{-\frac{n-2}{2}}$ (see (1.10), (1.11), (2.17) and §2f) and also the form of $\mathrm{H}_{\ge 1}$ as expressed in (5.13).

Hence we come to the conclusion that

(6.57)

$$\begin{aligned}&\Bigg|\, v_i\,(y) - \left(\frac{\lambda_i}{\lambda_i^2 \,+\, |\,y \,-\, \xi_i|^2}\right)^{\frac{n\,-\,2}{2}} - \left[\lambda_i^{\ell\,+1}\cdot\Gamma_{\mathbf{p}}\left(\frac{y\,-\,\xi_i}{\lambda_i}\right)\right]\cdot\left(\frac{\lambda_i}{\lambda_i^2 + |\,y\,-\,\xi_i|^2}\right)^{\frac{n}{2}}\\ &-\ \lambda_i^{\frac{n\,-\,2}{2}}\left[\sum_{j\,\ge\,1}\left(\frac{\mathcal{A}_j}{|\,(y\,-\,\xi_i)\,-\,\hat{Y}_j|^{\,n\,-\,2}}\ -\ \frac{\mathcal{A}_j}{|\,\hat{Y}_j|^{\,n\,-\,2}}\right) \,+\, \lambda_i\cdot\frac{h_o\cdot\tilde{\mathcal{R}}\,(\mathcal{Y})}{c}\right]\Bigg|\\ &=\ o_{\lambda_i}\left(\ell\ -\ \frac{n-2}{2}\right) \qquad\qquad \text{for} \quad |\,y\,|\,\le\,\rho_2 \qquad\qquad (i\,\gg\,1)\,.\end{aligned}$$

Here $\rho_2 > 0$ is a number slightly less than c. With (1.4) and

(6.58)

$$\begin{aligned}&O_{\mathrm{H}}\left(\lambda_i^{\frac{n\,-\,2}{2}}\right)\\ &\qquad := \ \lambda_i^{\frac{n\,-\,2}{2}}\left[\sum_{j\,\ge\,1}\left(\frac{\mathcal{A}_j}{|\,(y\,-\,\xi_i)\,-\,\hat{Y}_j|^{\,n\,-\,2}}\ -\ \frac{\mathcal{A}_j}{|\,\hat{Y}_j|^{\,n\,-\,2}}\right) + \lambda_i\cdot\frac{h_o\cdot\tilde{\mathcal{R}}\,(\mathcal{Y})}{c}\right],\end{aligned}$$

we arrive at (1.20). (As usual, $\mathcal{Y} \,=\, \lambda_i^{-1}\,(\,y\,-\,\xi_i)$.)

An **e-Appendix** is available at: `https://arxiv.org/pdf/1707.02401.pdf` (pp. 44–83) and from `https://doi.org/10.1090/tran/6983` (Supplementary appendix).

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