HAMILTONIAN GROUP ACTIONS ON EXACT SYMPLECTIC MANIFOLDS WITH PROPER MOMENTUM MAPS ARE STANDARD

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ABSTRACT. We give a complete characterization of Hamiltonian actions of compact Lie groups on exact symplectic manifolds with proper momentum maps. We deduce that every Hamiltonian action of a compact Lie group on a contractible symplectic manifold with a proper momentum map is globally linearizable.

1. The main result

Let G be a compact Lie group with Lie algebra $\mathfrak{g} = T_e G$ and dual space $\mathfrak{g}^* = T_e^* G$. A **momentum map** for a symplectic G action on a symplectic manifold (M, ω) is a map $\mu: M \to \mathfrak{g}^*$ that intertwines the G action on M with the coadjoint G action on $\mathfrak{g}^*$ and that satisfies Hamilton's equation

$$(1.1) \qquad d\mu^\xi = -\iota(\xi_M)\omega \qquad \text{for all } \xi \in \mathfrak{g},$$

where ξ_M is the vector field on M that corresponds to ξ and where $\mu^\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$ is the ξth component of μ. A symplectic G action is called **Hamiltonian** if it admits a momentum map. A **Hamiltonian G manifold** is a triple (M, ω, μ) where (M, ω) is a symplectic manifold with a symplectic G action and $\mu: M \to \mathfrak{g}^*$ is a momentum map. An **isomorphism** of Hamiltonian G manifolds is an equivariant symplectomorphism that intertwines the momentum maps. Throughout this paper, manifolds are assumed to be non-empty unless stated otherwise.

1.2. *Remark.* This terminology is not used consistently in the literature. We emphasize that throughout this paper we require momentum maps to be equivariant and we require the two-forms of Hamiltonian G manifolds to be non-degenerate.

1.3. *Remark.* A symplectic G action on a symplectic manifold (M, ω) is the same thing as a homomorphism from G to the group $\text{Symp}(M, \omega)$ of symplectomorphisms of (M, ω) that is smooth in the diffeological sense: the map $(g, x) \mapsto g \cdot x$ from $G \times M$ to M is smooth (cf. [8]). The action has a momentum map if and only if the image of the identity component of G is contained in the subgroup $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. (Being contained in $\text{Ham}(M, \omega)$ implies that there is a map $\mu: M \to \mathfrak{g}^*$ that satisfies Hamilton's equation (1.1). The map $x \mapsto \int_{a \in G} \text{Ad}^*(a)(\mu(a^{-1}x))da$, where da is the Haar probability measure on G, is then a momentum map.) It is natural to require the image of all of G, not only the identity

component, to be contained in $\operatorname{Ham}(M, \omega)$, but we will not need this stronger requirement.

We recall a particular construction of Hamiltonian G manifolds:

1.4. Definition. Let G be a compact Lie group with Lie algebra $\mathfrak{g} = T_e G$ and dual space $\mathfrak{g}^* = T_e^* G$. A **centred Hamiltonian G-model** is a Hamiltonian G manifold (Y, ω_Y, μ_Y) that is obtained by the following construction.

> Let H be a closed subgroup of G. Let (V, ω_V) be a symplectic vector space with a linear H action and quadratic momentum map $\mu_V \colon V \to \mathfrak{h}^*$. Consider the H action on T^*G that is induced from its action $h \colon a \mapsto ah^{-1}$ on G, with the fibrewise homogeneous momentum map. The model (Y, ω_Y) is the symplectic quotient at zero of $T^*G \times V$ with respect to the anti-diagonal H action. The G action and momentum map on Y are induced from the left action $g \colon a \mapsto ga$ on G and its homogeneous momentum map on T^*G.

We describe centred Hamiltonian G models in greater detail in Section 3. The purpose of this paper is to prove the following theorem.

1.5. Theorem. *Let a compact Lie group G act on a symplectic manifold (M, ω) with momentum map $\mu \colon M \to \mathfrak{g}^*$. Assume that M/G is connected. Assume that μ is proper and ω is exact. Then M is equivariantly symplectomorphic to a centred Hamiltonian G-model.*

We prove Theorem 1.5 in Section 6. Here are some implications on the topology, equivariant topology, and symplectic topology of the manifold.

1.6. Corollary. *Let a compact Lie group G act on a symplectic manifold (M, ω) with momentum map $\mu \colon M \to \mathfrak{g}^*$. Assume that M/G is connected. Assume that μ is proper and ω is exact. Then the following results hold.*

  (i) *The Euler characteristic of M is non-negative.*
 (ii) *M contains either no G fixed points or exactly one G fixed point.*
 (iii) *The Gromov width of (M, ω) is infinite.*

Here are two situations in which we get even more precise information.

1.7. Corollary. *Let the circle group S^1 act faithfully on a connected exact symplectic manifold with a proper momentum map. Then the manifold is equivariantly symplectomorphic either to the cylinder $S^1 \times \mathbb{R}$ with S^1 acting by rotations or to $\mathbb{C}^n$ with S^1 acting linearly.*

1.8. Corollary. *Let a compact Lie group G act on a symplectic manifold (M, ω) with a momentum map $\mu \colon M \to \mathfrak{g}^*$. Suppose that M is contractible and μ is proper. Then (M, ω) is equivariantly symplectomorphic to a symplectic vector space with a linear symplectic G action. In particular, a compact Lie group action on $\mathbb{R}^{2n}$ with proper momentum map is always linearizable.*

Proofs of Corollaries 1.6, 1.7, *and* 1.8. These corollaries follow from Theorem 1.5 and from properties of centred Hamiltonian G models that we list in Section 7. For Corollary 1.6, see Lemmas 7.2, 7.3, and 7.4. For Corollary 1.7, Lemma 7.5. For Corollary 1.8, note that ω is exact if M is contractible, and see Lemma 7.6. $\qquad \square$

1.9. **Example.** None of the following exact symplectic manifolds admit an action of a compact Lie group G with a proper momentum map. In particular, none of them admit a proper real valued function whose Hamiltonian flow is periodic.

(a) The cotangent bundle of any manifold with negative Euler characteristic, for example, T^*X where X is a closed oriented connected surface of genus ≥ 2 or the product of such a surface with a closed connected manifold of positive Euler characteristic.

(b) Any open subset of $\mathbb{R}^{2n}$, with coordinates $x_1, y_1, \ldots, x_n, y_n$ and the standard symplectic structure $\sum_{j=1}^n dx_j \wedge dy_j$, on which x_1 and y_1 are bounded.

This follows from Corollary 1.6: manifolds of the first form have a negative Euler characteristic, and manifolds of the second form have a finite Gromov width [6].

Properness of the momentum map cannot be dropped. Without this assumption, trivial actions provide counterexamples to the conclusions of Theorem 1.5 and of Corollaries 1.6 and 1.8. In Section 8 we give more interesting examples, of exotic Hamiltonian actions on symplectic vector spaces. For example, for any compact connected non-abelian Lie group G, there exists a symplectic vector space $\mathbb{R}^{2n}$ with a Hamiltonian G action that is not isomorphic to a linear action. Thus, the conclusion of Corollary 1.8 fails to hold in this situation. See Corollary 8.3.

Exactness of the symplectic form cannot be dropped. Without this assumption, actions on closed symplectic manifolds provide counterexamples to the conclusions of Theorem 1.5 and of Corollaries 1.6 and 1.7.

The strategy of the proof of Theorem 1.5 is as follows. After shifting the momentum map by a constant, we may assume that the momentum map is obtained from an invariant primitive of ω as described in Section 2. (Here we use our hypothesis that ω is exact.) Referring to Reyer Sjamaar's de Rham theory for singular symplectic quotients [26], we argue that ω descends to an exact symplectic form on $\mu^{-1}(0)/G$ and conclude (by compactness) that this quotient is a finite set. So $\mu^{-1}(0)$ is a finite union of G-orbits. By the local normal form theorem there exists a G-equivariant symplectomorphism F between some neighbourhood U of $\mu^{-1}(0)$ and an open subset V in a finite union $\bigsqcup Y_j$ of centred Hamiltonian G-models Y_j. The negative Liouville flows on M and on Y_j shrink any compact set into U and V, respectively. (The Liouville vector field X_M on M comes from our invariant primitive of ω. We use that μ intertwines X_M with the Euler vector field on $\mathfrak{g}^*$; see Lemma 5.6.) Combining these flows with F, we obtain a G-equivariant symplectomorphism between M and $\bigsqcup Y_j$. If M/G is non-empty and connected, there is exactly one Y_j.

In the literature, there are many classification results for Hamiltonian group actions whose complexity—half the dimension of a generic non-empty reduced space—is low. See [5, 12] (symplectic toric manifolds, even without compactness or properness), [23] (compact symplectic 2-manifolds, no action), [11] (Hamiltonian circle actions on compact symplectic four-manifolds), [15] ("tall" complexity one Hamiltonian torus actions, proper momentum maps), [17, 20] (complexity zero non-abelian group actions on compact symplectic manifolds), [2] (complexity one non-abelian group actions on compact symplectic six-manifolds). Additional references are listed in the introduction to [15].

What is special about our Theorem 1.5 is that it gives a situation in which we can characterize Hamiltonian G manifolds of *arbitrary* complexity. This work follows

two earlier results that apply to actions of arbitrary complexity:

    (i) Delzant [5, Section 1] proved that a Hamiltonian circle action on a compact symplectic $2n$-manifold whose fixed point set has exactly two connected components, of which one is an isolated fixed point, is equivariantly symplectomorphic to a standard circle action on $\mathbb{CP}^n$.

    (ii) Let a torus T act on a compact symplectic manifold (M, ω) with a momentum map $\mu \colon M \to \mathfrak{t}^*$. Let $\mathcal{T}$ be a convex open subset of $\mathfrak{t}^*$ that contains $\mu(M)$ and such that μ is proper as a map to $\mathcal{T}$. Let $x \in M$ be a point; suppose that $\mu^{-1}(\mu(x)) = \{x\}$ and that $\mu(x)$ is contained in the momentum image of every component of the fixed point set M^K for every subgroup K of T. Then M is equivariantly symplectomorphic to the open subset $\{z \in \mathbb{C}^n \mid \mu(x) + \pi \sum_j |z_j|^2 \eta_j \in \mathcal{T}\}$ of the standard $\mathbb{C}^n$, where $\eta_1, \ldots, \eta_n$ are the isotropy weights at p, and where our normalization convention is such that we identify the circle with $\mathbb{R}/\mathbb{Z}$ (rather than $\mathbb{R}/2\pi\mathbb{Z}$). This is proved in [15, Section 2].

We end this section with a discussion of the adjective "centred".

1.10. *Remark.* Let (M, ω, μ) be a Hamiltonian G manifold. Suppose that μ is proper as a map to some convex open subset $\mathcal{T}$ of $\mathfrak{g}^*$. Let $\alpha \in \mathfrak{g}^*$ be a point that is fixed under the coadjoint action. We say that the Hamiltonian G manifold is **centred** about α if, for every subgroup K of G, the point α is contained in the momentum map image of every component of the fixed point set M^K.

    (i) Every centred Hamiltonian G model whose momentum map is proper (to $\mathcal{T} := \mathfrak{g}^*$) is centred about $\alpha = 0$ according to this definition.

    (ii) This definition of "centred" is consistent with the notion of "centred" that was introduced in [13, Def. 1.4] and used in [14]. The only difference is that in [13, 14] we restricted to the special case that G is a torus and M is connected and we fixed a choice of $\mathcal{T}$. (The definition in [13, Def. 1.4] is phrased slightly differently from that of [14, Def. 2.2], but it is equivalent to it. See [14, Rmk. 2.7].)

2. Exact Hamiltonian G manifolds

A symplectic manifold (M, ω) is called **exact** if the symplectic form ω is exact, that is, if it has a primitive: a one-form λ such that $\omega = d\lambda$. If an exact form ω is preserved under the action of a compact Lie group G, then, by averaging, we can choose its primitive λ to be G invariant. In this situation, consider the map

$$\mu \colon M \to \mathfrak{g}^*$$

such that, for every $\xi \in \mathfrak{g}^*$, the component

$$\mu^\xi := \langle \mu, \xi \rangle \colon M \to \mathbb{R}$$

is given by plugging the corresponding vector field ξ_M into the one-form λ:

$$(2.1) \qquad\qquad\qquad \mu^\xi = \iota(\xi_M)\lambda.$$

This map is a momentum map for the G-action on (M, ω). (Indeed, by Cartan's formula $\mathcal{L}_{\xi_M}\lambda = (d\iota(\xi_M) + \iota(\xi_M)d)\,\lambda$. The right-hand side is $d\mu^\xi + \iota(\xi_M)\omega$, and the left-hand side is zero.) A Hamiltonian G manifold (M, ω, μ) is **exact** if μ is obtained in this way from an invariant primitive of ω.

The following lemma is used in the proof of Lemma 3.4. A variant of it for singular reduced spaces appears in Lemma 4.2.

2.2. **Lemma** (Reduction of exact symplectic manifolds). *Let G be a compact Lie group and let (M, ω, μ) be an exact Hamiltonian G manifold. Suppose that the G action on the level set $Z := \mu^{-1}(0)$ is free. Consider the inclusion-quotient diagram:*

$$
\begin{array}{ccc}
Z & \overset{i}{\longhookrightarrow} & M \\
{\scriptstyle\pi}\big\downarrow & & \\
M_{red} & &
\end{array}
$$

Then

    (i) *there exist unique smooth manifold structures on Z and M_{red} such that i is an immersion and π is a submersion;*

    (ii) *there exist a unique one-form λ_{red} and a unique two-form ω_{red} on M_{red} such that*

$$\pi^*\lambda_{red} = i^*\lambda \qquad and \qquad \pi^*\omega_{red} = i^*\omega;$$

    (iii) *the two-form ω_{red} on M_{red} is symplectic, and $\omega_{red} = d\lambda_{red}$.*

Proof. The only difference from the standard Marsden-Weinstein reduction [22] is that we also obtain a reduction of the one-form λ and not only of the two-form ω. Indeed, the one-form $i^*\lambda$ on Z is G invariant because λ is G-invariant, and it is G-basic (that is, descends to M_{red}) because, additionally, $\iota_{\xi_M} i^*\lambda = \mu^\xi \circ i = 0$ for all $\xi \in \mathfrak{g}$. $\qquad\square$

3. Centred Hamiltonian G models

In this section we describe centred Hamiltonian G models (Y, ω_Y, μ_Y) with more detail than in Definition 1.4, and on every such model we specify a particular G-invariant one-form λ_Y such that $d\lambda_Y = \omega_Y$ and whose corresponding momentum map is μ_Y.

The construction involves three ingredients: a compact Lie group G, a closed subgroup $H \subset G$, and a linear symplectic representation of H on a symplectic vector space (V, ω_V) with a quadratic momentum map $\mu_V \colon V \to \mathfrak{h}^*$.

We begin with some notation. We denote an element of the cotangent bundle T^*G by either β or (a, β), interchangeably, where $\beta \in T_a^*G$. For $g \in G$, we denote by $(a, \beta) \mapsto (ga, g\beta)$ the lifting to T^*G of the diffeomorphism $a \mapsto ga$ of G. Similarly, we denote by $(a, \beta) \mapsto (ah, \beta h)$ the lifting to T^*G of the diffeomorphism $a \mapsto ah$ of G. (That is, if $L_{g'} \colon G \to G$ and $R_{h'} \colon G \to G$ denote, respectively, the left and right translation maps $a \mapsto g'a$ and $a \mapsto ah'$, then $g\beta$ and βh are the images of β under the linear maps $L_{g^{-1}}^* \colon T_a^*G \to T_{ga}^*G$ and $R_{h^{-1}}^* \colon T_a^*G \to T_{ah}^*G$.) These transformations commute, so expressions such as $g\beta h$ are well defined. With this notation, the left invariant trivialization of the cotangent bundle carries $(a, \varphi) \in G \times \mathfrak{g}^*$ to $(a, a\varphi) \in T^*G$, and the coadjoint action of $a \in G$ on $\mathfrak{g}^* = T_e^*G$ is $\mathrm{Ad}^*(a) \colon \varphi \mapsto a\varphi a^{-1}$.

Let λ_{taut} be the tautological one-form on T^*G, so that $d\lambda_{\mathrm{taut}} = \omega_{T^*G}$ is the canonical symplectic form. The G action on T^*G that is induced from the action $g \mapsto L_g$ on G has the momentum map

$$\mu_L \colon T^*G \to \mathfrak{g}^*; \qquad (a, a\varphi) \mapsto \mathrm{Ad}^*(a)(\varphi).$$

The H action on T^*G that is induced from the action $h \mapsto R_{h^{-1}}$ on G has the momentum map

$$\mu_R \colon T^*G \to \mathfrak{h}^*; \qquad (a, a\varphi) \mapsto -\varphi|_{\mathfrak{h}},$$

where $\varphi \mapsto \varphi|_{\mathfrak{h}}$ is the natural map $\mathfrak{g}^* \to \mathfrak{h}^*$. Moreover, the momentum maps that are obtained from λ_{taut} vanish along the zero section, so they coincide with the momentum maps μ_L and μ_R.

Before proceeding, we recall the definition of the Euler vector field.

3.1. Definition.
The **Euler vector field** on a vector space W is the vector field that, under the natural trivialization of TW, is given by the identity map. Equivalently, it is the velocity vector field of the flow $t \mapsto e^t v$ on W.

Note that if X_V is the Euler vector field on a vector space V and α is a differential form on V with constant coefficients, then $\mathcal{L}_{X_V} \alpha = k\alpha$ where k is the degree of α.

We now turn to the symplectic vector space (V, ω_V) with the H action and with the quadratic momentum map $\mu_V \colon V \to \mathfrak{h}^*$. Let $\lambda_V = \frac{1}{2}\iota(X_V)\omega_V$, where X_V is the Euler vector field on V. Then λ_V is an H-invariant one-form on V, and $d\lambda_V = \omega_V$. (The latter follows from Cartan's formula, closedness of ω_V, and the identity $\mathcal{L}_{X_V}\omega_V = 2\omega_V$.) Moreover, the momentum map that is obtained from λ_V vanishes at the origin, so it coincides with the quadratic momentum map μ_V.

We now consider the product $T^*G \times V$, whose elements we write as $(a, a\varphi, v)$ with $a \in G$, $\varphi \in \mathfrak{g}^*$, and $v \in V$. We have the left G action $\tilde{L}_g \colon (a, a\varphi, v) \mapsto (ga, ga\varphi, v)$, with the momentum map

$$(3.2) \qquad \tilde{\mu}_L \colon T^*G \times V \to \mathfrak{g}^*; \qquad (a, a\varphi, v) \mapsto \mathrm{Ad}^*(a)(\varphi),$$

and the anti-diagonal H action $D_h \colon (a, a\varphi, v) \mapsto (ah^{-1}, a\varphi h^{-1}, h \cdot v)$, with the momentum map

$$(3.3) \qquad \tilde{\mu}_D \colon T^*G \times V \to \mathfrak{h}^*; \qquad (a, a\varphi, v) \mapsto -\varphi|_{\mathfrak{h}} + \mu_V(v).$$

The left G action and anti-diagonal H actions commute. The H momentum map $\tilde{\mu}_D$ is invariant with respect to the G action, and the G momentum map $\tilde{\mu}_L$ is invariant with respect to the anti-diagonal H action.

In Definition 1.4, we defined the centred Hamiltonian G model (Y, ω_Y) that is constructed from this data to be the symplectic reduction of $T^*G \times V$ with respect to the H action. Explicitly,

$$Y = (T^*G \times V)/\!/H := Z/H,$$

where

$$Z := \tilde{\mu}_D^{-1}(0) \left(= \left\{ (a, a\varphi, v) \mid \varphi(\xi) = \mu_V^\xi(v) \text{ for all } \xi \in \mathfrak{h} \right\} \right).$$

3.4. Lemma.

(1) *There exist unique manifold structures on Z and on Y such that the inclusion map $i \colon Z \to T^*G \times V$ is an immersion and the quotient map $\pi \colon Z \to Y$ is a submersion.*

(2) *There exist a unique one-form λ_Y and a unique two-form ω_Y on Y such that $\pi^*\lambda_Y = i^*(\lambda_{taut} \oplus \lambda_V)$ and $\pi^*\omega_Y = i^*(\omega_{T^*G} \oplus \omega_V)$.*

(3) *The two-form ω_Y is symplectic, and $d\lambda_Y = \omega_Y$.*

Moreover,

(4) *The left G action on $T^*G \times V$ descends to a G action on the model Y.*

(5) *The G momentum map $\tilde{\mu}_L \colon T^*G \times V \to \mathfrak{g}^*$ descends to a G momentum map*

$$\mu_Y \colon Y \to \mathfrak{g}^*.$$

(6) *The G momentum map μ_Y coincides with the momentum map that is obtained from the one-form λ_Y.*

Proof. For items (1)–(3), note that H acts freely on the zero level set Z, and apply Lemma 2.2. Item (4) follows from the fact that, on $T^*G \times V$, the left G action preserves the H momentum map $\tilde{\mu}_D$ and commutes with the anti-diagonal H action. Item (5) follows from the fact that $\tilde{\mu}_L$ is H invariant. Item (6) follows from the fact that, on $T^*G \times V$, the G momentum map $\tilde{\mu}_L$ coincides with the momentum map that is obtained from the one-form $\lambda_{\text{taut}} \oplus \lambda_V$. $\qquad\square$

We now record a property of centred models that we use in Lemma 6.1 below.

3.5. Definition. The **central orbit** in a centred Hamiltonian G-model $Y = (T^*G \times V)//H$ is the image in Y of (the zero section of T^*G)×(the origin of V).

3.6. Lemma. *Let (Y, ω_Y, μ_Y) be a centred Hamiltonian G model. Suppose that there exists a neighbourhood U' of the central orbit in Y such that $U' \cap \mu_Y^{-1}(0)$ consists of only the central orbit. Then $\mu_Y^{-1}(0)$ consists of only the central orbit, and the map $\mu_Y \colon Y \to \mathfrak{g}^*$ is proper.*

Proof. In the notation of (3.2) and (3.3), $\mu_Y^{-1}(0)$ is the quotient of $\tilde{\mu}_L^{-1}(0) \cap \tilde{\mu}_D^{-1}(0)$ by the anti-diagonal H action. Because $\tilde{\mu}_L^{-1}(0) \cap \tilde{\mu}_D^{-1}(0) =$ (the zero section in T^*G) × $\mu_V^{-1}(0)$ and μ_V is quadratic, we deduce that $\mu_Y^{-1}(0)/G \left(\cong \mu_V^{-1}(0)/H \right)$ is connected.

By assumption, the central orbit is an isolated point in $\mu_Y^{-1}(0)/G$. This implies that $\mu_Y^{-1}(0)$ consists of only the central orbit.

In particular, $\mu_V^{-1}(0) = \{0\}$. So $m := \min \left\{ \|\mu_V(v)\| \mid \|v\| = 1 \right\}$ (with respect to any norm on V) is positive. By homogeneity, $\|\mu_V(v)\| \geq m\|v\|^2$ for all v. This implies that $\mu_V \colon V \to \mathfrak{h}^*$ is proper. From this and the compactness of G we further deduce that $(\tilde{\mu}_L, \tilde{\mu}_D) \colon T^*G \times V \to \mathfrak{g}^* \times \mathfrak{h}^*$, and hence $\mu_Y \colon Y \to \mathfrak{g}^*$, is proper. $\quad\square$

We end this section with an alternative description of the model Y that we use in Section 7.

Fix an Ad-invariant inner product on $\mathfrak{g}$, and use it to embed $\mathfrak{h}^*$ in $\mathfrak{g}^*$ as the orthogonal complement of $\mathfrak{h}^0$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Consider the action of $h \in H$ on $G \times (\mathfrak{h}^0 \oplus V)$ by right multiplication by h^{-1} on the G factor, by the coadjoint action on the $\mathfrak{h}^0$ factor, and by the given representation on the V factor. Our alternative model is the quotient

$$G \times_H \left(\mathfrak{h}^0 \times V \right),$$

with G acting by left multiplication on the G factor. We have a pullback diagram:

$$
\begin{array}{ccccc}
G \times (\mathfrak{h}^0 \times V) & \longrightarrow & \tilde{\mu}_D^{-1}(0) & & \subset T^*G \times V \\
\downarrow & & \downarrow & & \\
G \times_H (\mathfrak{h}^0 \times V) & \longrightarrow & (T^*G \times V)//H & & = Y.
\end{array}
$$

The bottom right term is the model Y. The top arrow is the map

(3.7) $\qquad\qquad\qquad (a, \nu, v) \mapsto (a, \mathrm{Ad}^*(a)(\nu + \mu_V(v)), v).$

The bottom arrow is a G equivariant diffeomorphism; pulling back ω_Y by it, we get a G invariant symplectic form on $G \times_H (\mathfrak{h}^0 \oplus V)$ whose pullback to $G \times (\mathfrak{h}^0 \oplus V)$ coincides with the pullback of $\omega_{T^*G} \oplus \omega_V$ under the top arrow.

3.8. Lemma.

(1) *The pullback of this symplectic form on $G \times_H (\mathfrak{h}^0 \times V)$ under the inclusion map $v \mapsto [1, 0, v]$ of V is ω_V.*
(2) *The zero section $G \times_H \{0\}$ of $G \times_H (\mathfrak{h}^0 \times V)$ is isotropic.*
(3) *The momentum map for the G-action on $G \times_H (\mathfrak{h}^0 \times V)$ with respect to the symplectic form described above is*

$$[a, \nu, v] \mapsto \mathrm{Ad}^*(a)(\nu + \mu_V(v)).$$

Proof. To prove item (1), observe that this pullback form agrees with the pullback of $\omega_{T^*G} \oplus \omega_V$ under the map $V \ni v \mapsto (1, \mu_V(v), v) \in T^*G \times V$. This pullback equals ω_V, since the fiber T_1^*G is Lagrangian in T^*G. This proves (1).

Item (2) follows from the definition of the symplectic form. Item (3) follows from the fact that this momentum map is the composition of μ_Y (defined by (3.2) and Lemma 3.4(5)) with the map induced by (3.7).   □

The notion of a centred Hamiltonian G model comes from the local normal form theorem. (A special case of) the local normal form theorem can then be stated as follows.

3.9. Theorem. *Let a compact Lie group G act on a symplectic manifold (M, ω) with a (G-equivariant) momentum map $\mu \colon M \to \mathfrak{g}^*$, and let $G \cdot x$ be an orbit that is contained in the zero level set $\mu^{-1}(0)$. Then there exist a centred Hamiltonian G-model Y and a G-equivariant symplectomorphism from an invariant open neighbourhood of $G \cdot x$ in M to an invariant open subset in Y that takes $G \cdot x$ to the central orbit in Y.*

The local normal form theorem is the main ingredient in the work of Sjamaar-Lerman [25] and in Sjamaar's de Rham theory for symplectic quotients [26] (see Section 4). The theorem in its more general form applies to neighbourhoods of orbits that are not necessarily in the zero level set $\mu^{-1}(0)$ and, unless the orbit is isotropic, it involves Hamiltonian G models that are more general than the centred Hamiltonian G models of Definition 1.4. The theorem, due to Guillemin-Sternberg and Marle, can be found in [7, 21]. We use the local normal form theorem in Section 4 and in the proof of Lemma 6.1.

4. THE ZERO LEVEL SET OF THE MOMENTUM MAP
AND SJAMAAR'S DE RHAM THEORY FOR SYMPLECTIC QUOTIENTS

The purpose of this section is to prove Proposition 4.3, which is later used in the proof of Lemma 6.1.

We will use a de Rham theory for singular symplectic quotients that was introduced by Sjamaar in [26]. To explain it, we first recall some facts from the paper [25] of Sjamaar and Lerman; specifically, see [25, Theorems 2.1 and 5.9].

Let G be a compact Lie group, and let (M, ω, μ) be a Hamiltonian G manifold.

The reduced space $\mu^{-1}(0)/G$ is locally connected; this is a consequence of the local normal form theorem (Theorem 3.9). So the connected components of the reduced space are open and closed in the reduced space, and their preimages in M are closed in M.

Let $x \in \mu^{-1}(0)/G$. There exist a conjugacy class C of closed subgroups of G and a neighbourhood U of x in $\mu^{-1}(0)/G$, such that the set of orbits in U with stabilizers in C is open and dense in U. (We say that a G orbit has stabilizers in C if the stabilizers of the points in the orbit are in C. If this is true for one point in the orbit, then it is true for all the points in the orbit.)

Fix a connected component X of $\mu^{-1}(0)/G$.

It follows that there exists a unique conjugacy class C_X of closed subgroups of G such that the "principal stratum"

$$X_{\mathrm{princ}} := \{x \in X \mid x \text{ has stabilizers in } C_X\}$$

is open and dense in X. Moreover, X_{princ} is a manifold, in the following sense. Let Z_{princ} be the preimage of X_{princ} in $\mu^{-1}(0)$. Then there exist unique manifold structures on X_{princ} and on Z_{princ} such that the inclusion map $Z_{\mathrm{princ}} \to M$ is an immersion and such that the projection map $Z_{\mathrm{princ}} \to X_{\mathrm{princ}}$ is a submersion.

The **dimension** of X is the dimension of X_{princ}.

Following Sjamaar [26], we define a **differential form on** X to be a differential form on X_{princ} whose pullback to Z_{princ} extends to a differential form on M. (Because the preimage Z of X in $\mu^{-1}(0)$ is closed in M, a differential form on Z_{princ} extends to a differential form to M if and only if every point on Z has a neighbourhood on which this differential form extends.)

The space of differential forms on X is a differential graded algebra, with the usual operations of exterior derivative and wedge products. We denote it $\Omega(X)$.

The symplectic form ω on M descends to a two-form ω_X on X. When Z is a manifold and the G action on Z is free, this is exactly the Marsden-Weinstein reduction.

4.1. Lemma. *Let G be a compact Lie group, let (M, ω, μ) be a Hamiltonian G manifold, and let X be a connected component of $\mu^{-1}(0)/G$. If X is compact and of dimension ≥ 2, then the two-form ω_X is not exact in the differential complex $\Omega(X)$.*

Proof. Let $k = \frac{1}{2}\dim X$. Assume that $k \geq 1$. By [26, Cor. 7.6], the class of ω_X^k in the cohomology of the differential complex $\Omega(X)$ is non-zero. So ω_{red}^k, and hence ω_{red}, is not exact. $\square$

4.2. Lemma. *Let G be a compact Lie group, let (M, ω, μ) be an exact Hamiltonian G manifold, and let X be a connected component of $\mu^{-1}(0)/G$. Then the reduced form ω_X is exact in $\Omega(X)$.*

Proof. Let λ be a G invariant one-form on M such that $\omega = d\lambda$ and such that $\mu^\xi = \lambda(\xi_M)$ for all $\xi \in \mathfrak{g}$. Let Z_{princ} be the preimage of X_{princ} in $\mu^{-1}(0)$. Because $\mu = 0$ on Z_{princ}, the pullback of the one-form λ to Z_{princ} is horizontal and therefore G-basic. Hence this pullback descends to a one-form λ_X on X. Because $\omega = d\lambda$ on M, we have $\omega_X = d\lambda_X$ on X. $\square$

4.3. Proposition. *Let G be a compact Lie group, and let (M, ω, μ) be an exact Hamiltonian G manifold. Let X be a connected component of the reduced space $\mu^{-1}(0)/G$. Suppose that X is compact. Then X consists of one point, which is isolated in $\mu^{-1}(0)/G$.*

Proof. By Lemma 4.1, if X is compact and $\dim X > 0$, then ω_X is not exact in $\Omega(X)$. By Lemma 4.2, ω_{red} is exact in $\Omega(X)$. Thus, $\dim X = 0$. Because X is

connected, X consists of a single point. Because $\mu^{-1}(0)/G$ is locally connected, this point is isolated in $\mu^{-1}(0)/G$. $\square$

4.4. *Remark.* It follows that if M/G is connected and μ is proper, then $\mu^{-1}(0)$ consists of at most one G orbit. This is because these additional assumptions imply that $\mu^{-1}(0)/G$ is connected. (This follows from Kirwan's Morse-type theory for the norm-square of the momentum map, or it can be deduced from connectedness and convexity results for abelian groups as in [18]. In [18] it is assumed that G and M are connected; we get the general case by applying this special case to the action of the identity component of G on a connected component of M.)

5. Retracting M

In our main theorem, Theorem 1.5, we assumed that the momentum map $\mu\colon M \to \mathfrak{g}^*$ is proper. When (M, ω, μ) is exact, it is enough to assume that μ is proper as a map to an open subset $\mathcal{T}$ of $\mathfrak{g}^*$ that is starshaped about the origin. This weaker property comes up, for example, when we start with a Hamiltonian G action on a compact symplectic manifold and obtain M by restricting to the momentum map preimage of $\mathcal{T}$. From now on we will work with this weaker assumption, but the reader is welcome to restrict attention everywhere to the special case $\mathcal{T} = \mathfrak{g}^*$.

We recall some facts about vector fields and flows. Good references are the textbooks by John Lee [19] and by Bröcker and Jänich [1].

Recall that a **flow** on a manifold or manifold with boundary Y is a smooth map $(t, y) \mapsto \rho_t(y)$ to Y, defined on a **flow domain**—an open subset D of $\mathbb{R} \times Y$ such that for every $y \in Y$ the set $\{t \in \mathbb{R} \mid (t, y) \in D\}$ is an interval that contains the origin—such that $\rho_{t+s}(y) = \rho_t(\rho_s(y))$, in the sense that if the right-hand side is well defined, then so is the left-hand side and they are equal. We say that a flow is defined for all $t \geq 0$ if its flow domain contains $\mathbb{R}_{\geq 0} \times Y$.

By the fundamental theorem of ordinary differential equations, if X is a vector field on Y, then there exists a unique flow on Y, called the **flow of** X, that has the following properties. For any $x_0 \in X$, interval I of the form $[0, b)$ or $(a, 0]$, and smooth curve $x\colon I \to Y$, the curve satisfies $x(0) = x_0$ and $\dot{x} = X \circ x$ if and only if $I \subseteq \{t \in \mathbb{R} \mid (t, x_0) \in D\}$ and $x(t) = \rho_t(x_0)$ for all $t \in I$.

Given a map $f\colon Y_1 \to Y_2$, we say that a flow $\rho_t^{Y_1}$ on Y_1 **lifts** a flow $\rho_t^{Y_2}$ on Y_2 if the flow domain of $\rho_t^{Y_1}$ is the preimage under $(t, y) \mapsto (t, f(y))$ of the flow domain of $\rho_t^{Y_2}$ and if $f(\rho_t^{Y_1}(y)) = \rho_t^{Y_2}(f(y))$ whenever the left and right-hand sides of this equation are defined.

We recall that if an integral curve for a vector field does not exist for all times, then it leaves every compact set:

5.1. **Lemma.** *Let M be a smooth manifold without boundary, X a vector field on M, $x_0 \in M$, $t > 0$, and $x\colon [0, t) \to M$ a smooth curve such that*

(5.2) $x(0) = x_0, \quad \dot{x} = X \circ x.$

If the closure of $x([0, t))$ is compact, then x extends to a smooth solution of (5.2) that is defined on $[0, t')$ for some number $t' > t$.

Proof. This follows for example from the argument on p. 84 in [1] or in the "Escape Lemma" [19, Lemma 17.10]. $\square$

The next lemma will be used in the proofs of Lemmata 5.7 and 6.2 below.

5.3. Lemma. *Let X_1 be a vector field on Y_1, let X_2 be a vector field on Y_2, and let $f\colon Y_1 \to Y_2$ be a smooth map whose differential intertwines X_1 with X_2. Then the image of the flow domain of X_1 under the map $\mathrm{id}_{\mathbb{R}} \times f$ is contained in the flow domain of X_2.*

If f is proper, then the flow of X_1 lifts the flow of X_2. In particular, if the flow of X_2 is defined for all $t \geq 0$, then so is the flow of X_1.

Proof. Let $\rho^{Y_i}\colon D_i \to Y_i$ be the flow of X_i. Fix any $y_1^0 \in Y_1$ and $y_2^0 \in Y_2$ such that $f(y_1^0) = y_2^0$. Let $(a_i, b_i) = \{t \in \mathbb{R} \mid (t, y_i^0) \in D_i\}$, and define $y_i\colon (a_i, b_i) \to Y_i$ by $y_i(t) = \rho_t^{Y_i}(y_i^0)$. We claim that $(a_1, b_1) \subseteq (a_2, b_2)$ and that for all t in this interval we have $f(y_1(t)) = y_2(t)$.

Because $f(y_1^0) = y_2^0$ and the differential of f intertwines X_1 with X_2, and because the curve y_1 satisfies

(5.4) $$y_1(0) = y_1^0 \quad \text{and} \quad \dot{y}_1 = X_1 \circ y_1,$$

the curve $y_2' := f \circ y_1\colon (a_1, b_1) \to Y_2$ satisfies $y_2'(0) = y_2^0$ and $\dot{y}_2' = X_2 \circ y_2$. By the properties of the flow of X_2, it follows that $(a_1, b_1) \subseteq (a_2, b_2)$ and $y_2'(t) = y_2(t)$ for all $t \in (a_1, b_1)$.

Assume now that f is proper. It remains to show that $(a_1, b_1) = (a_2, b_2)$. Seeking a contradiction, assume that $b_1 < b_2$; the case $a_1 > a_2$ is similar. Then $[0, b_1] \subset [0, b_2)$. Since f is proper, the set $f^{-1}(y_2([0, b_1]))$ is compact. It contains $y_1([0, b_1))$. It follows from Lemma 5.1 that y_1 extends to a solution of (5.4) that is defined on $[0, b_1']$ for some $b_1' > b_1$. This contradicts the definition of b_1 and properties of the flow of X_2. $\qquad\square$

5.5. Definition. Let (M, ω) be an exact symplectic manifold and λ a one-form such that $d\lambda = \omega$. The corresponding **Liouville vector field** is the vector field X_M on M that satisfies $\iota(X_M)\omega = \lambda$.

The following lemma will be used in the proofs of Lemmata 5.8 and 6.2.

5.6. Lemma. *Let G be a compact Lie group, and let (M, ω, μ) be an exact Hamiltonian G manifold. Let λ be a G invariant one-form on M such that $d\lambda = \omega$ and $\mu^\xi = \iota(\xi_M)\lambda$ for all $\xi \in \mathfrak{g}$. Let X_M be the corresponding Liouville vector field. Then:*

- X_M *is G invariant.*
- $\mathcal{L}_{X_M}\lambda = \lambda.$
- *The momentum map μ intertwines X_M with the Euler vector field on $\mathfrak{g}^*$ (see Definition 3.1).*

Proof.

- The G invariance of X_M follows from the G invariance of λ and of ω.
- We have

$$\mathcal{L}_{X_M}\lambda = \iota(X_M)d\lambda + d\iota(X_M)\lambda = \iota(X_M)\omega + d\iota(X_M)\iota(X_M)\omega = \iota(X_M)\omega + 0 = \lambda.$$

- For each $\xi \in \mathfrak{g}$ we have

$$\mathcal{L}_{X_M}\mu^\xi = \iota(X_M)d\mu^\xi = -\iota(X_M)\iota(\xi_M)\omega = \iota(\xi_M)\iota(X_M)\omega = \iota(\xi_M)\lambda = \mu^\xi.$$

This implies that μ intertwines X_M with the Euler vector field on $\mathfrak{g}^*$. $\qquad\square$

Fix an Ad^*-invariant inner product on $\mathfrak{g}^*$. The next lemma will be used in the proof of Lemma 5.8.

5.7. Lemma. *Let M be a G manifold and $\mu\colon M \to \mathfrak{g}^*$ a map that intertwines the G action on M with the coadjoint G action on $\mathfrak{g}^*$. Let $\mathcal{T}$ be an open subset of $\mathfrak{g}^*$ that is Ad^*-invariant and is starshaped about the origin. Suppose that the image of μ is contained in $\mathcal{T}$ and that μ is proper as a map to $\mathcal{T}$.*

Let V be an open ball in $\mathfrak{g}^*$ centred at the origin and contained in $\mathcal{T}$, and let B be a closed ball in $\mathfrak{g}^*$ centred at the origin and contained in V. Suppose that there exists a vector field on $M \smallsetminus \mu^{-1}(0)$ that lifts the Euler vector field on $\mathfrak{g}^*$.

Then there exists a G equivariant map $\varphi\colon M \to M$ whose image is contained in $\mu^{-1}(V)$, whose restriction to some neighbourhood of $\mu^{-1}(B)$ is the identity map on that neighbourhood, and such that φ is G-equivariantly smoothly homotopic to the identity map on M.

Proof. Let ε_1 and ε_3 be, respectively, the radii of the balls B and V. Let ε_2 be strictly between ε_1 and ε_3, and let V' be the open ball of radius ε_2 in $\mathfrak{g}^*$ that is centred at the origin.

There exists a vector field on $[0, \infty)$ whose flow $(t, x) \mapsto \rho_t(x)$ is defined for all $t \geq 0$ and which satisfies $\rho_t|_{[0,\varepsilon_2]} = \mathrm{Identity}|_{[0,\varepsilon_2]}$ for all t and $\rho_1([0, \infty)) \subset [0, \varepsilon_3)$ when $t = 1$. The flow

$$\rho_t^{\mathcal{T}}(\beta) := \begin{cases} \rho_t(|\beta|) \cdot \frac{\beta}{|\beta|}, & \beta \neq 0, \\ 0, & \beta = 0, \end{cases}$$

on $\mathcal{T}$ is defined for all $t \geq 0$ and satisfies $\rho_t^{\mathcal{T}}|_{V'} = \mathrm{Identity}|_{V'}$ for all t and $\rho_1(\mathcal{T}) \subset V$ when $t = 1$.

Let $X^{\mathcal{T}}$ be the vector field on $\mathcal{T}$ that generates the flow $\rho^{\mathcal{T}}$. The assumptions on $\rho^{\mathcal{T}}$ imply that $X^{\mathcal{T}}$ is a multiple of the Euler vector field on $\mathfrak{g}^*$ by a smooth Ad^*-invariant function $h\colon \mathcal{T} \to \mathbb{R}$ that vanishes on B. Take a vector field on M that lifts the Euler vector field on $\mathfrak{g}^*$. By averaging, we may assume that it is G invariant; let X^M be its multiple by the function $h \circ \mu$. Then X^M is a G-invariant vector field on M that vanishes on $\mu^{-1}(B)$ and the map μ intertwines X^M with $X^{\mathcal{T}}$. By Lemma 5.3, the flow ρ_t^M of the vector field X^M is defined for all $t \geq 0$ and lifts the flow $\rho_t^{\mathcal{T}}$. The time one map $\varphi := \rho_1^M$ has the required properties. $\square$

The next lemma will be used in the proof of Proposition 6.3.

5.8. Lemma. *Let (M, ω, μ) be an exact Hamiltonian G manifold. Let $\mathcal{T}$ be an open subset of $\mathfrak{g}^*$ that is Ad^*-invariant and is starshaped about the origin. Suppose that the image of μ is contained in $\mathcal{T}$ and that μ is proper as a map to $\mathcal{T}$. Let B be a closed ball in $\mathfrak{g}^*$ centred at the origin and contained in $\mathcal{T}$. Then:*

(1) *Every connected component of M meets $\mu^{-1}(B)$.*
(2) *Let V be an open ball in $\mathcal{T}$ that is centred at the origin and that contains B. Let $\hat{\lambda}$ be a G invariant one-form on $\mu^{-1}(V)$ such that $d\hat{\lambda} = \omega$. Then there exists a G invariant one-form λ on M such that $d\lambda = \omega$ and such that $\lambda = \hat{\lambda}$ on $\mu^{-1}(B)$.*

Proof. By Lemma 5.6, the Liouville vector field on M lifts the Euler vector field on $\mathfrak{g}^*$. Part (1) then follows from Lemma 5.7.

Let λ' be a G invariant one-form on M such that $d\lambda' = \omega$; such a one-form exists because ω is exact. The difference $\hat{\lambda} - \lambda'$ is a closed one-form, defined on $\mu^{-1}(V)$. By Lemma 5.6, the Liouville vector field on M lifts the Euler vector field on $\mathfrak{g}^*$.

By Lemma 5.7, there exists a G-equivariant smooth map $\varphi\colon M \to M$ whose image is contained in $\mu^{-1}(V)$ and such that $\varphi|_{\mu^{-1}(B)} = \text{identity}_{\mu^{-1}(B)}$. Then

$$\lambda := \lambda' + \varphi^*(\hat{\lambda} - \lambda')$$

is defined on all of M. Because $d\lambda' = \omega$ and $d(\hat{\lambda} - \lambda') = 0$, we have $d\lambda = \omega$. Because $\varphi|_{\mu^{-1}(B)} = \text{Identity}_{\mu^{-1}(B)}$, we have $\lambda = \hat{\lambda}$ on $\mu^{-1}(B)$. This completes the proof of part (2). $\qquad\square$

6. Proof of the main result

In this section we finally prove Theorem 1.5. The proof is based on Theorem 6.8 below.

Recall that an isomorphism of Hamiltonian G manifolds is an equivariant symplectomorphism that intertwines the momentum maps. We call two Hamiltonian G manifolds **isomorphic** if there is an isomorphism between them. The **restriction** of a Hamiltonian G manifold (M, ω, μ) to an Ad^*-invariant open subset $\mathcal{T} \subset \mathfrak{g}^*$ is $\left(\mu^{-1}(\mathcal{T}), \omega|_{\mu^{-1}(\mathcal{T})}, \mu|_{\mu^{-1}(\mathcal{T})}\right)$.

The following lemma is used in the proof of Theorem 6.8.

6.1. Lemma (Local isomorphism). *Let G be a compact Lie group and let (M, ω, μ) be an exact Hamiltonian G manifold. Suppose that M/G is connected. Let $\mathcal{T} \subset \mathfrak{g}^*$ be an Ad^*-invariant open subset that is starshaped about the origin. Suppose that the image of μ is contained in $\mathcal{T}$ and the map $\mu\colon M \to \mathcal{T}$ is proper. Then there exist an Ad^*-invariant open neighbourhood W of the origin $0 \in \mathfrak{g}^*$ and a finite disjoint union of centred Hamiltonian G models, $\bigsqcup(Y_j, \omega_{Y_j}, \mu_{Y_j})$, whose restriction to W is isomorphic to the restriction of (M, ω, μ) to W.*

Proof. Since $\mu\colon M \to \mathcal{T}$ is proper, the level set $\mu^{-1}(0)$ is compact. By Proposition 4.3, $\mu^{-1}(0)$ is a finite union of G orbits, $\mu^{-1}(0) = \bigsqcup_{j=1}^{N} G \cdot x_j$.

Every orbit in the zero level set of the momentum map is isotropic. (More generally, in any Hamiltonian G manifold (M', ω', μ'), for any orbit $\mathcal{O} \subset M'$, its image is a coadjoint orbit $\mu(\mathcal{O}) \subset \mathfrak{g}^*$, and the pullback of the Kirillov-Kostant-Souriau form under the momentum map $\mu'|_{\mathcal{O}}\colon \mathcal{O} \to \mu'(\mathcal{O})$ coincides with the pullback of the ambient symplectic form ω' by the inclusion map $\mathcal{O} \hookrightarrow M'$. See Kazhdan-Kostant-Sternberg [16].)

The local normal form theorem for isotropic orbits (see Theorem 3.9) implies that, for each j, there exist a centred Hamiltonian G model $(Y_j, \omega_{Y_j}, \mu_{Y_j})$, a G-invariant neighbourhood U_j' of the central orbit in Y_j, and a G-invariant neighbourhood U_j of $G \cdot x_j$ in M such that $\left(U_j, \omega|_{U_j}, \mu|_{U_j}\right)$ is isomorphic to $\left(U_j', \omega_{Y_j}|_{U_j'}, \mu_{Y_j}|_{U_j'}\right)$.

In particular, each $U_j' \cap \mu_{Y_j}^{-1}(0)$ consists of a single orbit. By Lemma 3.6, for each j, the level set $\mu_{Y_j}^{-1}(0)$ consists of only the central orbit, and the momentum map $\mu_{Y_j}\colon Y_j \to \mathfrak{g}^*$ is proper. Because $\bigsqcup_j \mu_{Y_j}\colon \bigsqcup_j Y_j \to \mathfrak{g}^*$ and $\mu\colon M \to \mathfrak{g}^*$ are proper, there exists an Ad^*-invariant neighbourhood W of 0 in $\mathfrak{g}^*$ whose preimage under $\bigsqcup_j \mu_{Y_j}$ is contained in the neighbourhood $\bigsqcup_j U_j'$ of $\bigsqcup_j \mu_{Y_j}^{-1}(0)$ and whose preimage under μ is contained in the neighbourhood U of $\mu^{-1}(0)$. The restrictions of (M, ω, μ) and $\bigsqcup_j(Y_j, \omega_{Y_j}, \mu_{Y_j})$ to W are isomorphic. This proves Lemma 6.1. $\quad\square$

The following lemma will be used in the proof of Proposition 6.3.

6.2. **Lemma.** *Let G be a compact connected Lie group, and let (M, ω, μ) be an exact Hamiltonian G-manifold. Let $\mathcal{T} \subset \mathfrak{g}^*$ be an Ad^*-invariant open subset that is starshaped about the origin. Assume that the image of μ is contained in $\mathcal{T}$ and that the map $\mu \colon M \to \mathcal{T}$ is proper. Let λ be a G invariant one-form on M such that $d\lambda = \omega$ and $\mu^\xi = \iota(\xi_M)\lambda$ for all $\xi \in \mathfrak{g}$. Let X be the vector field on M that satisfies*

$$\iota(X)\omega = -\lambda;$$

this is the negative of the Liouville vector field corresponding to λ. Let Ψ_t be the flow of the vector field X. Then:

- *The flow Ψ_t is defined for all $t \geq 0$.*
- *The flow Ψ_t is G equivariant.*
- *We have $\Psi_t^* \lambda = e^{-t}\lambda$ for all $t \geq 0$.*
- *For each neighbourhood B' of the origin 0 in $\mathfrak{g}^*$ and for each $t \geq 0$, the flow Ψ_t restricts to a diffeomorphism*

$$\Psi_t \colon \mu^{-1}(e^t B' \cap \mathcal{T}) \to \mu^{-1}(B' \cap e^{-t}\mathcal{T})$$

   with inverse

$$\Psi_{-t} \colon \mu^{-1}(B' \cap e^{-t}\mathcal{T}) \to \mu^{-1}(e^t B' \cap \mathcal{T}).$$

Proof. By Lemma 5.6, μ intertwines X with the negative of the Euler vector field on $\mathfrak{g}^*$. Because $\mathcal{T}$ is starshaped about the origin, the flow of the restriction to $\mathcal{T}$ of the Euler vector field is defined for all $t \geq 0$. Because $\mu \colon M \to \mathcal{T}$ is proper and by Lemma 5.3, we deduce that the flow of X is also defined for all $t \geq 0$.

By Lemma 5.6, X is G invariant. This implies that Ψ_t is G equivariant.

By Lemma 5.6, $\mathcal{L}_X \lambda = -\lambda$. This implies that $\Psi_t^* \lambda = e^{-t}\lambda$.

The last claim follows from Lemma 5.3 and the fact that multiplication by e^{-t} restricts to a diffeomorphism from $e^t B' \cap \mathcal{T}$ to $B' \cap e^{-t}\mathcal{T}$. $\qquad\square$

The following proposition will be used in the proof of Theorem 6.8.

6.3. **Proposition** (Local to global). *Let G be a compact Lie group, and let (M, ω, μ) and (M', ω', μ') be exact Hamiltonian G-manifolds. Let $\mathcal{T} \subset \mathfrak{g}^*$ be an Ad^*-invariant open subset that is starshaped about the origin. Assume that the images of μ and μ' are contained in $\mathcal{T}$ and that the maps $\mu \colon M \to \mathcal{T}$ and $\mu' \colon M' \to \mathcal{T}$ are proper. Finally, assume that there exists an Ad^*-invariant neighbourhood W of 0 such that the restrictions of (M, ω, μ) and (M', ω', μ') to W are isomorphic. Then (M, ω, μ) and (M', ω', μ') are isomorphic.*

Proof. By assumption, there exist a neighbourhood W of the origin in $\mathfrak{g}^*$ and an isomorphism $F_0 \colon \mu^{-1}(W) \to \mu'^{-1}(W)$ between the restrictions of (M, ω, μ) and (M', ω', μ') to W.

Fix an Ad^*-invariant inner product on $\mathfrak{g}^*$. Without loss of generality, assume that the neighbourhood W is a ball in $\mathfrak{g}^*$ that is centred at the origin.

Choose a G-invariant primitive λ' of ω' that gives rise to the momentum map μ':

(6.4) $d\lambda' = \omega'$ and $\iota(\xi_{M'})\lambda' = \mu'^\xi$ for all $\xi \in \mathfrak{g}$.

The pullback

(6.5) $\hat\lambda := F_0^* \lambda'$

is a G-invariant one-form on $\mu^{-1}(W)$, satisfying

$$(6.6) \qquad d\hat{\lambda} = \omega|_{\mu^{-1}(W)} \quad \text{and} \quad \iota(\xi_M)\hat{\lambda} = \mu^{\xi}|_{\mu^{-1}(W)} \quad \text{for all } \xi \in \mathfrak{g}.$$

Let B be a closed ball that is centred at the origin and is contained in W. By part (2) of Lemma 5.8, we may find a G-invariant one-form λ on M such that $d\lambda = \omega$ and such that λ and $\hat{\lambda}$ coincide on $\mu^{-1}(B)$. By this and (6.6),

$$\iota(\xi_M)\lambda|_{\mu^{-1}(B)} = \mu^{\xi}|_{\mu^{-1}(B)} \quad \text{for all } \xi \in \mathfrak{g}.$$

By part (1) of Lemma 5.8 we deduce that the momentum map that is obtained from λ coincides with μ. So

$$(6.7) \qquad d\lambda = \omega \quad \text{and} \quad \iota(\xi_M)\lambda = \mu^{\xi} \quad \text{for all } \xi \in \mathfrak{g}.$$

We define X to be the unique vector field on M satisfying $\iota(X)\omega = -\lambda$ and take Ψ_t to be its flow. Similarly, we define X' to be the unique vector field on M' satisfying $\iota(X')\omega' = -\lambda'$ and take Ψ_t' to be its flow. By Lemma 6.2, these flows are defined for all $t \geq 0$.

Let B' be the interior of B. The restriction of F_0 to $\mu^{-1}(B')$ is a G equivariant diffeomorphism

$$F_B \colon \mu^{-1}(B') \to \mu'^{-1}(B')$$

that satisfies $F_B^*\lambda' = \lambda$. This implies that F_B intertwines X with X' and hence Ψ_t with Ψ_t'.

For every $t \geq 0$, define $\tilde{F}_t$ by the requirement that the following diagram commutes:

$$
\begin{array}{ccc}
\mu^{-1}(e^t B' \cap \mathcal{T}) & \xrightarrow{\ \tilde{F}_t\ } & \mu'^{-1}(e^t B' \cap \mathcal{T}) \\[4pt]
\Psi_t \downarrow & & \downarrow \Psi_t' \\[4pt]
\mu^{-1}(B' \cap e^{-t}\mathcal{T}) & \xrightarrow{\ F_B\ } & \mu'^{-1}(B' \cap e^{-t}\mathcal{T}).
\end{array}
$$

By Lemma 6.2, the vertical arrows in this diagram are G equivariant diffeomorphisms, and they satisfy $\Psi_t^*\lambda = e^{-t}\lambda$ and $\Psi_t'^*\lambda' = e^{-t}\lambda'$. The bottom arrow is also a G equivariant diffeomorphism, and it satisfies $F_B^*\lambda' = \lambda$. These facts imply that $\tilde{F}_t$ is an equivariant diffeomorphism and that $\tilde{F}_t^*\lambda' = \lambda$. By (6.7) and (6.4), $\tilde{F}_t$ is an isomorphism of Hamiltonian G manifolds.

For all $t \geq 0$ and $s \geq 0$ we have

$$
\begin{aligned}
\Psi_{t+s}' \circ \tilde{F}_t &= \Psi_s' \circ \Psi_t' \circ \tilde{F}_t & &\text{because } \Psi' \text{ is a flow} \\
&= \Psi_s' \circ F_B \circ \Psi_t & &\text{from the definition of } \tilde{F}_t \\
&= F_B \circ \Psi_s \circ \Psi_t & &\text{because } F_B \text{ intertwines the flows} \\
&= F_B \circ \Psi_{t+s} & &\text{because } \Psi \text{ is a flow.}
\end{aligned}
$$

By the definition of $\tilde{F}_{t+s}$, this implies that $\tilde{F}_{t+s}|_{e^t B' \cap \mathcal{T}} = \tilde{F}_t$.

So the union over $t \geq 0$ of the maps $\tilde{F}_t$ gives a map $\tilde{F} \colon M \to M'$. Because $\tilde{F}_t$ is a isomorphism for each t, we have that $\tilde{F}$ is also an isomorphism.   □

6.8. **Theorem** (Isomorphism). *Let G be a compact Lie group and let (M, ω, μ) be an exact Hamiltonian G manifold. Suppose that M/G is connected. Let $\mathcal{T} \subset \mathfrak{g}^*$ be an Ad^*-invariant open subset that is starshaped about the origin. Suppose that the image of μ is contained in $\mathcal{T}$ and the map $\mu \colon M \to \mathcal{T}$ is proper. Then there exists a centred Hamiltonian G model (Y, ω_Y, μ_Y) whose restriction to $\mathcal{T}$ is isomorphic to (M, ω, μ).*

Proof. By Lemma 6.1 and Proposition 6.3, there exists a finite disjoint union of centred Hamiltonian G models, $\bigsqcup_j (Y_j, \omega_{Y_j}, \mu_{Y_j})$, whose restriction to $\mathcal{T}$ is isomorphic to (M, ω, μ). If M/G is connected, since each $(Y_j|_{\mathcal{T}})/G$ is non-empty, there can be only one Y_j in this union. This proves Theorem 6.8.   □

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. Let a compact Lie group G act on a symplectic manifold (M, μ) with momentum map $\mu \colon M \to \mathfrak{g}^*$. Assume that M/G is connected, μ is proper, and ω is exact.

Let μ' be the momentum map that is obtained from some invariant primitive of ω. Then (M, ω, μ') is an exact Hamiltonian G manifold. By Theorem 6.8, there exists a centred Hamiltonian G model (Y, ω_Y, μ_Y) whose restriction to $\mathcal{T}$ is isomorphic to (M, ω, μ'), hence equivariantly sympectomorphic to (M, ω).   □

7. Properties of centred Hamiltonian G models

In this section we present properties of centred Hamiltonian G models that we used to prove Corollaries 1.6, 1.7, and 1.8 of Theorem 1.5.

Let G be a compact Lie group. Let (Y, ω_Y, μ_Y) be a centred Hamiltonian G-model associated with a closed subgroup H of G and a linear action of H on a symplectic vector space V with a quadratic momentum map $\mu_V \colon V \to \mathfrak{h}^*$.

We recall from Lemma 3.8 that the model Y can be identified with the vector bundle

$$(7.1) \qquad\qquad G \times_H \left(\mathfrak{h}^0 \times V \right)$$

over G/H, obtained as the quotient of $G \times \left(\mathfrak{h}^0 \times V \right)$ by the anti-diagonal H action, such that the embedding $v \mapsto [1, 0, v]$ of V as a subspace of the fibre is symplectic, the zero section $G \times_H \{0\}$ is isotropic, and the momentum map on the model is

$$[a, \nu, v] \mapsto \mathrm{Ad}^*(a)(\nu + \mu_V(v)),$$

where $\mathfrak{h}^0$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$ and where we identify $\mathfrak{h}^*$ with the orthogonal complement of $\mathfrak{h}^0$ in $\mathfrak{g}^*$ with respect to some Ad-invariant inner product.

7.2. **Lemma.** *The model Y is homotopy equivalent to G/H, and the Euler characteristic of Y is non-negative.*

Proof. In the model (7.1), the map $[a, \nu, v] \mapsto [a, t\nu, tv]$, for $t \in [0, 1]$, gives a deformation retraction of Y to the zero section. The zero section, in turn, is naturally identified with G/H. This proves the first sentence. The second sentence follows from the first because, by H. Hopf and H. Samelson [9, p. 241], the Euler characteristic of G/H is non-negative.   □

7.3. **Lemma.** *The fixed point set in the model Y is connected. Moreover, if the momentum map $Y \to \mathfrak{g}^*$ is proper, then this fixed point set is either empty or contains exactly one point.*

Proof. Consider the model as described in (7.1). If $H \neq G$, then the fixed point set is empty. Assume $H = G$. Then the model is isomorphic to the vector space V with the linear action of H and with the momentum map μ_V. The fixed point set V^H is connected because it is a linear subspace of V. Now assume that the momentum map μ is proper. Then its zero level set $\mu^{-1}(\{0\})$ is compact. The fixed point set

V^H then contains just one point, because it is contained in $\mu^{-1}(\{0\})$, and a vector subspace of V that is contained in a compact subset of V must be trivial.   □

Recall that the *Gromov width* of a $2n$-dimensional symplectic manifold (M, ω) is defined to be the number

$$\sup \left\{ \pi r^2 \,\big|\, B_r^{2n} \text{ symplectically embeds into } M \right\},$$

where $B_r^{2n} \subset \mathbb{R}^{2n}$ denotes the open ball of radius r centred at the origin, equipped with the standard symplectic structure.

7.4. Lemma. *The Gromov width of Y is infinite.*

Proof. Recall that the model Y is obtained as a symplectic reduction of $T^*G \times V$, where the two-form and momentum map on $T^*G \times V$ are induced from the one-form $\lambda_{\text{taut}} \oplus \lambda_V$ that is described in Section 3. Define a flow ρ_t on $T^*G \times V$ by fibrewise multiplication by e^t on T^*G and multiplication by $e^{\frac{1}{2}t}$ on V. Then $\rho_t^*(\lambda_{\text{taut}} \oplus \lambda_V) = e^t(\lambda_{\text{taut}} \oplus \lambda_V)$, and ρ_t commutes with the anti-diagonal H action by which we quotient to get Y. It follows that ρ_t descends to the symplectic reduction $Y = (T^*G \times V)//H$ and satisfies $\rho_t^* \lambda_Y = e^t \lambda_Y$, hence $\rho_t^* \omega_Y = e^t \omega_Y$.

By Darboux's theorem, for some $\varepsilon > 0$ there exists a symplectic embedding $i \colon B_\varepsilon^{2n} \to Y$. The composition $x \mapsto \rho_t(i(e^{-\frac{1}{2}t}x))$ is a symplectic embedding of B_r^{2n} into Y where $r = \varepsilon e^{\frac{1}{2}t}$. Because $\pi r^2 = \pi \varepsilon^2 e^t$ can be made arbitrarily large by appropriate choices of t, the Gromov width of Y is infinite.   □

We now present two situations in which we get even more precise information on the models that can occur in Theorem 1.5.

7.5. Lemma (Circle actions)**.** *Suppose that G is the circle group S^1 and that its action on Y is faithful. Then Y is equivariantly symplectomorphic to one of the following examples:*

(a) *The cylinder $S^1 \times \mathbb{R}$, with the standard symplectic form $dq \wedge dp$ where q mod $\mathbb{Z}$ is a coordinate on $S^1 \cong \mathbb{R}/\mathbb{Z}$ and p is a coordinate on $\mathbb{R}$. The circle S^1 acts by rotations on the S^1 factor.*

(b) *The vector space $\mathbb{C}^n$, with the standard symplectic form. The circle $S^1 \subseteq \mathbb{C}$ acts by $\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n)$, where $m_1, \ldots, m_n$ are either all positive integers or all negative integers.*

Proof. Because the S^1 action is faithful on Y, the H action is faithful on V. (This uses the fact that S^1 is abelian.) Because $\mu_Y \colon Y \to \mathfrak{g}^*$ is proper, the momentum map $\mu_V \colon V \to \mathfrak{h}^*$ is proper. These two facts imply that either $H = \{1\}$ and $V = \{0\}$ or $H = S^1$ and $V \cong \mathbb{C}^n$ where H acts on all the coordinates with positive weights or on all the coordinates with negative weights.   □

7.6. Lemma. *Suppose that the model Y is contractible. Then (Y, ω_Y, μ_Y) is equivariantly symplectomorphic to (V, ω_V, μ_V).*

Proof. By Lemma 7.2, since Y is contractible, so is G/H. Because G is a compact connected Lie group and H a closed subgroup, the quotient G/H is a closed manifold. Being a contractible closed manifold, G/H is a point. So $H = G$. It now follows from the description (7.1) that Y is isomorphic to V.   □

8. Exotic actions on symplectic vector spaces

If we drop the assumption that the momentum map be proper, the conclusions of Theorem 1.5 and of its consequences can fail dramatically, even when the symplectic manifold is a symplectic vector space. We show this through examples of exotic Hamiltonian actions on symplectic vector spaces. Corollaries 8.2, 8.3, and 8.4 give examples where, respectively, the conclusions of Corollary 1.6(ii), of Corollary 1.8, and of Theorem 1.5 fail.

Our main ingredient is the following result, which was communicated to us by K. Pawałowski.

8.1. **Proposition.** *Let G be a compact connected non-abelian Lie group. Then there exists a smooth G action on a Euclidean space without fixed points. Moreover, let F be a closed, stably parallelizable manifold. Then there exists a smooth G action on a Euclidean space whose fixed point set is diffeomorphic to F.*

Recall that a manifold is called stably parallelizable if the direct sum of its tangent bundle with some trivial vector bundle is trivializable. Examples include the empty set, finite sets, Lie groups, spheres of any dimension, and toric manifolds.

Proof of Proposition 8.1. Let Y be a finite dimensional countable G–CW complex, with finitely many orbit types and with no fixed points, which is contractible.

The construction of such a Y is described by Wu-chung Hsiang and Wu-yi Hsiang in [10, Theorem 1.9] and is based on earlier ideas of Conner, Floyd, and Montgomery [3, 4]. The idea is to take a representation V of G without a trivial summand and an equivariant map $f \colon S(V) \to S(V)$ of degree zero on its unit sphere, as constructed in [10, Proposition 1.4], and to take Y to be the tower of mapping cylinders of f.

Let X be the topological join of Y with F. Then X is a contractible finite dimensional countable G–CW complex with finitely many orbit types and with fixed point set F.

Let B be the union of F with the equivariant 0–cells in $X \smallsetminus F$. Let m be such that the direct sum of TF with a trivial vector bundle is isomorphic to the trivial bundle $F \times \mathbb{R}^m$, and let $E = X \times \mathbb{R}^m \to X$ be the trivial bundle. We now apply a theorem of Pawałowski [24, Theorem 3.1], which is based on Pawałowski's equivariant thickening procedure [24, Theorem 2.4]. Given these X, F, B, and E, this theorem yields a smooth G manifold M that contains B as a smooth invariant submanifold and such that $M^G = F$ and a homotopy equivalence $M \to X$. As noted in [24, Remarks 2.5 and 3.2], we can assume that dim $M \geq 5$, and the construction yields an M that is contractible and simply connected at infinity. By a result of Stallings [27, Corollary 5–1], M is diffeomorphic to a Euclidean space. $\square$

8.2. **Corollary.** *Let G be a compact connected non-abelian Lie group. Then there exists a Hamiltonian G action on a symplectic vector space without fixed points. Moreover, let F be a closed, stably parallelizable manifold. Then there exists a Hamiltonian G action on a symplectic vector space whose fixed point set is symplectomorphic to T^*F.*

Proof. By Proposition 8.1 there exists a smooth G action on a Euclidean space X whose fixed point set is diffeomorphic to F. The fixed point set of the induced G action on the cotangent bundle T^*X is symplectomorphic to the cotangent bundle T^*F of the fixed point set F. This is a consequence of the slice theorem for compact

group actions. We obtain Corollary 8.2 by taking V to be the direct sum $X \oplus X^*$, identified with the cotangent bundle T^*X, with the cotangent lifted G action. □

8.3. Corollary. *Let G be a compact connected non-abelian Lie group. Then there exists a Hamiltonian G action on a symplectic vector space that is not isomorphic to a linear action.*

Proof. In Corollary 8.2, take an action without fixed points. □

8.4. Corollary. *Let G be a compact connected non-abelian Lie group. Then there exists a Hamiltonian G action on a symplectic vector space (V, ω_V) with the following properties.*

(i) *There exist two points in V such that no G invariant subset of V that contains both points admits an equivariant open embedding into any centred Hamiltonian G model.*

(ii) *There exists a connected bounded G invariant open subset of V that does not admit an equivariant open embedding into any centred Hamiltonian G model.*

Proof. By Corollary 8.2, there exists a G action on a symplectic vector space (V, ω_V) with exactly two fixed points. By Lemma 7.3, the fixed point set of any centred Hamiltonian G model is connected. For (i), note that a manifold with a G action with two isolated fixed points does not admit an equivariant open embedding into any manifold with a G action whose fixed point set is connected. For (ii), take the invariant open subset to be $G \cdot B$ where B is an open ball that contains the two fixed points. □

Acknowledgments

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