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GENERIC SOLUTIONS OF EQUATIONS WITH ITERATED EXPONENTIALS

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Abstract. We study solutions of exponential polynomials over the complex field. Assuming Schanuel's Conjecture we prove that certain polynomials of the form

$$p(z, e^z, e^{e^z}, e^{e^{e^z}}) = 0$$

have generic solutions in $\mathbb{C}$.

1. Introduction

We consider analytic functions over $\mathbb{C}$ of the form

$$f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdot^{\cdot^{\cdot^{e^z}}}}}}) \tag{1}$$

where $p(x, y_1, \ldots, y_k) \in \mathbb{C}[x, y_1, \ldots, y_k]$, and we investigate the existence of a solution a which is generic over L (for L a finitely generated extension of $\mathbb{Q}$ containing the coefficients of p), i.e., such that

$$\mathrm{t.\,d.}_L(a, e^a, e^{e^a}, \ldots, e^{e^{e^{\cdot^{\cdot^{\cdot^{e^a}}}}}}) = k,$$

where k is the number of iterations of exponentation which appear in the polynomial p.

If the field L is not specified we will mean $L = \mathbb{Q}$.

Conjecture. *Let L be any finitely generated subfield of $\mathbb{C}$ and $p(x, y_1, \ldots, y_k)$ a nonzero irreducible polynomial in $L[x, y_1, \ldots, y_k]$, depending on x and the last variable y_k. Then*

$$p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}}) = 0$$

has a generic solution over L.

A result of Katzberg (see [11]) implies that (1) always has infinitely many zeros unless the polynomial is of a certain form; see Section 3. Hence, the main problem is to prove the existence of a solution which is generic. In this context a fundamental role is often played by a conjecture in transcendental number theory due to Schanuel which concerns the exponential function.

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Schanuel's Conjecture (SC). *Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \dots, \lambda_n, e^{\lambda_1}, \dots, e^{\lambda_n})$ has transcendence degree* $(\mathrm{t.\,d.}_{\mathbb{Q}})$ *at least n over $\mathbb{Q}$.*

(SC) includes the Lindemann-Weierstrass Theorem. The analogous statement for the ring of power series over $\mathbb{C}$ has been proved by Ax in [1]. Schanuel's Conjecture has played a crucial role in exponential algebra (see [15], [22], [3]) and in the model theory of exponential fields (see [16], [23], [18], [4], [5]).

Assuming Schanuel's Conjecture, we are able to prove some particular cases of the Conjecture.

Main Theorem (SC). *Let $p(x, y_1, y_2, y_3) \in \mathbb{Q}^{alg}[x, y_1, y_2, y_3]$ be a nonzero irreducible polynomial depending on x and the last variable. Then, there exists a generic solution of*

$$p(z, e^z, e^{e^z}, e^{e^{e^z}}) = 0.$$

In fact, we obtain infinitely many generic solutions. We prove analogous results for polynomials $p(z, e^{e^z})$ and $p(z, e^z, e^{e^z})$ (see Theorem 4.1 and Theorem 4.2). In the general case for $k > 3$ iterations of exponentiation we have only partial results (see Proposition 4.7).

One of the main ingredients in the proof of the above theorem is a result due to Masser on the existence of zeros of systems of exponential equations (see Section 2). Only very recently (in private correspondence with D. Masser) we have become aware that these ideas have been developed further in a recent article [2], where the authors show the existence of solutions of certain exponential polynomials. Some methodology is different from what we use in this paper, and moreover they are not interested in generic solutions.

One of our motivations for studying generic solutions of exponential polynomials comes from a fascinating analysis of the complex exponential field

$$(\mathbb{C}, +, \cdot, 0, 1, e^z),$$

due to Zilber [23]. Zilber identified a class of algebraically closed fields of characteristic 0 equipped with an exponential function. His axioms include Schanuel's Conjecture and are inspired by the complex exponential field and by Hrushovski's (1992) construction of strongly minimal structures (see [10]).

Zilber's idea is to have exponential structures which are as existentially closed as possible without violating Schanuel's Conjecture.

Zilber proved an important categoricity result for the class of his fields in every uncountable cardinality. He conjectured that the complex exponential field is the unique model of cardinality $2^{\aleph_0}$. The ideas contained in Zilber's axiomatization could provide new insights into the analysis of the complex exponential field.

One of the axioms of Zilber **(Strong Exponential Closure)** is concerned with generic solutions over any finitely generated subfield of systems of exponential polynomials, and it is the main obstruction to proving Zilber's conjecture modulo (SC). The Strong Exponential Closure implies the above Conjecture.

In this direction a first result was obtained by Marker for polynomials over $\mathbb{C}$ with only one iteration of exponentation. Using the Hadamard Factorization Theorem, Marker in [18] proved the existence of infinitely many solutions. By restricting the coefficients of the polynomial to $\mathbb{Q}^{alg}$ and assuming (SC) he showed the existence of infinitely many algebraically independent solutions over $\mathbb{Q}$. More recently, Mantova in [17] assuming (SC) improved Marker's result by eliminating the hypothesis on the coefficients of the polynomial.

In this paper we consider the next natural cases of exponential polynomials with two and three iterations of exponentations, and we obtain an analogous result to that of Marker. This is clearly a significant step in solving positively Zilber's Conjecture, but it is still far from proving the Strong Exponential Closure for $\mathbb{C}$.

Comparing the complex exponential field and Zilber's fields has been one of the main motivations in the recent papers [3], [5], [8], [13].

2. Masser's result

In some hand-written notes (see [19]) Masser proved the following result. For completeness we give the details of his proof.

Theorem 2.1. *Let $P_1(\overline{x}), \ldots, P_n(\overline{x}) \in \mathbb{C}[\overline{x}]$, where $\overline{x} = (x_1, \ldots, x_n)$, and $P_i(\overline{x})$ are nonzero polynomials in $\mathbb{C}[\overline{x}]$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that*

$$\begin{cases} e^{z_1} = P_1(z_1, \ldots, z_n), \\ e^{z_2} = P_2(z_1, \ldots, z_n), \\ \vdots \\ e^{z_n} = P_n(z_1, \ldots, z_n). \end{cases} \tag{2}$$

We have to show that the function $F : \mathbb{C}^n \to \mathbb{C}^n$ defined as

$$F(x_1, \ldots, x_n) = (e^{x_1} - P_1(x_1, \ldots, x_n), \ldots, e^{x_n} - P_n(x_1, \ldots, x_n)) \tag{3}$$

has a zero in $\mathbb{C}^n$. For the proof we need a result due to Kantorovich (see Theorem 5.3.1 in [7]) for vector functions in many variables over the reals. Kantorovich's theorem is a refinement of Newton's approximation method for vector functions over the reals; i.e., under a certain hypothesis the existence of a zero of the function in a neighbourhood of a fixed point is guaranteed. Here we need the following version of Kantorovich's theorem for $\mathbb{C}$.

Lemma 2.2. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an entire function, and let $\overline{p}_0$ be such that $J(\overline{p}_0)$, the Jacobian of F at $\overline{p}_0$, is nonsingular. Let $\eta = |J(\overline{p}_0)^{-1}F(\overline{p}_0)|$ and U the closed ball of center p_0 and radius 2η. Let $M > 0$ be such that $|H(F)|^2 \leq M^2$ (where $H(F)$ denotes the Hessian of F). If $2M\eta|J(\overline{p}_0)^{-1}| < 1$, then there is a zero of F in U.*

Proof. Using the canonical transformation ($z = x + iy \mapsto (x, y)$) that identifies $\mathbb{C}$ with $\mathbb{R}^2$ we will work with a function $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which satisfies the hypothesis of Kantorovich's theorem in the case of real variables. Hence (see Theorem 5.3.1 in [7]) G has a zero in $\mathbb{R}^{2n}$ which determines a zero of F in $\mathbb{C}^n$. □

Lemma 2.3. *Let $P_1(\bar{x}), \ldots, P_n(\bar{x}) \in \mathbb{C}[\bar{x}]$, where $\bar{x} = (x_1, \ldots, x_n)$, and let $d_1, \ldots, d_n$ be the total degrees of $P_1(\bar{x}), \ldots, P_n(\bar{x})$, respectively. There exists a constant $c > 0$ and an infinite set $S \subseteq \mathbb{Z}^n$ such that*

$$|P_j(2\pi i k_1, \ldots, 2\pi i k_n)| \geq c(1 + \sum_{l=1}^{n} |k_l|)^{d_j}$$

for all $\overline{k} = (k_1, \ldots, k_n) \in S$, $j = 1, \ldots, n$.

Proof. We prove the lemma for a single polynomial $P(\bar{x})$ of degree d. Let $P(\bar{x}) = Q_d(\bar{x}) + Q_{d-1}(\bar{x}) + \ldots + Q_0(\bar{x})$, where each $Q_h(\bar{x})$ is a homogenous polynomial of degree h. Fix $\bar{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ such that $Q_d(\bar{q}) \neq 0$, and let $S = \{t\bar{q} : t \in \mathbb{N}\}$. We now estimate $P(2\pi it\bar{q})$ for $t\bar{q} \in S$. Clearly,

$$|Q_d(2\pi it\bar{q}) + Q_{d-1}(2\pi it\bar{q}) + \ldots + Q_0(2\pi it\bar{q})| \geq |Q_d(2\pi it\bar{q})| - |Q_{d-1}(2\pi it\bar{q}) + \ldots + Q_0(2\pi it\bar{q})|.$$

Simple calculations give

$$|Q_d(2\pi it\bar{q})| = C_1|t\bar{q}|^d = C_1(|tq_1| + \ldots + |tq_n|)^d \tag{4}$$

for some constant C_1. By easy estimates we get

$$C_1(|tq_1| + \ldots + |tq_n|)^d \geq C_2(|tq_1| + \ldots + |tq_n| + 1)^d$$

for some constant C_2. Notice that all constants in the inequalities depend only on the total degree and on the coefficients of P. □

Proof of Theorem 2.1. Let S be as in Lemma 2.3. In order to apply Lemma 2.2 to $F(x_1, \ldots, x_n)$ as in (3) we choose a point $\bar{k} = (k_1, \ldots, k_n)$ in S and we look for a solution of F near $2\pi i\bar{k}$. We first transform the functions defining F by shifting the variables. Let $p_1 = P_1(2\pi i\bar{k}), \ldots, p_n = P_n(2\pi i\bar{k})$. Lemma 2.3 guarantees that $p_1, \ldots, p_n$ are different from 0. Let $a_1, \ldots, a_n$ be the principal values of $\log p_1, \ldots, \log p_n$, respectively. If $T = 1 + |k_1| + \ldots + |k_n|$, then

$$\max\{|a_1|, \ldots, |a_n|\} \leq C \log T, \tag{5}$$

for some constant C depending only on the coefficients and the degrees of the polynomials P_j, and not on the choice of $\bar{k}$ in S. We now make a change of variables by shifting each variable x_j by $2\pi ik_j + a_j$, and we solve the new system

$$\begin{cases} f_1(x_1, \ldots, x_n) = e^{x_1} - \frac{P_1(2\pi ik_1 + a_1 + x_1, \ldots, 2\pi ik_n + a_n + x_n)}{p_1} = 0, \\ f_2(x_1, \ldots, x_n) = e^{x_2} - \frac{P_2(2\pi ik_1 + a_1 + x_1, \ldots, 2\pi ik_n + a_n + x_n)}{p_2} = 0, \\ \vdots \\ f_n(x_1, \ldots, x_n) = e^{x_n} - \frac{P_n(2\pi ik_1 + a_1 + x_1, \ldots, 2\pi ik_n + a_n + x_n)}{p_n} = 0. \end{cases} \tag{6}$$

We now evaluate the Jacobian of the new system at the point $\overline{p}_0 = (0, \ldots, 0)$ (which corresponds to $(2\pi ik_1 + a_1, \ldots, 2\pi ik_n + a_n)$ after the shifting). We have

$$J(\overline{p}_0) = \begin{pmatrix} \partial_{x_1}(f_1) & \partial_{x_2}(f_1) & \ldots & \partial_{x_n}(f_1) \\ \partial_{x_1}(f_2) & \partial_{x_2}(f_2) & \ldots & \partial_{x_n}(f_2) \\ \vdots & \vdots & \vdots & \\ \partial_{x_1}(f_n) & \partial_{x_2}(f_n) & \ldots & \partial_{x_n}(f_n) \end{pmatrix} \tag{7}$$

where

$$\partial_{x_h}(f_h) = 1 - \frac{(\partial_{x_h} P_h)(2\pi ik_1 + a_1, \ldots, 2\pi ik_n + a_n)}{p_h}$$

for $h = 1, \ldots, n$, and

$$\partial_{x_h(f_j)} = -\frac{(\partial_{x_h} P_j)(2\pi ik_1 + a_1, \ldots, 2\pi ik_n + a_n)}{p_j}$$

for all $h \neq j$.

By Lemma 2.3 the quotients $-\frac{(\partial_{x_h} P_j)(2\pi i k_1 + a_1, \ldots, 2\pi i k_n + a_n)}{p_j}$ for all $h, j = 1, \ldots, n$ converge to 0 for large T. Hence, $J(\overline{p}_0)$ converges to the identity matrix, and so it is not singular. Moreover, also the inverse matrix $J(\overline{p}_0)^{-1}$ converges to the identity matrix, so $|J(\overline{p}_0)^{-1}|$ is bounded by a constant, say C_0. We need to evaluate the norm of $F(\overline{p}_0)$. By Lemma 2.3, equation (5), and the mean value theorem (see [14]) we obtain $|F(\overline{p}_0)| \leq C_1 \frac{\log T}{T}$, for some constant C_1.

Hence,

$$|J(\overline{p}_0)^{-1} F(\overline{p}_0)| \leq C_2 \frac{\log T}{T}, \tag{8}$$

for some constant C_2. Let $\eta = C_2 \frac{\log T}{T}$, and let U be the closed ball of center $\overline{p}_0 = (0, \ldots, 0)$ and radius 2η. In order to complete the proof, we need to satisfiy the last condition of Lemma 2.2, i.e., $2|J(\overline{p}_0)^{-1} F(\overline{p}_0)| M |J(\overline{p}_0)^{-1}| < 1$, for some $M > 0$ bounding the norm of the Hessian of the function F on U. This inequality follows from (8) and the boundness of $|J(\overline{p}_0)^{-1}|$. $\square$

2.1. **Generalization to algebraic functions.** Masser in his notes remarked that using the same argument the result can be generalized to algebraic functions. Here we give the proof following Masser's idea.

An algebraic function is a complex analytic function (in many variables) defined on some "cone" (at infinity) and satisfying a polynomial equation over $\mathbb{C}$. More precisely: for us, a cone is an open connected subset $U \subseteq \mathbb{C}^n$ such that for every $1 \leq t \in \mathbb{R}$, if $\overline{x} \in U$, then $t\overline{x} \in U$. We denote by $\overline{x} = (x_1, \ldots, x_n)$ an n-tuple, and by u a single variable.

Definition 2.4. An algebraic function is an analytic function $f : U \to \mathbb{C}$ such that there exists a nonzero polynomial $p(\bar{x}, u) \in \mathbb{C}[\bar{x}, u]$ with $p(\bar{x}, f(\bar{x})) = 0$ on all $\overline{x} \in U$. If, moreover, the polynomial p is monic in u, we say that f is integral algebraic.

Definition 2.5. Let $f : U \to \mathbb{C}$ (where U is a cone) be an algebraic function. We say that f is homogeneous of degree r if, for every $\bar{x} \in U$ and $1 \leq t \in \mathbb{R}$, we have $f(t\bar{x}) = t^r f(\bar{x})$.

For every algebraic function f there exists a unique $r \in \mathbb{Q}$ (the degree of f) and $h : U \to \mathbb{C}$ algebraic and homogeneous of degree r, such that $f(\bar{x}) - h(\bar{x}) = o(|\bar{x}|^r)$.

Fact. Notice that if f is a polynomial, then f is homogeneous in the above sense iff it is homogeneous as a polynomial and its degree is equal to the total degree as a polynomial. Moreover, every algebraic function can be expressed as the quotient of two integral functions (after shrinking the domain, if necessary), and the degree of an integral function is greater than or equal to 0.

Example 2.6. The function $(x_1, x_2) \mapsto \sqrt{x_1} + \sqrt[6]{x_1^3 + x_2^3}$ is integral and homogeneous of degree $1/2$, but is not analytic at infinity.

We now state and sketch a proof of a generalization of Theorem 2.1 to algebraic functions.

Theorem 2.7. *Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be nonzero algebraic functions, defined on some cone U. Assume that $U \cap (2\pi i \mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$. Then, the following system has a solution $\overline{a} \in U$:*

$$\begin{cases} e^{z_1} = f_1(\overline{z}), \\ \cdots \\ e^{z_n} = f_n(\overline{z}). \end{cases} \tag{9}$$

Sketch of the proof. The proof is quite similar to that of Theorem 2.1. We will only show the modifications that are needed in the case of algebraic functions.

First, we make a reduction to the case where all the f_i's are integral (and not only algebraic): it suffices to write $f_i = g_i/h_i$, where g_i and h_i are integral and solve the system in $2n$ equations and $2n$ variables:

$$\begin{cases} e^{x_1} = g_1(\overline{x} - \overline{y}), \\ \dots \\ e^{x_n} = g_n(\overline{x} - \overline{y}), \\ e^{y_1} = h_1(\overline{x} - \overline{y}), \\ \dots \\ e^{y_n} = h_n(\overline{x} - \overline{y}). \end{cases} \tag{10}$$

If $(\overline{a}, \overline{b}) \in U \times U$ is a solution of (10), then $\overline{a} - \overline{b}$ is a solution of (9).

Let $d_i \in \mathbb{Q}$ be the degree of f_i. Since we have assumed all the f_i's are integral, $d_i \geq 0$ for every i. Write $f_i = h_i + g_i$, with h_i homogeneous of degree d_i and $g_i(\bar{x}) = o(|\bar{x}|^{d_i})$. Choose $\overline{v} \in (2\pi i\mathbb{Z}^*)^n$ such that, for every $i \leq n$, $c_i = h_i(\overline{v}) \neq 0$. Pick $t \in \mathbb{N}$ large enough (we will see later how large), and denote $\overline{\omega} = t\overline{v}$. Notice that $f_i(\overline{\omega}) = t^{d_i}(c_i + o(1))$ and therefore, for some constant $c > 0$ and for t large enough, $|f_i(\overline{\omega})| \geq c(1 + |\overline{\omega}|)^{d_i}$ (Lemma 2.3).

Let $A_i = f_i(\overline{\omega})$ and a_i be the principal logarithm of A_i. It is easy to see that $a_i = O(\log t)$. As in the proof of Theorem 2.1, let $\overline{a} = a_1, \dots, a_n$. We make the change of variables $\overline{z} = \overline{\omega} + \overline{a} + \overline{x}$ and we are reduced to solving the equation $F(\bar{x}) = 0$, where

$$\begin{cases} F_1(\bar{x}) = e^{x_1} - \dfrac{f_1(\overline{\omega} + \overline{a} + \bar{x})}{A_1}, \\ \dots \\ F_n(\bar{x}) = e^{x_n} - \dfrac{f_n(\overline{\omega} + \overline{a} + \bar{x})}{A_n} \end{cases} \tag{11}$$

and $F(\bar{x}) = (F_1(\bar{x}), \dots, F_n(\bar{x}))$.

Finally, for t large enough, F satisfies the hypothesis of Lemma 2.2 on an open ball of center $\overline{0}$ contained in its domain, and we have finished. □

Remark 2.8. The above theorem can be generalized to the situation where, instead of being algebraic functions, $f_1, \dots, f_n$ are analytic on U and roots of some nonzero polynomials $P_i(\bar{x}, u) \in \mathcal{O}_n[u]$, where $\mathcal{O}_n$ is the ring of germs of functions on $\mathbb{C}^n$ analytic in a neigbourhood of infinity.

Given polynomials $p_1(\overline{x}, u), \dots, p_n(\overline{x}, u)$ of degree at least 1 in u, there exist a nonempty cone U and algebraic functions

$$f_1, \dots, f_n : U \to \mathbb{C},$$

such that $p_i(\bar{x}, f_i(\overline{x})) = 0$ on all U. Moreover, since $(2\pi i\mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$, we can also find U as above such that $(2\pi i\mathbb{Z}^*)^n \cap U$ is also Zariski dense. Thus, in order to find a solution of a system

$$p_1(\overline{x}, e^{x_1}) = 0, \dots, p_n(\overline{x}, e^{x_n}) = 0,$$

it suffices to find $\overline{a} \in U$ such that $e^{a_1} = f_1(\overline{a}), \dots, e^{a_n} = f_n(\overline{a})$, and we can apply the above theorem to find such $\overline{a}$.

Let $G_n(\mathbb{C}) = \mathbb{C}^n \times (\mathbb{C}^*)^n$ be the algebraic group. We have the following result.

Corollary 2.9. *Let $p_1, \dots, p_n \in \mathbb{C}[\overline{x}, u]$ be nonzero irreducible polynomials of degree at least 1 in u, and not of the form a constant times u. Let $V \subseteq G_n(\mathbb{C})$ be an irreducible component of the set*

$$\{(\bar{x}, \overline{y}) \in G_n(\mathbb{C}) : \bigwedge_{i=1}^{n} p_i(\bar{x}, y_i) = 0\}.$$

Assume that $\pi(V)$ is Zariski dense in $\mathbb{C}^n$ (where $\pi : G_n(\mathbb{C}) \to \mathbb{C}^n$ is the projection onto the first n coordinates). Then, the set $\{\overline{a} \in \mathbb{C}^n : (\overline{a}, e^{\overline{a}}) \in V\}$ is Zariski dense in $\mathbb{C}^n$.

Proof. Let $W \subset \mathbb{C}^n$ be a Zariski open subset. Let U be a cone and $f_1, \dots, f_n : U \to \mathbb{C}$ be algebraic functions, such that $U \cap (2\pi i\mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$, U is contained in W, and $p_i(\bar{x}, f_i(\bar{x})) = 0$ for every $\bar{x} \in U$. Choose $\overline{a}$ solving system (9) (the conditions on the polynomials p_i ensure that the f_i's exist and are nonzero). Then $(\overline{a}, e^{\overline{a}}) \in V$ and $\overline{a} \in W$. □

We can generalize the above lemma.

Lemma 2.10. *Let $W \subseteq G_n(\mathbb{C})$ be an irreducible algebraic variety such that $\pi(W)$ is Zariski dense in $\mathbb{C}^n$ (where $\pi : G_n(\mathbb{C}) \to \mathbb{C}^n$ is the projection onto the first n coordinates).*[1] *Then, the set $\{\overline{a} \in \mathbb{C}^n : (\overline{a}, e^{\overline{a}}) \in W\}$ is Zariski dense in $\mathbb{C}^n$.*

Proof. There exist polynomials $p_1, \dots, p_n \in \mathbb{C}[\overline{x}, u]$ and a V irreducible component of $\{(\overline{x}, \overline{y}) \in G_n(\mathbb{C}) : \bigwedge_{i=1}^{n} p_i(\overline{x}, y_i) = 0\}$ satisfying the hypothesis of Corollary 2.9 and moreover with $V \cap W$ Zariski dense in V. Thus, using Corollary 2.9 we complete the proof. □

3. Zeros of exponential polynomials over $\mathbb{C}$

Let (R, E) be an exponential ring. The ring of exponential polynomials over (R, E) in $z_1, \dots, z_n$ variables is defined by recursion and is denoted by $R[z_1, \dots, z_n]^E$ (for details see [6]).

Henson and Rubel in [9] gave a characterization of those exponential polynomials over $\mathbb{C}$ with no roots. Their proof is based on Nevanlinna theory.

Theorem 3.1 ([9])**.** *Let $F(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]^E$, so*

$$F(z_1, \dots, z_n) \text{ has no roots in } \mathbb{C} \text{ iff } F(z_1, \dots, z_n) = e^{G(z_1, \dots, z_n)},$$

where $G(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]^E$.

Katzberg in [11] using Nevanlinna theory and considering exponential polynomials in one variable proved the following result:

Theorem 3.2 ([11])**.** *A nonconstant exponential polynomial $F(z) \in \mathbb{C}[z]^E$ always has infinitely many zeros unless it is of the form*

$$F(z) = (z - \alpha_1)^{n_1} \cdot \ldots \cdot (z - \alpha_n)^{n_n} e^{G(z)},$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{C}, n_1, \dots, n_n \in \mathbb{N}$, and $G(z) \in \mathbb{C}[z]^E$.

[1] This is a nontrivial condition. A major problem is to replace this condition with much weaker ones while still retaining the conclusion of the lemma.

In [4], using purely algebraic methods, the two previous theorems have been proved for exponential polynomials over a Zilber field. Analogous results have been obtained independently by Shkop in [21].

We now investigate some special cases of the axiom of Strong Exponential Closure over $(\mathbb{C}, +, \cdot, 0, 1, e^z)$. Marker in [18] proved the first result in this direction for polynomials in z, e^z over $\mathbb{Q}^{alg}$.

The next natural case to consider is that of a polynomial $p(z, e^{e^z})$ with two iterations of exponentiation. The Hadamard Factorization Theorem cannot be applied anymore since the function $f(z) = p(z, e^{e^z})$ has infinite order.

Theorem 3.3. *Let $f(z) = p(z, e^z, e^{e^z}, \dots, e^{e^{e^{\cdots^{e^z}}}})$, where $p(x, y_1 \dots, y_k)$ is an irreducible polynomial over $\mathbb{C}[x, y_1, \dots, y_k]$. The function f has infinitely many solutions in $\mathbb{C}$ unless $p(x, y_1, \dots, y_k) = g(x) \cdot y_1^{n_{i_1}} \cdot \dots \cdot y_k^{n_{i_k}}$, where $g(x) \in \mathbb{C}[x]$.*

Proof. It is an immediate consequence of Theorem 3.2. An alternative proof is obtained easily by applying Theorem 2.7. □

In the sequel we will always assume that the polynomial p is not of the form $p(x, y_1, \dots, y_k) = c \cdot y_1^{n_{i_1}} \cdot \dots \cdot y_k^{n_{i_k}}$, where $c \in \mathbb{C}$, and we assume $p(x, y_1, \dots, y_k)$ is also an irreducible polynomial over $\mathbb{C}[x, y_1, \dots, y_k]$.

4. Generic solutions

Let $e_0(z) = z$, and for every $k \in \mathbb{N}$, define $e_{k+1}(z) = e^{e_k(z)}$. Fix $1 \le k \in \mathbb{N}$, let $\overline{x} = (x_0, \dots, x_k)$ and $p(\overline{x}) \in \mathbb{Q}^{alg}[\overline{x}]$. We assume the polynomial p is irreducible and depends on x_0 and the last variable. An element $a \in \mathbb{C}$ is a generic solution of

$$f(z) = p(z, e_1(z), \dots, e_k(z)) = 0 \tag{12}$$

if $\mathrm{t.d.}_{\mathbb{Q}}(a, e_1(a), \dots, e_k(a)) = k$.

In this section we investigate the existence of a generic solution a of (12). We will always assume that $p(\overline{x})$ is irreducible and depends on x_0 and x_k.

We will always assume Schanuel's Conjecture. Our proof is crucially based on Masser's result (see Section 2).

4.1. The function $f(z) = p(z, e^{e^z})$.

The first case we consider is when the exponential polynomial $f(z)$ has two iterations of exponentiation. In particular, we want to answer the following questions:

(1) Let $p(x, y) \in \mathbb{C}[x, y]$. Is there some $w \in \mathbb{C}$ so that (w, e^{e^w}) is a generic point of the curve $p(x, y) = 0$?
(2) What is the transcendence degree of the set of solutions of $f(z)$?

For this purpose we consider the corresponding system in four variables (z_1, z_2, w_1, w_2):

$$V = \begin{cases} p(z_1, w_2) = 0, \\ w_1 = z_2, \end{cases} \tag{13}$$

thought of as an algebraic set V in $G_2(\mathbb{C}) = \mathbb{C}^2 \times (\mathbb{C}^*)^2$.

Theorem 4.1 (SC). *If $p(x, y) \in \mathbb{Q}^{alg}[x, y]$, then the variety defined by V intersects the graph of exponentation in a generic point (w, e^w, e^w, e^{e^w}) (that is, $t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) = \dim V = 2$).*

Proof. By Theorem 3.3 the function $f(z) = p(z, e^{e^z})$ has a solution w in $\mathbb{C}$. If $w = 0$, then $e^{e^0} = e$, and from $p(0, e) = 0$ it follows that $p(x, y)$ is a polynomial in the variable y. Then e is algebraic over $\mathbb{Q}$, which is clearly a contradiction.

So, without loss of generality, $w \neq 0$.

We now assume (SC). The point (w, e^w, e^w, e^{e^w}) belongs to the variety V associated to system (13) which has dimension 2. We distinguish two cases.

Case 1. Assume that w and e^w are linearly independent. By Schanuel's Conjecture we have

$$t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) \geq 2.$$

Indeed, the transcendence degree is exactly 2 since w and e^{e^w} are algebraically dependent. Hence, $(w, e^w, e^w, e^{e^w}) \in V$ and $t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) = 2$, which is the dimension of V, and so the point (w, e^w, e^w, e^{e^w}) is generic for V.

Case 2. Suppose that w, e^w are linearly dependent over $\mathbb{Q}$. This means that

$$ne^w = mw \tag{14}$$

for some $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Since $w \neq 0$ then necessarily $n \neq 0$. Moreover, w is transcendental over $\mathbb{Q}$; otherwise we have a contradiction with the Lindemann Weierstrass Theorem. Applying exponentiation to relation (14) it follows that

$$e^{ne^w} = e^{mw},$$

i.e.,

$$(e^{e^w})^n = (e^w)^m = (\frac{m}{n}w)^m = (\frac{m}{n})^m w^m.$$

We now distinguish the cases when both n, m are positive, and the case when $n > 0$ and $m < 0$. We have that (w, e^{e^w}) is a root of either $q(x, y) = sx^m - y^n$ or $q(x, y) = x^{-m}y^n - r$, where $s, r \in \mathbb{Q}$. In both cases the polynomial $q(x, y)$ is irreducible, due to the fact that $(n, m) = 1$ (see Corollary of Lemma 2C in [20]).

Let $V(p)$ and $V(q)$ be the varieties associated to p and q, respectively. Clearly, $\dim V(p) = \dim V(q) = 1$. There is a point (w, e^{e^w}) which belongs to both varieties. Moreover, we know that every solution (w, e^{e^w}) of the polynomial p is such that w is transcendental, and this means that the point is generic for the variety $V(q)$. This implies that $V(q) \subseteq V(p)$, hence p divides q. By the irreducibility of both polynomials we have that p and q differ by a nonzero constant. Without loss of generality we can assume

$$p(x, y) = q(x, y) = sx^m - y^n \tag{15}$$

(the case of $p(x, y) = q(x, y) = x^{-m}y^n - r$ is treated in a similar way). Notice that for any solution (w, e^{e^w}) of $p(x, y) = 0$ the linear dependence between w and e^w is uniquely determined by the degrees of x and y in p; hence s in (15) is uniquely determined. We will show that it is always possible to find a solution (w, e^w, e^w, e^{e^w}) of system (13) with w, e^w linearly independent. Indeed, we consider the system

$$\begin{cases} p(z, e^{e^z}) = 0, \\ z \neq se^z, \end{cases} \tag{16}$$

that we can reduce to the following:

$$\begin{cases} e^z = A(z,t,u), \\ e^u = B(z,t,u), \\ e^t = C(z,t,u), \end{cases} \tag{17}$$

where $A(z,t,u) = \frac{t}{n}$, $B(z,t,u) = \frac{t}{n} - z$, and $C(z,t,u) = z^m s$. By Theorem 2.1 there exists a solution of system (17) which is generic since the second equation in (17) guarantees that there is no linear dependence between a solution z and its exponential e^z. □

4.2. **The function** $f(z) = p(z, e^z, e^{e^z})$**.** Now we examine the more general case of $f(z) = p(z, e^z, e^{e^z})$. For this purpose we consider the corresponding system in four variables (z_1, z_2, w_1, w_2):

$$V = \begin{cases} p(z_1, z_2, w_2) = 0, \\ w_1 = z_2, \end{cases} \tag{18}$$

thought of as an algebraic set V in $G_2(\mathbb{C})$.

Theorem 4.2 (SC)**.** *If $p(x,y,z) \in \mathbb{Q}^{alg}[x,y,z]$, then the variety V defined in* (18) *intersects the graph of exponentiation in a generic point.*

Proof. By Theorem 3.3, there exists $a \in \mathbb{C}$ such that $f(a) = 0$ and $a \neq 0$. Moreover, by the Lindemann-Weierstrass Theorem, a is transcendental over $\mathbb{Q}$. Also in this case $\dim V = 2$. We will show that $\operatorname{t.d.}_{\mathbb{Q}}(a, e^a, e^a, e^{e^a}) = 2$; then (a, e^a, e^a, e^{e^a}) is a generic point of V. If $\operatorname{t.d.}_{\mathbb{Q}}(a, e^a, e^a, e^{e^a}) = 1$, then by Schanuel's Conjecture, there exists $r \in \mathbb{Q}$ such that

$$e^a = ra. \tag{19}$$

We call $r \in \mathbb{Q}$ "bad" if there exists $a \in \mathbb{C}$ a solution of (18), such that $e^a = ra$.

We claim that there exist only finitely many bad $r \in \mathbb{Q}$. Let $r \in \mathbb{Q}$ be bad. Assume $r = n/m$, with $0 \neq n \in \mathbb{Z}$, $0 < m \in \mathbb{N}$, and $(n,m) = 1$. We have

$$me^a = na \tag{20}$$

for some $a \in \mathbb{C}$, and therefore

$$(e^{e^a})^m = (e^a)^n = (ra)^n.$$

For every "bad" rational r, the polynomial $p(x, rx, z)$ becomes two variables x, z, and we denote it by $p_r(x,z)$. Notice that $p(x, rx, z)$ may have become reducible.

Case 1. Assume $n > 0$. Let $q(x,z) = z^m - (r^n)x^n$ and $V(p_r)$ and $V(q)$ be the varieties associated respectively to p_r and q. We note that the polynomial $q(x,z)$ is irreducible (see Corollary of Lemma 2C [20]). The point (a, e^{e^a}) belongs to both varieties, and it is generic for the variety $V(q)$, since a is transcendental. This implies that $V(q) \subseteq V(p_r)$; hence the polynomial p_r divides q. In this case we cannot infer that q and p_r differ by a constant since p_r may be reducible. Thus, either $p_r \equiv 0$ or $\deg(p_r) \geq \max(n,m)$. In the first case, since p is nonzero, there exist only finitely many $r \in \mathbb{Q}$ such that $p(x, rx, z) \equiv 0$. In the second case, since $\deg(p_r) \leq \deg p$, we have that $\max(n,m) \leq \deg p$. Thus in both cases there are only finitely many bad r's.

Case 2. Assume $n < 0$. Let $q(x, z) = z^m x^{-n} - r^n$. Since q is an irreducible polynomial, we can argue as in the case $n > 0$ and conclude that there are only finitely many possible bad r's.

Let $\{r_1, \ldots, r_k\}$ be the set of bad rational numbers. Consider the system

$$
\begin{cases}
e^z = f_1(z, t, u_1, \ldots, u_k), \\
e^t = f_2(z, t, u_1, \ldots, u_k), \\
e^{u_1} = f_3(z, t, u_1, \ldots, u_k), \\
\ldots \\
e^{u_k} = f_{k+2}(z, t, u_1, \ldots, u_k),
\end{cases} \tag{21}
$$

where $f_1 = t, f_3 = t - r_1 z, \ldots, f_{k+2} = t - r_k z$, and f_2 is the algebraic function which solves z in the original polynomial $p(x, y, z) = 0$. By Theorem 2.7, (21) has a solution (b, e^b, e^b, e^{e^b}), which is a generic solution for (18), since the last k equations guarantee that there is no linear dependence between b and e^b. $\square$

4.3. **General case** $f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}})$**.** For the general case, assuming (SC), we have only partial results (see Proposition 4.7).

Lemma 4.3 (SC)**.** *Let $n \geq 2$ and let $f_1, \ldots, f_n$ be nonzero algebraic functions over $\mathbb{Q}(x_1, \ldots, x_n)$, defined over some cone U. Assume $U \cap (2\pi i \mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$ and* $\deg(f_1) \neq 0$. *The system*

$$
\begin{cases}
e^{x_1} = f_1(\overline{x}), \\
\ldots \\
e^{x_n} = f_n(\overline{x})
\end{cases} \tag{22}
$$

has a solution $\overline{a} \in \mathbb{C}^n$ satisfying $\mathrm{t.\,d.}_{\mathbb{Q}}(\overline{a}) \geq 2$.

Proof. For every $i \leq n$, let $d_i = \deg(f_i)$. Then

$$f_i = h_i + \epsilon_i$$

for a unique homogeneous algebraic function h_i of degree d_i and $\deg(\epsilon_i) < d_i$. Consider the system

$$
\begin{cases}
e^{x_1} = f_1(\overline{x}), \\
\ldots \\
e^{x_n} = f_n(\overline{x}), \\
h_1(\overline{x}) \neq 0, \\
\ldots \\
h_n(\overline{x}) \neq 0, \\
d_1 x_2 - d_2 x_1 \neq 0,
\end{cases} \tag{23}
$$

which can be easily reduced to a Masser system. Let $\overline{a}$ be a solution of system (22). We now prove that $\mathrm{t.\,d.}_{\mathbb{Q}}(\overline{a}) \geq 2$. Assume, by a contradiction, that $t.d._{\mathbb{Q}}(\overline{a}) \leq 1$. By the Lindemann-Weierstrass Theorem, necessarily we have $t.d._{\mathbb{Q}}(\overline{a}) = 1$, and by Schanuel's Conjecture, $\overline{a}$ has $\mathbb{Q}$-linear dimension 1. Thus, there exist $\overline{m}_1, \dots, \overline{m}_{n-1} \in \mathbb{Z}^n$ which are $\mathbb{Q}$-linearly independent and such that

$$\overline{m}_j \cdot \overline{a} = 0, \quad j = 1, \dots, n-1.$$

We have

$$\hat{f}(\overline{a})^{\overline{m}_j} = f_1(\overline{a})^{m_{j1}} \cdot \ldots \cdot f_n(\overline{a})^{m_{jn}} = e^{\overline{m}_j \cdot \overline{a}} = 1, \quad j = 1, \dots, n-1.$$

Let

$$L = \{\overline{z} \in \mathbb{C}^n : \bigwedge_{j=1}^{n-1} \overline{m}_j \cdot \overline{z} = 0\}.$$

Clearly, L is a $\mathbb{C}$-linear space of dimension 1, and $\overline{a} \in L$. Thus, L is the $\mathbb{C}$-linear span of $\overline{a}$. Moreover, since $t.d._{\mathbb{Q}}(\overline{a}) = 1$, for every $t \in \mathbb{C}$ such that $f_i(t\overline{a}) \neq 1$ and for every $i \leq n$, we have

$$\hat{f}(t\overline{a})^{\overline{m}_j} = 1, \quad j = 1, \dots, n-1.$$

For $t \in \mathbb{R}$, $t \gg 1$, since $h_i(\overline{a}) \neq 0$ for every i, we obtain

$$\overline{m}_j \cdot \overline{d} = 0, \quad j = 1, \dots, n-1$$

where $\overline{d} = (d_1, \dots, d_n)$. Thus, $\overline{d} \in L$. Since L has $\mathbb{C}$-linear dimension 1, we have $\overline{a} = \lambda \overline{d}$ for some $\lambda \in \mathbb{C}$, contradicting our choice $d_1 a_2 \neq d_2 a_1$. $\square$

Clearly Lemma 4.3 implies the following.

Corollary 4.4 (SC). *Let $n \geq 2$. Let $p_1(\overline{x}), \dots, p_n(\overline{x}) \in \mathbb{Q}^{alg}[\overline{x}]$ be nonconstant polynomials in $\overline{x} = (x_1, \dots, x_n)$. Then, the system*

$$\begin{cases} e^{x_1} = p_1(\overline{x}), \\ \dots \\ e^{x_n} = p_n(\overline{x}) \end{cases} \tag{24}$$

has a solution $\overline{a}$ such that $t.d._{\mathbb{Q}}(\overline{a}) \geq 2$. In particular, if $n = 2$, then (24) has a generic solution.

Remark 4.5. The hypotheses in Lemma 4.3 and Corollary 4.4 are minimal in order to ensure that $t.d._{\mathbb{Q}}(\overline{a}) \neq 0, 1$.

Adding some extra hypotheses we strengthen Corollary 4.4 as follows.

Lemma 4.6 (SC). *Let $p_1(\overline{x}), \dots, p_n(\overline{x}) \in \mathbb{Q}^{alg}[x_1, \dots, x_n]$. Let $c_i = p_i(\overline{0})$. Assume that the c_i's are nonzero and multiplicatively independent (i.e., for every $\overline{0} \neq \overline{m} \in \mathbb{Z}^n$, $\hat{c}^{\overline{m}} \neq 1$). Then, all solutions of the system*

$$\begin{cases} e^{x_1} = p_1(\overline{x}), \\ \dots \\ e^{x_n} = p_n(\overline{x}) \end{cases} \tag{25}$$

are generic.

Proof. Let $\overline{a} \in \mathbb{C}^n$ be a solution of (25) and let $k = n - t.d._{\mathbb{Q}}(\overline{a})$. Assume, by contradiction, that $k > 0$. By (SC), there exist $\overline{m}_1, \dots, \overline{m}_k \in \mathbb{Z}^n$ linearly independent, such that $\overline{m}_1 \cdot \overline{a} = \dots = \overline{m}_k \cdot \overline{a} = 0$. Thus, $(e^{\overline{a}})^{\overline{m}_1} = \dots = (e^{\overline{a}})^{\overline{m}_k} = 1$, and therefore $\hat{p}(\overline{a})^{\overline{m}_1} = \cdots = \hat{p}(\overline{a})^{\overline{m}_k} = 1$, where $\hat{p}(\overline{a})^{\overline{m}_j} = p_1(\overline{a})^{m_{j1}} \cdot \ldots \cdot p_n(\overline{a})^{m_{jn}}$, for $j = 1, \ldots, k$. Let $L = (\overline{m}_1)^{\perp} \cap \cdots \cap (\overline{m}_k)^{\perp}$. Thus, L is a linear space of dimension $n-k$ defined over $\mathbb{Q}$. Since $\overline{a} \in L$ and $\mathrm{t.d.}_{\mathbb{Q}}(\overline{a}) = \dim(L)$, we have that a is a generic point of L. Thus, $\hat{p}(\overline{x})^{\overline{m}_j} = 1$ on all L, for $j = 1, \ldots, k$. In particular, $\hat{c}^{\overline{m}_j} = \hat{p}(\overline{0})^{\overline{m}_j} = 1$, contradicting the assumption that the c_i's are multiplicatively independent. □

Now we are able to prove the following result.

Proposition 4.7 (SC)**.** *There is a solution $a \in \mathbb{C}$ of* (12) *such that*

$$t.d._{\mathbb{Q}}(a, e_1(a), \ldots, e_k(a)) \neq 0, 1, k-1.$$

Proof. As in the previous cases, $\mathrm{t.d.}(a, e_1(a), \ldots, e_k(a)) \neq 0$ because of the Lindemann-Weierstrass Theorem. In order to prove that $\mathrm{t.d.}_{\mathbb{Q}}(a, e_1(a), \ldots, e_k(a)) \neq 1$ it is enough to apply Lemma 4.3. Assume now that $\mathrm{t.d.}_{\mathbb{Q}}(a, e_1(a), \ldots, e_k(a)) = k-1$. By (SC) there exists a k-tuple $\overline{0} \neq (m_0, \ldots, m_{k-1}) \in \mathbb{Z}^k$ (and without loss of generality we can assume $m_{k-1} \neq 0$) such that

$$m_{k-1}e_{k-1}(a) = \sum_{i=0}^{k-2} m_i e_i(a) = \tilde{m} \cdot \tilde{a}$$

where $\tilde{m} = (m_0, \ldots, m_{k-2}) \in \mathbb{Z}^{k-1}$ and $\tilde{a} = (a, e_1(a), \ldots, e_{k-2}(a))$. Then the following relations hold:

(1) $e_{k-1}(a) = \sum_{i=0}^{k-2} \frac{m_i}{m_{k-1}} e_i(a) = \frac{\tilde{m}}{m_{k-1}} \cdot \tilde{a}$,
(2) $e_k(a)^{m_{k-1}} = e_1(a)^{m_0} e_2(a)^{m_1} \ldots e_{k-1}(a)^{m_{k-2}}$.

Let $\tilde{r} = (r_0, \ldots, r_{k-2}) = (\frac{m_0}{m_{k-1}}, \ldots, \frac{m_{k-2}}{m_{k-1}})$ and $\tilde{x} = (x_0, \ldots, x_{k-2})$. Let I_1, I_2 be the partition of $\{1, \ldots, k-2\}$ induced by $\tilde{m}$; i.e., I_1 is the set of those indices i corresponding to negative m_i's, and I_2 is the set of those indices j corresponding to positive m_j's. Define the following two polynomials:

$$g_{\tilde{r}}(\tilde{x}, z) = p(\tilde{x}, \tilde{r} \cdot \tilde{x}, z),$$

$$s_{\tilde{m}}(\tilde{x}, z) = z^{m_{k-1}} \prod_{i \in I_1} x_i^{-m_i} - \prod_{j \in I_2} x_j^{m_j}.$$

For convenient notation we consider the polynomial $s_{\tilde{m}}(\tilde{x}, z)$ also in the variable x_0 even if this variable does not appear. We notice that the polynomial $g_{\tilde{r}}$ may be reducible, while $s_{\tilde{m}}$ is irreducible (see [20]).

We call a tuple $(\frac{m_0}{m}, \ldots, \frac{m_{k-2}}{m}) \in \mathbb{Q}$ *bad* if there exists $a \in \mathbb{C}$ a solution of (12) such that

$$e_{k-1}(a) = \sum_{i=0}^{k-2} \frac{m_i}{m} e_i(a).$$

Notice that $(\tilde{a}, e_k(a))$ is a solution of both $g_{\tilde{r}}(\tilde{x}, z) = 0$ and $s_{\tilde{m}}(\tilde{x}, z) = 0$, and it is generic for $g_{\tilde{r}}(\tilde{x}, z) = 0$. Hence, $s_{\tilde{m}}$ divides $g_{\tilde{r}}$, and as in Theorem 4.2 there are only finitely many bad tuples of such rationals.

Arguing as in Theorem 4.2 we consider a new system as in (21) which has a solution that is a generic solution for (12). □

Corollary 4.8 (SC). *Let* $f(z) = p(z, e^z, e^{e^z}, e^{e^{e^z}})$, *where* $p(x, y, z, w) \in \mathbb{Q}^{alg}[x, y, z, w]$. *Then there is* $a \in \mathbb{C}$ *which is a generic solution for* $f(z) = 0$.

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