TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 2, February 2018, Pages 1377–1391
http://dx.doi.org/10.1090/tran/7196
Article electronically published on October 16, 2017

COMBINATORIAL CALABI FLOWS ON SURFACES

HUABIN GE

ABSTRACT. For triangulated surfaces, we introduce the combinatorial Calabi flow which is an analogue of the smooth Calabi flow. We prove that the solution to the combinatorial Calabi flow exists for all time and converges if and only if the Thurston's circle packing exists. As a consequence, the combinatorial Calabi flow provides a new algorithm to find circle packings with prescribed curvatures. The proofs rely on careful analysis of the combinatorial Calabi energy, combinatorial Ricci potential and discrete dual-Laplacians.

1. INTRODUCTION

An important question in modern geometry is to find canonical metrics on a given manifold. Seeking constant curvature metrics, E. Calabi studied the variational problem of minimizing the so-called "Calabi energy" in any fixed cohomology class of Kähler metrics and proposed the Calabi flow [2,3]. Hamilton introduced the Ricci flow ([20]), which has been used to solve the Poincaré conjecture. For dimension two, i.e., the smooth surface case, it is proved that both the Calabi flow and the normalized Ricci flow exist for all time and converge to a constant scalar curvature metric (see [4], [5], [6], [7], [9], [30], and [32]).

Given a triangulated surface, Thurston introduced the circle packing metric, which is a type of piecewise flat cone metric with singularities at the vertices. Thurston found that there are combinatorial obstructions for the existence of a circle packing metric with constant combinatorial curvatures (see section 13.7 in [33]). Motivated by the idea of Hamilton, Bennett Chow and Feng Luo [8] introduced the combinatorial Ricci flow. They proved that the combinatorial Ricci flow exists for all time and converges exponentially fast to Thurston's circle packing on surfaces. They also reproved the equivalence between Thurston's combinatorial condition (see (1.3) in [8], or (1.3) in this paper) and the existence of a constant curvature circle packing metric.

Inspired by the work of Bennett Chow and Feng Luo in [8], we consider the 2-dimensional combinatorial Calabi flow, which is the negative gradient flow of combinatorial Calabi energy. We interpret the Jacobian of the curvature map as a type of discrete Laplace operator, which comes from the dual structure of circle patterns. We get a uniform estimate of the bound for all entries of discrete

Received by the editors March 10, 2015, and, in revised form, July 5, 2016, and December 23, 2016.

2010 *Mathematics Subject Classification.* Primary 53C44, 52C26.

Key words and phrases. Circle packing, combinatorial Calabi flow, combinatorial Calabi energy, combinatorial Ricci potential.

This research was supported by the National Natural Science Foundation of China under grant No.11501027, and Fundamental Research Funds for the Central Universities (Nos. 2015JBM103, 2014RC028 and 2016JBM071).

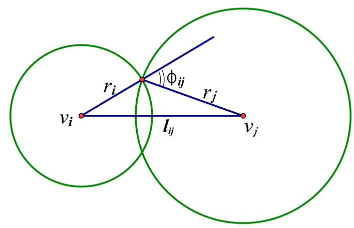

FIGURE 1. Circle packing metric

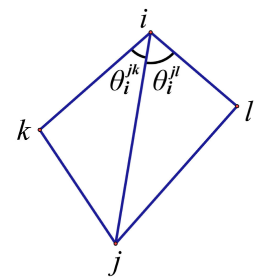

FIGURE 2. Two adjacent triangles

dual-Laplacians, which implies the long time existence of the solutions of the combinatorial Calabi flow. Then we prove that the combinatorial Calabi flow converges exponentially fast if and only if Thurston's combinatorial conditions are satisfied. It is shown that the combinatorial Calabi flow finds Thurston's circle patterns automatically. As a consequence, we can design algorithms to seek circle packing metrics with prescribed combinatorial curvatures. In fact, any algorithm minimizing the combinatorial Calabi energy or the combinatorial Ricci potential can achieve this goal.

1.1. **Circle packing metrics.** Suppose X is a closed surface with a triangulation $T = (V, E, F)$, where V, E, F denote the sets of vertices, edges, and faces respectively. A circle packing metric is defined to be a positive function $r : V \to (0, +\infty)$ on the vertices. A weight on the triangulation is defined to be a function $\Phi : E \to [0, \pi/2]$. Throughout this paper, a function defined on vertices is an N-dimensional column vector, where $N = |V|$ is the number of vertices. Moreover, all vertices, marked by $v_1, \ldots, v_N$, are ordered one by one and we often write i instead of v_i if there is no confusion. Thus we may think of circle packing metrics as points in $\mathbb{R}^N_{>0}$, where $\mathbb{R}^N_{>0}$ means N times of Cartesian product of $(0, \infty)$. A triangulated surface with a weight Φ is denoted as (X, T, Φ).

Let $l : E \to (0, +\infty)$ be a positive function assigning each edge $\{i, j\} \in E$ a length l_{ij}. We call l a piecewise linear metric if for every triangle $\{i, j, k\} \in F$, the three edge lengthes l_{ij}, l_{jk} and l_{ik} satisfy triangle inequalities. For a fixed triangulated surface (X, T, Φ), every circle packing metric r determines a piecewise linear metric on X by setting the length of edge $\{i, j\} \in E$ as (see Figure 1)

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_jcos(\Phi_{ij})}.$$

As a consequence, each face in F is isometric to a Euclidean triangle. More specifically, each face $\{i, j, k\} \in F$ is a Euclidean triangle with edge lengths l_{ij}, l_{jk}, l_{ki} because l_{ij}, l_{jk}, l_{ki} satisfy triangle inequalities ([33], Lemma 13.7.2). Furthermore, the triangulated surface (X, T) is composed by gluing Euclidean triangles coherently.

1.2. **Combinatorial curvatures and constant curvature metrics.** Given a triangulated surface (X, T, Φ) with a circle packing metric r, all inner angles of the triangles are determined by $r_1, \ldots, r_N$. Denote θ_i^{jk} as the inner angle at vertex i in the triangle $\{i, j, k\} \in F$; then the well-known combinatorial (or "discrete") Gauss curvature K_i at vertex i is defined as

$$K_i = 2\pi - \sum_{\{i,j,k\}\in F} \theta_i^{jk}, \tag{1.1}$$

where the sum is taken over each triangle with i as one of its vertices. Notice that θ_i^{jk} can be calculated by cosine law, thus θ_i^{jk} and K_i are explicit functions of the circle packing metric r.

For every circle packing metric r on (X, T, Φ), we have the combinatorial Gauss-Bonnet formula [8]

$$\sum_{i=1}^{N} K_i = 2\pi\chi(X). \tag{1.2}$$

Notice that, the combinatorial Gauss-Bonnet formula (1.2) is still valid for all piecewise linear surfaces (X, T, l), where $l : E \to (0, +\infty)$ is a piecewise linear metric, that is, the length structure l makes each face in F isometric to a Euclidean triangle. The average combinatorial curvature is $k_{av} = 2\pi\chi(X)/N$, which does not change as the metric (circle packing metric or piecewise linear metric) varies, and is an invariant that depends only on the topological (the Euler characteristic number $\chi(X)$) and combinatorial ($N = |V|$) information of (X, T).

Finding canonical metrics on manifolds is a central topic in geometry and topology. The constant curvature circle packing metric (denoted as r_{av}), a metric that determines the constant combinatorial curvature $K_{av} = K(r_{av}) = k_{av}(1, \ldots, 1)^T$, is a good candidate for privileged metrics. Thurston first studied this class of metrics, and found that there are combinatorial obstructions for the existence of constant curvature metrics [33]. Given a triangulated surface (X, T, Φ), for any nonempty proper subset $I \subset V$, let F_I be the subcomplex whose vertices are in I, and let $Lk(I)$ be the set of all such pairs (e, v), where (e, v) is made up of an edge e and a vertex v satisfying the following three conditions: (1) the end points of e are not in I; (2) v is in I; (3) e and v form a triangle. Thurston proved:

Theorem 1.1 (Thurston)**.** *Given a triangulated surface (X, T, Φ), there exists a constant combinatorial curvature circle packing metric if and only if the following combinatorial and topological condition is satisfied:*

$$2\pi\chi(X)\frac{|I|}{|V|} > -\sum_{(e,v)\in Lk(I)} (\pi - \Phi(e)) + 2\pi\chi(F_I), \quad \forall I : \phi \neq I \subsetneqq V. \tag{1.3}$$

Moreover, if a constant curvature metric exists, it is unique up to a scalar multiplication. In other words, if r_1 and r_2 are both constant curvature metrics, there is a positive real number c such that $r_1 = cr_2$.

In [8], Bennett Chow and Feng Luo introduced the combinatorial Ricci flow,

$$\frac{dr_i}{dt} = -K_i r_i \tag{1.4}$$

and its normalization

$$\frac{dr_i}{dt} = (K_{av} - K_i) r_i. \tag{1.5}$$

The combinatorial Ricci flow, which is an analogue of Hamilton's Ricci flow on surfaces [21], can be used to deform Thurston's circle pattern to a pattern with constant cone angles. It provides a new proof of Thurston's theorem on the existence of a constant circle packing metric, and suggests a natural algorithm to find circle packing metrics with prescribed curvatures. In fact, Chow and Luo proved

Theorem 1.2 (Chow-Luo)**.** *For any initial metric $r(0)$, the solution to the flow* (1.5) *exists for all time. Additionally, the flow* (1.5) *converges if and only if there exists a metric of constant curvature. Furthermore, if the flow converges, it converges exponentially fast to the metric of constant curvature.*

Remark 1. Unless specially stated, the convergence in this paper is defined according to the Euclidean topology. More concretely, we say $r(t)$ converges to some r^* as t goes to $+\infty$, if the Euclidean L^2 norm $\|r(t) - r^*\|$ converges to zero as t goes to $+\infty$.

In this paper, we introduce the combinatorial Calabi flow, which is the negative gradient flow of the combinatorial Calabi energy. The combinatorial Calabi flow is an analogue of a smooth Calabi flow on surfaces. We prove that the solution of the combinatorial Calabi flow exists for $t \in [0, +\infty)$ by a careful estimation of discrete dual-Laplacians. Moreover, if the solution to the combinatorial Calabi flow converges to some circle packing metric r^*, r^* has a constant curvature. On the other hand, if there exists a constant curvature metric r^* (that is, a metric r^* whose curvature $K(r^*)$ a constant), the solution to the combinatorial Calabi flow converges to a constant curvature metric r_{av}, which only differs from r^* by a scaling.

2. The 2-dimensional combinatorial Calabi flow

2.1. **Definition of the combinatorial Calabi flow.** For smooth surfaces, the Calabi flow [2–6, 9, 27–30, 32, 34] is defined as

$$\frac{\partial g}{\partial t} = \Delta K g,$$

where K is the Gaussian curvature. The Laplace-Beltrami operator plays an important role in the study of the smooth Calabi flow. Before giving the definition of the combinatorial Calabi flow, we need to define the combinatorial Laplace operator first, which is an analogue of the smooth Laplace-Beltrami operator.

Set $u_i = \ln r_i$, where $i = 1, \ldots, N$, then the coordinate transformation $u = u(r)$ maps $r \in \mathbb{R}^N_{>0}$ to $u \in \mathbb{R}^N$ homeomorphically. We denote this coordinate transformation as a map $\ln : \mathbb{R}^N_{>0} \to \mathbb{R}^N$, $r \mapsto u = \ln r$, and the inverse Exp. Similar to [12] and [13], we interpret the discrete Laplacian as the Jacobian of the curvature map $K = K(u)$.

Definition 2.1. Given a triangulated surface (X, T, Φ), the discrete dual-Laplacian "Δ", which is a special type of the discrete Laplacian, is defined as $-L^T$, where

$$L = (L_{ij})_{N\times N} = \frac{\partial(K_1, \ldots, K_N)}{\partial(u_1, \ldots, u_N)} = \begin{pmatrix} \frac{\partial K_1}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial K_1}{\partial u_N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial K_N}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial K_N}{\partial u_N} \end{pmatrix}.$$

Both Δ and L operate on functions f (defined on vertices, hence is a column vector) by a matrix multiplication, i.e.

$$\Delta f_i = (\Delta f)_i = -(L^T f)_i = -\sum_{j=1}^{N} \frac{\partial K_j}{\partial u_i} f_j = -\sum_{j=1}^{N} \frac{\partial K_j}{\partial r_i} r_i f_j. \tag{2.1}$$

Remark 2. As a special discrete Laplacian (Chung [10]), the discrete dual-Laplacian Δ defined above comes from the dual structure of the circle packings. Glickenstein [14, 16] studied this type of discrete Laplacians systematically. See Appendix A for more explanations.

Definition 2.2. Given a triangulated surface (X, T, Φ), the combinatorial Calabi flow is defined as

$$\frac{du_i}{dt} = \Delta K_i \tag{2.2}$$

with $u(0) \in \mathbb{R}^N$.

It is more convenient to write the combinatorial Calabi flow (2.2) in a matrix form as

$$\frac{du}{dt} = \Delta K = -L^T K. \tag{2.3}$$

Note that, the equation (2.3) is an autonomous ODE system.

Remark 3. We shall show that L is symmetric. Hence there is no need to use the transpose of L in (2.3). However, in higher dimensions or other situations, L may be nonsymmetric. We use L^T in equation (2.3) to avoid the negative gradient flow of the combinatorial Calabi energy undefined in those situations. See 5.4 in [12] for a more detailed description.

Remark 4. Using the matrix language, Bennett Chow and Feng Luo's combinatorial Ricci flow [8] is $\frac{du}{dt} = -K$. The normalized combinatorial Ricci flow is $\frac{du}{dt} = K_{av} - K$.

2.2. **The combinatorial Calabi flow is variational.**

Definition 2.3. Given a triangulated surface (X, T, Φ) with a circle packing metric r, the combinatorial Calabi energy is defined as

$$\mathcal{C}(r) = \|K - K_{av}\|^2 = \sum_{i=1}^{N} (K_i - k_{av})^2. \tag{2.4}$$

Consider the combinatorial Calabi energy $\mathcal{C}$ as a function of u. We then have

$$\nabla_u \mathcal{C} = \begin{pmatrix} \frac{\partial \mathcal{C}}{\partial u_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \mathcal{C}}{\partial u_N} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial K_1}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial K_N}{\partial u_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial K_1}{\partial u_N} & \cdot & \cdot & \cdot & \frac{\partial K_N}{\partial u_N} \end{pmatrix} \begin{pmatrix} K_1 \\ \cdot \\ \cdot \\ \cdot \\ K_N \end{pmatrix} = 2L^T K.$$

This implies the following proposition.

Proposition 2.4. *The combinatorial Calabi flow* (2.2) *is the negative gradient flow of combinatorial Calabi energy, and the Calabi energy* (2.4) *is descending along this flow.*

Now it is time to say a bit about the motivation to introduce the combinatorial Calabi flow. First, the combinatorial Calabi flow is a negative gradient flow of the L^2 norm of the discrete Gaussian curvature, which is more natural than the energy for the combinatorial Ricci flow, where the combinatorial Ricci flow is the negative gradient flow of the combinatorial Ricci potential (4.3). Compared with the combinatorial Ricci potential (4.3), the combinatorial Calabi energy is easier to calculate. Additionally, the combinatorial Calabi energy is nonnegative and attains its minimum at constant curvature metrics, which is unique (up to a scalar multiplication) by Thurston's Theorem 1.1. This gives natural monotonicity of the functional, which could be useful in higher dimensions or other situations; see for example the author and Xu's work [12]. Second, the combinatorial Calabi flow naturally finds the constant curvature metrics automatically without any renormalization. Third, the combinatorial Calabi flow is a fourth order flow (see 3.1, [14] for a definition of fourth order discrete differential operator), which is often employed in physical situations. Robinson-Trautman metrics [28] play an important role in the early understanding of gravitational radiation. It is pointed out by Tod [34] that the Robinson-Trautman equation, the essential part of Einstein field equations determined by Robinson-Trautman metrics, is equivalent to the Calabi flow equation (see [6, 9, 27, 29, 30]). The combinatorial Calabi flow seems to be the first discrete curvature flow studied in this way.

2.3. **Main properties of the combinatorial Calabi flow.** Because K_i and ΔK_i are explicit functions of $r_1, \ldots, r_N$, the local existence of the combinatorial Calabi flow (2.2) follows from Picard's existence and uniqueness theorem in the standard ODE theory. By a careful estimation of a discrete dual-Laplacian and a combinatorial Ricci potential, the long time existence is proved in Section 3, and the convergence is proved in Section 4. The main result in this paper is stated as follows.

Theorem 2.5. *Given a triangulated surface (X, T, Φ), for any initial circle packing metric $r(0) \in \mathbb{R}^N_{>0}$, the solution of the combinatorial Calabi flow* (2.2) *exists for $t \in [0, +\infty)$. Additionally, $r(t)$ converges if and only if there exists a constant curvature circle packing metric r_{av}. Furthermore, if the solution of the combinatorial Calabi flow* (2.2) *converges, it converges exponentially fast to a constant curvature circle packing metric.*

Combining Thurston's Theorem 1.1 and Chow-Luo's Theorem 1.2, we get the following corollary.

Corollary 2.6. *The following four statements are mutually equivalent:*

(1) *The solution of the combinatorial Calabi flow* (2.2) *converges.*
(2) *The solution of Chow-Luo's combinatorial Ricci flow* (1.5) *converges.*
(3) *There exists a constant curvature circle packing metric r_{av}.*
(4) $2\pi\chi(X)\frac{|I|}{|V|} > -\sum_{(e,v)\in Lk(I)}(\pi - \Phi(e)) + 2\pi\chi(F_I), \quad \forall I : \phi \neq I \subsetneqq V.$

Furthermore, if any of the four statements is satisfied, the solutions of the combinatorial Calabi flow (2.2) *and the combinatorial Ricci flow* (1.5) *converge exponentially fast to the unique (up to a scalar multiplication of r_{av}) constant curvature circle packing metric.*

3. Long time existence

3.1. **Discrete Laplacians for circle packing metrics.** Any circle packing metric r determines an intrinsic metric structure on fixed (X, T, Φ) by Euclidean cosine law. The lengths l_{ij}, angles θ_i^{jk} and curvatures K_i are elementary functions of $r = (r_1, \ldots, r_N)^T$. We denote $j \sim i$ if the vertices i and j are adjacent. For any vertex i and any edge $j \sim i$, set (see Figure 2)

$$B_{ij} = \frac{\partial(\theta_i^{jk} + \theta_i^{jl})}{\partial r_j} r_j; \tag{3.1}$$

then $B_{ij} = B_{ji}$, since $\frac{\partial \theta_i^{jk}}{\partial r_j} r_j = \frac{\partial \theta_j^{ik}}{\partial r_i} r_i$ (see Lemma 2.3 in [8]).

Proposition 3.1. *For any $1 \leq i, j \leq N$ and $i \sim j$, we have*

$$0 < B_{ij} < 2\sqrt{3}. \tag{3.2}$$

Proof. We just need to prove $0 < \frac{\partial \theta_i^{jk}}{\partial r_j} r_j < \sqrt{3}$. Since it is not the main interest of this paper, we defer the details to Appendix A. □

Proposition 3.2. *Define $L = (L_{ij})_{1 \leq i,j \leq N}$ as in Definition 2.1; then*

$$L_{ij} = \begin{cases} \sum\limits_{k \sim i} B_{ik}\,, & j = i, \\ -B_{ij}\,, & j \sim i, \\ 0\,, & \textit{else}. \end{cases} \tag{3.3}$$

Proof. This can be proved by direct calculations; therefore we omit the details. □

Proposition 3.3. *L is a semi-positive definite $N \times N$ matrix, whose rank is $N-1$. Moreover, the null space of L is $Ker(L) = \{t(1, \ldots, 1)^T | t \in \mathbb{R}\}$.*

Proof. This follows directly from Lemma 3.10 in [8]. □

The differential form $\omega = \sum_{i=1}^N (K_i - k_{av}) du_i$ is closed, for $L_{ij} = L_{ji}$. Thus the integral

$$\int_{u_0}^{u} \sum_{i=1}^{N} \big(K_i - k_{av}\big) du_i$$

is well defined [8], where u_0 is an arbitrary point in $\mathbb{R}^N$. This integral is crucial for the proof of our main theorem. For convenience, we call this integral the combinatorial Ricci potential.

For any smooth closed manifold (M, g) with Riemannian metric g, we know that $\int_M \Delta f = 0$, where f is an arbitrary smooth function on M. For a combinatorial surface (X, T, Φ) with a circle packing metric r, we have $\sum_{i=1}^N \Delta K_i = -(1, \ldots, 1)^T L K = 0$. Thus we obtain the following proposition.

Proposition 3.4. *As long as the combinatorial Calabi flow exists, both $\prod_{i=1}^N r_i(t) \equiv \prod_{i=1}^N r_i(0)$ and $\sum_{i=1}^N u_i(t) \equiv \sum_{i=1}^N u_i(0)$ are constants.*

3.2. **Long time existence of the combinatorial Calabi flow.** Notice that

$$\Delta K_i = \sum_{j\sim i} B_{ij}(K_j - K_i). \tag{3.4}$$

Using the estimation of B_{ij} in (3.2), we obtain the following theorem.

Theorem 3.5. *Given a triangulated surface (X, T, Φ), for any initial circle packing metric $r(0)$, the solution of the combinatorial Calabi flow* (2.2) *exists for all time $t \in [0, +\infty)$.*

Proof. Let d_i denote the degree at vertex v_i, which is the number of edges adjacent to v_i. Set $d = max(d_1, \dots, d_N)$; then $(2-d)\pi < K_i < 2\pi$. By the estimation of B_{ij} in Propositon 3.1, all $|\Delta K_i|$ are uniformly bounded by a positive constant $c = 2\sqrt{3} \cdot N \cdot d \cdot \pi$, which depends only on the triangulation. Then we have

$$c_0 e^{-ct} \le r_i(t) \le c_0 e^{ct}$$

where $c_0 = c\,(r(0))$, which implies that the combinatorial Calabi flow has a solution for all time $t \in [0, \infty)$ for any $r(0) \in \mathbb{R}^N_{>0}$. $\square$

Remark 5. The long time existence of $r(t)$ can be deduced without using the uniform estimation of B_{ij} in (3.2). See Remark 6.

4. Convergence to a constant curvature metric

From the calculation above, it's easy to see the following proposition is true.

Proposition 4.1. *Along the combinatorial Calabi flow, the discrete Gauss curvature evolves according to*

$$\frac{dK}{dt} = -L^2 K. \tag{4.1}$$

For any $c > 0$, denote $\mathscr{P}_c = \left\{ r = (r_1, \dots, r_N)^T \in \mathbb{R}^N_{>0} \;\middle|\; \prod_{i=1}^N r_i = c \right\}$, and denote $\mathscr{U}_a = \left\{ u = (u_1, \dots, u_N)^T \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^N u_i = a \right\}$; then $\mathscr{P}_c = Exp(\mathscr{U}_a)$, where $a = \ln c$. Note that the constant curvature $K_{av} = k_{av}(1, \dots, 1)^T$ may not belong to $K(\mathbb{R}^N_{>0})$. By Proposition 3.4 we know that $\{r(t)\} \subset \mathscr{P}_c$ along the combinatorial Calabi flow (2.2), where $c = \prod_{i=1}^N r_i(0)$. Now we are at the stage of proving convergence results.

4.1. **Proof of Theorem 2.5.** We have proved the long time existence of the flow (2.2) in Section 3. Denote $r(t)$, $t \in [0, \infty)$ as the solution of the combinatorial Calabi flow (2.2).

First we prove the "only if" part. If $r(t)$ converges, i.e., $r(+\infty) = \lim\limits_{t\to+\infty} r(t) \in \mathbb{R}^N_{>0}$ exists, then both $K(+\infty) = \lim\limits_{t\to+\infty} K(t) \in K(\mathbb{R}^N_{>0})$ and $L(+\infty) = \lim\limits_{t\to+\infty} L(t)$ exist. This leads to the existence of $\mathcal{C}(+\infty)$ and $\mathcal{C}'(+\infty)$. Combining with the fact that $\mathcal{C}(t)$ is uniformly bounded and using Proposition 3.3, we have

$$\mathcal{C}'(t) = 2\sum_{i=1}^N K_i' K_i = 2K^T K' = -2K^T L^2 K \le 0,$$

and then

$$\mathcal{C}'(+\infty) = -2K^T(+\infty) L^2(+\infty) K(+\infty) = 0.$$

Hence

$$K(+\infty) \in Ker(L^2) = Ker(L).$$

By Proposition 3.3, $K(+\infty)$ is a constant, and $r(+\infty)$ is a constant curvature metric.

Next we prove the "if" part. Assume there exists a constant curvature circle packing metric r_{av}, which implies $K_{av} \in K(\mathbb{R}^N_{>0})$. We want to show $r(t)$ converges, i.e.

$$r(+\infty) = \lim_{t \to +\infty} r(t) \in \mathbb{R}^N_{>0}.$$

We carry out the proof in three steps.

Step 1. Denote λ_1 as the minimum positive eigenvalue of L. Since the matrix L is semi-positive definite by Proposition 3.3, λ_1^2 is the minimum positive eigenvalue of L^2. By standard tricks in basic linear algebra theory, we have

$$K^T L^2 K = (K - K_{av})^T L^2 (K - K_{av}) \geq \lambda_1^2 \|K - K_{av}\|^2 = \lambda_1^2 \mathcal{C}. \tag{4.2}$$

Step 2. we show that $\{r(t) \,|\, t \in [0, \infty)\} \Subset \mathbb{R}^N_{>0}$. Consider the combinatorial Ricci potential

$$f(u) = \int_{u_{av}}^{u} \sum_{i=1}^{N} \big(K_i - k_{av}\big) du_i\,,\ u \in \mathbb{R}^N, \tag{4.3}$$

where $u_{av} = \ln r_{av}$. This integral is well defined, because $\sum_{i=1}^N (K_i - k_{av}) du_i$ is a closed differential form. $f\big|_{\mathscr{U}_a}$ is strictly convex (Theorem B.2), hence the following map:

$$\begin{aligned} \nabla f\big|_{\mathscr{U}_a} : \mathscr{U}_a &\to \mathbb{R}^N \\ u &\mapsto K - K_{av} \end{aligned}$$

is injective. Therefore u_{av} is the unique critical point of $f\big|_{\mathscr{U}_a}$ and f is bounded below by zero ($f(u) \geq f(u_{av}) = 0$). Consider $\varphi(t) = f\big(u(t)\big)$; then

$$\varphi'(t) = (\nabla f)^T \cdot \frac{du}{dt} = (K - K_{av})^T(-LK) = -K^T L K \leq 0,$$

hence $\varphi(t)$ is descending as t increases and then $0 \leq \varphi(t) \leq \varphi(0) = f\big(u(0)\big)$, for any $t \in [0, +\infty)$. Hence $\big\{u(t) \,|\, t \in [0, +\infty)\big\} \subset \Big(f\big|_{\mathscr{U}_a}\Big)^{-1}\big([0, \varphi(0)]\big)$. Because $f\big|_{\mathscr{U}_a}$ is proper by Theorem B.2, $\Big(f\big|_{\mathscr{U}_a}\Big)^{-1}\big([0, \varphi(0)]\big)$ is a compact subset of $\mathscr{U}_a$. Therefore $\big\{u(t) \mid t \in [0, +\infty)\big\} \Subset \mathscr{U}_a$, or equivalently,

$$\{r(t) \,|\, t \in [0, \infty)\} \Subset \mathscr{P}_c \subset \mathbb{R}^N_{>0}.$$

Step 3. we show that $r(t)$ converges exponentially fast to r_{av}. Due to step 2, $\lambda_1^2(t)$, the first eigenvalue of $L^2(t)$, has a uniform lower bound along the Calabi flow, i.e., $\lambda_1^2(t) \geq \lambda/2 > 0$, where λ is a positive constant. Using (4.2), we obtain

$$\mathcal{C}'(t) = -2K^T L^2 K \leq -2\lambda_1^2(t)\mathcal{C} \leq -\lambda \mathcal{C},$$

which implies

$$\mathcal{C}(t) \leq \mathcal{C}(0) e^{-\lambda t}, \tag{4.4}$$

and the curvature $K_i(t)$ converges exponentially fast to k_{av}. Moreover, we can prove that $r(t)$ and $u(t)$ converge exponentially fast to r_{av} and u_{av} respectively by similar methods in the proof of Lemma 4.1 in [13]. Finally, we finish the proof. □

Remark 6. We can prove $\{r(t)\,|\,t \in [0,T)\} \Subset \mathbb{R}^N_{>0}$ similarly without assuming $T=+\infty$. Using the classical extension theorem of solutions in ODE theory, we obtain $T=+\infty$, which gives a new proof of Theorem 3.5.

4.2. **The convergence to admissible curvatures.** Theoretically, a metric with a constant combinatorial Gaussian curvature seems to be the best candidate for a "good" metric. This is because people often like geometric objects with symmetry. However, for practical applications, especially in medical imaging and computer graphics fields (see [24, 35–37]), any prescribed curvatures that meet the needs of users can be the best candidate for a "good" metric.

Definition 4.2. We call any $\overline{K} \in \mathbb{R}^N$ a prescribed or target curvature. If $\overline{K} \in \mathbb{R}^N$ is realized by some $\bar{r} \in \mathbb{R}^N_{>0}$, or say, $\overline{K} = K(\bar{r})$ is exactly the curvature at $\bar{r}$, then $\overline{K}$ is admissible or attainable.

The combinatorial Calabi flow can be used to deform any circle packing metric to one with admissible curvatures. To do so, it is natural to minimize the modified discrete Calabi energy

$$\overline{\mathcal{C}}(u) = \|K - \overline{K}\|^2. \tag{4.5}$$

This inspires us to consider the user prescribed combinatorial Calabi flow

$$\frac{du}{dt} = -\frac{1}{2}\nabla_u \overline{\mathcal{C}} = L(\overline{K} - K), \tag{4.6}$$

where $\overline{K} \in \mathbb{R}^N$ is any prescribed curvature. Similar to Subsection 4.1, we can prove the following theorem.

Theorem 4.3. *For any initial circle packing metric $r(0) \in \mathbb{R}^N_{>0}$, the solution of* (4.6) *exists for $t \in [0, +\infty)$. Moreover, the following three statements are mutually equivalent:*

1. *The solution of* (4.6) *converges.*
2. *The prescribed curvature $\overline{K}$ is admissible, i.e., $\overline{K} \in K(\mathbb{R}^N_{>0})$.*
3. *The solution of the user prescribed Ricci flow $\frac{du}{dt} = \overline{K} - K$ converges.*

Furthermore, if any of the above statements is true, the solution of the flow (4.6) *converges exponentially fast to $\bar{r}$.*

By Andreev-Thurston's classical work, the space of all admissible curvatures $K(\mathbb{R}^N_{>0})$ can be described completely by the combinatorial and topological information of (X, T, Φ). In fact, they get the following theorem.

Theorem 4.4 (Andreev-Thurston)**.** *Consider K as a map from $\mathbb{R}^N_{>0}$ to $\mathbb{R}^N$; then K is injective when restricted to the subset $\left\{r \in \mathbb{R}^N_{>0} \,\middle|\, \prod_{i=1}^N r_i = 1\right\}$. In other words, the metric is determined by its curvature up to a scalar multiplication. Moreover, $K(\mathbb{R}^N_{>0})$ equals*

$$\Big\{x \in \mathbb{R}^N \,\Big|\, \sum_{i=1}^N x_i = 2\pi\chi(X)\Big\} \bigcap \Big(\bigcap_{\phi \neq I \subsetneqq V} Y_I\Big),$$

where

$$Y_I = \Big\{x \in \mathbb{R}^N \,\Big|\, \sum_{i \in I} x_i > -\sum_{(e,v)\in Lk(I)} \big(\pi - \Phi(e)\big) + 2\pi\chi(F_I)\Big\}.$$

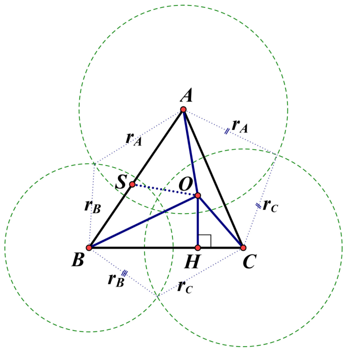

FIGURE 3. O is inside $\triangle ABC$, such that H lies between B and C.

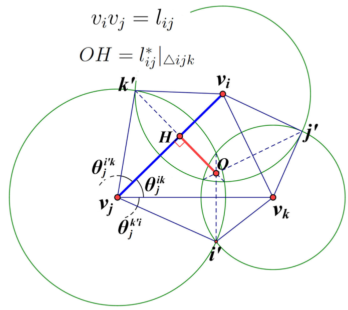

FIGURE 4. Discrete dual-Laplacian

For a proof of Andreev-Thurston's theorem, see Thurston [33], Marden-Rodin [26], Colin de Verdière [11], He [22], Chow-Luo [8] and Stephenson [31].

APPENDIX A. PROOF OF PROPOSITION 3.1

We first prove an elementary Euclidean geometry result related to circle packing.

Lemma A.1. *Consider a triangle $\triangle ABC$ coming from a configuration of circle patterns with weight $\Phi \in [0, \frac{\pi}{2}]$. Assume O is inside $\triangle ABC$. OH is the altitude from O onto side BC. Assume the point H lies between B and C (see Figure 3). Then $|OH| < \sqrt{3}|BC|$.*

Proof. Suppose this lemma is false. Then $|OH| \geq \sqrt{3}|BC|$, and

$$|OH| \geq \sqrt{3}|BC| \Rightarrow \begin{cases} |OH| \geq \sqrt{3}|BH| \Rightarrow \measuredangle BOH \leq \pi/6, \\ |OH| \geq \sqrt{3}|CH| \Rightarrow \measuredangle COH \leq \pi/6. \end{cases}$$

Hence

$$\measuredangle BOC = \measuredangle BOH + \measuredangle COH \leq \pi/3,$$

and

$$\measuredangle AOC = 2\pi - (\measuredangle AOB + \measuredangle BOC) \geq 2\pi - \left(\pi + \frac{\pi}{3}\right) = \frac{2\pi}{3},$$

$$\measuredangle AOB = 2\pi - (\measuredangle AOC + \measuredangle BOC) \geq 2\pi - \left(\pi + \frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

In $\triangle AOB$, we have $|AB| > |AO|$, for AB faces to a bigger angle. Thus we can select a unique point S between A and B so that $|AS| = |AO|$. using

$$\frac{\measuredangle AOB}{2} < \frac{\pi - \measuredangle BAO}{2} < \frac{\pi}{2},$$

we get

$$\measuredangle SOB = \measuredangle AOB - \frac{\pi - \measuredangle BAO}{2} \in \left[\measuredangle AOB - \frac{\pi}{2}, \measuredangle AOB - \frac{\measuredangle AOB}{2}\right] \subset \left[\frac{\pi}{6}, \frac{\pi}{2}\right],$$

and

$$\measuredangle OSB = \pi - \frac{\pi - \measuredangle BAO}{2} \in \left[\pi - \frac{\pi}{2}, \pi - \frac{\measuredangle AOB}{2}\right] \subset \left[\frac{\pi}{2}, \frac{2\pi}{3}\right].$$

Then we obtain

$$\frac{|OB|}{|AB| - |AO|} = \frac{|OB|}{|BS|} = \frac{\sin \measuredangle BSO}{\sin \measuredangle SOB} \leq \frac{1}{(\frac{1}{2})} = 2,$$

which implies

$$|AB| - |AO| \geq \frac{1}{2}|OB|.$$

Similarly, we have

$$|AC| - |AO| \geq \frac{1}{2}|OC|.$$

Next we show

$$r_B \geq \frac{1}{2}|OB|$$

and

$$r_C \geq \frac{1}{2}|OC|.$$

If $r_B \leq |OB|$, we know $r_A \leq |OA|$ by use of $r_A^2 - r_B^2 = |OA|^2 - |OB|^2$. We already know $r_A + r_B \geq |AB|$. Hence

$$r_B \geq |AB| - r_A \geq |AB| - |OA| \geq \frac{1}{2}|OB|.$$

Thus we always have $r_B \geq \frac{1}{2}|OB|$, no matter $r_B \geq |OB|$ or $r_B \leq |OB|$. Similarly, we have $r_C \geq \frac{1}{2}|OC|$. Then it is easy to see

$$r_B^2 + r_C^2 \geq \frac{1}{4}(|OB|^2 + |OC|^2) \geq \frac{1}{4} \cdot 2|OH|^2 = \frac{1}{2}|OH|^2 \geq \frac{3}{2}|BC|^2 > |BC|^2,$$

which contradicts the fact

$$|BC| = \sqrt{r_B^2 + r_C^2 + 2r_Br_C\cos\Phi_{BC}} \geq \sqrt{r_B^2 + r_C^2}.$$

□

(The above lemma belongs to Ruixiang Zhang and Chenjie Fan.)

Corollary A.2. *Given* (X, T, Φ), *where* X *is a closed surface,* T *is a triangulation,* $\Phi \in [0, \pi/2]$ *is a weight. Then*

$$0 < \frac{\partial \theta_i^{jk}}{\partial r_j} r_j < \sqrt{3}.$$

Proof. Let O be the unique (A2, [8]) intersection point in Figure 4. Denote $l_{ij}^*|_{\triangle ijk}$ as the directed distance from O to H. The directed distance from O to edge v_iv_j is positive (negative), when O is inside (outside) $\angle v_iv_jv_k$. Thurston [33] claimed that $l_{ij}^*|_{\triangle ijk}$ is positive. Hence O is inside the $\triangle v_iv_jv_k$. Note it was shown (A2, [8]) that $\frac{\partial \theta_i^{jk}}{\partial r_j} r_j = \frac{l_{ij}^*|_{\triangle ijk}}{l_{ij}}$. Using Lemma A.1 we get the conclusion above. □

Corollary A.3. *Proposition* 3.1 *is true, that is,* $0 < B_{ij} < 2\sqrt{3}$.

Remark 7. Let $l^*_{ij} \doteqdot l^*_{ij}|_{\triangle ijk}+l^*_{ij}|_{\triangle ijl}$. Then it's easy to see $B_{ij} = \frac{l^*_{ij}}{l_{ij}}$. Furthermore,

$$\Delta f_i = -\sum_{j=1}^{N} \frac{\partial K_j}{\partial u_i} f_j = -\sum_{j=1}^{N} L_{ij} f_j = \sum_{j\sim i} B_{ij}(f_j - f_i) = \sum_{j\sim i} \frac{l^*_{ij}}{l_{ij}}(f_j - f_i), \tag{A.1}$$

where $f : V \to \mathbb{R}$ is a function defined on vertices. Generally, discrete Laplace operator [10] "Δ" is often written as $\Delta f_i = \sum_{j\sim i} \omega_{ij}(f_j - f_i)$, where ω_{ij} is a weight defined on each edge $i \sim j$. Here B_{ij} comes from the dual structure of circle patterns (see [8, 14, 33] for more details) and this is why we call (2.1) and (A.1) discrete dual-Laplacian. There are other kinds of discrete Laplacians, such as the cotangent-Laplacian, which is a special kind of discrete dual-Laplacian showed by Glickenstein [16]. For more details related to discrete Laplacians, see [16–18, 22, 23].

Appendix B. Combinatorial Ricci potential is proper

Lemma B.1. *Assume that $\psi \in C^2(\mathbb{R}^n)$ is a function satisfying*

(1) *ψ is strictly convex;*
(2) *there exists at least one point p such that $\nabla\psi(p) = 0$.*

Then $\lim\limits_{x\to\infty} \psi(x) = +\infty$. Moreover, ψ is proper.

Proof. Set $h(t) = \inf\limits_{|x|=t} \psi(x)$, $t \geq 0$. Then $h(t)$ is a nondecreasing function. We just need to prove $h(t) \to +\infty$, as $t \to +\infty$. The process is almost the same with Lemma B.1 in [13]; we omit the details. Notice that there is another way to prove Lemma B.1 by remark 2.7 in [1]. □

Theorem B.2. *Given (X, T, Φ), where X, T and Φ are defined as before. Assume there exists a constant curvature metric $r_{av} \in \mathbb{R}^N_{>0}$. Thus $K(r_{av})$, the discrete curvature at r_{av}, is a constant, which is denoted as k_{av}. Consider the combinatorial Ricci potential*

$$f(u) = \int_{u_{av}}^{u} \sum_{i=1}^{N} \big(K_i - k_{av}\big) du_i\,,\ u \in \mathbb{R}^N,$$

where $u_{av} = \ln r_{av}$. Then for arbitrary constant a, $f\big|_{\mathscr{U}_a}$ is proper and

$$\lim_{u\to\infty,\, u\in\mathscr{U}_a} f(u) = +\infty.$$

Proof. Note the kernel of $Hess f$ is $(1, \ldots, 1)$, which is perpendicular to the hyperplane $\mathscr{U}_a$. Hence $Hess f$ is strictly positive definite when constrained on $\mathscr{U}_a$, implying that f is strictly convex when constrained on $\mathscr{U}_a$. Further note that ∇f has a zero point on $\mathscr{U}_a$; then by Lemma B.1, we get the conclusion above. □

Acknowledgments

The author would like to show gratitude to his advisor, Gang Tian, for introducing the area of combinatorial curvature flows. The author thanks Professor Feng Luo for reading the paper carefully and giving numerous suggestions which greatly improved the paper, and thanks Guanxiang Wang, Haozhao Li, Yiyan Xu and Wenshuai Jiang for helpful discussions. The author also thanks Dongfang Li who did numerical simulations for the combinatorial Calabi flows. The author is grateful for the suggestions given by anonymous referees.

References

[1] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49, DOI 10.2307/1995057. MR0251664

[2] Eugenio Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR645743

[3] Eugenio Calabi, *Extremal Kähler metrics. II*, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 95–114. MR780039

[4] Shu-Cheng Chang, *The 2-dimensional Calabi flow*, Nagoya Math. J. **181** (2006), 63–73. MR2210710

[5] Shu-Cheng Chang, *Global existence and convergence of solutions of Calabi flow on surfaces of genus $h \geq 2$*, J. Math. Kyoto Univ. **40** (2000), no. 2, 363–377, DOI 10.1215/kjm/1250517718. MR1787876

[6] X. X. Chen, *Calabi flow in Riemann surfaces revisited: a new point of view*, Internat. Math. Res. Notices **6** (2001), 275–297, DOI 10.1155/S1073792801000149. MR1820328

[7] Bennett Chow, *The Ricci flow on the* 2*-sphere*, J. Differential Geom. **33** (1991), no. 2, 325–334. MR1094458

[8] Bennett Chow and Feng Luo, *Combinatorial Ricci flows on surfaces*, J. Differential Geom. **63** (2003), no. 1, 97–129. MR2015261

[9] Piotr T. Chruściel, *Semi-global existence and convergence of solutions of the Robinson-Trautman (*2*-dimensional Calabi) equation*, Comm. Math. Phys. **137** (1991), no. 2, 289–313. MR1101689

[10] Fan R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR1421568

[11] Yves Colin de Verdière, *Un principe variationnel pour les empilements de cercles* (French), Invent. Math. **104** (1991), no. 3, 655–669, DOI 10.1007/BF01245096. MR1106755

[12] Huabin Ge and Xu Xu, *Discrete quasi-Einstein metrics and combinatorial curvature flows in 3-dimension*, Adv. Math. **267** (2014), 470–497, DOI 10.1016/j.aim.2014.09.011. MR3269185

[13] Huabin Ge and Xu Xu, *2-dimensional combinatorial Calabi flow in hyperbolic background geometry*, Differential Geom. Appl. **47** (2016), 86–98, DOI 10.1016/j.difgeo.2016.03.011. MR3504921

[14] David Glickenstein, *A combinatorial Yamabe flow in three dimensions*, Topology **44** (2005), no. 4, 791–808, DOI 10.1016/j.top.2005.02.001. MR2136535

[15] David Glickenstein, *A maximum principle for combinatorial Yamabe flow*, Topology **44** (2005), no. 4, 809–825, DOI 10.1016/j.top.2005.02.002. MR2136536

[16] D. Glickenstein, *Geometric triangulations and discrete Laplacians on manifolds*, Preprint at arXiv:math/0508188v1 [math.MG].

[17] David Glickenstein, *A monotonicity property for weighted Delaunay triangulations*, Discrete Comput. Geom. **38** (2007), no. 4, 651–664, DOI 10.1007/s00454-007-9009-y. MR2365828

[18] David Glickenstein, *Discrete conformal variations and scalar curvature on piecewise flat two- and three-dimensional manifolds*, J. Differential Geom. **87** (2011), no. 2, 201–237. MR2788656

[19] X. D. Gu, R. Guo, F. Luo, and W. Zeng, *Discrete Laplace-Beltrami operator determines discrete Riemannian metric*, Preprint at arXiv:1010.4070v1 [cs.DM].

[20] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR664497

[21] Richard S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262, DOI 10.1090/conm/071/954419. MR954419

[22] Zheng-Xu He, *Rigidity of infinite disk patterns*, Ann. of Math. (2) **149** (1999), no. 1, 1–33, DOI 10.2307/121018. MR1680531

[23] Anil Nirmal Hirani, *Discrete exterior calculus*, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–California Institute of Technology. MR2704508

[24] M. Jin, J. Kim, and X. D. Gu, *Discrete surface Ricci flow: theory and applications*, Mathematics of Surfaces XII, Lecture Notes in Computer Science Vol. 4647, 209-232.

[25] Feng Luo, *Combinatorial Yamabe flow on surfaces*, Commun. Contemp. Math. **6** (2004), no. 5, 765–780, DOI 10.1142/S0219199704001501. MR2100762

[26] Al Marden and Burt Rodin, *On Thurston's formulation and proof of Andreev's theorem*, Computational methods and function theory (Valparaíso, 1989), Lecture Notes in Math., vol. 1435, Springer, Berlin, 1990, pp. 103–115, DOI 10.1007/BFb0087901. MR1071766

[27] Alan D. Rendall, *Existence and asymptotic properties of global solutions of the Robinson-Trautman equation*, Classical Quantum Gravity **5** (1988), no. 10, 1339–1347. MR964979

[28] I. Robinson and A. Trautman, *Spherical gravitational waves*, Phys. Rev. Lett. 4 (1960), 431-432.

[29] Bernd G. Schmidt, *Existence of solutions of the Robinson-Trautman equation and spatial infinity*, Gen. Relativity Gravitation **20** (1988), no. 1, 65–70, DOI 10.1007/BF00759256. MR925329

[30] David Singleton, *On global existence and convergence of vacuum Robinson-Trautman solutions*, Classical Quantum Gravity **7** (1990), no. 8, 1333–1343. MR1064183

[31] Kenneth Stephenson, *Introduction to circle packing*, Cambridge University Press, Cambridge, 2005. The theory of discrete analytic functions. MR2131318

[32] Michael Struwe, *Curvature flows on surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), no. 2, 247–274. MR1991140

[33] W. P. Thurston, *Geometry and topology of 3-manifolds*, Chapter 13, Princeton lecture notes 1976, http://www.msri.org/publications/books/gt3m.

[34] K. P. Tod, *Analogues of the past horizon in the Robinson-Trautman metrics*, Classical Quantum Gravity **6** (1989), no. 8, 1159–1163. MR1005655

[35] W. Zeng, D. Samaras, and X. D. Gu, *Ricci flow for 3D shape analysis*, IEEE Transactions on Pattern Analysis & Machine Intelligence, 32(4) (2010), 662-677.

[36] W. Zeng, R. Shi, Z. Su, and X. D. Gu, *Colon surface registration using Ricci flow*, Abdomen and Thoracic Imaging, An Engineering & Clinical Perspective (2014), 389-419.

[37] M. Zhang, F. Li, Y. He, S. Li, D. Wang, and L. M. Lui, *Registration of brainstem surfaces in adolescent idiopathic scoliosis using discrete Ricci flow*, 15th International Conference, Nice, France, October 1-5, 2012, Proceedings, Part II, Medical Image Computing and Computer-Assisted Intervention CMICCAI (2012), 146-154.

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, People's Republic of China

E-mail address: hbge@bjtu.edu.cn