

CAMBRIAN FRAMEWORKS FOR CLUSTER ALGEBRAS OF AFFINE TYPE

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ABSTRACT. We give a combinatorial model for the exchange graph and \mathbf{g} -vector fan associated to any acyclic exchange matrix B of affine type. More specifically, we construct a reflection framework for B in the sense of [N. Reading and D. E. Speyer, “Combinatorial frameworks for cluster algebras”] and establish good properties of this framework. The framework (and in particular the \mathbf{g} -vector fan) is constructed by combining a copy of the Cambrian fan for B with an antipodal copy of the Cambrian fan for $-B$.

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1. INTRODUCTION

In a series of papers [24, 25, 28], the authors studied cluster algebras of finite type in terms of the combinatorics and geometry of finite Coxeter groups, and in particular sortable elements and Cambrian lattices. In [29–31], these constructions were extended to infinite Coxeter groups. Sortable elements and Cambrian semilattices were shown in [31] to produce combinatorial models of cluster algebras of infinite type, with an important limitation: Sortable elements can only model the part of the exchange graph that corresponds to clusters whose \mathbf{g} -vector cones intersect the interior of the Tits cone.

In this paper, we show how to extend the sortable/Cambrian setup to obtain a complete combinatorial model when B is acyclic and its Cartan companion $\text{Cart}(B)$ is of affine type. The basic idea can be illustrated by a very simple example.

Suppose $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Its Cartan companion $\text{Cart}(B) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ is of type \tilde{A}_1 . Figure 1 shows the \mathbf{g} -vector fan associated to B . The left picture of Figure 2 shows the Cambrian fan associated to B , while the right picture shows the Cambrian fan associated to $-B$. In each Cambrian fan picture, the Tits cone is identified

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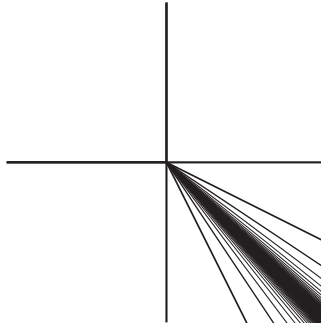


FIGURE 1. The \mathbf{g} -vector fan for B with $\text{Cart}(B)$ of type \tilde{A}_1

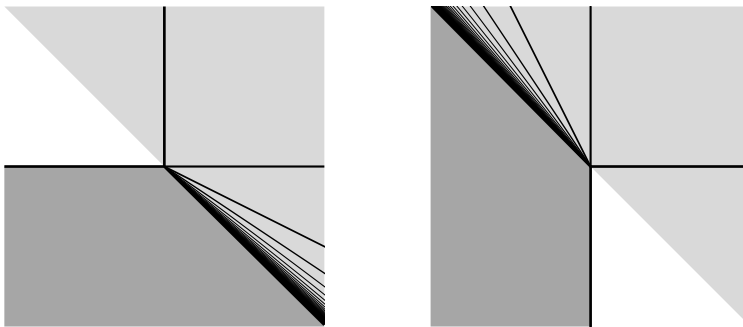


FIGURE 2. The Cambrian fans for B and for $-B$

by light gray shading and the area outside the Cambrian fan is identified by dark gray shading. Two surprising things happen. First, the Cambrian fan for B is “compatible” with the image under the antipodal map of the Cambrian fan for $-B$, in the sense that the union of these two fans is again a fan (the **doubled Cambrian fan**). Second, the doubled Cambrian fan coincides with the \mathbf{g} -vector fan, so that the dual graph to the doubled Cambrian fan is isomorphic to the exchange graph.

The main results of the paper are that the first of these surprising things happens for all acyclic B and that the second happens whenever B is acyclic and $\text{Cart}(B)$ is of affine type. For richer examples than the rank-two example described above, see Examples 3.22, 3.29, 3.36 and 4.10.

We now state our main results more formally. For each acyclic exchange matrix B , we construct a doubled Cambrian fan \mathcal{DF}_c as the union of the Cambrian fan for B with the antipodal Cambrian fan for $-B$. The **doubled Cambrian framework** is the pair $(\text{DCamb}_c, \text{DC}_c)$, where DCamb_c is the dual graph to \mathcal{DF}_c and DC_c is a certain labeling of DCamb_c by roots. In the language of [31], the key result is the following.

Theorem 1.1. *Suppose that B is acyclic and $\text{Cart}(B)$ is of affine type. Then $(\text{DCamb}_c, \text{DC}_c)$ is a complete, exact, well-connected, polyhedral, simply connected reflection framework.*

Let $\mathcal{A}_\bullet(B)$ be the principal-coefficients cluster algebra associated to B . Using results of [31] (as we explain in Section 2.4), the following corollaries are easily obtained.

Corollary 1.2. *If B is acyclic and $\text{Cart}(B)$ is of affine type, then the exchange graph of the cluster algebra $\mathcal{A}_\bullet(B)$ is isomorphic to DCamb_c . The isomorphism sends a vertex v of DCamb_c to a seed whose \mathbf{c} -vectors are (simple-root coordinates of) $\text{DC}_c(v)$.*

Corollary 1.3. *If B is acyclic and $\text{Cart}(B)$ is of affine type, then \mathcal{DF}_c coincides with the fan of \mathbf{g} -vector cones for the cluster algebra $\mathcal{A}_\bullet(B)$.*

Corollary 1.3 refers to the \mathbf{g} -vector cones defined by interpreting \mathbf{g} -vectors as fundamental-weight coordinates of vectors in the weight lattice. See [31, Remark 5.17] and Section 2.3 of the present paper. When B is of infinite nonaffine type, the doubled Cambrian fan may be a proper subfan of the \mathbf{g} -vector fan. (See Example 3.36.) However, the doubled Cambrian fan coincides with the \mathbf{g} -vector fan for all 2×2 exchange matrices, even those of nonaffine infinite type. (See Remark 3.33.)

One of the initial motivations for this work was to prove the affine-type case of many of the standard structural conjectures on cluster algebras. Indeed, theorems about frameworks from [31] combined with the results stated above accomplish that task. (See Corollary 2.5.) Many of the standard conjectures have been turned into theorems in full generality (not merely the affine case) using the machinery of scattering diagrams [14]. See also the table at the end of [31, Section 3.3] for more on the previous status of these conjectures.

The approach to cluster algebras via scattering diagrams makes it *more* important, not less important, to make explicit constructions in key special cases, like the affine case. Scattering diagrams consist of a rational fan, decorated with certain rational functions, and containing the \mathbf{g} -vector fan as a subfan. Other constructions, notably semi-invariants (see [18] for a detailed treatment of the affine case, [16] for the acyclic case and [17] for the beginnings of an investigation of the general case) and the mutation fan [27] also yield fans that contain the \mathbf{g} -vector fan.

However, none of these methods give a direct combinatorial description of the fan, and it is difficult to use them to study the regions of the fan. Scattering diagrams natively construct the walls of the fan and it is difficult to use them to discuss the chambers cut out by those walls. (Not impossible though! See [23] for some progress along these lines.) Semi-invariants also naturally construct the walls, although connections have been found to cluster tilting and τ -tilting modules which correspond to the regions [16, 19]. Neither method naturally connects the fan directly to lattice theory, although torsion classes form a lattice which has been related to semi-invariants [19].

In contrast, our methods provide direct descriptions of the \mathbf{g} -vector fan in the affine types in terms of the combinatorics of affine Coxeter groups and root systems. We label the regions of the \mathbf{g} -vector fan by sortable elements in those groups, which can be described either in terms of reduced words or in terms of pattern avoidance ([29]). And our methods are built on lattice theoretic and geometric properties which are well suited for proving global properties of the fans we study.

The remainder of the paper is devoted to constructing the doubled Cambrian framework and fan and proving Theorem 1.1. We begin with background in Section 2. We define the doubled Cambrian framework and fan for any acyclic exchange matrix in Section 3, where we also prove the part of Theorem 1.1 that does not need the hypothesis the $\text{Cart}(B)$ is of affine type. The proof of Theorem 1.1 is completed in Section 4.

Most of the affine Dynkin diagrams are trees, and thus admit no nonacyclic orientations. The exception is \tilde{A}_{n-1} ($n \geq 3$), which is an n -cycle. Thus, in almost every case when $\text{Cart}(B)$ is of affine type, B is acyclic and the doubled Cambrian framework construction provides a complete framework for B . However, the cyclic orientation in type \tilde{A}_{n-1} ($n \geq 3$) does not fit into the doubled Cambrian framework. In this case, the associated cluster algebra is of finite type D_n . In [32], we construct a complete framework for the cyclically oriented n -cycle using the \tilde{A}_{n-1} root system and a variation on the doubled Cambrian framework idea.

2. BACKGROUND

2.1. Reflection frameworks. In this section, we review the background material on frameworks from [31]. The doubled Cambrian frameworks belong to a special class of frameworks called *reflection frameworks*. For that reason, in this paper, we review only the definition of reflection frameworks, and not the general definition of frameworks that is required to model cluster algebras in general.

The starting point is an *exchange matrix* $B = [b_{ij}]$, with rows and columns indexed by a set I , with $|I| = n$. This is a skew-symmetrizable integer matrix, meaning that there exists a positive real-valued function d on I with $d(i)b_{ij} = -d(j)b_{ji}$ for all $i, j \in I$. The *Cartan companion* $\text{Cart}(B)$ of B is the square matrix $A = [a_{ij}]$ with diagonal entries 2 and off-diagonal entries $a_{ij} = -|b_{ij}|$. This is a symmetrizable generalized Cartan matrix, and specifically $d(i)a_{ij} = d(j)a_{ji}$ for all $i, j \in I$. (See [20] for more on Cartan matrices. Or, for an exposition tailored to the present purposes, see [29, Section 2.2].)

We take V to be a real vector space of dimension n with a basis $\Pi = \{\alpha_i : i \in I\}$. We write V^* for the dual vector space to V and write $\langle x, y \rangle$ for the canonical pairing between $x \in V^*$ and $y \in V$. The vectors α_i are the *simple roots*. The vectors $\alpha_i^\vee = d(i)^{-1}\alpha_i$ are the *simple co-roots*. The set of simple co-roots is written Π^\vee . We define a bilinear form ω whose matrix is B , written in terms of the simple co-root basis on one side and the simple root basis on the other. Specifically, we set $\omega(\alpha_i^\vee, \alpha_j) = b_{ij}$. It is easily checked that ω is skew-symmetric.

The Cartan matrix A defines a Coxeter group W and a specific reflection representation of W on V . The group is generated by the set $S = \{s_i : i \in I\}$ of *simple reflections*, where each s_i acts on a simple root α_j by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. It follows that the action on a simple co-root α_j^\vee is $s_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee$. Furthermore, A encodes a symmetric bilinear form K on V with $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. The action of W can be rewritten as $s_i(\alpha_j) = \alpha_j - K(\alpha_i^\vee, \alpha_j)\alpha_i$, so that s_i acts as a reflection with respect to the form K . In particular, the action of W preserves the form K .

When it is convenient, we will replace the indexing set I with the set $S = \{s_i : i \in I\}$ of simple reflections in W . For example, the simple root α_i may appear as α_s for $s = s_i$.

The *real roots* are the vectors in the W -orbit of Π , and the *real co-roots* are the vectors in the W -orbit of Π^\vee . (Imaginary roots make an appearance in Section 4.1.)

The set of all real roots constitutes the (*real*) **root system** Φ associated to A . The root system is a subset of the **root lattice**, the lattice in V generated by Π . The root system is the disjoint union of **positive roots** (roots in the nonnegative linear span of Π) and **negative roots** (roots in the nonpositive linear span of Π). Each root β has a corresponding co-root β^\vee , related by the scaling $\beta^\vee = 2\frac{\beta}{K(\beta,\beta)}$. The set of all co-roots is a root system in its own right, associated to the Cartan matrix A^T . The integer span of the simple co-roots is a lattice called the **co-root lattice**. A **reflection** in W is an element conjugate to an element of S . For each root β , there is a reflection t in W such that $tx = x - K(\beta^\vee, x)\beta$ for all $x \in V$. This defines a bijection between positive roots and reflections in W . We write β_t for the positive root associated to a reflection t .

A **quasi-graph** is a graph with ordinary or **full** edges and also **half-edges**. Half-edges should be thought of as edges that dangle from a vertex without connecting that vertex to any other. More formally, a quasi-graph is a hypergraph with edges of size 1 or 2. We will assume our quasi-graphs to be **simple**: no two edges connect the same pair of vertices, and every full edge connects two distinct vertices. We will also consider only quasi-graphs that are **regular of degree n** , meaning that each vertex is incident to exactly n -edges (i.e. k half-edges and $n - k$ full edges for some k from 0 to n). Finally, we will consider **connected** quasi-graphs, meaning that, deleting half-edges, the remaining graph is connected in the usual sense. A pair (v, e) consisting of a vertex v and an edge e containing v , is called an **incident pair**. Let $I(v)$ denote the set of edges e containing the vertex v .

We will define a reflection framework to be a connected n -regular quasi-graph G together with a labeling C which labels each incident pair in G by a vector $C(v, e)$ in V . The set $\{C(v, e) : e \in I(v)\}$ of labels on a vertex v will be written $C(v)$. The labeling must satisfy certain conditions that we now explain.

Base condition. There exists a vertex v_b of G such that $C(v_b)$ is the set of simple roots of Φ .

In light of the Base condition, we identify the indexing set I with $I(v_b)$. Specifically, we identify $e \in I(v_b)$ with the index $i \in I$ such that $C(v_b, e) = \alpha_i$.

Root condition. Each label $C(v, e)$ is a real root in Φ .

In particular, each label $C(v, e)$ has a corresponding **co-label** $C^\vee(v, e)$, which is the co-root corresponding to the root $C(v, e)$. The set of co-labels on a vertex v is denoted by $C^\vee(v)$.

For the benefit of the reader who is reading this work together with [31], we recall that, in a general framework, a label $C(v, e)$ is not necessarily a root, so there is no reasonable notion of a “co-root corresponding to $C(v, e)$ ”. Instead, the co-labels are defined separately to obey a Co-transition condition that is dual to the Transition condition. In a reflection framework, we can define co-labels more simply as described above. See [31, Proposition 2.13].

The **Euler form** E associated to B is defined by:

$$E(\alpha_i^\vee, \alpha_j) = \begin{cases} \min(b_{ij}, 0) & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j. \end{cases}$$

The symmetric bilinear form K obeys $K(\alpha, \beta) = E(\alpha, \beta) + E(\beta, \alpha)$ for any $\alpha, \beta \in V$. The skew-symmetric bilinear form ω satisfies $\omega(\alpha, \beta) = E(\alpha, \beta) - E(\beta, \alpha)$.

Supposing the Root condition holds for (G, C) , for each vertex v of G we define $C_+(v)$ to be the set of positive roots in $C(v)$ and $C_-(v)$ to be the set of negative roots in $C(v)$. We also define a directed graph $\Gamma(v)$ whose vertex set is $C(v)$ and which has an edge $\beta \rightarrow \beta'$ whenever $\beta \neq \beta'$ and $E(\beta, \beta') \neq 0$.

Euler conditions. Let v be a vertex of G and let e and f be distinct edges incident to v . Write $\beta = C(v, e)$ and $\gamma = C(v, f)$. Then

- (E1) If $\beta \in C_+(v)$ and $\gamma \in C_-(v)$, then $E(\beta, \gamma) = 0$.
- (E2) If $\text{sgn}(\beta) = \text{sgn}(\gamma)$, then $E(\beta, \gamma) \leq 0$.
- (E3) The graph $\Gamma(v)$ is acyclic.

Here $\text{sgn}(\beta) = 1$ if β is a positive root and -1 if β is a negative root.

Reflection condition. Suppose v and v' are distinct vertices incident to the same edge e . If $\beta = C(v, e) = \pm\beta_t$ for some reflection t and $\gamma \in C(v)$, then $C(v')$ contains the root

$$\gamma' = \begin{cases} t\gamma & \text{if } \omega(\beta_t^\vee, \gamma) \geq 0, \text{ or} \\ \gamma & \text{if } \omega(\beta_t^\vee, \gamma) < 0. \end{cases}$$

A **reflection framework** for B is a pair (G, C) consisting of a connected, n -regular quasi-graph G , with a labeling C satisfying the Base condition, the Root condition, the Reflection condition, and the Euler conditions (E1), (E2), and (E3). If a reflection framework exists for B , then in particular the graph $\Gamma(v_b)$ is acyclic. This condition on $\Gamma(v_b)$ is equivalent to saying that we can reorder the rows and columns of B so that $b_{ij} > 0$ when $i < j$. When such a reordering is possible, we say that B is **acyclic**. The more general notion of a **framework** applies to arbitrary exchange matrices B with no requirement of acyclicity. See [31, Section 2] for details.

The following is [31, Proposition 2.4], specialized to reflection frameworks.

Proposition 2.1. *Suppose (G, C) is a reflection framework for B and let v be a vertex of G . Then the label set $C(v)$ is a basis for the root lattice and the co-label set $C^\vee(v)$ is a basis for the co-root lattice.*

In particular, no label occurs twice in $C(v)$. Accordingly, the Reflection condition lets us relate the sets $I(v)$ and $I(v')$ of edges incident to adjacent vertices v and v' . Specifically, if an edge e connects v to v' , we define a function μ_e from $I(v)$ to $I(v')$. We set $\mu_e(e)$ to be e , and for each $f \in I(v) \setminus \{e\}$, taking $\gamma = C(v, f)$, we define $\mu_e(f)$ to be the edge $f' \in I(v')$ such that $C(v', f') = \gamma'$ in the Reflection condition. The following version of the Reflection condition is useful, although it amounts only to a restatement of the definition of μ_e :

Reflection condition, restated. Suppose v and v' are distinct vertices incident to the same edge e . Let $C(v, e) = \pm\beta_t$ for some reflection t . Let $f \in I(v)$ and write γ for $C(v, f)$. Then

$$C(v', \mu_e(f)) = \begin{cases} t\gamma & \text{if } \omega(\beta_t^\vee, \gamma) \geq 0, \text{ or} \\ \gamma & \text{if } \omega(\beta_t^\vee, \gamma) < 0. \end{cases}$$

The **fundamental weights** associated to the root system Φ are the vectors in the basis of V^* that is dual to Π^\vee . The notation ρ_i stands for the fundamental weight that is dual to α_i^\vee in this dual basis. The lattice in V^* generated by the fundamental weights is called the **weight lattice**. Proposition 2.1 implies that

the dual basis to $C^\vee(v)$ is a basis for the weight lattice. Write $R(v)$ for this dual basis and write $R(v, e)$ for the vector dual to $C^\vee(v, e)$ in $R(v)$. The following is [31, Proposition 2.8], specialized to reflection frameworks.

Proposition 2.2. *Let (G, C) be a reflection framework for B and let v and v' be adjacent vertices of G . Then $R(v) \cap R(v')$ contains exactly $n-1$ vectors. Specifically, if e is the edge connecting v to v' and $f \in I(v) \setminus \{e\}$, then $R(v, f) = R(v', \mu_e(f))$. Also, $R(v, e)$ and $R(v', e)$ lie on opposite sides of the hyperplane spanned by $R(v) \cap R(v')$.*

2.2. Polyhedral geometry in V^* . A **closed polyhedral cone** (or simply a **cone**) in V^* is a set of the form $\bigcap \{x \in V^* : \langle x, \beta_i \rangle \geq 0\}$ for vectors β_1, \dots, β_k in V . A subset of V^* is a closed polyhedral cone if and only if it is the nonnegative linear span of a finite set of vectors in V^* . A **simplicial cone** is a cone such that β_1, \dots, β_k form a basis for V . Equivalently, a simplicial cone is the nonnegative linear span of a basis for V^* .

A **face** of a cone F is a subset G of F such that there is a linear functional on V^* that is 0 on G and nonnegative on F . Taking the functional to be zero, we see that F is a face of itself. A face G of F with $\dim(G) = \dim(F) - 1$ is called a **facet** of F . The **relative interior** of a cone is the cone minus (set-theoretically) its proper faces.

Two cones **meet nicely** if their intersection is a face of each of them. A **fan** is a set \mathcal{F} of cones such that (1) if F is in \mathcal{F} , then all faces of F are in \mathcal{F} , and (2) any two cones F_1 and F_2 in \mathcal{F} meet nicely. The fan is **simplicial** if its maximal cones are simplicial. We will use some well-known easy facts: Each face of a cone is itself a cone. The relation “is a face of” is transitive. Two faces of the same cone meet nicely. A set \mathcal{F} of cones satisfying (1) is a fan if and only if any two of its *maximal* cones meet nicely. References for these facts can be found in [31, Section 3.2].

2.3. Cluster algebras of geometric type. As before, take I to be an indexing set with $|I| = n$. Let J be an indexing set, disjoint from I , with $|J| = m$. An **extended exchange matrix** is an $(n+m) \times n$ integer matrix with columns indexed by I and rows indexed by the disjoint union $I \sqcup J$, such that the $n \times n$ submatrix whose rows are indexed by I is skew-symmetrizable. This skew-symmetrizable submatrix is the **underlying exchange matrix**. Let x_i be indeterminates indexed by i in $I \sqcup J$, and let \mathbb{F} be the field of rational functions in these indeterminates, with rational coefficients. A **cluster** is an n -tuple of algebraically independent elements of \mathbb{F} . The individual elements of the cluster are called **cluster variables**. A **seed** is a pair consisting of an extended exchange matrix and a cluster.

Let T be the n -regular tree. For each edge e of T , connecting vertices v and v' , fix a pair of inverse bijections between the set $I(v)$ of edges incident to v and the set $I(v')$. We will call both maps μ_e and let the context distinguish the two. Now fix an **initial seed** (\tilde{B}, X) where \tilde{B} is an extended exchange matrix with underlying exchange matrix B and X is a cluster, which we may as well take to be $x_i : i \in I$. We will associate to each vertex v of T a seed (\tilde{B}^v, X^v) . In each seed (\tilde{B}^v, X^v) , the rows of \tilde{B}^v are indexed by $I(v) \sqcup J$, while the columns of \tilde{B} , as well as the cluster variables, are indexed by $I(v)$. Choose some vertex v_b of T , identify I with $I(v_b)$, and take $(\tilde{B}^{v_b}, X^{v_b})$ to be the initial seed (\tilde{B}, X) . The remaining seeds are defined recursively by **seed mutations**. Let e be an edge connecting v to v' . We extend the map μ_e to a bijection from $I(v) \sqcup J$ to $I(v') \sqcup J$ by letting the map fix J pointwise.

Matrix mutation. The matrices \tilde{B}^v and $\tilde{B}^{v'}$ are related by

$$(2.1) \quad b_{\mu_e(p)\mu_e(q)}^{v'} = \begin{cases} -b_{pq}^v & \text{if } p = e \text{ or } q = e, \\ b_{pq}^v + \text{sgn}(b_{pe}^v)[b_{pe}^v b_{eq}^v]_+ & \text{otherwise,} \end{cases}$$

for $p \in I(v) \sqcup J$ and $q \in I(v)$. The notation $[a]_+$ means $\max(a, 0)$.

Cluster mutation. The clusters X^v and $X^{v'}$ are related by

$$(2.2) \quad x_{\mu_e(q)}^{v'} = \begin{cases} \frac{1}{x_q^v} \left(\prod_p (x_p^v)^{[b_{pe}^v]_+} + \prod_p (x_p^v)^{[-b_{pe}^v]_+} \right) & \text{if } q = e, \\ x_q^v & \text{if } q \neq e, \end{cases}$$

for $q \in I(v)$. The index p runs over the set $I(v) \sqcup J$.

Let $\mathcal{A}(\tilde{B}, X)$ be the subalgebra of \mathbb{F} generated by all of the cluster variables x_e^v , where v runs over all vertices of T and e runs over $I(v)$, and by the x_i for $i \in J$. This is called the **cluster algebra (of geometric type)** associated to the initial seed (\tilde{B}, X) . To see how the cluster algebras of geometric type are a special case of a more general construction, see [13, Section 2]. For a treatment more notationally compatible with the treatment here, see [31, Section 3.1].

Two seeds (\tilde{B}^v, X^v) and $(\tilde{B}^{v'}, X^{v'})$ are **equivalent** if there exists a bijection $\lambda : I(v) \rightarrow I(v')$ such that $b_{\lambda(e)\lambda(f)}^{v'} = b_{ef}^v$ for $e, f \in I(v)$, such that $b_{j,\lambda(f)}^{v'} = b_{jf}^v$ for $j \in J$ and $f \in I(v)$, and such that $x_{\lambda(e)}^{v'} = x_e^v$ for $e \in I(v)$. Such a bijection induces, by seed mutation, a bijection from the neighbors of v to the neighbors of v' with each neighbor of v defining a seed equivalent to the seed at the corresponding neighbor of v' . The **exchange graph** $\text{Ex}(\tilde{B})$ is the quotient of T obtained by identifying vertices v and v' if they define equivalent seeds, and identifying edges of v with edges of v' by the bijection λ . In all of the notation defined above (e.g. the maps μ_e), we can correctly use $\text{Ex}(\tilde{B})$ in place of T .

Specializing, suppose that J is a disjoint copy of I and construct an extended exchange matrix \tilde{B} whose rows indexed by I are B and whose rows indexed by J constitute an identity matrix. A cluster algebra associated to this initial extended exchange matrix is said to have **principal coefficients** at the initial seed. Up to isomorphism, it depends only on B , and thus is denoted $\mathcal{A}_\bullet(B)$. The associated exchange graph is $\text{Ex}_\bullet(B)$. In the principal coefficients case, we write each extended exchange matrix \tilde{B}^v as $\begin{bmatrix} B^v \\ H^v \end{bmatrix}$ such that B^v is the exchange matrix associated to v and H^v is a matrix with rows indexed by I and columns indexed by $I(v)$. The **c-vectors** at the seed v are the vectors whose simple-root coordinates are given by the columns of H^v . Specifically, \mathbf{c}_e^v is the vector given by the column of H^v indexed by $e \in I(v)$.

To each cluster variable x_e^v in a cluster algebra with principal coefficients, there is an associated **g-vector** \mathbf{g}_e^v . Here, we construct the **g-vectors** as vectors in the weight lattice: A cluster variable $x_i^{v_b} = x_i$ in the initial cluster has as its **g-vector** the fundamental weight ρ_i . The other **g-vectors** are defined recursively as follows.

g-vector mutation. Let e be an edge v to v' . The **g-vectors** of the clusters X^v and $X^{v'}$ are related by

$$(2.3) \quad \mathbf{g}_{\mu_e(q)}^{v'} = \begin{cases} -\mathbf{g}_q^v + \sum_{p \in I(v)} [-b_{pq}^v]_+ \mathbf{g}_p^v - \sum_{i \in I} [-b_{iq}^v]_+ \mathbf{b}_i & \text{if } q = e, \\ \mathbf{g}_q^v & \text{if } q \neq e. \end{cases}$$

The notation \mathbf{b}_i stands for the vector in V^* with fundamental-weight coordinates given by the i^{th} column of B .

It is not immediately obvious that this recursion yields a well-defined \mathbf{g} -vector, but \mathbf{g}_e^v is indeed well-defined because it is the degree of x_e^v with respect to a certain \mathbb{Z}^n -grading on the cluster algebra with principal coefficients. See [13, Sections 6–7].

2.4. Frameworks and cluster algebras. We now introduce some additional conditions on frameworks and quote some key results on the connection between frameworks and combinatorial models for cluster algebras. For a more detailed development of these and related results, see [31]. The results *quoted* in this section (Theorems 2.3 and 2.4 and Corollary 2.6) are stated for reflection frameworks, but are proved in [31] for the more general frameworks defined there. In addition, two results are *proved* in this section for reflection frameworks: Proposition 2.7, which holds for general frameworks by essentially the same proof, and Proposition 2.8, which is special to reflection frameworks. (See Remark 2.10.)

A reflection framework (G, C) is **complete** if G has no half-edges. It is **injective** if $v \mapsto C(v)$ is an injective map from vertices of G to subsets of Φ .

Define $\text{Cone}(v)$ to be the cone $\bigcap_{e \in I(v)} \{x \in V^* : \langle x, C^\vee(v, e) \rangle \geq 0\}$ in V^* . We will see soon that this notation is compatible with the earlier definition of $\text{Cone}(v)$ as the cone spanned by the \mathbf{g} -vectors of cluster variables associated to $v \in \text{Ex}_\bullet(B)$. A reflection framework (G, C) is **polyhedral** if (1) the cones $\text{Cone}(v)$ are the maximal cones of a fan in V^* and (2) distinct vertices v of G define distinct cones $\text{Cone}(v)$. In particular, every polyhedral reflection framework is injective. A **well-connected** polyhedral reflection framework has the following additional property: Suppose $\text{Cone}(v)$ and $\text{Cone}(v')$ intersect in a face F . Then there is a path $v = v_0, v_1, \dots, v_k = v'$ in G such that F is a face of $\text{Cone}(v_i)$ for all i from 0 to k .

We now prepare to define a notion of simple connectivity for frameworks, which is described more fully (for general frameworks) in [31, Section 4.2]. Let (G, C) be a reflection framework. Let v_0 be a vertex of G and suppose e_0 and e_1 are two full edges incident to v_0 . Let v_{-1} be the vertex connected to v_0 by e_0 , let $e_{-1} = \mu_{e_0}(e_1)$, let v_{-2} be the vertex connected to v_{-1} by e_{-1} , let $e_{-2} = \mu_{e_{-1}}(e_0)$, etc. Similarly, let v_1 be the vertex connected to v_0 by e_1 , let $e_2 = \mu_{e_1}(e_0)$, let e_2 connect v_1 to a vertex v_2 , etc. We can continue in this manner as long as the edges involved are full edges. If we never encounter half-edges (in particular, if G is complete), then we have defined a path $\dots \xrightarrow{e_{-1}} v_{-1} \xrightarrow{e_0} v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots$ that is either bi-infinite or closes up into a cycle. We refer to $\dots \xrightarrow{e_{-1}} v_{-1} \xrightarrow{e_0} v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots$ as a **rank-two path** or **rank-two cycle** accordingly.

A reflection framework (G, C) is **simply connected** if the fundamental group $\pi_1(G, v_0)$ is generated by paths of the form $\sigma\tau\sigma^{-1}$ where τ travels around a rank-two cycle and σ is some path from the basepoint v_0 to that rank-two cycle. In other words, take the graph G and build a regular CW -complex Σ by filling in 2-dimensional cells whose boundaries are the rank-two cycles. Then (G, C) is simply connected if the topological space Σ is simply connected. See the end of [31, Section 4.2] for discussion of the relationship between Σ and the geometry of the polyhedral fan \mathcal{F} that occurs when (G, C) is polyhedral.

Another condition called **ampleness** is considered in [31], and a framework is called **exact** if it is injective and ample. Here, we wish to avoid defining ampleness, and we can safely do so because of [31, Proposition 4.18], which states that if a

framework is simply connected, then it is ample. Instead, we describe the role that ampleness will play below in the assertion in Theorem 2.3 that a complete, exact reflection framework is isomorphic to the exchange graph: Ampleness insists that the framework not be properly covered by the exchange graph. (Similarly, injectivity forces the exchange graph not to be properly covered by the framework.)

For the present purposes, the following two theorems (stated in the special case of reflection frameworks) are the most important results about frameworks and cluster algebras. For statements that are closer to minimizing the hypotheses, see [31, Sections 3–4]. The first of the two theorems is obtained by combining [31, Theorem 3.25], [31, Theorem 3.26], and [31, Theorem 4.2].

Theorem 2.3. *Suppose (G, C) is a complete, exact reflection framework for B . Then there exists a graph isomorphism $v \mapsto \text{Seed}(v) = (\tilde{B}^v, X^v)$ from G to the principal coefficients exchange graph $\text{Ex}_\bullet(B)$, such that the base vertex v_b maps to the initial seed. Furthermore, if v is any vertex of G ,*

- (1) *The exchange matrix $B^v = [b_{ef}^v]_{e,f \in I(v)}$ associated to $\text{Seed}(v)$ has $b_{ef}^v = \omega(C^\vee(v, e), C(v, f))$.*
- (2) *For each $e \in I(v)$, the \mathbf{c} -vector \mathbf{c}_e^v is $C(v, e)$.*
- (3) *Each \mathbf{c} -vector \mathbf{c}_e^v has a definite sign: it is either in the nonnegative span of the simple roots or in the nonpositive span of the simple roots.*
- (4) *If $X^v = (x_e^v : e \in I(v))$ is the cluster in $\text{Seed}(v)$, then for each $e \in I(v)$, the \mathbf{g} -vector $\mathbf{g}_e^v = \mathbf{g}(x_e^v)$ is $R(v, e)$.*

The second of the two theorems is obtained by combining [31, Theorem 4.1], [31, Corollary 4.4], and [31, Corollary 4.6]. It states that the existence of a framework with good properties implies many of the standard structural conjectures.

Theorem 2.4. *If a complete, exact, well-connected polyhedral reflection framework exists for B , then Conjectures 3.9–3.13, 3.15–3.18, and 3.20 of [31] all hold for principal coefficients at B . If, in addition, a complete reflection framework exists for $-B$, then Conjecture 3.19 of [31] also holds for principal coefficients at B . Furthermore, the fan defined by the framework is identical to the fan defined by \mathbf{g} -vectors of clusters in $\mathcal{A}_\bullet(B)$.*

The results quoted here show how Corollaries 1.2 and 1.3 follow from Theorem 1.1. Specifically, Theorems 1.1 and 2.3 immediately imply Corollary 1.2. We will construct the doubled Cambrian framework $(\text{DCamb}_c, \text{DC}_c)$ and the doubled Cambrian fan \mathcal{DF}_c in such a way that \mathcal{DF}_c is the fan defined by the framework $(\text{DCamb}_c, \text{DC}_c)$. Thus Theorem 1.1 and the last assertion of Theorem 2.4 immediately imply Corollary 1.3. We also have the following corollary.

Corollary 2.5. *If B is acyclic and $\text{Cart}(B)$ is of affine type, then 3.9–3.20 of [31] all hold for principal coefficients at B .*

Almost all of the assertions of Corollary 2.5 follow from Theorem 2.4 and from the existence of a framework satisfying the properties established in Theorem 1.1. The exception is the assertion that Conjecture 3.14 of [31] holds. This is very easily argued using Theorem 2.3, but requires information about the actual construction of the doubled Cambrian framework. We provide this argument in Section 4.6.

We close this section with some results related to the polyhedral property and to rank-two cycles and paths. First, a weakening of the polyhedral property holds

in all frameworks, as stated in the following corollary. This result is a corollary of Proposition 2.2 and appears as [31, Corollary 2.9].

Corollary 2.6. *Let (G, C) be a reflection framework for B and let v and v' be adjacent vertices of G . Then $\text{Cone}(v)$ and $\text{Cone}(v')$ intersect in a common facet.*

Next, we prove two propositions about rank-two cycles in complete, well-connected polyhedral reflection frameworks. The following is a slight strengthening of [31, Proposition 4.17], stated in the special case of reflection frameworks.

Proposition 2.7. *Suppose (G, C) is a complete, well-connected polyhedral reflection framework with corresponding fan \mathcal{F} . If τ is a rank-two cycle or path, then there exists a unique codimension-2 face F of \mathcal{F} such that τ is the set of vertices v of G with $F \subset \text{Cone}(v)$. If F is a codimension-2 face of \mathcal{F} , then the set of vertices v of G with $F \subset \text{Cone}(v)$ forms a rank-two cycle or path.*

Proof. Suppose $\tau = (\dots, v_{-1}, v_0, v_1, \dots)$ is a rank-two cycle/path. The Reflection condition in restated form implies that

$$\text{Span}_{\mathbb{R}}(C(v_i, e_i), C(v_i, e_{i+1})) = \text{Span}_{\mathbb{R}}(C(v_{i+1}, e_{i+1}), C(v_{i+1}, e_{i+2})).$$

So the 2-plane $\text{Span}_{\mathbb{R}}(C(v_i, e_i), C(v_i, e_{i+1}))$ is the same for all i and, dually, the $(n - 2)$ -dimensional subspace $C(v_i, e_i)^\perp \cap C(v_{i+1}, e_{i+1})^\perp$ is the same for all i . Call this $(n - 2)$ -dimensional subspace R . Then $\text{Cone}(v_i) \cap R$ is an $(n - 2)$ -dimensional face of $\text{Cone}(v_i)$. Moreover, this face of $\text{Cone}(v_i)$ is a boundary face of the common facet of $\text{Cone}(v_i)$ and $\text{Cone}(v_{i+1})$, so that $\text{Cone}(v_i) \cap R$ is also a face of $\text{Cone}(v_{i+1})$, and $\text{Cone}(v_i) \cap R = \text{Cone}(v_{i+1}) \cap R$. Setting $F = \text{Cone}(v_0) \cap R$, we see that F is a face of every $\text{Cone}(v_i)$.

We now want to show that, if u is any vertex of G , with $\text{Cone}(u) \supseteq F$, then u is in τ . By well-connectedness, there is a path $u = u_0, \dots, u_N = v_0$ such that F is contained in all of the $\text{Cone}(u_i)$. If u is not of the form v_i , then there must be an index j for which $u_j \notin \{v_i\}$ but $u_{j+1} = v_i$ for some i . But the only neighbors of v_i whose cones contain F are $v_{i\pm 1}$. This contradiction shows that τ is the set of vertices v of G with $F \subset \text{Cone}(v)$.

Finally, let F be any codimension-2 face of \mathcal{F} . We want to show that the set $\{v \in G : F \subset \text{Cone}(v)\}$ forms a rank-two path or cycle. Let $\text{Cone}(v_0)$ be a maximal cone of \mathcal{F} containing F . Since (G, C) is complete, by Proposition 2.2, there are two distinct vertices v' with $\text{Cone}(v')$ adjacent to $\text{Cone}(v_0)$ and $F \subseteq \text{Cone}(v')$. Write v_{-1} and v_1 for these vertices and write $\tau = (\dots, v_{-1}, v_0, v_1, \dots)$ for the rank-two cycle/path through v_{-1} , v_0 , and v_1 . (Since (G, C) is complete, there are no half-edges and we can continue this cycle/path indefinitely.) Then, by the preceding paragraphs, τ is precisely the set of vertices v of G for which $F \subset \text{Cone}(v)$. \square

Proposition 2.8. *Suppose (G, C) is a complete, well-connected polyhedral reflection framework with corresponding fan \mathcal{F} . Suppose F is a codimension-2 face of \mathcal{F} and let τ be the corresponding rank-two path or cycle. Let Φ' be the rank-two root system $\{\beta \in \Phi : \langle \beta, F \rangle = 0\}$. Then τ is a cycle if and only if Φ' is of finite type.*

Remark 2.9. The set Φ' is the set of roots of Φ contained in a 2-dimensional linear subspace of V . The results of [8, 9] show in particular that the intersection of Φ with any linear subspace is a root system in a broader sense than that of this paper (Section 2.1). However, when the subspace is 2-dimensional, the intersection is a root system in our sense. (See [29, Remark 2.13] for an example of what can go wrong when the subspace has dimension greater than 2.)

Remark 2.10. Proposition 2.8 is special to reflection frameworks, and need not hold for a general framework. (See [31, Remark 4.11].) We emphasize that, in any framework, a rank-two path or cycle $\dots, v_{-1}, v_0, v_1, \dots$ is a rank-two cycle, by definition, if it forms a finite cycle in the graph G , whether or not Φ' is of finite type.

Remark 2.11. In Proposition 2.8 when τ is a cycle and Φ' is of finite type, one can describe the link of F in \mathcal{F} using [31, Lemmas 4.9–4.10]. The link of F in \mathcal{F} is isomorphic to the \mathfrak{g} -vector fan associated to an exchange matrix B' such that $\text{Cart}(B')$ is the Cartan matrix for Φ' .

To prove Proposition 2.8, we quote the following technical result, which is part of [31, Corollary 4.13] restricted to reflection frameworks.

Lemma 2.12. *Let (G, C) be a reflection framework and let $\tau = (\dots, v_{-1}, v_0, v_1, \dots)$ be a rank-two cycle. Write e and f for the edges in τ incident to v_0 . Define $\beta = C(v_0, e)$ and $\gamma = C(v_0, f)$ and $b = \omega(\beta^\vee, \gamma)$ and $b' = \omega(\gamma^\vee, \beta)$. Then $\begin{bmatrix} 2 & -|b'| \\ -|b| & 2 \end{bmatrix}$ is a Cartan matrix of finite type.*

Proof of Proposition 2.8. If $\tau = (\dots, v_{-1}, v_0, v_1, \dots)$ is a rank-two cycle, define $e, f, \beta, \gamma, b,$ and b' as in Lemma 2.12. That lemma says that the Cartan matrix $\tilde{A} := \begin{bmatrix} 2 & -|b'| \\ -|b| & 2 \end{bmatrix}$ is of finite type. The Euler condition (E3) implies that $K(\beta^\vee, \gamma) = \pm b$ and $K(\gamma^\vee, \beta) = \pm b'$, with the same sign. Proposition 2.1 implies that β and γ are an integer basis for the lattice spanned by Φ' . Therefore \tilde{A} is the Cartan matrix of Φ' . We see that if τ is a cycle, then Φ' is of finite type.

Now suppose that $\tau = (\dots, v_{-1}, v_0, v_1, \dots)$ is an infinite path. Let X be the set of roots β such that β lies in some $C(v_i)$ and $F \subset \beta^\perp$. Since (G, C) is polyhedral, the set X is infinite. However, the Reflection condition implies that $X \subset \Phi'$, and thus Φ' is infinite. □

3. DOUBLED CAMBRIAN FRAMEWORKS

In this section, we review the definition of Cambrian frameworks, and use them to construct the doubled Cambrian framework $(\text{DCamb}_c, \text{DC}_c)$ for any acyclic exchange matrix B . We then prove the part of Theorem 1.1 that doesn't depend on the hypothesis that $\text{Cart}(B)$ is of affine type.

3.1. Cambrian frameworks. Let B be an acyclic exchange matrix with Cartan companion $A = \text{Cart}(B)$ and let W be the associated Coxeter group. We now review, from [31], the construction of a reflection framework for B called the **Cambrian framework**. Cambrian frameworks are defined using a Coxeter element which encodes the orientation of B , as we will now explain.

A **Coxeter element** of W is an element c that can be expressed as a product of the simple reflections S , in some order, with each simple reflection occurring exactly once. The information contained in B is equivalent to the information (A, c) , where c is the Coxeter element obtained by multiplying S in an order such that s_i precedes s_j whenever $b_{ij} > 0$. The acyclicity of B implies that S can be ordered by that rule. Two different orders satisfying that rule multiply to the same Coxeter element because $b_{ij} = 0$ implies $A_{ij} = 0$, which implies that s_i and s_j commute.

TABLE 1. Sign conventions for Coxeter elements and bilinear forms

If $b_{ij} > 0$, then $c = \cdots s_i \cdots s_j \cdots$. $E_c(\alpha_i^\vee, \alpha_j) = 0$. $\omega_c(\alpha_i^\vee, \alpha_j) > 0$.	If $b_{ij} < 0$, then $c = \cdots s_j \cdots s_i \cdots$. $E_c(\alpha_i^\vee, \alpha_j) < 0$. $\omega_c(\alpha_i^\vee, \alpha_j) < 0$.
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We write ω_c and E_c for the bilinear forms ω and E to emphasize their dependence on c . Table 1 emphasizes our convention on how B defines a Coxeter element together with our other related conventions.

An *initial* letter in a Coxeter element c is an element s of S such that c has an expression as a product of the elements of S , beginning with s . Similarly, a *final* letter in c is a simple reflection s that can occur last in an expression for c as a product of the elements of S . If s is an initial or final letter of c , then scs is another Coxeter element.

A *reduced word* for an element $w \in W$ is a word in the generators S that is of minimal length among words for w . The *length* of w is defined to be the length of a reduced word for w , and is written $\ell(w)$. The *(right) weak order* is a partial order on W defined as the transitive closure of all relations of the following form: $w < ws$ for $w \in W$ and $s \in S$ with $\ell(w) < \ell(ws)$. The weak order is a meet semilattice in general and a lattice if W is finite. Furthermore, every nonempty set U has a meet $\bigwedge U$, and if U has an upper bound, then it has a join $\bigvee U$.

For each subset $J \subseteq S$, the subgroup W_J of W generated by J is called a *standard parabolic subgroup*. The subgroup W_J is a Coxeter group in its own right, with simple reflections J , and is an order ideal in the weak order on W . The root system Φ_J associated to W_J is the intersection $\Phi \cap V_J$, where V_J is the span of $\{\alpha_s : s \in J\}$. Given $w \in W$ and $J \subseteq S$, there exists a unique maximal element w_J among elements of W_J below w in the weak order. We call w_J the *projection* of w to W_J .

A Coxeter element c of W induces a Coxeter element of W_J called the *restriction* of c to W_J . (This is usually not the projection c_J .) The restriction of c to W_J is obtained by deleting the letters in $S \setminus J$ from a reduced word for c . We define the notation $\langle s \rangle$ to stand for $S \setminus \{s\}$.

We give here a recursive definition of *c-sortable elements*, by induction on the rank of W (the cardinality n of S) and on the length of elements of W . First, the identity element is c -sortable for any c . If s is initial in c , then we can decide recursively whether w is c -sortable, by considering two cases: If $w \not\geq s$, then w is c -sortable if and only if it is in $W_{\langle s \rangle}$ and is sc -sortable. If $w \geq s$, then w is c -sortable if and only if sw is scs -sortable. The condition $w \geq s$ is equivalent to $\ell(sw) = \ell(w) - 1$. The recursion involves either deciding sortability in a Coxeter group $W_{\langle s \rangle}$ of rank $n - 1$ or deciding sortability for an element sw of length $\ell(w) - 1$, and thus terminates. The consistency of these recursive requirements follows from a nonrecursive definition of c -sortable elements, found for example in [31, Section 5.1].

The vertices of the c -Cambrian framework are the c -sortable elements. Before defining the quasi-graph structure on c -sortable elements, we recursively define a label set for each c -sortable element. Let v be c -sortable and let s be initial in c .

Define

$$(3.1) \quad C_c(v) = \begin{cases} C_{sc}(v) \cup \{\alpha_s\} & \text{if } v \not\geq s, \\ sC_{scs}(sv) & \text{if } v \geq s. \end{cases}$$

The set $C_{sc}(v)$ is a set of roots in $\Phi_{(s)}$, defined by induction on the rank of W . The set $C_{scs}(sv)$ is defined by induction on the length of v . The base of the inductive definition is that the unique element of the trivial Coxeter group has empty label set or, for those who dislike such stark minimalism, that the identity is labeled by the set of simple roots.

The nonrecursive definition of c -sortable elements constructs a special reduced word for each c -sortable element, called its *c -sorting word*. The set $C_c(v)$ can also be defined nonrecursively in terms of the combinatorics of the c -sorting word for v . For details, see [29, Section 5] or [31, Section 5.1].

The quasi-graph structure on c -sortable elements is obtained from the restriction of the weak order to c -sortable elements. This restriction is called the *c -Cambrian semilattice* and denoted Camb_c . The c -Cambrian semilattice is a sub-meet-semilattice [29, Theorem 7.1] of the weak order on W . We will also use the symbol Camb_c for the undirected Hasse diagram of the c -Cambrian semilattice.

The following lemma, which is part of [31, Lemma 5.11], is the key to constructing and labeling an n -regular quasi-graph from the c -Cambrian semilattice. The symbol \prec denotes a cover relation.

Lemma 3.1. *If $v' \prec v$ in the c -Cambrian semilattice, then there exists a unique root β such that $\beta \in C_c(v')$ and $-\beta \in C_c(v)$. The root β is positive.*

Suppose v is a c -sortable element. In light of Lemma 3.1, we can label each incident pair (v, e) , where e is an edge $v' \prec v$ in Camb_c , by the unique negative root $\beta \in C_c(v)$ such that $-\beta \in C_c(v')$. Similarly, we can label each incident pair (v, e) , where e is an edge $v \prec v'$ in Camb_c by the unique positive root $\beta \in C_c(v)$ such that $-\beta \in C_c(v')$. As explained in the paragraphs before [31, Theorem 5.12], the vertices in the graph Camb_c all have degree at most n . To each vertex, we add the appropriate number of half-edges, to make an n -regular quasi-graph. The new half-edges on v get the roots in $C_c(v)$ that were not assigned to full edges. We will need one more condition on a framework.

Full edge condition. If (v, e) is an incident pair and the sign of $C(v, e)$ is negative, then e is a full edge.

Most of the following theorem is stated as [31, Corollary 5.14], but the assertion about the Full edge condition follows from [31, Theorem 5.12]. (That theorem states that (Camb_c, C_c) is a “descending” framework, and the Full edge condition is part of the definition of a descending framework.)

Theorem 3.2. *The pair (Camb_c, C_c) is an exact, well-connected, polyhedral, simply connected reflection framework for B and satisfies the Full edge condition.*

For each c -sortable element v , define a cone

$$\text{Cone}_c(v) = \bigcap_{\beta \in C_c(v)} \{x \in V^* : \langle x, \beta \rangle \geq 0\}.$$

The cone $\text{Cone}_c(v)$ coincides with the cone $\text{Cone}(v)$ defined in Section 2.4. The *c -Cambrian fan* \mathcal{F}_c is the collection consisting of the cones $\text{Cone}_c(v)$ and all of

their faces, with v running over all c -sortable elements. The following result is part of [31, Corollary 5.15].

Theorem 3.3. *The c -Cambrian fan \mathcal{F}_c is a simplicial fan, and distinct vertices of Camb_c label distinct maximal cones of \mathcal{F}_c .*

We now fill in some additional results on sortable elements, Cambrian semilattices, and the Cambrian framework. The following propositions are [29, Proposition 2.30] and [29, Proposition 3.13].

Proposition 3.4. *Let $J \subseteq S$ and let c' be the restriction of c to W_J . Then an element $v \in W_J$ is c -sortable if and only if it is c' -sortable.*

Proposition 3.5. *If v is c -sortable and $J \subseteq S$, then v_J is c' -sortable, where c' is the restriction of c to W_J .*

Here v_J is the projection of v to the standard parabolic subgroup W_J . See Section 3.1.

The following theorems are [29, Theorem 7.1] and [29, Theorem 7.4].

Theorem 3.6. *Let U be a collection of c -sortable elements of W . If U is nonempty, then $\bigwedge U$ is c -sortable. If U has an upper bound, then $\bigvee U$ is c -sortable.*

Theorem 3.7. *If s is initial in c , then the map*

$$v \mapsto \begin{cases} sv & \text{if } v \geq s, \\ s \vee v & \text{if } v \not\geq s, \end{cases}$$

is a bijection from the set $\{c\text{-sortable elements } v \text{ such that } s \vee v \text{ exists}\}$ to the set of all scs -sortable elements. The inverse map is

$$x \mapsto \begin{cases} sx & \text{if } x \not\geq s, \\ x_{\langle s \rangle} & \text{if } x \geq s. \end{cases}$$

An **inversion** of $w \in W$ is a reflection t such that $\ell(tw) < \ell(w)$. Inversions interact nicely with the map $w \mapsto w_J$ for any $J \subseteq S$. Specifically, $\text{inv}(w_J) = \text{inv}(w) \cap W_J$.

A **cover reflection** of $w \in W$ is an inversion t of w such that $tw = ws$ for some $s \in S$. The name refers to the fact that $tw < w$ in the weak order if and only if t is a cover reflection of w . We write $\text{inv}(w)$ for the set of inversions of w and $\text{cov}(w)$ for the set of cover reflections of w . We have $\text{inv}(tw) = \text{inv}(w) \setminus \{t\}$ when t is a cover reflection of w .

The following propositions are first, [29, Proposition 5.2], second, the combination of parts of [29, Proposition 5.1] and [29, Proposition 5.4], and third, the combination of parts of [29, Proposition 5.1] and [29, Proposition 5.3].

Proposition 3.8. *Let v be a c -sortable element. The set of negative roots in $C_c(v)$ is $\{-\beta_t : t \in \text{cov}(v)\}$.*

Proposition 3.9. *Let s be initial in c and let v be a c -sortable element of W such that s is a cover reflection of v . Then $\text{cov}(v) = \{s\} \cup \text{cov}(v_{\langle s \rangle})$, and the set of positive roots in $C_c(v)$ is obtained by applying the reflection s to each positive root in $C_{sc}(v_{\langle s \rangle})$.*

Proposition 3.10. *Let s be final in c and let v be c -sortable with $v \geq s$. Then $v = s \vee v_{\langle s \rangle}$ and $C_c(v) = C_{cs}(v_{\langle s \rangle}) \cup \{-\alpha_s\}$.*

We will also need a certain downward projection map from W to c -sortable elements in W . Given $w \in W$, by [29, Corollary 6.2], there is an element $\pi_{\downarrow}^c(w)$ that is maximal among the c -sortable elements that are $\leq w$. The map π_{\downarrow}^c lets us describe the cover relations in Camb_c . The description is proved as [31, Lemma 5.9].

Lemma 3.11. *Let v be a c -sortable element. Then $v' \triangleleft v$ in the c -Cambrian semilattice if and only if $v' = \pi_{\downarrow}^c(tv)$ for some $t \in \text{cov}(v)$. Furthermore if t_1 and t_2 are distinct cover reflections of v , then $\pi_{\downarrow}^c(t_1v) \neq \pi_{\downarrow}^c(t_2v)$.*

Recall that Lemma 3.1 is part of [31, Lemma 5.11]. The rest of [31, Lemma 5.11] gives some more detail:

Lemma 3.12. *If $v' \triangleleft v$ in the c -Cambrian semilattice, then the unique root β such that $\beta \in C_c(v')$ and $-\beta \in C_c(v)$ is the positive root β_t associated to the cover reflection t of v such that $v' = \pi_{\downarrow}^c(tv)$.*

The following theorem is [29, Theorem 6.1].

Theorem 3.13. π_{\downarrow}^c is order preserving.

Let D be the cone $\bigcap_{s \in S} \{x \in V^* : \langle x, \alpha_s \rangle \geq 0\}$. The cones wD , for $w \in W$, are distinct, and form a fan whose support is called the **Tits cone**. (Here the action of W on V^* is dual to its action on V .)

The following theorem is [29, Theorem 6.3].

Theorem 3.14. *Let v be c -sortable. Then $\pi_{\downarrow}^c(w) = v$ if and only if $wD \subseteq \text{Cone}_c(v)$.*

We record the following simple observation.

Proposition 3.15. *If $w \in W$ and $J \subseteq S$, then $\pi_{\downarrow}^c(w_J) = (\pi_{\downarrow}^c(w))_J$.*

Proof. Since $w \geq w_J$, Theorem 3.13 says that $\pi_{\downarrow}^c(w) \geq \pi_{\downarrow}^c(w_J)$. Since $x \mapsto x_J$ is order preserving, since $\pi_{\downarrow}^c(w_J) \in W_J$, and since W_J is an order ideal in the weak order on W , we have $(\pi_{\downarrow}^c(w))_J \geq (\pi_{\downarrow}^c(w_J))_J = \pi_{\downarrow}^c(w_J)$. On the other hand, since $\pi_{\downarrow}^c(w) \leq w$, we have $(\pi_{\downarrow}^c(w))_J \leq w_J$, and $(\pi_{\downarrow}^c(w))_J$ is c -sortable by Propositions 3.4 and 3.5. But $\pi_{\downarrow}^c(w_J)$ is the maximal c -sortable element below w_J , so $\pi_{\downarrow}^c(w_J) \geq (\pi_{\downarrow}^c(w))_J$. \square

Let $w \in W$. Earlier, we used the notation $\text{inv}(w)$ to denote the inversions of W as a set of reflections. It is convenient to use the bijection between reflections and positive roots to also think of $\text{inv}(w)$ as a set of positive roots. We observe that $\text{inv}(w)$ is the set of positive roots β such that wD is contained in $\{x \in V^* : \langle x, \beta \rangle \leq 0\}$.

Let β and γ be distinct, positive, real roots in Φ such that all of the positive roots in the linear span of β and γ are in the *nonnegative* span of β and γ . We call $\Phi' = \text{Span}_{\mathbb{R}}(\beta, \gamma) \cap \Phi$ a **generalized rank-two parabolic** sub root system of Φ and call β and γ the **canonical roots** of Φ' . An element $w \in W$ is **c -aligned with respect to Φ'** if one of the following cases holds:

- (i) $\omega_c(\beta, \gamma) = 0$ and $\text{inv}(w) \cap \Phi'$ is \emptyset or $\{\beta\}$ or $\{\gamma\}$ or $\{\beta, \gamma\}$.
- (ii) $\omega_c(\beta, \gamma) > 0$ and either $\gamma \notin \text{inv}(w)$ or $\text{inv}(w) \cap \Phi' = \{\gamma\}$ or $\text{inv}(w) \supseteq \Phi'$.
- (iii) $\omega_c(\beta, \gamma) < 0$ and either $\beta \notin \text{inv}(w)$ or $\text{inv}(w) \cap \Phi' = \{\beta\}$ or $\text{inv}(w) \supseteq \Phi'$.

The last option in Case (i) can only occur if β and γ are perpendicular. However, it is possible to have $\omega_c(\beta, \gamma) = 0$ even when β and γ are not perpendicular. (For example, looking forward to Examples 3.22 and 4.10, define $\beta = s_3s_1\alpha_2$ and

$\gamma = s_2 s_2 \alpha_2$. Then $\omega_c(\beta^\vee, \gamma) = \omega_c(3\alpha_1^\vee + \alpha_2^\vee + 3\alpha_3^\vee, \alpha_1 + 2\alpha_2) = 0$, but $K(\beta^\vee, \gamma) = -2$. The last options in Cases (ii) and (iii) can only occur if Φ' is finite.

The following is a slight weakening of [29, Theorem 4.3].

Theorem 3.16. *An element w of W is c -sortable if and only if w is c -aligned with respect to every generalized rank-two parabolic sub root system of Φ .*

We now prove three key lemmas that relate the cones $\text{Cone}_c(v)$ to the hyperplanes α_s^\perp . Given a set $U \subseteq V^*$ and a root $\beta \in \Phi$, we say that U is **above** the hyperplane β^\perp if every point in U is either in β^\perp or on the opposite side of β^\perp from D . If U is above β^\perp and does not intersect β^\perp , then we say that U is **strictly above** β^\perp . Similarly, U is **below** β^\perp if every point in U is either contained in β^\perp or on the same side of β^\perp as D , and U is **strictly below** β^\perp if U is below β^\perp and does not intersect β^\perp .

Lemma 3.17. *Let c be a Coxeter element, let $s \in S$, and let v be c -sortable. If $v \not\geq s$, then $\text{Cone}_c(v)$ is below α_s^\perp , and if $v \geq s$, then $\text{Cone}_c(v)$ is above α_s^\perp .*

We prove Lemma 3.17 by quoting two nontrivial results and giving an easy argument. Proposition 3.18 is [31, Corollary 5.21], and Proposition 3.19 appears as the second assertion of [31, Corollary 5.15]. See also Theorem 2.3(4).

Proposition 3.18. *Given any vertex v in the Cambrian framework (Camb_c, C_c) and any $s \in S$, all of the \mathbf{g} -vectors of cluster variables in $\text{Seed}(v)$ are weakly on the same side of the hyperplane α_s^\perp in V^* .*

Proposition 3.19. *Given any vertex v in the Cambrian framework (Camb_c, C_c) , the extreme rays of the cone $\text{Cone}_c(v)$ are spanned by the \mathbf{g} -vectors of cluster variables in $\text{Seed}(v)$.*

Proof of Lemma 3.17. Propositions 3.19 and 3.18 combine to establish the following assertion: For any c -sortable element v and any $s \in S$, the cone $\text{Cone}_c(v)$ is either above or below α_s^\perp . If $v \not\geq s$, then vD is below α_s^\perp , so Theorem 3.14 implies that $\text{Cone}_c(v)$ is below α_s^\perp . Similarly, if $v \geq s$, then vD is above α_s^\perp , so Theorem 3.14 implies that $\text{Cone}_c(v)$ is above α_s^\perp . \square

Lemma 3.20. *Let s be initial in c . Let v be c -sortable with $v \geq s$. Then $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(v_{\langle s \rangle})$.*

Proof. Proposition 3.5 says that $v_{\langle s \rangle}$ is sc -sortable, and therefore it is c -sortable by Proposition 3.4. In particular, the notation $\text{Cone}_c(v_{\langle s \rangle})$ makes sense.

By Lemma 3.17, $\text{Cone}_c(v)$ is above α_s^\perp . In particular, $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(v)$, which we call F . We first claim that $F \subseteq \text{Cone}_c(v_{\langle s \rangle})$. To prove the claim, we argue by induction on $\ell(v)$ and break into two cases. The base case $v = s$ falls into Case 1, where we do not appeal to induction.

Case 1. F is a facet of $\text{Cone}_c(v)$. Then $-\alpha_s \in C_c(v)$, so s is a cover reflection of v by Proposition 3.8. Equation (3.1) says that $C_c(v_{\langle s \rangle}) = C_{sc}(v_{\langle s \rangle}) \cup \{\alpha_s\}$. By Proposition 3.9, each element of $C_{sc}(v_{\langle s \rangle})$ either coincides with an element of $C_c(v) \setminus \{-\alpha_s\}$ or is related to an element of $C_c(v) \setminus \{-\alpha_s\}$ by the reflection s . Thus the defining inequalities of $\text{Cone}_c(v)$ and $\text{Cone}_c(v_{\langle s \rangle})$ coincide for points on α_s^\perp , so the claim is true in this case.

Case 2. $\dim(F) < n - 1$. Write $F = \text{Cone}_c(v) \cap (\beta_1^\perp \cap \cdots \cap \beta_k^\perp)$ for distinct $\beta_1, \dots, \beta_k \in C_c(v)$. Let y be in the relative interior of F and let x be in the interior of D . If every β_i is positive, then in particular, x is strictly below each β_i^\perp . But also, $\text{Cone}_c(v)$ is weakly below each β_i^\perp , and therefore for small enough $\varepsilon > 0$, the point $y + \varepsilon x$ is in the interior of $\text{Cone}_c(v)$. But $y + \varepsilon x$ is strictly below α_s^\perp , and this contradiction to Lemma 3.17 shows we can choose i such that $\beta_i = -\beta_t$ for some reflection t . Let $u = \pi_\downarrow^c(tv)$, which, by Lemma 3.11, is covered by v in Camb_c . Corollary 2.6 (in light of Theorem 3.2) says that $\text{Cone}_c(v)$ and $\text{Cone}_c(u)$ intersect in a facet, so that in particular F is a face of $\text{Cone}_c(u)$. By induction, $F' = \text{Cone}_c(v') \cap \alpha_s^\perp$ is contained in $\text{Cone}_c(u_{\langle s \rangle})$, but $F \subseteq F'$, so $F \subseteq \text{Cone}_c(u_{\langle s \rangle})$.

Now, Proposition 3.15 says that $u_{\langle s \rangle} = \pi_\downarrow^c((tv)_{\langle s \rangle})$ and that $v_{\langle s \rangle} = (\pi_\downarrow^c(v))_{\langle s \rangle} = \pi_\downarrow^c(v_{\langle s \rangle})$. If $(tv)_{\langle s \rangle} = v_{\langle s \rangle}$, then $u_{\langle s \rangle} = v_{\langle s \rangle}$, and we are done. If not, then since $\text{inv}(v_{\langle s \rangle}) = \text{inv}(v) \cap W_{\langle s \rangle}$ and $\text{inv}((tv)_{\langle s \rangle}) = \text{inv}(tv) \cap W_{\langle s \rangle}$, and since $\text{inv}(tv) = \text{inv}(v) \setminus \{t\}$, we conclude that $\text{inv}((tv)_{\langle s \rangle}) = \text{inv}(v_{\langle s \rangle}) \setminus \{t\}$. In particular, the reflection t is in $W_{\langle s \rangle}$, and $t(v_{\langle s \rangle}) = (tv)_{\langle s \rangle}$ is covered by $v_{\langle s \rangle}$. Thus $u_{\langle s \rangle} \prec v_{\langle s \rangle}$ in Camb_c by Lemma 3.11, and Corollary 2.6 says that $\text{Cone}_c(u_{\langle s \rangle})$ and $\text{Cone}_c(v_{\langle s \rangle})$ intersect in a mutual facet defined by β_t^\perp . Therefore $F \subseteq \text{Cone}_c(v_{\langle s \rangle})$.

Having proven the claim, we prove the lemma. Since $\text{Cone}_c(v) \cap \alpha_s^\perp \subseteq \text{Cone}_c(v_{\langle s \rangle})$, the intersection $\text{Cone}_c(v) \cap \alpha_s^\perp$ equals $(\text{Cone}_c(v) \cap \text{Cone}_c(v_{\langle s \rangle})) \cap (\text{Cone}_c(v_{\langle s \rangle}) \cap \alpha_s^\perp)$. Theorem 3.3 implies that $\text{Cone}_c(v) \cap \text{Cone}_c(v_{\langle s \rangle})$ is a face of $\text{Cone}_c(v_{\langle s \rangle})$. By Lemma 3.17, $\text{Cone}_c(v_{\langle s \rangle})$ is below α_s^\perp , so $\text{Cone}_c(v_{\langle s \rangle}) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(v_{\langle s \rangle})$. We have written $\text{Cone}_c(v) \cap \alpha_s^\perp$ as an intersection of faces of $\text{Cone}_c(v_{\langle s \rangle})$. \square

Lemma 3.21. *Let s be final in c . Let v be c -sortable with $v \not\geq s$. Then $s \vee (sv)_{\langle s \rangle}$ exists, and is c -sortable, and $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(s \vee (sv)_{\langle s \rangle})$.*

Proof. Since $sv \geq s$ and $sv > (sv)_{\langle s \rangle}$, we know that $s \vee (sv)_{\langle s \rangle}$ exists. By the recursive definition of sortability, the element sv is scs -sortable, so $(sv)_{\langle s \rangle}$ is cs -sortable by Proposition 3.5, and thus c -sortable by Proposition 3.4. Now $s \vee (sv)_{\langle s \rangle}$ is c -sortable by Theorem 3.6.

By Lemma 3.20, $\text{Cone}_{scs}(sv) \cap \alpha_s^\perp$ is a face of $\text{Cone}_{scs}((sv)_{\langle s \rangle})$. Because $\text{Cone}_c(v)$ equals $s\text{Cone}_{scs}(sv)$ by (3.1), the intersection $\text{Cone}_c(v) \cap \alpha_s^\perp$ equals $\text{Cone}_{scs}(sv) \cap \alpha_s^\perp$. Thus $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_{scs}((sv)_{\langle s \rangle})$.

By (3.1), we have $C_{scs}((sv)_{\langle s \rangle}) = C_{cs}((sv)_{\langle s \rangle}) \cup \{\alpha_s\}$. On the other hand, Proposition 3.10 (with $s \vee (sv)_{\langle s \rangle}$ replacing v) says that $C_c(s \vee (sv)_{\langle s \rangle})$ equals $C_{cs}((sv)_{\langle s \rangle}) \cup \{-\alpha_s\}$. We see that $\text{Cone}_{scs}((sv)_{\langle s \rangle})$ and $\text{Cone}_c(s \vee (sv)_{\langle s \rangle})$ intersect in a common facet F defined by α_s^\perp . Since $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_{scs}((sv)_{\langle s \rangle})$ contained in α_s^\perp , it is also a face of F , and thus a face of $\text{Cone}_c(s \vee (sv)_{\langle s \rangle})$. \square

3.2. Doubling the Cambrian fan. We construct the doubled Cambrian framework by “gluing” together two opposite Cambrian frameworks. We begin by doubling the Cambrian fan.

We continue the notation B for an acyclic exchange matrix with $A = \text{Cart}(B)$, Coxeter group W , and Coxeter element c . The notation $\text{Tits}(A)$ denotes the Tits cone and $-\text{Tits}(A)$ is its negation. We write \mathcal{DF}_c for the collection consisting of the cones in the Cambrian fan \mathcal{F}_c and the negations of cones in the Cambrian fan $\mathcal{F}_{c^{-1}}$. In symbols, $\mathcal{DF}_c = \mathcal{F}_c \cup (-\mathcal{F}_{c^{-1}})$. We call \mathcal{DF}_c the **doubled Cambrian fan** and justify this name in Theorem 3.24.

Example 3.22. We consider the doubled Cambrian fan for the exchange matrix $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. The corresponding Coxeter group is of type \tilde{G}_2 and the corresponding Coxeter element is $c = s_1s_2s_3$. (Note that the indices 1, 2, and 3 do *not* give the standard linear numbering of the Dynkin diagram.) Figure 3 shows the c -Cambrian fan \mathcal{F}_c for this example. (Compare [29, Figure 6].) The picture should be interpreted as follows: The c -Cambrian fan, intersected with a unit sphere about the origin, is a collection of spherical triangles and their faces. The picture shows this collection of triangles stereographically projected to the plane. The red circle marks the boundary of the Tits cone. Figure 4 is a similar projection of the antipodal image $-\mathcal{F}_{c^{-1}}$ of the c^{-1} -Cambrian fan. Figure 5 shows the doubled Cambrian fan \mathcal{DF}_c , the union of the two fans pictured in Figures 3 and 4. In each of Figures 4 and 5, the triangular exterior region in the stereographic projection represents a cone in the fan. Figure 6 shows the same doubled Cambrian fan in a different stereographic projection. The boundary of the Tits cone is the red vertical line in the center and the Tits cone projects to the left of the line.

Remark 3.23. Preferences for visualizing 3-dimensional fans vary, even among coauthors. Readers who find the stereographic projections referenced in Example 3.22 difficult to interpret will find a visualization of the same example, via affine slices, in Example 4.10.

Theorem 3.24. *For any acyclic exchange matrix B , the collection \mathcal{DF}_c of cones is a simplicial fan.*

Proof. Once we know that \mathcal{DF}_c is a fan, the assertion that the fan is simplicial is immediate from the definition, in light of Theorem 3.3. As explained in Section 2.2, to see that \mathcal{DF}_c is a fan, we only need to check that the maximal cones meet nicely. Moreover, we already know (Theorem 3.3) that two maximal cones of \mathcal{F}_c meet nicely, as do two cones of $-\mathcal{F}_{c^{-1}}$. So we need to show that, if v is c -sortable and u is c^{-1} -sortable, then $\text{Cone}_c(v)$ and $-\text{Cone}_{c^{-1}}(u)$ meet nicely.

We argue by induction first on the rank of W and then, holding the rank constant, on $\ell(v)$. Our base case is the vacuous case where W has rank 0. If the rank is positive but $v = e$, so $\ell(v) = 0$, this is not a base case but rather a case where we induct on rank; this scenario is covered by Cases 2 and 4 below.

We now consider the inductive case where the rank of W is positive, so c is not e , and let s be initial in c . We break into four cases.

Case 1. $v \geq s$ and $u \not\geq s$. Then $\text{Cone}_c(v) = s\text{Cone}_{scs}(sv)$ and $-\text{Cone}_{c^{-1}}(u) = -s\text{Cone}_{sc^{-1}s}(su)$ by (3.1). By induction on $\ell(v)$ we see that $\text{Cone}_{scs}(sv)$ and $-\text{Cone}_{sc^{-1}s}(su)$ meet nicely, so $\text{Cone}_c(v)$ and $-\text{Cone}_{c^{-1}}(u)$ meet nicely.

Case 2. $v \not\geq s$ and $u \geq s$. Then v is in $W_{\langle s \rangle}$ and $C_c(v) = C_{sc}(v) \cup \{\alpha_s\}$ by (3.1). Also, $u = s \vee u_{\langle s \rangle}$ and $-C_{c^{-1}}(u) = -C_{c^{-1}s}(u_{\langle s \rangle}) \cup \{\alpha_s\}$ by Proposition 3.10. By induction on rank, $\text{Cone}_{sc}(v)$ and $-\text{Cone}_{c^{-1}s}(u_{\langle s \rangle})$ meet nicely in $V_{\langle s \rangle}$. Therefore the cones in V defined by $C_{sc}(v)$ and $-C_{c^{-1}s}(u_{\langle s \rangle})$ also meet nicely, and this nice meeting is preserved by intersecting with the half-space consisting of points below α_s^\perp .

Case 3. $v \geq s$ and $u \geq s$. Then $\text{Cone}_c(v)$ is above α_s^\perp and $-\text{Cone}_{c^{-1}}(u)$ is below α_s^\perp , by Lemma 3.17, used twice. Thus

$$(3.2) \quad \text{Cone}_c(v) \cap (-\text{Cone}_{c^{-1}}(u)) = (\text{Cone}_c(v) \cap \alpha_s^\perp) \cap (-\text{Cone}_{c^{-1}}(u)).$$

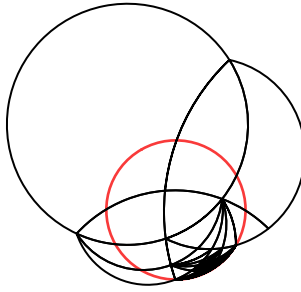


FIGURE 3. The c -Cambrian fan for B as in Example 3.22

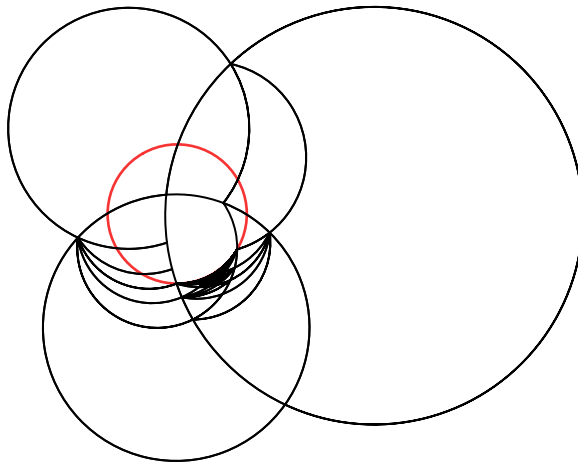


FIGURE 4. The antipodal c^{-1} -Cambrian fan for B as in Example 3.22

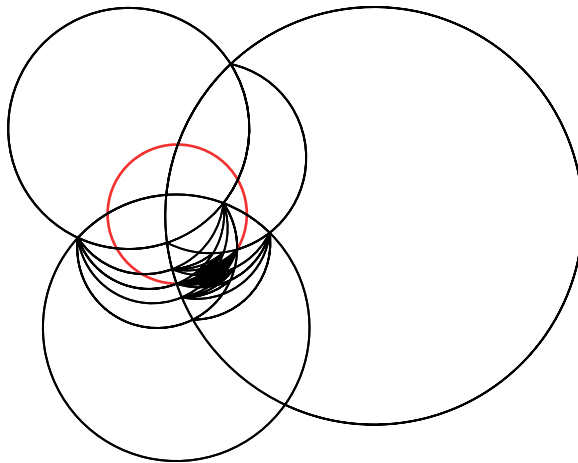


FIGURE 5. The doubled Cambrian fan for B as in Example 3.22

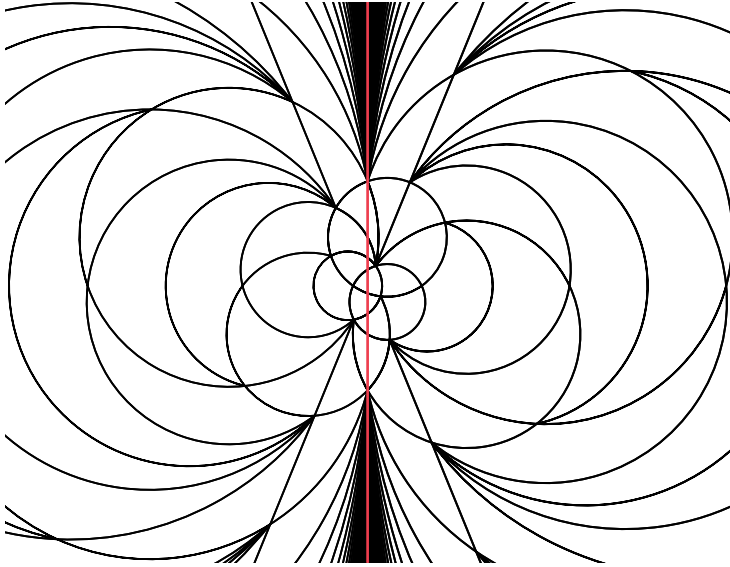


FIGURE 6. Another view of the doubled Cambrian fan

We now apply the Case 2 argument to $(v_{\langle s \rangle}, u)$ to conclude that $\text{Cone}_c(v_{\langle s \rangle})$ and $-\text{Cone}_{c-1}(u)$ meet nicely. This uses induction on rank. By Lemma 3.20, $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(v_{\langle s \rangle})$, so $\text{Cone}_c(v) \cap \alpha_s^\perp$ and $-\text{Cone}_{c-1}(u)$ also meet nicely. That is, $(\text{Cone}_c(v) \cap \alpha_s^\perp)$ is a face of $\text{Cone}_c(v) \cap \alpha_s^\perp$ and of $-\text{Cone}_{c-1}(u)$. But then by (3.2), $\text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u))$ is a face of $\text{Cone}_c(v) \cap \alpha_s^\perp$ and of $-\text{Cone}_{c-1}(u)$. Since $\text{Cone}_c(v) \cap \alpha_s^\perp$ is a face of $\text{Cone}_c(v)$, we see that $\text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u))$ is a face of $\text{Cone}_c(v)$ as well.

Case 4. $v \not\geq s$ and $u \not\geq s$. Then $\text{Cone}_c(v)$ is below α_s^\perp and $-\text{Cone}_{c-1}(u)$ is above, so

$$(3.3) \quad \text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u)) = \text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp).$$

By Lemma 3.21, $-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp$ is a face of $-\text{Cone}_{c-1}(s \vee (su)_{\langle s \rangle})$. We first argue as in Case 2, replacing (v, u) by $(v, s \vee (su)_{\langle s \rangle})$, and conclude, using induction on rank, that $\text{Cone}_c(v)$ and $-\text{Cone}_{c-1}(s \vee (su)_{\langle s \rangle})$ meet nicely. By Lemma 3.21, $-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp$ is a face of $-\text{Cone}_{c-1}(s \vee (su)_{\langle s \rangle})$, so $\text{Cone}_c(v)$ and $-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp$ meet nicely. By (3.3), $\text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u))$ is a face of $\text{Cone}_c(v)$ and of $-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp$. Since $-\text{Cone}_{c-1}(u) \cap \alpha_s^\perp$ is a face of $-\text{Cone}_{c-1}(u)$, we see that $\text{Cone}_c(v) \cap (-\text{Cone}_{c-1}(u))$ is a face of $-\text{Cone}_{c-1}(u)$ as well. \square

Remark 3.25. There is an alternate strategy for proving Theorem 3.24. The fact that the \mathbf{g} -vector fan is indeed a fan is one of the structural conjectures mentioned in the introduction. It was proved in the acyclic case (and beyond) in [7] and in general in [14]. As a consequence of Theorem 2.3(4), the Cambrian fan \mathcal{F}_c is a subfan of the \mathbf{g} -vector fan. Thus, to prove Theorem 3.24, it would be enough to show that $-\mathcal{F}_{c-1}$ is also a subfan of the \mathbf{g} -vector fan. This proof (or an even simpler approach using the mutation fan of [27] in place of the \mathbf{g} -vector fan) can indeed be carried out. However, here we prefer to construct the doubled Cambrian fan

completely in terms of the combinatorics and geometry of Coxeter groups and root systems, without relying on any other heavy machinery.

The same cone may occur in both \mathcal{F}_c and $-\mathcal{F}_{c^{-1}}$. In fact, since the Cambrian fan covers the Tits cone, we have the following immediate corollary to Theorem 3.24.

Corollary 3.26. *For every c -sortable element v such that the interior of $\text{Cone}_c(v)$ meets $-\text{Tits}(A)$, there is a c^{-1} -sortable element v' with $\text{Cone}_c(v) = -\text{Cone}_{c^{-1}}(v')$.*

Remark 3.27. When W is finite, Corollary 3.26 becomes the statement that \mathcal{F}_c and $-\mathcal{F}_{c^{-1}}$ are identical. This is essentially [26, Proposition 1.3], although that result is phrased in terms of Cambrian congruences. In that case, $v' = \pi_{\downarrow}^{c^{-1}}(vw_0)$.

We pause to record some additional useful facts, which reproduce the finite case of [29, Theorem 9.6] and [29, Proposition 9.7] (cf. [28, Proposition 7.4]) and strengthen the general case of the same theorem and proposition. These are not needed for the remainder of the paper, but add insight into the structure of \mathcal{DF}_c . For s in S , define $\mathcal{DF}_c^{\uparrow(s)}$ to be the set of cones in \mathcal{DF}_c that are above α_s^{\perp} and define $\mathcal{DF}_c^{\downarrow(s)}$ to be the set of cones in \mathcal{DF}_c below α_s^{\perp} . Suppose $J \subseteq S$ and recall that the restriction of c to W_J is the Coxeter element c' of W_J obtained by deleting the letters in $S \setminus J$ from a reduced word for c . Let A_J be the Cartan matrix obtained by deleting from A the rows and columns indexed by $S \setminus J$, and let Φ_J be the corresponding sub root system of Φ . Let V_J be the subspace of V spanned by $\{\alpha_s : s \in J\}$. The parabolic subgroup W_J fixes V_J as a set. Let Proj_J stand for the surjection from V^* onto $(V_J)^*$ that is dual to the inclusion of V_J into V . Recall that $\langle s \rangle$ means $S \setminus \{s\}$. When s is initial in c , we write \mathcal{DF}_{sc} for the doubled Cambrian fan constructed in $V_{\langle s \rangle}$ using the Coxeter element sc of $W_{\langle s \rangle}$.

Proposition 3.28. *Let s be initial in c . Then*

- (1) $\mathcal{DF}_c = \mathcal{DF}_c^{\uparrow(s)} \cup \mathcal{DF}_c^{\downarrow(s)}$.
- (2) $\mathcal{DF}_c^{\uparrow(s)} = s[\mathcal{DF}_{scs}^{\downarrow(s)}]$.
- (3) *The map $F \mapsto \text{Proj}_{\langle s \rangle}^{-1}(F) \cap \{x \in V^* : \langle x, \alpha_s \rangle \geq 0\}$ is a bijection from the set of maximal cones of \mathcal{DF}_{sc} to the set of maximal cones of $\mathcal{DF}_c^{\downarrow(s)}$.*
- (4) *The map $F \mapsto \text{Proj}_{\langle s \rangle}^{-1}(F) \cap \{x \in V^* : \langle x, \alpha_s \rangle \leq 0\}$ is a bijection from the set of maximal cones of \mathcal{DF}_{sc} to the set of maximal cones of $\mathcal{DF}_{scs}^{\uparrow(s)}$.*

Proof. Assertion (1) is the statement that every cone of \mathcal{DF}_c is either above or below α_s^{\perp} . This is true, independent of the hypothesis that s is initial, by Lemma 3.17 applied to \mathcal{F}_c and to $-\mathcal{F}_{c^{-1}}$. Theorem 3.7 and equation (3.1) show that the reflection s takes the cones of \mathcal{F}_c above α_s^{\perp} bijectively to the cones of \mathcal{F}_{scs} below α_s^{\perp} . The same theorem and equation show that s takes the cones of $-\mathcal{F}_{c^{-1}}$ above α_s^{\perp} bijectively to the cones of $-\mathcal{F}_{sc^{-1}s}$ below α_s^{\perp} . Thus (2) holds. Assertion (3) follows from (3.1) for cones of \mathcal{F}_c below α_s^{\perp} and from Proposition 3.10 for cones of $-\mathcal{F}_{c^{-1}}$ below α_s^{\perp} . Assertion (4) holds similarly with the roles of (3.1) and Proposition 3.10 reversed. □

3.3. Doubling the Cambrian framework. Theorem 3.24 is the key to constructing the doubled Cambrian reflection framework ($\text{DCamb}_c, \text{DC}_c$). We define DCamb_c to be the dual graph to \mathcal{DF}_c . More directly, let $-\text{Camb}_{c^{-1}}$ be $\text{Camb}_{c^{-1}}$ realized as a graph on formal negations $-v$ of c^{-1} -sortable elements v with an edge $(-u, -v)$ in $-\text{Camb}_{c^{-1}}$ if and only if (u, v) is an edge in $\text{Camb}_{c^{-1}}$. Since

there is typically no abelian group structure on c -sortable elements, this notation should cause no confusion. We start with a disjoint union of Camb_c and $-\text{Camb}_{c-1}$ with labels $C_c(v)$ on each vertex v of Camb_c and labels $-C_{c-1}(v)$ on each vertex $-v$ of $-\text{Camb}_{c-1}$. We identify a vertex v of Camb_c with a vertex $-v'$ of Camb_{c-1} if $\text{Cone}_c(v) = -\text{Cone}_{c-1}(v')$. In this case, since $\text{Cone}_c(v)$ and $\text{Cone}_{c-1}(v')$ are both defined by roots in the root system associated to A , we must have $C_c(v) = -C_{c-1}(v')$. We think of C_c as a labeling of incident pairs as in Section 3.1, and use the equality $C_c(v) = -C_{c-1}(v')$ of sets to identify the sets $I(v)$ and $I(-v')$ of edges in the obvious way. If v has a full edge connecting v to u but the corresponding edge of $-v'$ is a half-edge, then the identified edge is a full edge connecting the identified vertex $v = -v'$ to the vertex u , and similarly if v has a half-edge where $-v'$ has a full edge.

This way of defining DCamb_c also yields a labeling DC_c of incident pairs of DCamb_c . For an incident pair (\tilde{v}, \tilde{e}) in DCamb_c corresponding to an incident pair (v, e) in Camb_c , we set $\text{DC}_c(\tilde{v}, \tilde{e}) = C_c(v, e)$. For an incident pair (\tilde{v}, \tilde{e}) in DCamb_c corresponding to an incident pair $(-v, -e)$ in $-\text{Camb}_{c-1}$, we set $\text{DC}_c(\tilde{v}, \tilde{e}) = -C_{c-1}(v, e)$. If a given incident pair in DCamb_c takes both of these forms, then, by construction, the two ways of computing DC_c on that incident pair agree.

Example 3.29. This example continues Example 3.22. Figure 7 shows the graph DCamb_c . The vertices and edges that exist both in Camb_c and $-\text{Camb}_{c-1}$ are shown in red. These correspond to the triangles in Figure 6 whose interiors intersect the equator (the red line).

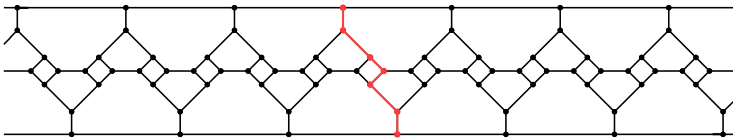


FIGURE 7. DCamb_c

Theorem 3.30. *If DCamb_c is connected, then $(\text{DCamb}_c, \text{DC}_c)$ is a polyhedral reflection framework.*

Proof. The Base condition holds in DCamb_c because it holds in Camb_c . The Root condition and the Euler conditions hold in DCamb_c because they hold in Camb_c and Camb_{c-1} , because $E_{c-1}(\beta, \gamma) = E_c(\gamma, \beta)$, and because passing from Camb_c to $-\text{Camb}_c$ corresponds to changing the signs of all labels. Finally, each full edge e in DCamb_c comes from a full edge in Camb_c or a full edge in $-\text{Camb}_{c-1}$, or both. If e comes from a full edge in Camb_c , then the Reflection condition holds in DCamb_c because it holds in Camb_c . If e comes from a full edge in Camb_{c-1} , then the Reflection condition holds in Camb_{c-1} . Passing from the Reflection condition in Camb_{c-1} to the Reflection condition in DCamb_c , the root β_t does not change sign because it is positive by definition, but γ does change sign because we pass from label sets C_{c-1} in Camb_{c-1} to label sets $-C_{c-1}$ in DCamb_c . However, $\omega_c(\beta_t, -\gamma) = \omega_{c-1}(\beta_t, \gamma)$, so the Reflection condition is preserved. Thus DCamb_c satisfies the definition of a reflection framework. By construction, \mathcal{DF}_c is the collection of cones associated to vertices of DCamb_c . By Theorem 3.24, \mathcal{DF}_c is a fan. Also by construction, if a cone of \mathcal{F}_c and a cone of $-\mathcal{F}_{c-1}$ coincide, then the corresponding

vertices are identified in DCamb_c . In light of Theorem 3.2, we see that distinct vertices of DCamb_c label distinct maximal cones of \mathcal{DF}_c . Thus $(\text{DCamb}_c, \text{DC}_c)$ is polyhedral. \square

The hypothesis of Theorem 3.30 only holds when \mathcal{F}_c and $-\mathcal{F}_{c^{-1}}$ have full-dimensional cones in common. The following proposition tells us that this happens in many cases. As special cases of the proposition, DCamb_c is connected in the affine case and, more generally, whenever every proper parabolic subgroup of W is finite.

Proposition 3.31. *Suppose $s_1 \cdots s_n$ is a reduced word for a Coxeter element c of W , and suppose that, for some $i \in \{1, \dots, n - 1\}$, the parabolic subgroups $W_{\{s_1, \dots, s_i\}}$ and $W_{\{s_{i+1}, \dots, s_n\}}$ are both finite. Then DCamb_c is connected.*

We first need the following lemma. We write $(w_0)_J$ for the longest element of W_J , when W_J is finite.

Lemma 3.32. *Let $c = s_1 s_2 \cdots s_n$ and let $J = \{s_{m+1}, \dots, s_n\}$ for some $m \geq 1$. Suppose that W_J is finite. Then $C_c((w_0)_J) = \{\alpha_1, \alpha_2, \dots, \alpha_m, -\alpha_{m+1}, \dots, -\alpha_n\}$.*

Proof. We argue by induction on $n - m$. If $n - m = 0$, so that $(w_0)_J$ is the identity, this is immediate. If $n - m > 0$, then let s stand for s_n and write $J' = \{s_{m+1}, \dots, s_{n-1}\}$. Then $C_{cs}((w_0)_{J'}) = \{\alpha_1, \alpha_2, \dots, \alpha_m, -\alpha_{m+1}, \dots, -\alpha_{n-1}\}$ by induction. Since $(w_0)_J \geq s$ and $[(w_0)_J]_{\langle s \rangle} = (w_0)_{J'}$, we apply Proposition 3.10 to conclude that $C_c((w_0)_J) = C_{cs}((w_0)_{J'}) \cup \{-\alpha_s\}$. \square

Proof of Proposition 3.31. Write $J = \{s_{i+1}, \dots, s_n\}$. The element $(w_0)_J$ is c -sortable by Theorem 3.6 because it is the join of J . Similarly, $(w_0)_{S \setminus J}$ is c^{-1} -sortable. Lemma 3.32 implies that

$$C_c((w_0)_J) = \{\alpha_1, \dots, \alpha_i, -\alpha_{i+1}, \dots, -\alpha_n\} = -C_{c^{-1}}((w_0)_{S \setminus J}).$$

We have identified a cone that is in \mathcal{F}_c and in $-\mathcal{F}_{c^{-1}}$. \square

Remark 3.33. If B is 2×2 , then by Proposition 3.31 (or by an easy direct check), DCamb_c is connected. Thus $(\text{DCamb}_c, \text{DC}_c)$ is a polyhedral reflection framework by Theorem 3.30. One can also easily check that $(\text{DCamb}_c, \text{DC}_c)$ is complete, exact, simply connected, and well-connected.

Remark 3.34. The identity element is c^{-1} -sortable for any c , so the formal negative of the identity is a vertex of DCamb_c . The labels on the negative of the identity are the negative simple roots. Consider a path v_0, \dots, v_p in DCamb_c such that v_0 is the identity and write e_i for the edge connecting v_{i-1} to v_i . As pointed out in [31, Remark 4.19] for general frameworks, such a path is a **green sequence** in the sense of [21] (see also [5]) if and only if all of the roots $\text{DC}_c(v_i, e_i)$ are positive. We also observe that this green sequence is a **maximal green sequence** if and only if v_p is the negative of the identity. The following proposition is worth mentioning, even though it does not add to the list of exchange matrices B for which maximal green sequences are known to exist. (Maximal green sequences are known [5, Lemma 2.20] to exist for all acyclic B .)

Proposition 3.35. *If DCamb_c is connected, then it contains a maximal green sequence.*

Proof. If DCamb_c is connected, then there exists a c -sortable element u and a c^{-1} -sortable element v such that $\text{Cone}_c(u) = -\text{Cone}_{c^{-1}}(v)$. Lemma 3.1 and the definition of the Cambrian framework (Camb_c, C_c) imply that any unrefinable chain u_0, \dots, u_k from the identity to u is a green sequence in Camb_c . Similarly, any unrefinable chain v_0, \dots, v_ℓ from the identity to v is a green sequence in $\text{Camb}_{c^{-1}}$. Using the symbols u_i and $-v_i$ to label the corresponding vertices in DCamb_c , the sequence $u_0, \dots, u_k = -v_\ell, \dots, -v_0$ is a maximal green sequence. \square

In the next section, we establish some additional properties of the doubled Cambrian framework when B is acyclic and $\text{Cart}(B)$ is of affine type, including completeness. The Cartan companion in Examples 3.22 and 3.29 is of affine type, and the completeness of the doubled Cambrian framework is seen in the fact that the dual graph to \mathcal{DF}_c depicted in Figure 7 is 3-regular. Before we confine our attention to the affine case, we give an example where the doubled Cambrian framework is not complete.

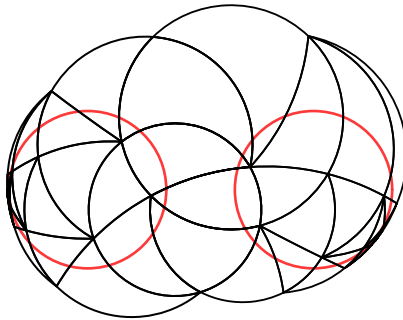


FIGURE 8. A fan \mathcal{DF}_c whose corresponding framework is not complete

Example 3.36. Figure 8 shows the doubled Cambrian fan for $B = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$. The two red circles show the boundary of the Tits cone and the boundary of the negative Tits cone. The Cambrian fan \mathcal{F}_c consists of the cones whose projections (curvilinear triangles) intersect the inside of the left circle. The fan $-\mathcal{F}_{c^{-1}}$ consists of cones whose projections intersect the inside of the right circle. The dual graph DCamb_c to \mathcal{DF}_c is not 3-regular, so $(\text{DCamb}_c, \text{DC}_c)$ is not a complete framework in this case. Specifically, the exterior of the figure is not a cone of \mathcal{DF}_c , so the cones that border the exterior correspond to vertices of degree less than 3. However, the dual graph is connected because there are cones that meet both the interior of $\text{Tits}(A)$ and the interior of $-\text{Tits}(A)$.

4. AFFINE TYPE

In this section, we discuss root systems of affine type, gather some useful facts about them, and complete the proof of Theorem 1.1. This theorem says that when B is acyclic and its Cartan companion $\text{Cart}(B)$ is of affine type, $(\text{DCamb}_c, \text{DC}_c)$ is a complete, exact, well-connected, polyhedral, simply connected reflection framework.

By Theorem 3.30, if $(\text{DCamb}_c, \text{DC}_c)$ is connected, then it is a polyhedral reflection framework. Recall that the polyhedral property implies injectivity, that simple connectivity implies ampleness, and that exact means injective and ample. Thus

the task is to show connectedness, completeness, well-connectedness, and simple connectivity. We do this in Propositions 4.2, 4.3, 4.17, and 4.20.

4.1. Affine type and affine root systems. A good reference on affine root systems is [20, Chapters 4-6]. See also [22].

As before, let B be a skew-symmetrizable matrix, with Cartan companion $A = \text{Cart}(B)$, with root system Φ and symmetric bilinear form K . The Cartan matrix A is of **affine type** if K is positive semidefinite, K is not positive definite and, for any proper subset $J \subsetneq S$, the restriction of K to V_J is positive definite. This definition of affine type is equivalent to that of [20]. The equivalence is [20, Proposition 4.7.b], up to a technical point: Whereas [20] explicitly requires that affine Cartan matrices are indecomposable, our definition (and in particular the positive definiteness of restrictions to V_J) easily implies indecomposability. When A is of affine type, we say that the Coxeter group W and the root system Φ are affine.

The affine Coxeter groups are in bijection with the indecomposable finite (crystallographic) root systems. Specifically, if W is an affine Coxeter group, then there is an $(n - 1)$ -element subset S_0 of S , and an indecomposable finite root system Φ_0 with Coxeter group W_0 , such that W is isomorphic to a semidirect product of W_0 with the lattice generated by the co-roots Φ_0^\vee . The subset S_0 may not be uniquely determined, but we will fix a choice of S_0 . Write s_{aff} for the unique element of $S \setminus S_0$. We say that W has type \tilde{X} , where X is the type of Φ_0 . (The type of Φ_0 does not depend on the choice of S_0 .)

Since our goal is to construct a framework for the exchange matrix B , we must talk about roots. This raises an issue which is sometimes overlooked in discussions of affine Coxeter groups: Just as the finite Coxeter group of type B_n comes from two different root systems (B_n and C_n), there are affine Coxeter groups that come from more than one root system. For example, the Cartan matrices $\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$ both specify the Coxeter group of type \tilde{A}_1 . The root systems for these two Cartan matrices are different and this difference can be seen when we construct frameworks for corresponding exchange matrices.

We now review the theory of affine root systems. Let Φ be an affine (real) root system associated to a Cartan matrix A . The associated symmetric bilinear form K has a one-dimensional kernel. The nonzero elements of the root lattice contained in the kernel are called the **imaginary roots**, and like the real roots, these consist of two classes, the **positive imaginary roots** contained in the nonnegative span of the simple roots Π and the **negative imaginary roots** contained in the nonpositive span of Π . Let δ be the shortest positive imaginary root (the nonzero element of the root lattice closest to the origin in the ray spanned by positive imaginary roots). The vector δ has strictly positive coordinates with respect to the basis Π of simple roots. Since δ is in the kernel of K , it is fixed by the action of W .

Corresponding to the decomposition $S = S_0 \sqcup \{s_{\text{aff}}\}$ of the simple reflections in affine Coxeter groups is a decomposition $\Pi = \Pi_0 \sqcup \{\alpha_{\text{aff}}\}$ with the following properties: The subset Π_0 spans an indecomposable finite root system Φ_0 ; and the simple root α_{aff} is a positive scaling of $\delta - \theta$, where θ is a positive root in Φ_0 . (The root θ is either the highest root of Φ_0 or the highest short root of Φ_0 .) Every root in Φ is a positive scaling of a vector of the form $\beta + k\delta$, where $\beta \in \Phi_0$ and $k \in \mathbb{Z}$. A root obtained as a positive scaling of $\beta + k\delta$ is a positive root if and only if either k is positive or $k = 0$ and β is positive. In some affine root systems Φ , there exist vectors $\beta + k\delta$ with $\beta \in \Phi_0$ and $k \in \mathbb{Z}$ that are not scalings of

roots. However, for each $\beta \in \Phi_0$, there are infinitely many positive integers k and infinitely many negative integers k such that $\beta + k\delta$ is a positive scaling of a root. See [20, Proposition 6.3] for more details.

Let W and W_0 be the Coxeter groups associated to Φ and Φ_0 respectively, let V and V_0 be their respective reflection representations, with $V_0 \subset V$, and let V^* and V_0^* be the respective dual representations. Thus $V = V_0 \oplus \mathbb{R}\delta$. We write π for the projection onto V_0 with kernel $\mathbb{R}\delta$. The Tits cone $\text{Tits}(A)$ is $\{x \in V^* : \langle x, \delta \rangle > 0\} \cup \{0\}$. The boundary $\partial\text{Tits}(A)$ of the Tits cone is δ^\perp . The inclusion of V_0^* into V^* , dual to the projection π , identifies $\partial\text{Tits}(A)$ with V_0^* .

The action of W preserves the set $V_1^* := \{x \in V^* : \langle x, \delta \rangle = 1\}$. The action of W on V_1^* is generated by reflections in affine hyperplanes $\{x \in V_1^* : \langle x, \beta \rangle = k\}$ in V_1^* for $k \in \mathbb{Z}$ and $\beta \in \Phi_0$. Thus we can view W as a group of affine Euclidean motions of an $(n - 1)$ -dimensional vector space. In this context, we define D_1 to be $D \cap V_1^*$ where, as before, D is the cone $\bigcap_{s \in S} \{x \in V^* : \langle x, \alpha_s \rangle \geq 0\}$. Then V_1^* is tiled by translates of the simplex D_1 . For an element w of W , the inversions of w correspond to the affine hyperplanes separating wD_1 from D_1 . For a presentation from this perspective, see [2, Section VI.2].

The best-known affine root systems are the “standard” affine root systems, in bijection with, and constructed directly from, the finite root systems. The Dynkin diagrams of these root systems are shown in Table Aff 1 of [20, Chapter 4]. In the standard affine root systems, the roots are exactly the vectors of the form $\beta + k\delta$ and the reflecting hyperplanes in V^* are precisely the hyperplanes of the form $\langle \beta, x \rangle = k\langle \delta, x \rangle$ for $k \in \mathbb{Z}$ and $\beta \in \Phi_0$. Every affine root system can be obtained from a standard affine root system by rescaling roots (necessarily with consistent rescaling within W -orbits of roots) and leaving the underlying bilinear form K unchanged. Rescaling the root system does not change the Coxeter group or the reflection representation, so the reader may ignore this issue in any discussion which refers only to these concepts.

Example 4.1. As an example of a “nonstandard” affine root system, take $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ and construct a root system Φ with simple roots α_1 and α_2 and simple co-roots $\alpha_1^\vee = \alpha_1$ and $\alpha_2^\vee = 4\alpha_2$. Set $\Phi_0 = \{\pm\alpha_1\}$, so that $S_0 = \{s_1\}$ and $\alpha_{\text{aff}} = \alpha_2$. The kernel of A is the real span of $\delta = \alpha_1 + 2\alpha_2$. The highest root of Φ_0 is $\theta = \alpha_1$, so $\alpha_{\text{aff}} = \alpha_2 = \frac{1}{2}(\delta - \theta)$. The roots in the W -orbit of α_1 are of the form $\pm\alpha_1 + 2k\delta$ for integers k , while roots in the W -orbit of α_2 are $\frac{1}{2}(\pm\alpha_1 + (2k + 1)\delta)$ for integers k .

This root system can be rescaled to a standard affine root system with simple roots $\alpha'_1 = \alpha_1$ and $\alpha'_2 = 2\alpha_2$. We have $K(\alpha'_1, \alpha'_1) = K(\alpha_1, \alpha_1) = 2$ and $K(\alpha'_2, \alpha'_2) = K(2\alpha_2, 2\alpha_2) = 2$, so $(\alpha'_1)^\vee = \alpha'_1$, and $(\alpha'_2)^\vee = \alpha'_2$. The Cartan matrix A' for Φ' is $A' = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. The vector δ is $\alpha'_1 + \alpha'_2$, and the roots in Φ' are $\pm\alpha_1 + k\delta$ for integers k .

One more technical observation will be useful. Write t_θ for the reflection with respect to the root θ . Since θ is a root in $\Phi_0 \subset \Phi$, the reflection t_θ makes sense in both reflection representations V and V_0 . Since α_{aff} is a positive scaling of $\delta - \theta$ and since $K(\delta, x) = 0$ for all $x \in V$, we can rewrite the action of s_{aff} on V in terms of t_θ and δ as follows:

$$(4.1) \quad s_{\text{aff}}(x) = t_\theta(x) + \langle \theta^\vee, x \rangle \delta.$$

4.2. Connectedness and completeness. We now establish the properties of connectedness and completeness for $(\text{DCamb}_c, \text{DC}_c)$.

Proposition 4.2. *If A is of affine type, then DCamb_c is connected.*

Proof. When $\text{Cart}(B)$ is of affine type, the corresponding Coxeter group W is of affine type. Thus every proper parabolic subgroup of W is of finite type, so the result follows from Proposition 3.31. \square

Proposition 4.3. *If A is of affine type, then $(\text{DCamb}_c, \text{DC}_c)$ is complete.*

Proof. The assertion is that DCamb_c has no half-edges. Given a vertex \tilde{v} in DCamb_c , we may as well (up to swapping c and c^{-1} and negating all labels) assume that the vertex is represented by a c -sortable element v , i.e. a vertex of Camb_c . Let \tilde{e} be an edge incident to \tilde{v} in DCamb_c , and let e be the corresponding edge in Camb_c . If $C_c(v, e)$ is a negative root, then by the Full edge condition on Camb_c , the edge e is a full edge, so \tilde{e} is a full edge. Suppose $C_c(v, e)$ is positive. If the interior of $\text{Cone}_c(v)$ intersects $-\text{Tits}(A)$, then Theorem 3.26 says that there exists a c^{-1} -sortable element v' with $\text{Cone}_c(v) = -\text{Cone}_{c^{-1}}(v')$. The edge \tilde{e} corresponds to an edge e' of $-\text{Camb}_{c^{-1}}$, which must be a full edge, by the Full edge condition, since $C_{c^{-1}}(v', e') = -C_c(v, e)$ is negative. Thus \tilde{e} is a full edge. If the interior of $\text{Cone}_c(v)$ is disjoint from $-\text{Tits}(A)$, then since the closure of $\text{Tits}(A) \cup (-\text{Tits}(A))$ is V , the interior of $\text{Cone}_c(v)$ is contained in $\text{Tits}(A)$. Thus, since $C_c(v, e)$ is a positive root, there are elements $w_1 < w_2$ in W such that $w_1 D \in \text{Cone}_c(v)$ and $w_2 D \notin \text{Cone}_c(v)$, with the cones $w_1 D$ and $w_2 D$ separated by $C_c(v, e)^\perp$. Then $v = \pi_\downarrow^c(w_1)$, and we define $u = \pi_\downarrow^c(w_2)$. Since \mathcal{F}_c is a fan (Theorem 3.3) and because $C(v, e)$ is an inward facing normal to a facet of $\text{Cone}_c(v)$, the vector $-C_c(v, e)$ is an inward facing normal to a facet of $\text{Cone}_c(u)$. Thus $-C_c(v, e)$ is $C_c(u, f)$ for some $f \in I(u)$. The Full edge condition says that f connects u to a vertex w , and thus $\text{Cone}_c(u)$ and $\text{Cone}_c(w)$ share a facet defined by $C_c(u, f)$. Since \mathcal{F}_c is a fan, we must have $w = v$ and $f = e$. In particular, \tilde{e} is a full edge. \square

As consequences of Proposition 4.3, we obtain some important facts about \mathcal{DF}_c when $\text{Cart}(B)$ is of affine type. First, since the maximal cones of \mathcal{DF}_c are simplicial and n -dimensional, every maximal cone in \mathcal{DF}_c is adjacent to exactly n other maximal cones. Also, every codimension-1 cone in \mathcal{DF}_c is contained in exactly two full-dimensional cones.

4.3. Faces of \mathcal{DF}_c and the boundary of $\text{Tits}(A)$. We now prove some key facts about how faces of the affine doubled Cambrian fan intersect the boundary of the Tits cone. We will use repeatedly the fact that each cone of \mathcal{DF}_c is the intersection of half-spaces of the form $\{x \in V^* : \langle x, \beta \rangle \geq 0\}$ for $\beta \in \Phi$.

Proposition 4.4. *If A is of affine type, then every face of \mathcal{DF}_c contained in $\partial\text{Tits}(A)$ has dimension less than $n - 1$. All $(n - 1)$ -dimensional cones in $\mathcal{DF}_c \cap \partial\text{Tits}(A)$ are of the form $F \cap \partial\text{Tits}(A)$ where F is an n -dimensional cone of \mathcal{DF}_c and $\partial\text{Tits}(A)$ passes through the interior of F .*

Proof. Every $(n - 1)$ -dimensional cone of $\mathcal{DF}_c \cap \partial\text{Tits}(A)$ is either (1) of the form $F \cap \partial\text{Tits}(A)$ for F an n -dimensional cone of \mathcal{DF}_c whose interior meets $\partial\text{Tits}(A)$ or (2) an $(n - 1)$ -dimensional cone of \mathcal{DF}_c . So the first sentence implies the second.

To see the truth of the first sentence, let G be an $(n - 1)$ -dimensional cone of \mathcal{DF}_c . Then the normal vector to G is a real root. But the normal to $\partial\text{Tits}(A)$ is δ , which is not a scaling of a real root. \square

Recall that π is the projection of V onto V_0 with kernel $\mathbb{R}\delta$. The intersection $\{x \in V^* : \langle x, \beta \rangle \geq 0\} \cap \partial\text{Tits}(A)$ is $\{x \in V_0^* : \langle x, \pi(\beta) \rangle \geq 0\}$, and $\pi(\beta)$ is a scaling of a root in Φ_0 . Thus each cone of \mathcal{DF}_c meets $\partial\text{Tits}(A)$ in a cone of V_0^* defined by half-spaces corresponding to roots in Φ_0 . In other words, each cone of \mathcal{DF}_c meets $\partial\text{Tits}(A)$ in a cone that is a union of cones in the Coxeter fan of W_0 . (This is the fan whose maximal cones are the closures of the connected components of the complement, in V_0^* , of the union of the reflecting hyperplanes of W_0 .) Since there are only finitely many cones in the Coxeter fan of W_0 , we obtain the following fact.

Proposition 4.5. *If A is of affine type, then there are only finitely many different cones of the form $F \cap \partial\text{Tits}(A)$ where F is a cone in \mathcal{DF}_c .*

By Proposition 4.4, the intersection of a maximal cone F with $\partial\text{Tits}(A)$ is $(n-1)$ -dimensional if and only if the interior of F intersects $\partial\text{Tits}(A)$. Since distinct maximal cones of \mathcal{DF}_c have disjoint relative interiors (by Theorem 3.24), we also have the following observation.

Proposition 4.6. *If A is of affine type, then there are only finitely many maximal cones in \mathcal{DF}_c whose interior intersects $\partial\text{Tits}(A)$.*

Define $D_0 := \bigcap_{s \in S_0} \{x \in V_0^* : \langle x, \alpha_s \rangle \geq 0\}$. The cone D_0 is to Φ_0 as D is to Φ . The following proposition records a special case of the observation, made above, that each cone of \mathcal{DF}_c meets $\partial\text{Tits}(A)$ in a cone that is a union of cones in the Coxeter fan of W_0 .

Proposition 4.7. *For A of affine type, if F is a maximal cone of \mathcal{DF}_c with $F \cap \partial\text{Tits}(A)$ of dimension $n - 1$, then $F \cap \partial\text{Tits}(A)$ is $\bigcup_{u_0} u_0 D_0$, with u_0 running over some subset of W_0 .*

4.4. The support of the doubled Cambrian fan. The *support* $|\mathcal{F}|$ of a fan \mathcal{F} is $\bigcup_{F \in \mathcal{F}} F$. We now present a theorem and a corollary which describe the support of the doubled Cambrian fan \mathcal{DF}_c in the affine case. We define

$$\begin{aligned} \Phi_0^{\omega^+} &= \{\beta \in \Phi_0 : \omega_c(\beta, \delta) > 0\}, \\ \Phi_0^{\omega^0} &= \{\beta \in \Phi_0 : \omega_c(\beta, \delta) = 0\}, \\ \Phi_0^{\omega^-} &= \{\beta \in \Phi_0 : \omega_c(\beta, \delta) < 0\}. \end{aligned}$$

If we use the skew-symmetric form ω to define a map $V \rightarrow V^*$, then $\Phi_0^{\omega^+}$, $\Phi_0^{\omega^0}$ and $\Phi_0^{\omega^-}$ are the roots of Φ_0 which are sent to the interior of the Tits cone, boundary of the Tits cone, and interior of the negative Tits cone respectively.

The map $\gamma \mapsto -\omega_c(\gamma, \delta)$ is a linear functional on V_0 and thus a point in V_0^* . We call this point x_c .

With this notation in place, we state the main results of this section.

Theorem 4.8. *If A is of affine type, then $|\mathcal{F}_c| \cap \partial\text{Tits}(A) = |-\mathcal{F}_{c-1}| \cap \partial\text{Tits}(A)$, and thus both equal $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$. Furthermore, $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$ is nonempty and*

$$|\mathcal{DF}_c| \cap \partial\text{Tits}(A) = \bigcup_{\beta \in \Phi_0^{\omega^+}} \{x \in \partial\text{Tits}(A) : \langle x, \beta \rangle \geq 0\}.$$

Corollary 4.9. *If A is of affine type, then $V^* \setminus |\mathcal{DF}_c|$ is a nonempty open cone in $\partial\text{Tits}(A)$, and thus is of codimension 1 in V^* . This cone is*

$$V^* \setminus |\mathcal{DF}_c| = \bigcap_{\beta \in \Phi_0^{\omega^+}} \{x \in \partial\text{Tits}(A) : \langle x, \beta \rangle < 0\}.$$

Also, $V^* \setminus |\mathcal{DF}_c|$ is the union of the relative interiors of those cones in the W_0 -Coxeter fan that contain x_c .

Example 4.10. We illustrate these results by continuing Examples 3.22 and 3.29, with exchange matrix $B = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. The reader may wish to compare Figures 10, 11 and 12 with Figures 5 and 6, which display the same fan in stereographic projection.

The associated Coxeter group is of type \tilde{G}_2 . The Coxeter element defined by B is $c = s_1 s_2 s_3$. The subgroup W_0 is generated by s_1 and s_2 . Figure 9 shows the root system Φ_0 for W_0 and part of the weight lattice, with some weights labeled. (Recall from Section 2.1 that the fundamental weights ρ_i are the dual basis to the simple co-roots.) In this example, $\delta = 2\alpha_1 + 3\alpha_2 + \alpha_3$ and $\Phi_0^{\omega^0} = \{\pm(\alpha_1 + 2\alpha_2)\}$. The set $\Phi_0^{\omega^+}$ is $\{2\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_1, -\alpha_2, -\alpha_1 - 3\alpha_2\}$ and x_c is $-4\rho_1 + 6\rho_2$.

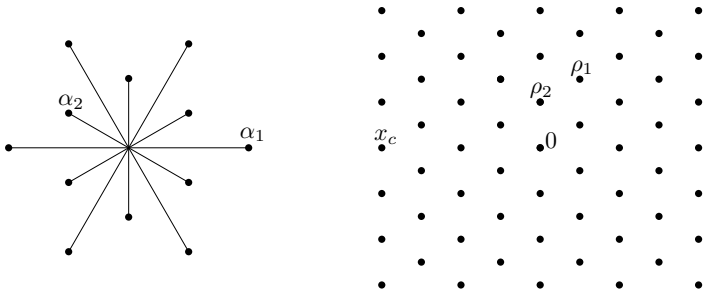


FIGURE 9. The root system Φ_0 and weight lattice for Example 4.10

Figure 10 shows \mathcal{DF}_c intersected with the affine plane $V_1^* = \{x \in V^* : \langle x, \delta \rangle = 1\}$. Nonzero cones in \mathcal{DF}_c intersect V_1^* if and only if they intersect the interior of $\text{Tits}(A)$, so Figure 10 is a representation of \mathcal{F}_c . (Essentially the same picture appears as [29, Figure 4].) Several of the maximal cones are labeled by the corresponding c -sortable element. Figure 11 shows the intersection of \mathcal{DF}_c with the opposite affine plane $V_{-1}^* = \{x \in V^* : \langle x, \delta \rangle = -1\}$. This is a representation of $-\mathcal{F}_{c^{-1}}$, and several maximal cones are labeled by the corresponding c^{-1} -sortable element. Both pictures are from the point of view of a point in $\text{Tits}(A)$ separated from the origin by V_1^* .

Figure 12 shows the intersection of \mathcal{DF}_c with $V_0^* = \partial\text{Tits}(A)$, from the same point of view. The shaded region indicates $V^* \setminus |\mathcal{DF}_c|$, and the vector x_c is indicated. Black lines indicate the boundaries of the intersections $F \cap \partial\text{Tits}(A)$ for maximal cones F of \mathcal{DF}_c . Each maximal cone F of \mathcal{DF}_c such that $F \cap \partial\text{Tits}(A)$ is 2-dimensional is labeled in black by the c -sortable element v such that $F = \text{Cone}_c(v)$ and in red by the c^{-1} -sortable element v' such that $F = -\text{Cone}_{c^{-1}}(v')$. Gray dashed lines indicate the decomposition of cones $F \cap \partial\text{Tits}(A)$ into cones in the W_0 -Coxeter fan. A gray dashed line also indicates the decomposition of the cone $V^* \setminus |\mathcal{DF}_c|$ into two cones in the W_0 -Coxeter fan.

We now prepare to prove Theorem 4.8 by establishing a sequence of lemmas. The first is [29, Lemma 3.8] when x and y are roots, and holds for all x and y by linearity.

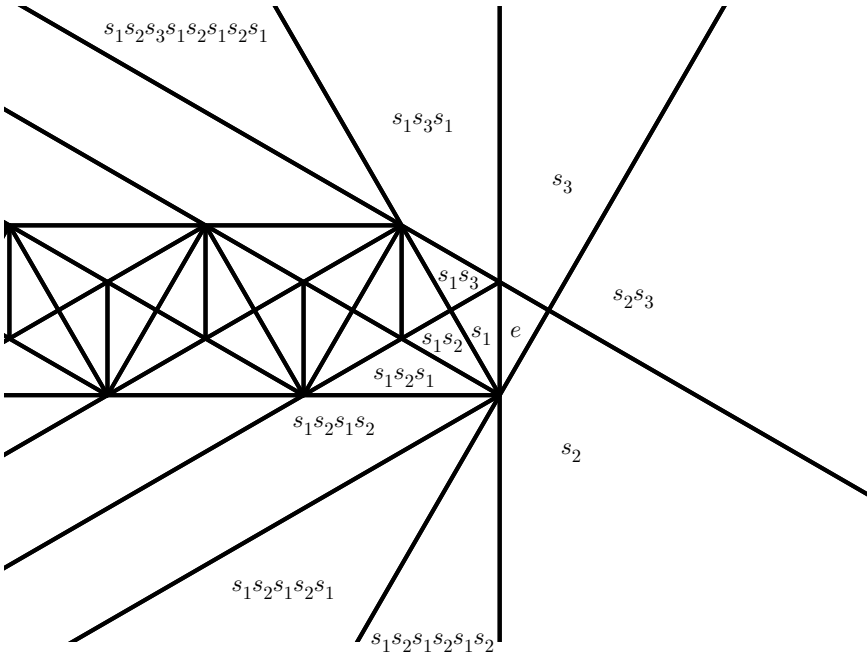


FIGURE 10. The Cambrian fan \mathcal{F}_c of Example 4.10

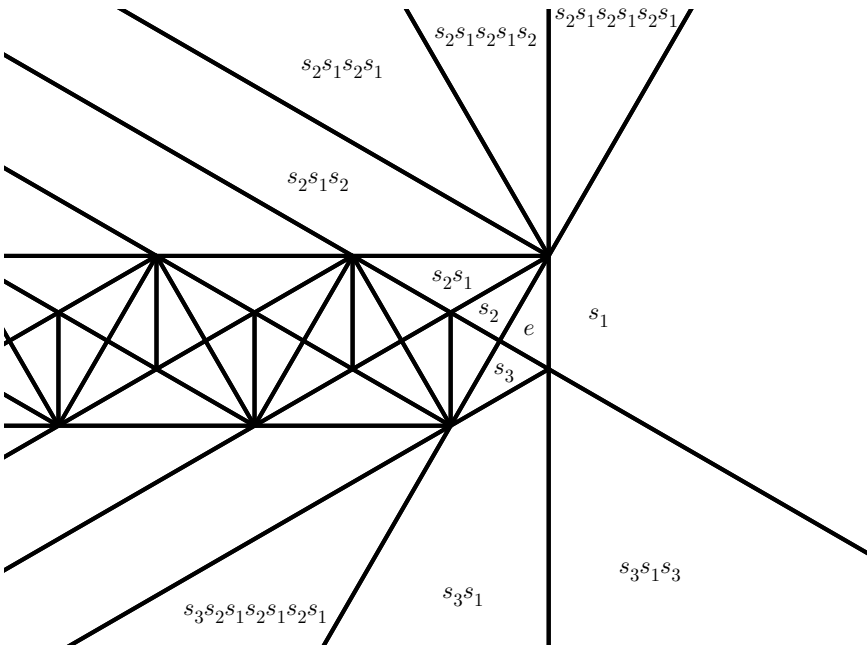


FIGURE 11. The opposite Cambrian fan $-\mathcal{F}_{c-1}$ of Example 4.10

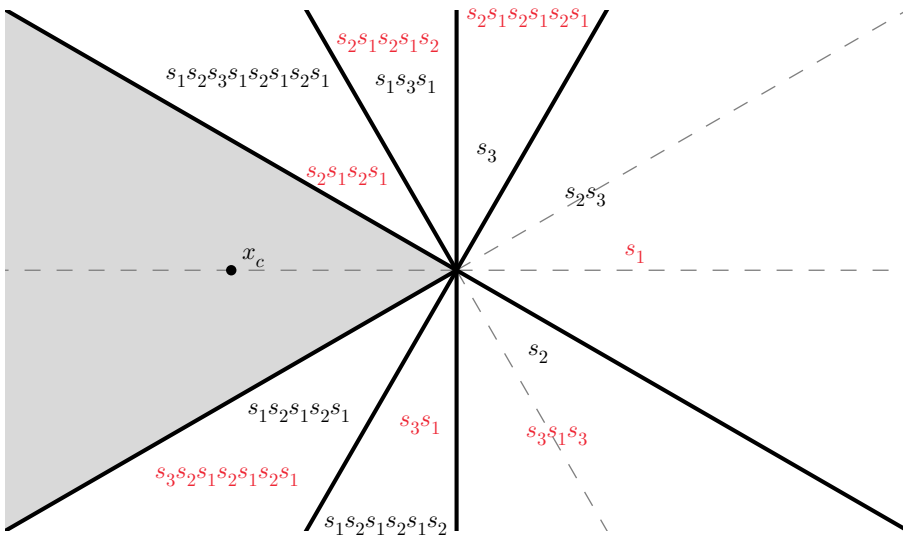


FIGURE 12. $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$ for Example 4.10

Lemma 4.11. *If s is initial or final in c , then $\omega_c(x, y) = \omega_{scs}(sx, sy)$ for all vectors x and y in V .*

Lemma 4.12. *If v is c -sortable, then there exists $\beta \in C_c(v)$ such that $\pi(\beta)$ is a positive scaling of a root in $\Phi_0^{\omega^+}$.*

Proof. Since every root β has $\pi(\beta) = \beta - r\delta$ for some r , we need to show that there exists $\beta \in C_c(v)$ such that $\omega_c(\beta, \delta) > 0$. We do so by induction on $\ell(v)$. Let s be initial in c .

If $v \not\geq s$, then $\alpha_s \in C_c(v)$ and $\omega_c(\alpha_s, \alpha_{s'}) \geq 0$ for all $s' \in S$, with strict inequality for at least one s' . Since δ has strictly positive simple-root coordinates, we have $\omega_c(\alpha_s, \delta) > 0$.

If $v \geq s$, then by induction there exists $\beta \in C_{scs}(sv)$ with $\omega_{scs}(\beta, \delta) > 0$. Equation (3.1) says $s\beta \in C_c(v)$ and Lemma 4.11 implies that $\omega_c(s\beta, s\delta) > 0$. The element δ is fixed by every element of W , so $\omega_c(s\beta, \delta) > 0$. \square

Lemma 4.13. *The sets $\Phi_0^{\omega^+}$ and $\Phi_0^{\omega^-}$ are nonempty.*

Proof. Let s be initial in c . Since δ is a strictly positive combination of the simple roots, we have $\omega_c(\alpha_s, \delta) > 0$. If $s \neq s_{\text{aff}}$, then $\alpha_s \in \Phi_0^{\omega^+}$ and $-\alpha_s \in \Phi_0^{\omega^-}$. If $s = s_{\text{aff}}$, then take s' final in c , so that $\omega_c(\alpha_{s'}, \delta) < 0$. Since s_{aff} cannot also be final in c , we have $\alpha_{s'} \in \Phi_0^{\omega^-}$ and $-\alpha_{s'} \in \Phi_0^{\omega^+}$. \square

It is less obvious whether $\Phi_0^{\omega^0}$ is empty. As a consequence of the following lemma, $\Phi_0^{\omega^0}$ is nonempty except when Φ has rank 2. That is to say, except when the Cartan matrix is $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$, or $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$.

Lemma 4.14. *The real span of $\Phi_0^{\omega^0}$ is the hyperplane $\{x \in V_0 : \omega_c(x, \delta) = 0\}$ in V_0 .*

In order to prove Lemma 4.14, we recall some results on writing elements of W_0 as products of reflections. The **reflection length** ℓ_T of an element $w \in W_0$

is the length of a minimal-length expression for w as a product of reflections. An expression $t_1 \cdots t_k$ for w as a product of reflections has minimal length if and only if the roots β_{t_i} are linearly independent. (This is a result of [6]; see also [1, Lemma 2.4.5].) When the roots β_{t_i} are linearly independent, their real span is the image of the map $w - \text{Id}$ from V_0 to itself. (This follows from results of [3, 4] and is established explicitly in the proof of [1, Theorem 2.4.7].) In particular, $\ell_T(w)$ is the dimension of the image of $w - \text{Id}$. Finally, if t is a reflection, then $\ell_T(tw) = \ell_T(w) \pm 1$; this follows from the well-known fact that there is a homomorphism from W to $\{\pm 1\}$ sending each reflection to -1 .

Proof of Lemma 4.14. Define U_c to be $\{x \in V_0 : \omega_c(x, \delta) = 0\}$. By Lemma 4.13, U_c is not all of V_0 , so it is a hyperplane in V_0 . The claim is that this hyperplane is spanned by the roots it contains.

By Lemma 4.11 and since δ is fixed by the action of W , if s is initial in c , we see that $U_{scs} = sU_c$. Since the action of s preserves the set of roots, the lemma for c is equivalent to the lemma for scs . Accordingly, we may as well assume that $c = s_1 s_2 \cdots s_{n-1} s_{\text{aff}}$, where s_1, \dots, s_{n-1} are the elements of S_0 and s_{aff} is the remaining simple reflection. Write c_0 for the Coxeter element $s_1 s_2 \cdots s_{n-1}$ of W_0 . We have $\ell_T(c_0) = n - 1$ because the vectors α_s for $s \in S_0$ are independent.

Recall that t_θ is the reflection with respect to the root $\theta \in \Phi_0$ such that α_{aff} is a positive scaling of $\delta - \theta$. Since t_θ is a reflection, the element $c_0 t_\theta$ has reflection length $\ell_T(c_0) \pm 1 = (n - 1) \pm 1$, but $n - 1$ is the maximum reflection length of an element in W_0 , so $\ell_T(c_0 t_\theta) = n - 2$. Thus $\text{Im}(c_0 t_\theta - \text{Id})$ in V_0 is a hyperplane in V_0 which is spanned by $n - 2$ roots of Φ_0 ; call this hyperplane U . We will show that $U = U_c$, so U_c is also spanned by roots of Φ_0 .

Let $x \in U$, so that $x = c_0 t_\theta y - y$ for some $y \in V_0$. Then using (4.1), we compute

$$cy = c_0 s_{\text{aff}} y = c_0 (t_\theta y + \langle \theta^\vee, y \rangle \delta) = c_0 t_\theta y + \langle \theta^\vee, y \rangle \delta = x + y + \langle \theta^\vee, y \rangle \delta.$$

Thus $x = cy - y - \langle \theta^\vee, y \rangle \delta$, and therefore

$$\omega_c(x, \delta) = \omega_c(cy, \delta) - \omega_c(y, \delta).$$

By n applications of Lemma 4.11, we see that $\omega_c(cy, \delta) = \omega_c(y, c^{-1}\delta)$. But δ is fixed by the action of W , so $\omega_c(cy, \delta) = \omega_c(y, \delta)$ and thus $\omega_c(x, \delta) = 0$. We have shown that $U \subseteq U_c$. But U and U_c are both hyperplanes in V_0 , so $U = U_c$ and we are done. \square

We pause to point out an important consequence of Lemma 4.14. Recall from Section 4.3 the definition of the Coxeter fan of W_0 in V_0^* .

Proposition 4.15. *The point x_c is contained in the relative interior of a ray of the Coxeter fan of W_0 .*

Proof. Lemma 4.14 says that there are $n - 2$ linearly independent roots $\beta_1, \dots, \beta_{n-2}$ in $\Phi_0^{\omega_0}$. The intersection $V_0^* \cap \bigcap_{i=1}^{n-2} \beta_i^\perp$ is a line in V_0^* and since it is an intersection of reflecting hyperplanes for W_0 , it is the union of two rays of the Coxeter fan. The point $x_c \in V_0^*$ is contained in this line because $\langle x_c, \beta_i \rangle = -\omega_c(\beta_i, \delta) = 0$ for all i . Since $x_c \neq 0$ by Lemma 4.13, it is contained in the relative interior of a ray of the Coxeter fan. \square

The following lemma will assist us in applying Theorem 3.16 to the proof of Theorem 4.8.

Lemma 4.16. *Let $(\beta_1, \dots, \beta_{n-1})$ be linearly independent roots in Φ_0 . Then there are only finitely many elements w of W such that, for each i , the size of $\text{inv}(w) \cap \text{Span}_{\mathbb{R}}(\beta_i, \delta)$ is at most 1.*

Proof. Given a root $\beta \in \Phi_0$, if a positive scaling of $\beta + k\delta$ is a root for some integer k , then we write $\beta^{(k)}$ for this root. The set $\Phi \cap \text{Span}_{\mathbb{R}}(\beta, \delta)$ consists of the roots of the form $\beta^{(k)}$ and $(-\beta)^{(k)}$. For every root β in Φ_0 , there are infinitely many nonnegative integers k for which there is a root of the form $\beta^{(k)}$.

Recall the notation $V_1^* := \{x \in V^* : \langle x, \delta \rangle = 1\}$ and $D_1 := D \cap V_1^*$. A positive root $\beta^{(k)}$ is an inversion of $w \in W$ if and only if D_1 and wD_1 are on opposite sides of the hyperplane $\{x : \langle x, \beta \rangle = -k\}$. Similarly, a positive root $(-\beta)^{(k)}$ is an inversion of w if and only if D_1 and wD_1 are on opposite sides of the hyperplane $\{x : \langle x, \beta \rangle = k\}$. Thus the condition that $\text{inv}(w) \cap \text{Span}_{\mathbb{R}}(\beta_i, \delta)$ is at most 1 restricts wD_1 to lie between the two hyperplanes $\{x : \langle x, \beta_i \rangle = \text{sgn}(\beta_i)\}$ and $\{x : \langle x, \beta_i \rangle = -2 \text{sgn}(\beta_i)\}$.

Since the β_i are a basis for V_0 , imposing that wD_1 lies between the $n - 1$ pairs of parallel hyperplanes normal to the β_i is equivalent to imposing that wD_1 lies in a parallelepiped. Since W acts by Euclidean motions on V_1^* , each of the simplices wD_1 is congruent. But the parallelepiped is bounded and so has finite volume, and thus contains only finitely many simplices wD_1 . □

We can now prove that the doubled Cambrian fan has the promised support.

Proof of Theorem 4.8. We first show that $|\mathcal{F}_c| \cap \partial\text{Tits}(A)$ is contained in or equal to $\bigcup_{\beta \in \Phi_0^{\omega^+}} \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$. Since $|\mathcal{F}_c| = \bigcup_v \text{Cone}_c(v)$, it is enough to check that $\text{Cone}_c(v) \cap \partial\text{Tits}(A)$ is contained in the right hand side for any c -sortable v . Fix v , and let β be as in Lemma 4.12. Let β_0 be the root in $\Phi_0^{\omega^+}$ that is a positive scaling of $\pi(\beta)$. Then $\text{Cone}_c(v) \subseteq \{x \in V^* : \langle x, \beta \rangle \geq 0\}$ and thus $\text{Cone}_c(v) \cap \partial\text{Tits}(A) \subseteq \{x \in V_0^* : \langle x, \beta_0 \rangle \geq 0\}$.

The rest of the proof establishes the reverse containment. Let $\beta \in \Phi_0^{\omega^+}$; we must show that $\{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$ is contained in $|\mathcal{F}_c|$. The set $\{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$ is a union of cones of u_0D_0 , for u_0 in W_0 , where D_0 is $\bigcap_{s \in S_0} \{x \in V_0^* : \langle x, \alpha_s \rangle \geq 0\}$. Thus it is enough to show that, if $u_0D_0 \subseteq \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$, then $u_0D_0 \subseteq |\mathcal{F}_c|$.

Lemma 4.14 implies (see, for example, [10]) that $\Phi_0^{\omega^0}$ is a root system of rank $n - 2$, so there exist linearly independent elements $\beta_1, \dots, \beta_{n-2}$ of $\Phi_0^{\omega^0}$ such that u_0D_0 is contained in $\{x \in V_0^* : \langle x, \beta_i \rangle \geq 0, 1 \leq i \leq n - 2\}$. Defining β_{n-1} to be β (the given root in $\Phi_0^{\omega^+}$) we have $u_0D_0 \subseteq \{x \in V_0^* : \langle x, \beta_i \rangle \geq 0, 1 \leq i \leq n - 1\}$. Note that the roots $\beta_1, \dots, \beta_{n-1}$ are linearly independent because $\beta \in \Phi_0^{\omega^+}$.

We retain the notation $\beta^{(k)}$ from the proof of Lemma 4.16. For each β_i , there is a smallest integer k_i such that $(-\beta_i)^{(k_i)}$ is a positive root. Define

$$\Lambda := \{x \in V^* : \langle x, \delta \rangle \geq 0\} \cap \bigcap_{i=1}^{n-1} \{x \in V^* : \langle x, (-\beta_i)^{(k_i)} \rangle \leq 0\}.$$

Note that $(-\beta_i)^{(k_i)}$ is a linear combination of δ and a nonzero multiple of β_i , and that the vectors β_i are linearly independent. So the n normal vectors defining Λ are linearly independent and Λ is a simplicial cone, one of whose faces contains u_0D_0 . Since Λ lies on the $\text{Tits}(A)$ side of δ^\perp , the interior of Λ is contained in $\text{Tits}(A)$ and thus is covered by its intersections with cones of \mathcal{F}_c . We claim that only finitely many cones of \mathcal{F}_c meet the interior of Λ .

We pause to illustrate the claim by returning to Example 4.10. Let $\beta = 2\alpha_1 + 3\alpha_2$ and let u_0D_0 be the cone spanned by ρ_2 and $-\rho_1 + 3\rho_2$. This is the cone that is marked by (black) $s_1s_3s_1$ and (red) $s_2s_1s_2s_1s_2$ in Figure 12. Because $\Phi_0^{\omega_0} = \{\pm(\alpha_1 + 2\alpha_2)\}$, we must take $\beta_1 = \alpha_1 + 2\alpha_2$ and by construction β_2 is $\beta = 2\alpha_1 + 3\alpha_2$. We have shaded $\Lambda \cap V_1^*$ in Figure 13, a miniature version of Figure 10. The claim is that only finitely many cones $\text{Cone}_c(v)$ meet Λ . In this case, we see that 14 cones $\text{Cone}_c(v)$ in \mathcal{F}_c meet Λ .

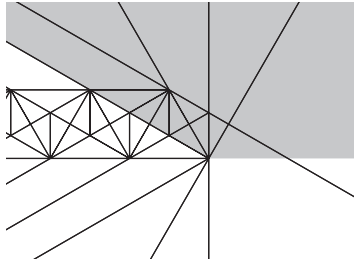


FIGURE 13. The shaded region is the set $\Lambda \cap V_1^*$ used in the proof of Theorem 4.8. Compare to Figure 10

To prove the claim, suppose that $v \in W$ is c -sortable and that $\text{Cone}_c(v)$ meets the interior of Λ . Then there exists $w \in W$ with $wD \subseteq \Lambda$ and $\pi_{\downarrow}^c(w) = v$. Since wD is contained in $\{x \in \text{Tits}(A) : \langle x, (-\beta_i)^{(k_i)} \rangle \leq 0\}$, we know that wD is *not* contained in $\{x \in \text{Tits}(A) : \langle x, (\beta_i)^{(k)} \rangle \leq 0\}$ for any positive root $\beta_i^{(k)}$. So such a $\beta_i^{(k)}$ is not an inversion of w and, since $w \geq \pi_{\downarrow}^c(w) = v$ (Theorem 3.14), no positive root $\beta_i^{(k)}$ can be an inversion of v .

But also, v is c -sortable and $\omega_c(\beta_i, \delta) \geq 0$. Let Φ' be the generalized rank-two parabolic sub root system $\Phi \cap \text{Span}_{\mathbb{R}}(\beta_i, \delta)$. Theorem 3.16 says in particular that v is c -aligned with respect to Φ' , and therefore at most one positive root of the form $(-\beta_i)^{(k)}$ can be an inversion of v for each i . We have shown that, if $\text{Cone}_c(v)$ meets the interior of Λ , then (for each i) there is at most one inversion of v of the form $(\pm\beta_i)^{(k)}$. Now Lemma 4.16 proves the claim.

Let \mathcal{V} be $\bigcup \text{Cone}_c(v)$, where the union ranges over those v such that $\text{Cone}_c(v)$ meets the relative interior of Λ . We have just shown that this union is finite, so \mathcal{V} is a finite union of closed cones and is itself closed. By Theorem 3.14, \mathcal{V} contains the interior of Λ , and since \mathcal{V} is closed, it also contains the facet of Λ containing u_0D_0 . We have shown that $u_0D_0 \subseteq |\mathcal{F}_c|$, as desired.

We have shown that $|\mathcal{F}_c| \cap \partial\text{Tits}(A)$ equals $\bigcup_{\beta \in \Phi_0^{\omega_0}} \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$, which is nonempty by Lemma 4.13. An analogous argument shows that $|\mathcal{F}_{c^{-1}}| \cap \partial\text{Tits}(A)$ equals $\bigcup_{\beta \in \Phi_0^{\omega_0}} \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$. (The negative signs introduced by the antipodal map cancel the negative signs introduced by passing from c to c^{-1} .) Since $|\mathcal{F}_c| \cap \partial\text{Tits}(A)$ and $|\mathcal{F}_{c^{-1}}| \cap \partial\text{Tits}(A)$ coincide, they both equal $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$, which therefore equals $\bigcup_{\beta \in \Phi_0^{\omega_0}} \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$. \square

Proof of Corollary 4.9. By Theorem 3.14, the interiors of $\text{Tits}(A)$ and $-\text{Tits}(A)$ lie in $|\mathcal{F}_c|$ and $|\mathcal{F}_{c^{-1}}|$. So $V^* \setminus |\mathcal{DF}_c| = \partial\text{Tits}(A) \setminus (|\mathcal{DF}_c| \cap \partial\text{Tits}(A))$. The first description of $V^* \setminus |\mathcal{DF}_c|$ follows by taking the complement of the expression for $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$ in Theorem 4.8.

We now prove the second description. Every cone of \mathcal{DF}_c intersects $\partial\text{Tits}(A)$ in a union of cones of the W_0 -Coxeter fan. We must show that a cone F of the W_0 -Coxeter fan is contained in $|\mathcal{DF}_c|$ if and only if $x_c \notin F$.

For each $\beta \in \Phi_0$, the cone F lies entirely to one side of β^\perp or the other, or on β^\perp . Thus we deduce from Theorem 4.8 that $F \subseteq |\mathcal{DF}_c|$ if and only if there exists $\beta \in \Phi_0^{\omega^+}$ such that $F \subseteq \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$. But also $\beta \in \Phi_0^{\omega^+}$ if and only if $\omega_c(\beta, \delta) > 0$, or equivalently $\langle x_c, \beta \rangle < 0$. So $F \subseteq |\mathcal{DF}_c|$ if and only if there exists $\beta \in \Phi_0$ with both (a) $F \subseteq \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$ and (b) $\langle x_c, \beta \rangle < 0$. If both (a) and (b) hold for some β , then necessarily $x_c \notin F$, and conversely, if $x_c \notin F$ then some facet-defining hyperplane of F separates x_c from F . This hyperplane is β^\perp for some $\beta \in \Phi_0$. Therefore either β or $-\beta$ satisfies (a) and (b).

The first description writes $V^* \setminus |\mathcal{DF}_c|$ as a finite intersection of open cones in $\partial\text{Tits}(A)$, so $V^* \setminus |\mathcal{DF}_c|$ is an open cone in $\partial\text{Tits}(A)$. The second description implies that $V^* \setminus |\mathcal{DF}_c|$ is nonempty. □

4.5. Well-connectedness. In this section, we prove that $(\text{DCamb}_c, \text{DC}_c)$ is well-connected and prove a corollary to well-connectedness which is helpful in proving simple connectivity.

Proposition 4.17. *If A is of affine type, then $(\text{DCamb}_c, \text{DC}_c)$ is well-connected.*

Proof. The polyhedral property was already established. It remains to show that if F is a proper face of two maximal cones in \mathcal{DF}_c , then there exists a path in DCamb_c connecting the two corresponding vertices, such that, along the path, each corresponding cone has F as a face. We already know that (Camb_c, C_c) and $(\text{Camb}_{c^{-1}}, C_{c^{-1}})$ have this property by Theorem 3.2. By Corollary 3.26, we need only check the case where v is c -sortable, v' is c^{-1} -sortable, the interior of $\text{Cone}_c(v)$ is in $\text{Tits}(A)$ and the interior of $-\text{Cone}_{c^{-1}}(v')$ is in $-\text{Tits}(A)$. In particular, F is in the boundary of $\text{Tits}(A)$.

In the first paragraph of the proof of Theorem 4.8, we showed that there exists $\beta \in \Phi_0^{\omega^+}$ such that $\text{Cone}_c(v) \cap \partial\text{Tits}(A) \subseteq \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$. Since F is a face of $\text{Cone}_c(v)$ and is contained in $\partial\text{Tits}(A) = V_0^*$, we have $F \subseteq \{x \in V_0^* : \langle x, \beta \rangle \geq 0\}$ for the same $\beta \in \Phi_0^{\omega^+}$. Choose x in the relative interior of F . By Proposition 4.4, F has dimension less than $n - 1$ as do all other faces of \mathcal{F}_c contained in $\partial\text{Tits}(A)$. Thus there exists a vector $y \in V_0^*$ such that, for small enough $\varepsilon > 0$, the vector $x + \varepsilon y$ is not contained in any nonmaximal face of \mathcal{F}_c . Furthermore, we can choose y such that $\langle y, \beta \rangle \geq 0$, so that $x + \varepsilon y \in |\mathcal{F}_c|$ for all $\varepsilon > 0$ by Theorem 4.8. We conclude that for small enough ε , the point $x + \varepsilon y$ is in the interior of some $\text{Cone}_c(u)$ containing F in its boundary.

Since \mathcal{F}_c is a fan and F is a face of $\text{Cone}_c(v)$, we see that F is a face of $\text{Cone}_c(u)$. Since (Camb_c, C_c) is well-connected, there is a path in Camb_c connecting u to v such that, along the path, each corresponding cone has F as a face. Furthermore, by Corollary 3.26, there exists a c^{-1} -sortable element u' such that $\text{Cone}_c(u) = -\text{Cone}_{c^{-1}}(u')$. Thus there is a path in $-\text{Camb}_{c^{-1}}$ from u' to v' such that, along the path, each corresponding cone has F as a face. Concatenating the two paths, we obtain the desired path in DCamb_c . □

Corollary 4.18. *Suppose A is of affine type and let F be a face of \mathcal{DF}_c of dimension $n - 2$. Then the maximal cones of \mathcal{DF}_c containing F form a rank-two cycle or path in DCamb_c . If the relative interior of F is in the interior of $|\mathcal{DF}_c|$, then they form a cycle. Otherwise they form a path. Moreover, in the former case, the*

rank-two root system $\Phi' := \{\beta \in \Phi : \langle \beta, F \rangle = 0\}$ is finite and, in the latter case, Φ' is infinite of affine type.

Proof. The first assertion of the corollary is immediate by Propositions 2.7 and 4.17. Let τ be the rank-two cycle or path in DCamb_c associated to F . We write $\text{relint}(F)$ for the relative interior of F and break into cases based on the relationship between $\text{relint}(F)$ and $\partial\text{Tits}(A)$.

If the relative interior of F intersects $\text{Tits}(A)$, then let $x \in \text{relint}(F) \cap \text{Tits}(A)$. The stabilizer of x with respect to the action of W on V^* is a proper (not necessarily standard) parabolic subgroup of W generated by reflections in the roots $\Phi'' = \{\beta \in \Phi : \langle \beta, x \rangle = 0\}$. (This is a standard fact. Since $\text{Tits}(A)$ is the union of cones uD for $u \in W$, there exists $w \in W$ such that $wx \in D$. The stabilizer of wx is then described by [15, Theorem 5.13].) Since W is of affine type, its proper parabolic subgroups are all finite, and we conclude that Φ'' is finite. The root system Φ' described in the statement of the corollary is a sub root system of Φ'' , and thus Φ' is also finite. Proposition 2.8 now implies that τ is a cycle. Therefore the relative interior of F lies in the interior of $|\mathcal{DF}_c|$, so the proposition holds in this case.

If the relative interior of F intersects $-\text{Tits}(A)$, then the analogous argument works. Thus it remains to consider the case where F is contained in $\partial\text{Tits}(A)$. In this case, $\delta \in \text{Span}_{\mathbb{R}}(\Phi')$. We conclude that Φ' is infinite, and since Φ' is a sub root system of Φ , it is necessarily of affine type. Proposition 2.8 now implies that τ is a path. To complete the proof, we must show that F is not in the interior of $|\mathcal{DF}_c|$. Supposing to the contrary, if we view F as a cone in the fan $\mathcal{DF}_c \cap \partial\text{Tits}(A)$, the cone F is shared by two $(n - 1)$ -dimensional faces G_1 and G_2 of $\mathcal{DF}_c \cap \partial\text{Tits}(A)$. Proposition 4.4 says that each G_i is the intersection with $\partial\text{Tits}(A)$ of a maximal cone C_i of \mathcal{DF}_c . Every other maximal cone of \mathcal{DF}_c containing F has two facets defined by roots in Φ' . But then, since the hyperplanes β^\perp for $\beta \in \Phi'$ accumulate to $\delta^\perp = \partial\text{Tits}(A)$, there are only finitely many different maximal cones in \mathcal{DF}_c containing F as a face, contradicting the fact that τ is an infinite path. The situation is illustrated schematically in Figure 14. In the picture, the bold lines indicate facets of C_1 and C_2 and the lighter lines indicate hyperplanes β^\perp for $\beta \in \Phi'$. □

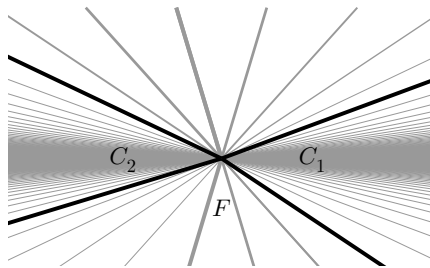


FIGURE 14. An illustration for the proof of Corollary 4.18

The proof of Corollary 4.18 also establishes the following useful fact, which augments Proposition 4.4.

Corollary 4.19. *If A is of affine type, then every $(n - 2)$ -dimensional face of \mathcal{DF}_c contained in $\partial\text{Tits}(A)$ is in the boundary of $|\mathcal{DF}_c|$.*

4.6. Simple connectivity. The final piece in the proof of Theorem 1.1 is the following proposition.

Proposition 4.20. *If A is of affine type, then $(\text{DCamb}_c, \text{DC}_c)$ is simply connected.*

In preparation for the proof of the proposition, consider a loop $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p = v_0$ in DCamb_c . Choose generic points y_i in the interior of the cones $\text{Cone}_c(v_i)$ so that the line segment $\overline{y_i y_{i+1}}$ crosses between $\text{Cone}_c(v_i)$ and $\text{Cone}_c(v_{i+1})$ in the relative interior of the wall $\text{Cone}_c(v_i) \cap \text{Cone}_c(v_{i+1})$. In particular, $\overline{y_i y_{i+1}}$ is contained entirely in the interior of $|\mathcal{DF}_c|$. We write γ for the loop $\bigcup_{i=0}^{p-1} \overline{y_i y_{i+1}}$ in V^* .

Let σ be the union of the closed cones of the W_0 -Coxeter fan that contain x_c . From Corollary 4.9, $V^* \setminus \sigma$ is the interior of $|\mathcal{DF}_c|$. Choose z to be a generic point in the relative interior of $-\sigma$. For each i from 0 to $p - 1$, write Δ_i for the triangle $\text{ConvexHull}(z, y_i, y_{i+1})$.

Lemma 4.21. *The cone σ does not intersect Δ_i .*

Proof. If the line segment $\overline{y_i y_{i+1}}$ does not cross $\partial\text{Tits}(A)$, then the claim is obvious, since z is the only point of Δ_i in $\partial\text{Tits}(A)$ while σ is contained in $\partial\text{Tits}(A)$ and we constructed z to lie outside σ .

If $\overline{y_i y_{i+1}}$ crosses $\partial\text{Tits}(A)$, let w be the point $\overline{y_i y_{i+1}} \cap \partial\text{Tits}(A)$. Then $\Delta_i \cap \partial\text{Tits}(A)$ is the line segment \overline{wz} . Now $\overline{y_i y_{i+1}}$ lies in the interior of $|\mathcal{DF}_c|$, so w is in the relative interior of $|\mathcal{DF}_c| \cap \partial\text{Tits}(A)$. By Theorem 4.8, there is some β in Φ_0^+ so that $\langle w, \beta \rangle > 0$. But z is in the relative interior of $-\sigma$, which equals $\bigcap_{\beta \in \Phi_0^+} \{x \in \partial\text{Tits}(A) : \langle x, \beta \rangle > 0\}$, so also $\langle z, \beta \rangle > 0$. Therefore the entire segment \overline{wz} lies strictly to one side of β^\perp , and thus misses σ . □

Assuming we have chosen our points y_i and z generically enough, Δ_i only meets the faces of dimension $\geq n - 2$ in \mathcal{DF}_c , and meets them transversely, $\partial\Delta_i$ only meets the faces of dimension $\geq n - 1$, and z lies in a maximal cone of \mathcal{DF}_c .

Lemma 4.22. *Only finitely many cones of \mathcal{DF}_c meet Δ_i .*

Proof. Suppose otherwise. Since Δ_i is transverse to \mathcal{DF}_c , if Δ_i meets a cone, it meets its relative interior. For each such cone C , choose a point $q(C)$ in the intersection of Δ and the relative interior of C . Since Δ_i is compact, the set of such $q(C)$ must have an accumulation point $r \in \Delta_i$. Let F be the cone of \mathcal{DF}_c in whose relative interior r lies.

Since Δ_i is disjoint from all faces of \mathcal{DF}_c of dimension less than $n - 2$, we know that $\dim F \geq n - 2$. If $\dim F = n$, then we have a contradiction, as there is an open neighborhood of r in V^* which meets no cones of \mathcal{DF}_c other than F . If $\dim F = n - 1$, we likewise have a contradiction, as there is an open neighborhood of r in V^* which meets only three cones of \mathcal{DF}_c (namely, F and the two cones containing it). Finally, suppose that $\dim F = n - 2$. Since Δ_i lies in the interior of \mathcal{DF}_c (Lemma 4.21), the fact that $\text{relint}(F)$ meets Δ_i means that $\text{relint}(F)$ lies in the interior of \mathcal{DF}_c . From Corollary 4.18, F lies in finitely many cones of \mathcal{DF}_c and we have a contradiction as in the other cases. □

Proof of Proposition 4.20. Continuing the notation from above, we want to show that the loop $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p = v_0$ is trivial modulo relations coming from finite rank-two cycles in DCamb_c . We chose z to lie in $\text{Cone}_c(w)$ for some

vertex w of DCamb_c . We write $v_i \rightsquigarrow w$ for the path from v_i to w in DCamb_c following the edge $y_i z$ of Δ_i ; the wavy line reminds us that we may pass through other cones of \mathcal{DF}_c on the way. We also write $w \rightsquigarrow v_i$ for the reverse path.

Pulling the path from y_i to y_{i+1} across the triangle Δ_i , from the edge $\overline{y_i y_{i+1}}$ of the triangle to the two edges $\overline{y_i z} \cup \overline{z y_{i+1}}$, we cross over codimension-2 faces of \mathcal{DF}_c . By Lemma 4.22, we only cross finitely many of them. From Corollary 4.18, all of these codimension-2 faces correspond to finite rank-two cycles in DCamb_c . So $v_i \rightarrow v_{i+1}$ is equivalent to $v_i \rightsquigarrow w \rightsquigarrow v_{i+1}$ modulo rank-two cycles. Thus the cycle $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p = v_0$ is equivalent to the following cycle, which is obviously trivial: $v_0 \rightsquigarrow w \rightsquigarrow v_1 \rightsquigarrow w \rightsquigarrow v_2 \rightsquigarrow \cdots \rightsquigarrow v_{p-1} \rightsquigarrow w \rightsquigarrow v_p = v_0$. \square

Having proved Theorem 1.1, we conclude by paying an old debt. Almost all of Corollary 2.5 follows from Theorems 1.1 and 2.4. The remaining piece is Conjecture 3.14 of [31] (the sign coherence of \mathbf{g} -vectors), which is now easy now that we have the definition of the doubled Cambrian framework.

End of the proof of Corollary 2.5. Proposition 3.18 implies that the conclusion of Conjecture 3.14 of [31] holds in the Cambrian reflection frameworks (Camb_c, C_c) and $(\text{Camb}_{c-1}, C_{c-1})$. The antipodal map preserves sign-coherence of \mathbf{g} -vectors, so Conjecture 3.14 of [31] holds for all vertices of $(\text{DCamb}_c, \text{DC}_c)$. \square

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