

SEMI-PARABOLIC TOOLS FOR HYPERBOLIC HÉNON MAPS AND CONTINUITY OF JULIA SETS IN \mathbb{C}^2

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ABSTRACT. We prove some new continuity results for the Julia sets J and J^+ of the complex Hénon map $H_{c,a}(x, y) = (x^2 + c + ay, ax)$, where a and c are complex parameters. We look at the parameter space of dissipative Hénon maps which have a fixed point with one eigenvalue $(1 + t)\lambda$, where λ is a root of unity and t is real and small in absolute value. These maps have a semi-parabolic fixed point when t is 0, and we use the techniques that we have developed in a prior work for the semi-parabolic case to describe nearby perturbations. We show that for small nonzero $|t|$, the Hénon map is hyperbolic and has connected Julia set. We prove that the Julia sets J and J^+ depend continuously on the parameters as $t \rightarrow 0$, which is a two-dimensional analogue of radial convergence from one-dimensional dynamics. Moreover, we prove that this family of Hénon maps is stable on J and J^+ when t is non-negative.

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1. INTRODUCTION

Complex analytic maps with a parabolic fixed point or cycle have generated much interest in dynamics in one complex variable as they play a fundamental role in understanding the parameter space of rational maps. Moreover, they provide important models for understanding non-hyperbolic behavior.

In [RT] we studied the family of Hénon maps with a semi-parabolic fixed point or cycle, and showed that the family has nice stability properties. In this paper we want to unravel the mystery about how these semi-parabolic maps sit in the parameter space of Hénon maps and describe the Julia sets of nearby perturbations.

Chapter 2 provides a useful digression to dynamics in one complex variable. Consider a quadratic polynomial $p(x) = x^2 + c$ with a parabolic fixed point and denote by J_p its Julia set. The parameter c lies in the boundary of the Mandelbrot set. It is well-known from the work of P. Lavaurs [L] and A. Douady [D] that on a neighborhood of the parameter c in \mathbb{C} the Julia set does not vary continuously in the Hausdorff topology. Parabolic implosion represents the source of discontinuity

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and of obtaining limit Julia sets with enriched dynamics. Using quasiconformal techniques, C. McMullen [Mc] (and also P. Haïssinsky [Hai]) showed that J_{p_n} converges to J_p when p_n converges to p horocyclically or radially (i.e. non-tangentially with respect to the boundary of a Mandelbrot set in the quadratic case). These tools are harder, if not impossible, to apply to several complex variables, where an analogue of the Uniformization Theorem does not exist. We first set out to give a topological proof of the continuity result for polynomial Julia sets under a stronger radial convergence assumption. The proof involves recovering the Julia set as the image of the unique fixed point f^* of a (weakly) contracting operator in an appropriate function space. In Section 2.5 we prove that f^* depends continuously on the parameter, and thus the corresponding Julia sets converge to the Julia set of the parabolic polynomial, in the Hausdorff topology. After gaining some valuable insight from the study of the one-dimensional problem, notably from Lemma 2.14, we pursue the two-dimensional problem and prove some new continuity results in Chapter 3 for the Julia sets J and J^+ of a complex Hénon map.

We consider the family of complex Hénon maps $H_{c,a}(x,y) = (p(x) + ay, ax)$, where p is a quadratic polynomial, $p(x) = x^2 + c$, and a is a complex parameter. When $a \neq 0$, this is a polynomial automorphism of \mathbb{C}^2 . The dynamics of Hénon maps bears some resemblance to the dynamics of 1-D polynomials, however extending results from one to several variables requires envisioning new techniques and approaches, and in many cases the emerging picture is substantially different and contains new and thrilling phenomena not present in the one-dimensional world. In order to describe the dynamics of the Hénon map, one studies the sets K^+ and K^- of points which do not escape to infinity under forward and respectively backward iterations. The topological complements of K^\pm in \mathbb{C}^2 are denoted by U^\pm and called the escaping sets. The most interesting dynamics occur on the boundary of the sets K^\pm and U^\pm where chaotic behavior is present. The sets $J^\pm = \partial K^\pm = \partial U^\pm$ and $J = J^+ \cap J^-$ are called the Julia sets of the Hénon map. The sets J and $K = K^+ \cap K^-$ are compact, while the sets J^\pm are closed, connected and unbounded [BS1].

A quadratic Hénon map is uniquely determined by the eigenvalues λ and ν at a fixed point so we will sometimes write $H_{\lambda,\nu}$ in place of $H_{c,a}$ to mark this dependence. The precise formula for $H_{\lambda,\nu}$ is given at the beginning of Chapter 3. We say that a Hénon map is semi-parabolic if it has a fixed point (or cycle) with one eigenvalue λ , a root of unity, and one eigenvalue smaller than one in absolute value. Unlike hyperbolic Hénon maps, semi-parabolic ones are not stable under perturbations. E. Bedford, J. Smillie and T. Ueda have described some semi-parabolic bifurcations in \mathbb{C}^2 for $\lambda = 1$ in [BSU17]. In particular, they show that at a parameter value with a semi-parabolic fixed point with the eigenvalues $\lambda = 1$ and $|\nu| < 1$, the sets J , J^+ , K and K^+ vary discontinuously with the parameters, while J^- and K^- vary continuously with the parameters. The phenomenon described in [BSU17] is a two-dimensional analogue of parabolic implosion that occurs in complex dimension one.

In order to state our main results, consider a primitive root of unity $\lambda = e^{2\pi i p/q}$ and let $\lambda_t = (1 + t)\lambda$. For t real and small in absolute value, we look at the parameter space \mathcal{P}_{λ_t} of complex Hénon maps which have a fixed point with one eigenvalue λ_t . The equation of the curve \mathcal{P}_{λ_t} is given in Proposition 3.2. When $t = 0$ these maps are semi-parabolic; when $t \neq 0$, we regard the maps corresponding

to parameters from \mathcal{P}_{λ_t} as perturbations of the semi-parabolic ones. We show in Section 3.8 that there exists $\delta > 0$ such that for $(c, a) \in \mathcal{P}_{\lambda_t}$ and $0 < |a| < \delta$, the Julia sets J and J^+ depend continuously on the parameters as t approaches 0. An equivalent formulation is given in the theorem below. These results can be regarded as a natural extension of the concept of radial convergence of Julia sets [Mc] to higher dimensions, in the context of polynomial automorphisms of \mathbb{C}^2 .

Theorem 1.1 (Continuity). *There exists $\delta > 0$ such that if $|\nu_t| < \delta$ and $\nu_t \rightarrow \nu$ as $t \rightarrow 0$, then the Julia sets J and J^+ depend continuously on the parameters, i.e.,*

$$J_{(\lambda_t, \nu_t)}^+ \rightarrow J_{(\lambda, \nu)}^+ \quad \text{and} \quad J_{(\lambda_t, \nu_t)} \rightarrow J_{(\lambda, \nu)}$$

in the Hausdorff topology.

For the set J^+ we are taking the Hausdorff topology on the one-point compactification of \mathbb{C}^2 . What we prove in Theorem 1.1 is the continuity of Julia sets J and J^+ as we approach a semi-parabolic parameter from the interior of a hyperbolic component of the Hénon connectedness locus, similar to radial convergence from 1-D dynamics. Our next theorem describes the dynamical nature of the perturbed semi-parabolic maps.

Theorem 1.2 (Hyperbolicity). *There exist $\delta, \delta' > 0$ such that in the parametric region*

$$\mathcal{HR}_{\delta, \delta'} = \{(c, a) \in \mathcal{P}_{\lambda_t} : 0 < |a| < \delta \text{ and } -\delta' < t < \delta', t \neq 0\}$$

the Julia set $J_{c,a}$ is connected and the Hénon map $H_{c,a}$ is hyperbolic.

By definition, the connectedness locus for the Hénon family is the set of parameters $(c, a) \in \mathbb{C}^2$ such that the Julia set $J_{c,a}$ is connected. Theorem 1.2 shows that the parametric region $\{(c, a) \in \mathcal{P}_{\lambda} : |a| < \delta\}$ of semi-parabolic Hénon maps lies in the boundary of a hyperbolic component of the Hénon connectedness locus. In fact, when $\lambda \neq 1$, it lies in the boundary of two such hyperbolic components. A mechanism for loss of hyperbolicity at the boundary of the horseshoe region through the development of tangencies between stable and unstable manifolds is described by E. Bedford and J. Smillie in [BS], and more recently by Z. Arai and Y. Ishii in [AI]. In Theorem 1.2 we describe a different mechanism for loss of hyperbolicity, through the creation of a semi-parabolic fixed point. We first do a local analysis in Sections 3.1, 3.2 and 3.3, and show how to deform the local semi-parabolic structure into a hyperbolic structure; these sections are applicable to holomorphic germs of diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-parabolic fixed point, and their perturbations. We complete the proof of Theorem 1.2 in Section 3.5.

Theorem 1.2 proves the existence of a larger region of hyperbolicity for complex Hénon maps than what was previously known. It is in general very hard to exhibit regions of hyperbolicity for Hénon maps. Z. Arai developed a computer program for detecting hyperbolicity, that relies on heavy numerical computations. Otherwise, the only Hénon maps proven to be hyperbolic using only theoretical arguments correspond to the horseshoe region and to perturbations of 1-D hyperbolic maps. It is also known from early works of J. E. Fornæss and N. Sibony [FS], and J. Hubbard and R. Oberste-Vorth [HOV1, HOV2], that Hénon maps that come from perturbations of hyperbolic polynomials with connected Julia sets inherit both of these properties. However, the proof gave no control on the admissible size of perturbations as we approach the boundary of the Mandelbrot set, i.e., it was not

known that the size of the region $\mathcal{HR}_{\delta,\delta'}$ does not decrease to 0 as $t \rightarrow 0$. Z. Arai [A] gave a computer-assisted proof for the existence of hyperbolic plateaus for the family of complex Hénon maps $H_{c,a}$ with both parameters c and a real. In our language, these regions correspond to strips on the right/left side of the real curves $\mathcal{P}_{\pm 1}^n \cap \mathbb{R}^2$, where $\mathcal{P}_{\pm 1}^n$ is the set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a cycle of order n with one eigenvalue ± 1 . The existence of these regions is established in Theorem 1.2.

Corollary 1.2.1. *There exists an $\epsilon > 0$ such that the real parametric region*

$$\left\{ (c, a) \in \mathbb{R} \times (-\epsilon, \epsilon) : a \neq 0, \frac{(1+a)^2}{4} - \epsilon < c < \frac{(1+a)^2}{4} \right\}$$

is a region of hyperbolicity for the Hénon family $H_{c,a}(x, y) = (x^2 + c - ay, x)$. The Hénon map is written in the standard parametrization and has Jacobian a .

To compare our results with [B] and [BSU17], suppose λ is 1. Theorems 1.1 and 1.2 answer Questions 3 and 4 of E. Bedford, from [B]. Corollary 1.2.1 is formulated as a specific answer to Question 3 and the set $c = (a + 1)^2/4$ from the corollary is simply the defining equation of parabola \mathcal{P}_1 . When $(c, a) \in \mathcal{P}_1$, the Hénon map has a double fixed point with one eigenvalue 1. From the “right” of the real parabola $\mathcal{P}_1 \cap \mathbb{R}^2$ we have semi-parabolic implosion described in [BSU17]. More specifically, in [BSU17] it is shown that there exists a sequence $\epsilon_n \rightarrow 0$ which converges to 0 tangentially to the positive real axis ($Re(\epsilon_n) > 0$ and $Im(\epsilon_n) < \text{const.} |\epsilon_n|^2$) such that the Julia set $J_{c_n,a}$ corresponding to the sequence $c_n = (a + 1)^2/4 + \epsilon_n$ does not converge to the Julia set $J_{c,a}$ in the Hausdorff topology. By comparison, Theorem 1.1 shows that we have continuity of J and J^+ from the “left” of the real parabola $\mathcal{P}_1 \cap \mathbb{R}^2$. When $(c, a) \in \mathcal{P}_1 \cap \mathbb{R}^2$ and $0 < |a| < \delta$ we get that $J_{c-\epsilon,a} \rightarrow J_{c,a}$ and $J_{c-\epsilon,a}^+ \rightarrow J_{c,a}^+$ as $\epsilon \rightarrow 0^+$. Note that Theorem 1.1 gives no information on what happens to the “right” of parabola \mathcal{P}_1 ; indeed, when $\lambda = 1$, both curves \mathcal{P}_{1+t} and \mathcal{P}_{1-t} are to the left of \mathcal{P}_1 . This can be seen from the fact that for $a = 0$ and $t \neq 0$, the polynomial p_t has two distinct fixed points, with multipliers $1 \pm t$; therefore regardless of whether t is positive or negative, p_t has an attracting fixed point, and belongs to the interior of the main cardioid of the Mandelbrot set.

This paper is built on previous work done by the authors in [R] and [T]. We use the tools that we have developed for the study of semi-parabolic germs/Hénon maps in [RT] to extend the results from [RT] to nearby perturbations of semi-parabolic germs/Hénon maps. We can actually say more about the stability properties of our family of Hénon maps when the parameter t is non-negative:

Theorem 1.3 (Stability). *The family of complex Hénon maps $\mathcal{P}_{\lambda_t} \ni (c, a) \rightarrow H_{c,a}$ is a structurally stable family on J and J^+ for $0 < |a| < \delta$ and $0 \leq t < \delta'$.*

We say that the family of Hénon maps $\mathcal{P}_{\lambda_t} \ni (c, a) \rightarrow H_{c,a}$ is structurally stable on J when $t \in [0, \delta']$ and $|a| < \delta$ if for any two pairs $(c_i, a_i) \in \mathcal{P}_{\lambda_{t_i}}$, with $|a_i| < \delta$ and $t_i \in [0, \delta']$ for $i = 1, 2$, we have $(H_{c_1,a_1}, J_{c_1,a_1})$ conjugate to $(H_{c_2,a_2}, J_{c_2,a_2})$. Consequently, the Julia sets J_{c_1,a_1} and J_{c_2,a_2} are homeomorphic. We explain structural stability on J^+ in a similar way, by replacing J with J^+ . We complete the proof of Theorem 1.3 in Section 3.8.

Another notion of stability (called weak stability) was introduced by R. Dujardin and M. Lyubich [DL] for holomorphic families $(f_z)_{z \in \Lambda}$ of moderately dissipative polynomial automorphisms of \mathbb{C}^2 , where Λ is a connected complex manifold. Weak

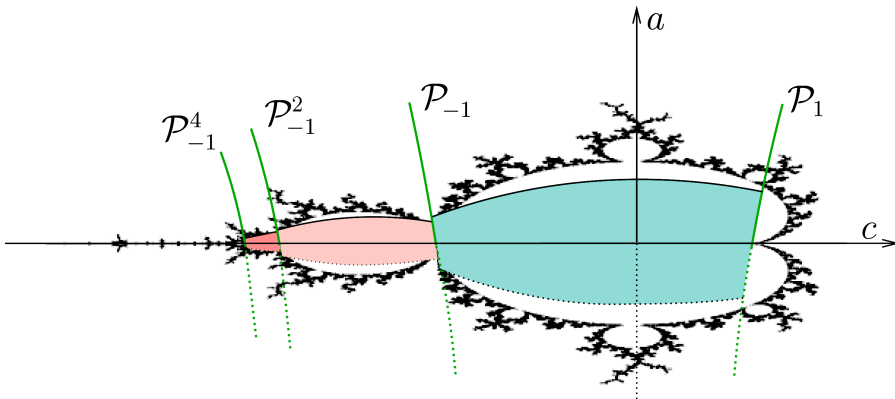


FIGURE 1. The curves $\mathcal{P}_1, \mathcal{P}_{-1}, \mathcal{P}_{-1}^2, \mathcal{P}_{-1}^4$ are examples of parametric curves containing semi-parabolic Hénon maps. There exist regions of hyperbolicity between \mathcal{P}_1 and \mathcal{P}_{-1} and \mathcal{P}_{-1} and \mathcal{P}_{-1}^2 which belong to hyperbolic components of the Hénon connectedness locus.

stability is defined in terms of branched holomorphic motions of the set J^* (the closure of the saddle periodic points), but an equivalent easier definition is the following: the family is weakly stable if periodic orbits do not bifurcate. The equivalence between weak stability and continuity of J^* is discussed in [DL], and the relation between weak stability and uniform hyperbolicity on J^* is analyzed by P. Berger and R. Dujardin in [BD]. These results do not apply to our context, but they are of independent interest.

2. CONTINUITY OF POLYNOMIAL JULIA SETS

In this section we focus only on one-dimensional dynamics. We first discuss continuity of polynomial Julia sets, which will prove useful in understanding continuity of Julia sets for Hénon maps. This will be treated in Section 3.

Assume that p is a quadratic polynomial. The *filled Julia set* of the polynomial p is

$$K_p = \{z \in \mathbb{C} : |p^{\circ n}(z)| \text{ bounded as } n \rightarrow \infty\},$$

and the *Julia set* of p is $J_p = \partial K_p$. The filled Julia set K_p is connected iff the orbit of the unique critical point is bounded. If K_p is connected (or equivalently J_p is connected), then there exists a unique analytic isomorphism

$$(1) \quad \Psi_p : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - K_p$$

such that $\Psi_p(z^2) = p(\Psi_p(z))$ and $\Psi_p(z)/z \rightarrow 1$ as $z \rightarrow \infty$. If J_p is locally connected, then Ψ_p extends to the boundary \mathbb{S}^1 and defines a continuous surjection (see [Mi])

$$(2) \quad \gamma_p : \mathbb{S}^1 \rightarrow J_p.$$

The Julia set of a hyperbolic or parabolic polynomial is connected and locally connected (see [DH]). The map Ψ_p^{-1} is the *Böttcher coordinate* of the polynomial p , while the map Ψ_p is called the *inverse Böttcher isomorphism* (or the *Böttcher chart* [H]). The boundary map γ_p is called the *Carathéodory loop* of p .

The continuous map $G_p : \mathbb{C} \rightarrow \mathbb{R}$, defined by $G_p(z) = \log |\Psi_p^{-1}(z)|$ for $z \in \mathbb{C} - K_p$ and $G_p(z) = 0$ for $z \in K_p$, is called the Green function of the polynomial p . Each level set of the Green function $\{z : G_p(z) = \log(R)\}$ with $R > 1$ is called an equipotential for the polynomial p . This is the image of the circle of radius R under Ψ_p .

2.1. Horocyclic and radial convergence. The topic of convergence of Julia sets in the Hausdorff topology (of compact sets in \mathbb{P}^1) is very vast and has been covered by many authors (A. Douady [D], P. Lavaurs [L], C. McMullen [Mc], P. Haïssinsky [Hai], etc.). We only recall here a theorem from [Mc] about horocyclic and radial convergence of rational maps, and give the simplified form of the theorem for quadratic polynomials.

Definition 2.1 (Hausdorff topology). The compact sets K_n converge to the compact set K in the Hausdorff topology if the following conditions hold:

- (a) Every neighborhood of a point $x \in K$ meets all but finitely many K_n .
- (b) If every neighborhood of x meets infinitely many K_n , then $x \in K$.

Theorem 2.2 ([Mc]). *Let f be a geometrically finite rational map and suppose that $f_n \rightarrow f$ horocyclically (or radially), preserving critical relations. Then $J_{f_n} \rightarrow J_f$ in the Hausdorff topology.*

Theorem 2.2 can be expressed in a simplified form when we restrict to the family of quadratic polynomials. Let p be a quadratic polynomial. The sequence of polynomials p_n converges to p algebraically if $\deg(p_n) = \deg(p)$ for all n , and the coefficients of p_n converge to the coefficients of p .

Definition 2.3 (Horocyclic/radial convergence of multipliers). Let $\lambda_n \rightarrow 1$ in \mathbb{C}^* ,

$$\lambda_n = e^{L_n + i\theta_n} \quad \text{and} \quad \theta_n \rightarrow 0.$$

The sequence λ_n converges to 1 horocyclically if $\theta_n^2/L_n \rightarrow 0$. The sequence λ_n converges to 1 radially if $\theta_n = \mathcal{O}(|L_n|)$, that is, there exists $M > 0$ such that $|\theta_n| \leq ML_n$ for $n > 0$.

Theorem 2.4. *Let p be a quadratic polynomial with a parabolic fixed point α_0 with multiplier $e^{2\pi ip/q}$. Let p_n be a sequence of quadratic polynomials, such that $p_n \rightarrow p$ algebraically. Assume that each p_n^q has a fixed point α_n , such that $\alpha_n \rightarrow \alpha_0$ and such that the sequence of multipliers $\lambda_n = (p_n^q)'(\alpha_n)$ converges to 1 horocyclically (or radially). Then*

$$J_{p_n} \rightarrow J_p$$

in the Hausdorff topology.

The proof of Theorem 2.2 is quite involved and uses quasiconformal theory and it is not very clear how one could extend it to higher dimensions. We would therefore like to first outline a more topological proof of continuity in one dimension.

2.2. A topological proof of continuity in dimension one. Let $\lambda = e^{2\pi ip/q}$ be a primitive root of unity of order q . Set

$$\lambda_t = (1 + t)\lambda,$$

for t real and sufficiently small. Consider the family of quadratic polynomials

$$(3) \quad p_t(x) = x^2 + c_t, \quad \text{where} \quad c_t = \frac{\lambda_t}{2} - \frac{\lambda_t^2}{4}.$$

For $t > 0$ the polynomial p_t is hyperbolic and has a repelling fixed point $\alpha_t = \lambda_t/2$ of multiplier λ_t and a q -periodic attractive orbit. For $t = 0$ the polynomial p_0 has a parabolic fixed point α_0 of multiplier λ . The multiplicity of the fixed point α_0 as a solution of the equation $p_0^q(x) = x$ is $q + 1$. Finally, when $t < 0$, p_t has an attracting fixed point α_t of multiplier λ_t and a q -periodic repelling orbit.

We have $p_t \rightarrow p_0$ uniformly as $t \rightarrow 0$. The continuity of the corresponding Julia sets (that we state below as Theorem 2.5) is an easy consequence of McMullen's Theorem 2.2. The sequence of multipliers $\lambda_t^q = (1 + t)^q$ has no imaginary part, therefore it converges horocyclically and radially to 1.

Theorem 2.5. *The Julia set J_{p_t} of the polynomial p_t depends continuously on the parameter t , that is, $J_{p_t} \rightarrow J_{p_0}$ in the Hausdorff topology.*

We give a new proof of the continuity result for the family p_t which does not use quasiconformal theory. The proof relies on the techniques developed by Douady and Hubbard in [DH] for proving the local connectivity of Julia sets of polynomials where all critical points are attracted to attracting or parabolic cycles. An adaptation of this technique was also used in [Kw] to build semi-conjugacies between Julia sets of geometrically finite rational maps.

We build a continuous family of bounded metrics μ_t on the neighborhood of the Julia set J_{p_t} , with respect to which the polynomial p_t is weakly expanding. Then we will recover the Julia set J_{p_t} as the image of the unique fixed point f_t^* of a weakly contracting operator F_t in an appropriate function space. We will show that the fixed point f_t^* is continuous with respect to t , and conclude that the Julia sets J_{p_t} converge in the Hausdorff topology as $t \rightarrow 0$ to the Julia set of the parabolic polynomial.

We will illustrate the technique for $t \geq 0$. The case when t is negative is almost identical, but it requires a small technical adjustment, which we will discuss in a more general setting in Chapter 3, where we adapt the construction to a family of polynomial automorphisms of \mathbb{C}^2 .

2.3. Normalizing coordinates at a repelling fixed point. When $t > 0$, the polynomial p_t is hyperbolic and expanding with respect to the Poincaré metric on a suitable neighborhood of J_{p_t} . In order to make the choice of metrics continuous with respect to the parameter t when $t \rightarrow 0$, we need to correct the metric μ_t near the repelling fixed point α_t , which becomes parabolic when $t = 0$. A naive idea would be to take a small disk D_t around the repelling point α_t on which the polynomial p_t is analytically conjugated to its linear part $z \rightarrow \lambda_t z$, $|\lambda_t| > 1$, hence naturally expanding with respect to the Euclidean metric. This is not very helpful however, because the radius of D_t converges to 0 as t converges to 0. This issue can be dealt with by constructing “normalizing coordinates” around α_t , similar to the parabolic case $t = 0$. We will build a larger neighborhood \mathbb{D}_ρ , with ρ independent of t , around α_t , on which the polynomial is not fully linearized, but rather conjugated to a “normal form”.

Let $\epsilon_0 = \tan(2\pi/9)$ and $\epsilon_1 = \epsilon_0/\sqrt{\epsilon_0^2 + 1}$. The meaning of these constants is fully explained by equation (30) and the discussion following it.

Proposition 2.6. *There exist $\delta' > 0$ and $\rho > 0$ such that for all t with $|t| < \delta'$ there exists a coordinate transformation $\phi_t : \mathbb{D}_{\rho'}(\alpha_t) \rightarrow \mathbb{D}_\rho(0)$ defined in a neighborhood of the repelling fixed point α_t such that in the new coordinates the polynomial p_t*

can be written as $\tilde{p}_t(x) = \lambda_t(x + x^{q+1} + C_t x^{2q+1} + \mathcal{O}(x^{2q+2}))$. Suppose $t \in [0, \delta']$. In the region

$$\Delta^- = \{|x| \leq \rho : \operatorname{Re}(x^q) > \epsilon_0 |\operatorname{Im}(x^q)|\}$$

the derivative \tilde{p}_t' is expanding, with a factor of

$$|\lambda_t| (1 + (q + 3/2)\epsilon_1 |x|^q) > |\lambda_t| \geq 1.$$

The compact region

$$\Delta^+ = \{|x| \leq \rho : \operatorname{Re}(x^q) \leq \epsilon_0 |\operatorname{Im}(x^q)|\}$$

satisfies $\Delta^+ \subset \operatorname{int}(K_{\tilde{p}_t}) \cup \{0\}$ and $\tilde{p}_t(\Delta^+) \subset \operatorname{int}(\Delta^+) \cup \{0\}$.

Proof. One performs for the family p_t the same sequence of coordinate transformations as the ones done in [BH] or [DH] in the parabolic case. After a global coordinate change that brings the fixed point α_t to the origin, we can assume that $p_t(x) = \lambda_t x + x^2$. Suppose by induction that for $k \geq 2$ the maps p_t have the form

$$x_1 = \lambda_t x + a_t x^k + \mathcal{O}(x^{k+1})$$

where $a_t \neq 0$ for $|t| < \delta'$.

Consider the coordinate transformation

$$X = x + b_t x^k \quad \text{with inverse} \quad x = X - b_t X^k + \dots$$

In the new coordinate system, we get

$$\begin{aligned} X_1 &= x_1 + b_t x_1^k = \lambda_t x + (a_t + b_t \lambda_t^k) x^k + \dots \\ &= \lambda_t (X - b_t X^k + \dots) + (a_t + b_t \lambda_t^k) (X - b_t X^k + \dots)^k + \dots \\ &= \lambda_t X + (a_t + b_t (\lambda_t^k - \lambda_t)) X^k + \dots \end{aligned}$$

When $t \neq 0$, we have $|\lambda_t| \neq 1$, so $\lambda_t \neq \lambda_t^k$ for all k with $2 \leq k \leq q$. If k is not congruent to 1 modulo q , then $\lambda_0^k \neq \lambda_0$ as well, so we can set

$$b_t = \frac{a_t}{\lambda_t - \lambda_t^k}$$

and eliminate the term $a_t x^k$. The transformations $x + b_t x^k$ are injective on a uniform neighborhood of 0.

This proves that by successive coordinate transformations of the form $X_t = x + b_t x^k$ we can eliminate terms with powers that are not congruent to 1 modulo q , so the first term that cannot be eliminated in this way will have a power of the form $a_t x^{\nu q+1}$, for some integer $\nu \geq 1$. The parabolic multiplicity of the fixed point α_0 is 1, so $\nu = 1$.

Thus the map takes the form

$$(4) \quad x_1 = \lambda_t(x + a_t x^{q+1} + \mathcal{O}(x^{q+2})).$$

Of course, when $t \neq 0$, we could use the same map $X_t = x + b_t x^{q+1}$ to eliminate the term $a_t x^{q+1}$, however this would require shrinking the domain of injectivity of X_t to 0 as $t \rightarrow 0$, as well as losing the continuity of X_t with respect to t at $t = 0$.

We can further reduce equation (4) to

$$(5) \quad x_1 = \lambda_t(x + x^{q+1} + \mathcal{O}(x^{q+2}))$$

by considering a linear map $X = A_t x$, where A_t is a constant chosen such that $A_t^q = a_t$.

In equation (5) we can eliminate all terms of the form $a_t x^k$, with $q + 1 < k < 2q + 1$, using the same coordinate transformation as before, $X_t = x + b_t x^k$, where $b_t = \frac{a_t}{\lambda_t - \lambda_t^k}$. We thus arrive at the normal form

$$\tilde{p}_t(x) = \lambda_t(x + x^{q+1} + C_t x^{2q+1} + \mathcal{O}(x^{2q+2})).$$

When $t = 0$ the regions Δ^+ and Δ^- represent attracting and respectively repelling sectors for the (normalized) parabolic polynomial \tilde{p}_0 . The attractive sector Δ^+ belongs to the interior of the filled-in Julia set $\text{int}(K_{\tilde{p}_0}) \cup \{0\}$ and all points in Δ^+ converge under forward iterations to the parabolic fixed point 0, which lies in the Julia set. When $t < 0$, the sector Δ^+ belongs to the basin of attraction of the attracting fixed point 0. When $t > 0$, the sector Δ^+ belongs to $\text{int}(K_{\tilde{p}_t}) \cup \{0\}$, because Δ^+ (with 0 removed) is a trapping region for a q -periodic attractive orbit [BH]. We prove these facts directly for Hénon maps in Propositions 3.6 and 3.7, and the proofs apply also to the family of polynomials considered in this lemma.

To show that the derivative \tilde{p}'_t is expanding on Δ^- when $t > 0$, we perform the same computations as in the parabolic case [DH]. The choice of ϵ_0 and ϵ_1 guarantees that if $\text{Re}(x^q) > \epsilon_0 \text{Im}(x^q)$, then $\text{Re}(x^q) > \epsilon_1 |x|^q$. Consider a constant m so that $|\tilde{p}'_t(x) - \lambda_t(1 + (q + 1)x^q)| < m|x|^{2q}$ on \mathbb{D}_ρ . Using the normal form for \tilde{p}_t , we get

$$\begin{aligned} |\tilde{p}'_t(x)| &= |\lambda_t| |1 + (q + 1)x^q + \mathcal{O}(x^{2q})| \geq |\lambda_t| (|1 + (q + 1)x^q| - m|x|^{2q}) \\ &\geq (1 + t) (1 + (q + 1)\epsilon_1|x|^q - m|x|^{2q}) > (1 + t) (1 + (q + 3/2)\epsilon_1|x|^q) \end{aligned}$$

for $|x|$ sufficiently small. Hence $|\tilde{p}'_t(x)| > |\lambda_t|$ throughout Δ^- , for all $t \in [0, \delta']$. \square

Let $\Delta_t^+ = \phi_t^{-1}(\Delta^+)$ and $\Delta_t^- = \phi_t^{-1}(\Delta^-)$. Recall that $\alpha_t = \phi_t^{-1}(0)$. By Proposition 2.6, the set $\Delta_t^+ - \{\alpha_t\}$ belongs to the interior of the filled-in Julia set K_{p_t} . Moreover, when $t > 0$, the q -periodic attractive orbit of the polynomial p_t is contained in the sector Δ_t^+ . The Julia set J_{p_t} near α_t is completely contained in the repelling sectors:

$$J_{p_t} \cap \mathbb{D}_{\rho'}(\alpha_t) = J_{p_t} \cap \Delta_t^-.$$

When $t \in [-\delta', \delta']$, the polynomial p_t has connected Julia set; the critical point 0 of p_t is attracted to the q -periodic orbit when $t > 0$, respectively to the parabolic fixed point when $t = 0$, and to the attracting fixed point when $t < 0$. So there exists a first iterate $n_t \in \mathbb{N}$ such that $p_t^{\circ(n_t+1)}(0) \in \Delta_t^+$, otherwise said, there exists a first iterate for which $p_t^{\circ n_t}(c_t) \in \Delta_t^+$ and $p_t^{\circ n_t}(0) \notin \Delta_t^+$. The function $t \rightarrow n_t$ is locally constant and we can assume without loss of generality that when δ' is small, the number n_t is the same for all $t \in [-\delta', \delta']$. Therefore we can remove the dependence on t and denote n_t by N . Denote further by $p_t^{-\circ N}(\Delta_t^+)$ the connected component of the N^{th} preimage of the set Δ_t^+ that contains the fixed point α_t .

2.4. A continuous family of bounded metrics. For each value of the parameter t we construct a neighborhood U_t of the Julia set J_{p_t} and a metric μ_t on U_t with respect to which the polynomial p_t is expanding. The family (U_t, μ_t) will be continuous with respect to the parameter t .

The outer boundary of the set U_t is an equipotential of the Julia set J_{p_t} . The inner boundary is $\partial p_t^{-\circ N}(\Delta_t^+)$, where N is defined above. Formally, choose $R > 2$ and set

$$(6) \quad U_t = \mathbb{C} - \overline{p_t^{-\circ N}(\Delta_t^+)} - \{z \in \mathbb{C} - K_{p_t} : |\Psi_{p_t}^{-1}(z)| \geq R\}.$$

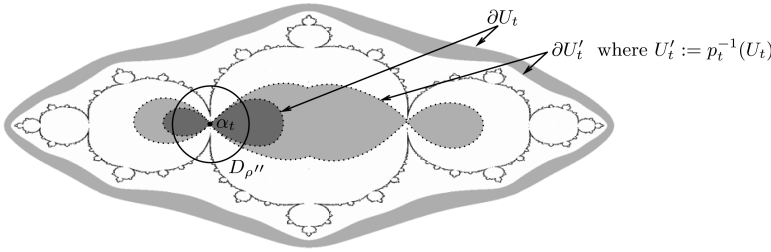


FIGURE 2. The polynomial p_t has a fixed point at α_t . The corresponding neighborhoods U_t and U'_t are also shown.

Let $U'_t = p_t^{-1}(U_t)$. The set U'_t is contained in U_t by construction (see Figure 2), and we can put on U'_t the Poincaré metric of U_t . The map $p_t : U'_t \rightarrow U_t$ is a covering map, hence expanding:

$$|(z, \xi)|_{U_t} < |(z, \xi)|_{U'_t} = |(p(z), p'(z)\xi)|_{U_t} \text{ for } z \in U'_t \text{ and } \xi \in T_z U'_t.$$

However U'_t is not relatively compact in U_t because $\partial U'_t \cap \partial U_t = \{\alpha_t\}$, so there is no constant of uniform expansion. On the repelling sectors Δ_t^- , one can define a metric $\mu_{\Delta_t^-}$ as the pull-back of the Euclidean metric from the normalizing coordinates Δ^- .

$$|(z, \xi)|_{\Delta_t^-} := |\phi'_t(z)\xi| \text{ for } z \in \Delta_t^- \text{ and } \xi \in T_z \Delta_t^-,$$

where the latter length is the modulus of the complex number $\phi'_t(z)\xi$.

Definition 2.7. Let $\mu_t = \inf(\mu_{U_t}, M\mu_{\Delta_t^-})$, where M is a positive real number and

$$M > \sup \left\{ \frac{2\mu_{U_t}(z, \xi)}{\mu_{\Delta_t^-}(p_t(z), p'_t(z)\xi)} : z \in p_t^{-1}(\Delta_t^-), z \notin \Delta_t^- \text{ and } t \in [0, \delta'] \right\}.$$

By choosing M sufficiently large, one can assure that on the boundary of V_t the infimum is attained by the Poincaré metric μ_{U_t} . So the metric μ_t is continuous on U'_t . Note also that the Poincaré metric is infinite at α_t while the Euclidean metric is bounded; therefore there exists a neighborhood of α_t , uniform with respect to t , for which the infimum in Definition 2.7 is attained by the Euclidean metric $M\mu_{\Delta_t^-}$.

Lemma 2.8. *The family of metrics μ_t depends continuously on the parameter t and it is dominated above and below by the Euclidean metric. There exist $m_1 > 0$ and $m_2 > 0$ such that*

$$m_1|x - y| < d_{\mu_t}(x, y) < m_2|x - y| \text{ for any } x, y \in U'_t.$$

Proof. By construction, the neighborhood U_t and the repelling sectors Δ_t^- depend continuously on t . Let ρ_t denote the density function of the Poincaré metric on U_t , $\mu_{U_t}(z, dz) = \rho_t(z)|dz|$. The map ρ_t is positive, C^∞ -smooth on U'_t and continuous with respect to t . Hence μ_{U_t} is bounded below by the Euclidean metric on U'_t . The metric μ_{U_t} on this set is also bounded above on U'_t , except on a small neighborhood of the fixed point $\alpha_t \in \partial U_t$.

The metric $\mu_{\Delta_t^-}$ is the pull-back of the Euclidean metric by a holomorphic injective map ϕ_t , continuous with respect to t . We have $\mu_{\Delta_t^-}(z, dz) = |\phi'_t(z)||dz|$, where $|\phi'_t(z)| > 0$ is bounded above and below on Δ_t^- . Therefore the infimum metric μ_t is bounded above and below with respect to the Euclidean metric. \square

Lemma 2.9. *The polynomial p_t is strictly expanding with respect to the metric μ_t on the set U'_t when $t > 0$.*

Proof. Let $z, z' \in U'_t$ and $\xi \in T_z U'_t, \xi' \in T_{z'} U'_t$ such that $z' = p_t(z)$ and $\xi' = p'_t(z)\xi$.

We will show that for each $t > 0$ there exists a constant $k_t > 1$ such that

$$\mu_t(z', \xi') > k_t \cdot \mu_t(z, \xi).$$

There are four cases to consider:

- (a) $\mu_t(z, \xi) = \mu_{U_t}(z, \xi)$ and $\mu_t(z', \xi') = \mu_{U_t}(z', \xi')$.

This happens only if both z and z' are outside a small neighborhood of the point α_t . Outside this neighborhood, the set U'_t is compactly contained in U_t , so p_t expands strongly with respect to the Poincaré metric. For all $t \in [0, \delta']$ there exists $\kappa_t > 1$ such that

$$(7) \quad \mu_{U_t}(z', \xi') > \kappa_t \cdot \mu_{U_t}(z, \xi).$$

The constant κ_t depends only on the distance between the boundaries ∂U_t and $\partial U'_t$ outside a disk of fixed size around the fixed point α_t , therefore $\inf_{t \in [0, \delta']} \kappa_t > 1$.

- (b) $\mu_t(z, \xi) = M\mu_{\Delta_t^-}(z, \xi)$ and $\mu_t(z', \xi') = M\mu_{\Delta_t^-}(z', \xi')$.

The normalized polynomial \tilde{p}_t expands with respect to the Euclidean metric, so by Proposition 2.6 we have

$$\mu_{\Delta_t^-}(z', \xi') > (1 + t) \cdot \mu_{\Delta_t^-}(z, \xi).$$

Notice that the constant of expansion is 1 if and only if $t \rightarrow 0$ and $\phi_t(z) \rightarrow 0$ (that is, z approaches the parabolic fixed point α_0).

- (c) $\mu_t(z, \xi) = M\mu_{\Delta_t^-}(z, \xi)$ and $\mu_t(z', \xi') = \mu_{U_t}(z', \xi')$.

Similar to case (a), the point z' cannot be too close to the fixed point α_t , so

$$\mu_{U_t}(z', \xi') > \kappa_t \cdot \mu_{U_t}(z, \xi) \geq \kappa_t \cdot M\mu_{\Delta_t^-}(z, \xi).$$

- (d) $\mu_t(z, \xi) = \mu_{U_t}(z, \xi)$ and $\mu_t(z', \xi') = M\mu_{\Delta_t^-}(z', \xi')$.

There are two sub-cases to consider

- (i) If $z \in p_t^{-1}(\Delta_t^-) \cap \Delta_t^-$, then

$$M\mu_{\Delta_t^-}(z', \xi') > (1 + t) \cdot M\mu_{\Delta_t^-}(z, \xi) > (1 + t) \cdot \mu_{U_t}(z, \xi).$$

- (ii) If $z \in p_t^{-1}(\Delta_t^-)$ but $z \notin \Delta_t^-$, then the conclusion follows from the choice of the constant M , as shown below:

$$2\mu_{U_t}(z, \xi) = \frac{2\mu_{U_t}(z, \xi)}{\mu_{\Delta_t^-}(z', \xi')} \cdot \mu_{\Delta_t^-}(z', \xi') < M\mu_{\Delta_t^-}(z', \xi').$$

Set $k_t := \min(1 + t, \kappa_t)$. From estimates (a), (b), (c) and (d) we can easily see that

$$\mu_t(z', \xi') > k_t \cdot \mu_t(z, \xi).$$

We get uniform expansion when $t > 0$ because k_t is strictly greater than 1. □

The metric μ_t induces a natural path metric on U'_t . If $\eta : [0, 1] \rightarrow U'_t$ is a rectifiable path, then its length with respect to the metric μ_t is given by the formula

$$(8) \quad \ell_{\mu_t}(\eta) = \int_0^1 \mu_t(\eta(s), \eta'(s)) ds.$$

The distance between two points x and y from U'_t with respect to the metric μ_t is

$$(9) \quad d_{\mu_t}(x, y) = \inf \ell_{\mu_t}(\eta),$$

where the infimum is taken after all rectifiable paths $\eta : [0, 1] \rightarrow U'_t$ with $\eta(0) = x$ and $\eta(1) = y$.

2.5. Contraction in the space of functions. For each value of the parameter t , we will construct a sequence of equipotentials in the complement of the filled Julia set K_{p_t} and show that they converge to the Julia set J_{p_t} , uniformly with respect to t . In our setting, the filled Julia set K_{p_t} is connected. Let $\Psi_{p_t} : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C} - K_{p_t}$ be the inverse Böttcher isomorphism of the polynomial p_t as in (1) and let $\gamma_t : \mathbb{S}^1 \rightarrow J_{p_t}$ be the Carathéodory loop of p_t as in (2) (i.e., the continuous extension of Ψ_{p_t} to the boundary). We write γ_t instead of γ_{p_t} to simplify notation. By the definition of the isomorphism Ψ_{p_t} , we have $\Psi_{p_t}(z^2) = p_t(\Psi_{p_t}(z))$, for $z \in \mathbb{C} - \overline{\mathbb{D}}$.

Let $R > 2$ be a fixed constant, chosen as in equation (6). For each $t \in [0, \delta']$, consider the space of functions

$$\mathcal{F}_t = \left\{ \gamma_{t,n} : \mathbb{S}^1 \rightarrow U'_t : \gamma_{t,n}(s) = \Psi_{p_t} \left(R^{1/2^{n+1}} e^{2\pi i s} \right), n \in \mathbb{N} \right\}.$$

For each t , the space \mathcal{F}_t is just a sequence of parametrized equipotentials $\{\gamma_{t,n}\}_{n \geq 0}$, corresponding to the polynomial p_t . The Green function G_{p_t} of the polynomial p_t is continuous with respect to t and z . Therefore each map $(t, s) \mapsto \gamma_{t,n}(s)$ is continuous with respect to t and s . The polynomial p_t maps each equipotential $\gamma_{t,n}$ to the equipotential $\gamma_{t,n-1}$ by a two-to-one covering map. We can select a branch of p_t^{-1} by using the inverse Böttcher isomorphism and setting

$$p_t^{-1} \left(\Psi_{p_t} \left(R^{1/2^n} e^{2\pi i (2s)} \right) \right) := \Psi_{p_t} \left(R^{1/2^{n+1}} e^{2\pi i s} \right) \text{ for } s \in \mathbb{S}^1 \text{ and } n \geq 1.$$

Therefore, the space \mathcal{F}_t comes with a natural operator $p_t^{-1} : \mathcal{F}_t \rightarrow \mathcal{F}_t$, given by the rule

$$(10) \quad p_t^{-1}(\gamma_{t,n-1}(2s)) = \gamma_{t,n}(s), \quad s \in \mathbb{S}^1, \quad n \geq 1.$$

Endow the function space \mathcal{F}_t with the supremum metric

$$d_{\mu_t}(\gamma_{t,n}, \gamma_{t,k}) = \sup_{s \in \mathbb{S}^1} d_{\mu_t}(\gamma_{t,n}(s), \gamma_{t,k}(s))$$

and let $\overline{\mathcal{F}}_t$ be the completion of \mathcal{F}_t with respect to the supremum metric d_{μ_t} . Notice also that the metric d_{μ_t} is bounded, by Lemma 2.8.

Theorem 2.10 (Browder [Br],[KS]). *Let (X, d) be a complete metric space and suppose $f : X \rightarrow X$ satisfies*

$$d(f(x), f(y)) < h(d(x, y)) \quad \text{for all } x, y \in X,$$

where $h : [0, \infty) \rightarrow [0, \infty)$ is increasing, continuous from the right, and $h(s) < s$ for all $s > 0$. Then f has a unique fixed point x^* and $f^n(x) \rightarrow x^*$ for each $x \in X$.

Definition 2.11. We will call a function h that verifies the hypothesis of Theorem 2.10 a *Browder function*.

Remark 2.12. Assume that the space X from Theorem 2.10 is bounded. The rate of convergence to the fixed point is controlled by the function h , namely if we choose $L > 0$ such that $L - h(L) > \text{diam}(X)$, then the following estimate holds:

$$d(f^{\circ n}(x), x^*) < h^{\circ n}(L) \text{ for any } x \in X, n \in \mathbb{N}.$$

We apply Browder’s theorem to the complete metric space $(\overline{\mathcal{F}}_t, d_{\mu_t})$ with the operator p_t^{-1} , to show that for each $t \geq 0$, the sequence $\{\gamma_{t,n}\}_{n \geq 0}$ converges uniformly as $n \rightarrow \infty$ to a continuous function γ_t . As a consequence of the same Theorem 2.10 and Remark 2.12, we obtain the continuity of the map $t \rightarrow \gamma_t$ with respect to the parameter t .

For each $t \in [0, \delta']$, consider the function $h_t : [0, \infty) \rightarrow [0, \infty)$ given by

$$h_t(s) := \sup \{d_{\mu_t}(x, y) : x, y \in p_t^{-1}(U'_t) \text{ and } d_{\mu_t}(p_t(x), p_t(y)) \leq s\}.$$

Clearly the function h_t is increasing (i.e., $s_1 < s_2 \Rightarrow h_t(s_1) \leq h_t(s_2)$) and satisfies the inequality

$$d_{\mu_t}(\gamma_{t,n+1}, \gamma_{t,k+1}) < h_t(d_{\mu_t}(\gamma_{t,n}, \gamma_{t,k})), \quad k, n \in \mathbb{N}.$$

The family of metrics μ_t is continuous with respect to t , so the map $t \rightarrow h_t$ is continuous with respect to $t \in [0, \delta']$. Moreover, by Lemma 2.9, when $t > 0$ we have

$$(11) \quad h_t(s) < \frac{s}{k_t} \quad \text{for } s > 0.$$

Inequality (11) implies that p_t^{-1} is strictly contracting with respect to the metric d_{μ_t} when $t > 0$. Banach Fixed Point Theorem assures that for each $t > 0$, the operator p_t^{-1} has a unique fixed point $\gamma_t : \mathbb{S}^1 \rightarrow \overline{U'_t}$, and the sequence $\gamma_{t,n}$ converges to γ_t as $n \rightarrow \infty$.

In [DH] (and also in [H]) it is shown that the function h_0 verifies the hypothesis of Theorem 2.10, hence the sequence $\gamma_{0,n}$ converges to the unique fixed point of the operator $p_0^{-1} : \mathcal{F}_0 \rightarrow \mathcal{F}_0$, that is, to a continuous function $\gamma_0 : \mathbb{S}^1 \rightarrow \overline{U'_0}$. The image of γ_0 is invariant under the parabolic polynomial p_0 and it parametrizes its Julia set J_{p_0} .

Notice that the constant k_t goes to 1 when t goes to 0, so we haven’t obtained any information yet about the continuity of the map $t \rightarrow \gamma_t$ with respect to t when $t = 0$.

To provide a unified approach to the hyperbolic and parabolic cases, we define a new map $h : [0, \infty) \rightarrow [0, \infty)$, $h(s) := \sup_{t \in [0, \delta']} h_t(s)$.

Lemma 2.13. *The map h is increasing and $h(s) < s$ for all $s > 0$.*

Proof. When $t = 0$, it is proven in [DH] and [H] that $h_0(s) < s$ for all $s > 0$. When $t > 0$, inequality (11) yields that $h_t(s) < s$ for all $s > 0$. For a fixed $s \in \mathbb{R}^+$, the map $t \rightarrow h_t(s)$ is continuous with respect to t , thus it attains its supremum on $[0, \delta']$, so $h(s) < s$. For each t , the function h_t is increasing, by definition. It is obvious then that the function h is also increasing. □

The function h is increasing, so $h(s+) = \lim_{\epsilon \rightarrow 0^+} h(s + \epsilon)$ is well defined. The function $h^+ : s \mapsto h(s+)$ is right continuous and the following lemma holds:

Lemma 2.14. *The function $h^+ : [0, \infty) \rightarrow [0, \infty)$ is a Browder function, i.e., it is right continuous, increasing, and $h^+(s) < s$ for all $s > 0$. Moreover*

$$d_{\mu_t}(\gamma_{t,n+1}, \gamma_{t,k+1}) < h^+(d_{\mu_t}(\gamma_{t,n}, \gamma_{t,k})) \quad \text{for all } t \in [0, \delta'] \text{ and } k, n \in \mathbb{N}.$$

Proof. The only non-trivial property to check is the fact that $h(s+) < s$ for all $s > 0$. By Lemma 2.13 we know that $h(s) < s$ for all $s > 0$, so $h(s+) \leq s$ for all $s > 0$.

Suppose that $h(s+) = s$ for some $s > 0$. Let $\epsilon_n \searrow 0$ be a decreasing sequence of positive numbers such that $h(s+) = \lim_{\epsilon_n \rightarrow 0} h(s + \epsilon_n)$. From the definition of h we have

$$h(s + \epsilon_n) = \sup_{t \in [0, \delta'] } h_t(s + \epsilon_n).$$

From the definition of the supremum, for every $n > 0$ there exists $t_n \in [0, \delta']$ such that

$$(12) \quad h_{t_n}(s + \epsilon_n) > \sup_{t \in [0, \delta'] } h_t(s + \epsilon_n) - \epsilon_n = h(s + \epsilon_n) - \epsilon_n.$$

The function h_{t_n} satisfies

$$(13) \quad h_{t_n}(s + \epsilon_n) < (s + \epsilon_n) \cdot \frac{1}{k_{t_n}} < s + \epsilon_n.$$

The sequence t_n is bounded, so after passing to a convergent subsequence, we may assume that $t_n \rightarrow \tau$ for some $\tau \in [0, \delta']$. Let us show that $\tau = 0$. Assume that $\tau \neq 0$. From inequalities (12) and (13) we know that

$$h(s + \epsilon_n) - \epsilon_n < h_{t_n}(s + \epsilon_n) < \frac{1}{k_{t_n}} \cdot (s + \epsilon_n).$$

Taking the limit as $n \rightarrow \infty$, we get

$$s = h(s+) \leq \frac{1}{k_\tau} \cdot s.$$

Since $s > 0$ we get $k_\tau \leq 1$. Then $\tau = 0$, otherwise we would have $k_\tau > 1$.

Pick m an integer, and let n be any integer $n \geq m$. The sequence ϵ_n is decreasing, so $\epsilon_n < \epsilon_m$. The function h_{t_n} is increasing, so

$$(14) \quad h_{t_n}(s + \epsilon_n) \leq h_{t_n}(s + \epsilon_m) \text{ for any } n \geq m.$$

From the inequalities (12) and (14) we obtain

$$h(s + \epsilon_n) - \epsilon_n < h_{t_n}(s + \epsilon_m) \text{ for any } n \geq m.$$

After passing to the limit as $n \rightarrow \infty$ and using the continuity of h_t with respect to t , we get

$$s = h(s+) \leq h_0(s + \epsilon_m) \text{ for every } m \in \mathbb{N}.$$

Letting $m \rightarrow \infty$ we get $s \leq h_0(s+)$. In the parabolic case $h_0(s+) < s$ for every $s > 0$. This yields $s = 0$, which is a contradiction. \square

Remark 2.15. Lemma 2.14 is very important, because it provides a *reduction of the hyperbolic case to the parabolic case*, hence allowing a uniform treatment of both cases. Lemma 2.14 uses only minimum information about the parabolic case, that is, the fact that $h_0(s+) < s$ for all $s > 0$. Another remark is that in the one-dimensional setting, one could presumably prove that the maps h_t are already right continuous, so $h_t(s) = h_t(s+)$. However, we will apply this lemma in higher dimensions (where the maps h_t will not necessarily be right continuous), so the existence of the Browder function h^+ from Lemma 2.14 bypasses this problem and is central for the application of Browder’s Fixed Point Theorem.

Theorem 2.16. *For each t , the sequence $\gamma_{t,n}$ converges to a fixed point $\gamma_t : \mathbb{S}^1 \rightarrow \overline{U}'_t$ of the operator p_t^{-1} . The rate of convergence to the fixed point is uniform in t .*

Proof. By Lemma 2.14, we can use the same Browder function h^+ for any parameter $t \in [0, \delta']$ and apply Theorem 2.10 to show the existence of a unique fixed point γ_t for the operator p_t^{-1} . Each map $\gamma_t : \mathbb{S}^1 \rightarrow \overline{U_t}$ is continuous.

We show that $\gamma_{t,n}$ converges to γ_t uniformly with respect to $t \in [0, \delta']$. By Lemmas 2.14 and 2.8 and Remark 2.12 there exists $L > 0$ such that

$$m_1 \|\gamma_{t,n} - \gamma_t\| < d_{\mu_t}(\gamma_{t,n}, \gamma_t) \leq (h^+)^{\circ n}(L) \searrow 0 \text{ for any } t \in [0, \delta'],$$

so the rate of convergence of $\gamma_{t,n}$ to the fixed point γ_t is bounded by the rate at which the sequence $(h^+)^{\circ n}(L)$ decreases to 0. Therefore the sequence of functions $t \rightarrow \gamma_{t,n}$ converges as $n \rightarrow \infty$ to γ_t , uniformly with respect to t . \square

In Theorem 2.16, we have constructed a sequence of functions $\gamma_{t,n}$, continuous with respect to t , and proved that it converges uniformly as $n \rightarrow \infty$ to γ_t . Hence the limit function γ_t is continuous with respect to t on $[0, \delta]$. The continuity of the Julia sets J_{p_t} from Theorem 2.5 follows immediately as $J_{p_t} = \text{Im}(\gamma_t)$.

3. CONTINUITY AND STABILITY OF JULIA SETS FOR HÉNON MAPS

Inspired by the one-dimensional setting from Chapter 2, we now turn back to dynamics in two complex variables and prove a continuity result in Theorem 1.1 for Hénon maps with a semi-parabolic fixed point. We will consider the Hénon map written in the form

$$H_{c,a}(x, y) = (p(x) + ay, ax), \text{ where } p(x) = x^2 + c.$$

When $a \neq 0$, this map is a biholomorphism of constant Jacobian $-a^2$, whose inverse is

$$H_{c,a}^{-1}(x, y) = (y/a, (x - p(y/a))/a).$$

As in [HOV1], for $r > 0$ large enough, the dynamical space \mathbb{C}^2 can be divided into three regions: the bidisk $\mathbb{D}_r \times \mathbb{D}_r = \{(x, y) : |x| \leq r, |y| \leq r\}$,

$$(15) \quad V^+ = \{(x, y) : |x| \geq \max(|y|, r)\} \text{ and } V^- = \{(x, y) : |y| \geq \max(|x|, r)\}.$$

The escaping sets U^\pm can be described in terms of V^\pm as follows:

$$U^+ = \bigcup_{k \geq 0} H^{-\circ k}(V^+)$$

and $U^- = \bigcup_{k \geq 0} H^{\circ k}(V^-)$. By taking their complements in \mathbb{C}^2 we obtain $K^+ = \mathbb{C}^2 - U^+$, the set of points that do not escape to infinity in forward time, and $K^- = \mathbb{C}^2 - U^-$, the set of points that do not escape to infinity in backward time. The Julia set J^+ is the common boundary of K^+ and U^+ . Similarly J^- is the common boundary of K^- and U^- . In fact, in the dissipative case, $J^- = K^-$ (see [FM]). The sets $J = J^+ \cap J^-$ and $K = K^+ \cap K^-$ are contained in $\mathbb{D}_r \times \mathbb{D}_r$.

Definition 3.1. Let \mathbf{q} be a fixed point of H and λ and ν be the two eigenvalues of $DH_{\mathbf{q}}$. The fixed point \mathbf{q} is called:

- (a) *hyperbolic* if $|\nu| < 1$ and $|\lambda| > 1$;
- (b) *semi-parabolic* if $|\nu| < 1$ and $\lambda = e^{2\pi ip/q}$;
- (c) *attracting* if $|\nu| < 1$ and $|\lambda| < 1$.

Let $\lambda = e^{2\pi ip/q}$ be a root of unity of order q and set $\lambda_t := (1 + t)\lambda$. In Section 2.2 we considered the family p_t of polynomials, $p_t(x) = x^2 + c_t$, with a fixed point $\alpha_t = \lambda_t/2$ of multiplier λ_t . The exact formula for the coefficient c_t is given by Proposition 3.2 below. In Theorem 2.5 we showed that the Julia sets J_{p_t} converge to the Julia set J_{p_0} as $t \rightarrow 0$.

In 1-D, the multiplier of a fixed point of a quadratic polynomial uniquely identifies the polynomial. The following proposition provides a description of the parameter space of Hénon maps for which one eigenvalue of the fixed point is known.

Proposition 3.2. *The set \mathcal{P}_{λ_t} of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a fixed point with one eigenvalue λ_t is a curve of equation*

$$(16) \quad c = (1 - a^2) \left(\frac{\lambda_t}{2} - \frac{a^2}{2\lambda_t} \right) - \left(\frac{\lambda_t}{2} - \frac{a^2}{2\lambda_t} \right)^2.$$

Notation and conventions. The curve \mathcal{P}_{λ_t} has degree 4 in the variable a , and degree 2 as a function of the Jacobian, which is $-a^2$. For this reason, we will sometimes call the curves \mathcal{P}_{λ_t} complex parabolas. When $t = 0$, the curve \mathcal{P}_{λ} contains the Hénon maps that have a semi-parabolic fixed point with one eigenvalue λ , a root of unity.

For the rest of the paper, we denote by $c_t(a)$ the right hand side of equation (16). The Hénon map is completely determined by the choice of a and t , so we will use $H_{a,t}$ in place of $H_{c_t(a),a}$ when there is no danger of confusion. We write

$$(17) \quad H_{a,t}(x, y) = (x^2 + c_t(a) + ay, ax) = (x^2 + c_t + a^2w + ay, ax),$$

where the residual term w is bounded and depends only on a and λ_t ,

$$w := \frac{-1 + \lambda_t - \lambda_t^2}{2\lambda_t} + \frac{a^2}{2\lambda_t} \left(1 - \frac{1}{2\lambda_t} \right).$$

The Hénon map is also determined by the eigenvalues λ and ν at a fixed point and we will sometimes write $H_{\lambda,\nu}$ in place of $H_{c,a}$ to stress this dependency. The formula for $H_{\lambda,\nu}$ is the following:

$$H_{\lambda,\nu}(x, y) = \left(x^2 + (\lambda + \nu)(2 + 2\lambda\nu - \lambda - \nu)/4 \pm i\sqrt{\lambda\nu}y, \pm i\sqrt{\lambda\nu}x \right).$$

It may seem that there are two choices, but they are in fact conjugated by the affine change of variables $(x, y) \mapsto (x, -y)$.

A simple analysis shows that any constant $r > 3$ works in the definition of sets V^\pm from (15) for the whole family of Hénon maps $H_{a,t}$ for $|a|$ and $|t|$ small. From now on, we assume that $r > 3$ is a fixed constant. Moreover, we assume that $|t| < 1/(2q)$ and $|a| < 1/2$ as minimal requirements and we will specify other restrictions when necessary.

Let $\mathbf{q}_{a,t}$ denote the fixed point of $H_{a,t}$ which has one eigenvalue λ_t . Suppose $|a|$ and $|t|$ are sufficiently small. We will see that the following bifurcation occurs:

- (a) if $t = 0$, then $H_{a,0}$ has a semi-parabolic fixed point $\mathbf{q}_{a,0}$ of multiplicity $q + 1$;
- (b) if $t > 0$, then $H_{a,t}$ has a hyperbolic fixed point $\mathbf{q}_{a,t}$ and a q -periodic attractive orbit;
- (c) if $t < 0$, then $H_{a,t}$ has an attracting fixed point $\mathbf{q}_{a,t}$ and a q -periodic hyperbolic cycle.

In the degenerate case $a = 0$, the fixed point $\mathbf{q}_{0,t}$ is $(\alpha_t, 0)$, where α_t is the fixed point of the polynomial p_t . The Hénon maps become $H_{0,t}(x, y) = (p_t(x), 0)$. The

Julia set of the Hénon map $H_{0,t}$ is just the Julia set of the polynomial p_t , so we write $J_{0,t} = J_{p_t}$. Moreover $J_{0,t}^+ = J_{p_t} \times \mathbb{C}$.

3.1. Local dynamics – perturbed normal forms. In this section we give a normal form for perturbations of semi-parabolic germs with semi-parabolic multiplicity 1. This provides an analogue of Proposition 2.6 for the two-dimensional setting. Proposition 3.4 is more general. Theorem 3.5 is specialized to the case of Hénon maps $H_{a,t}$ that come from perturbations of the family of polynomials p_t discussed in the previous chapter.

Definition 3.3. Let H be a holomorphic germ of $(\mathbb{C}^2, \mathbf{q})$ whose eigenvalues at the fixed point \mathbf{q} are λ and ν , with $0 < |\nu| < 1$ and $|\nu| < |\lambda|$. The strong stable manifold of the fixed point \mathbf{q} corresponding to the eigenvalue ν is

$$(18) \quad W^{ss}(\mathbf{q}) = \{z \in \mathbb{C}^2 : \lim_{n \rightarrow \infty} |\nu|^{-n} \text{dist}(H^{\circ n}(z), \mathbf{q}) = \text{const.}\}.$$

We refer to [S] and [MNTU] for a consistent treatment of strong stable manifolds. For the Hénon map $H_{a,t}$, the strong stable manifold $W^{ss}(\mathbf{q}_{a,t})$ contains points that get attracted to the fixed point $\mathbf{q}_{a,t}$ at an exponential rate $|\nu_{a,t}|^n$. When $\mathbf{q}_{a,t}$ is hyperbolic or semi-parabolic, $W^{ss}(\mathbf{q}_{a,t})$ lives in J^+ . When the fixed point is attracting, or semi-Siegel, $W^{ss}(\mathbf{q}_{a,t})$ belongs to the interior of K^+ . Note also that when $a = 0$, $W^{ss}(\mathbf{q}_{0,t}) = \mathbf{q}_{0,t} \times \mathbb{C}$.

We will denote by $W_{loc}^{ss}(\mathbf{q}_{a,t})$ the local strong stable manifold of $\mathbf{q}_{a,t}$ relative to the polydisk $\mathbb{D}_r \times \mathbb{D}_r$ that is, the connected component of $W^{ss}(\mathbf{q}_{a,t}) \cap \mathbb{D}_r \times \mathbb{D}_r$ that contains the point $\mathbf{q}_{a,t}$.

Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with an isolated fixed point at 0 with eigenvalues $|\nu| < 1$ and $\lambda = e^{2\pi ip/q}$. The multiplicity of 0 as a solution of the equation $H^{\circ q}(x) = x$ is a number congruent to 1 modulo q . Suppose therefore that $x = 0$ is a fixed point of $H^{\circ q}$ of multiplicity $mq + 1$; we call m the *semi-parabolic multiplicity* of H .

Proposition 3.4. Let $\{H_t\}_{|t| < \delta'}$ be an analytic family of germs of diffeomorphisms of $(\mathbb{C}^2, 0)$ whose eigenvalues at 0 are $\lambda_t = (1 + t)\lambda$ and ν_t , with $|\nu_t| < \min(1, |\lambda_t|)$ and $|\nu_t||\lambda_t|^{2q} < 1$. If the semi-parabolic multiplicity of H_0 is 1, then there exist local coordinates (x, y) in which H_t has the form $H_t(x, y) = (x_1, y_1)$, with

$$(19) \quad \begin{cases} x_1 = \lambda_t(x + x^{q+1} + Cx^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots), \\ y_1 = \nu_t y + xh(x, y), \end{cases}$$

where C is a constant depending on t , and $a_i(\cdot)$ and $h(\cdot, \cdot)$ are germs of holomorphic functions from $(\mathbb{C}, 0)$ to \mathbb{C} , respectively from $(\mathbb{C}^2, 0)$ to \mathbb{C} , such that $a_1(0) = \lambda_t$ and $h(0, 0) = 0$. The coordinate transformations depend smoothly on t .

Proof. The case $\lambda = 1$ and $t = 0$ was proved by Ueda [U] and Hakim [Ha]. In [RT, Proposition 3.3], this was stated for any primitive root of unity $\lambda = e^{2\pi ip/q}$ and $t = 0$.

By straightening the local strong stable manifold of the fixed point 0 we can assume that H_t is written in the form:

$$(20) \quad \begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots, \\ y_1 = \nu_t y + xh(x, y), \end{cases}$$

where $a_j(\cdot)$ and $h(\cdot, \cdot)$ are holomorphic functions with $a_1(0) = \lambda_t$ and $h(0, 0) = 0$.

One can make a holomorphic change of coordinates to make the first $2q + 1$ coefficients of the power series in the first coordinate constants. We proceed in three steps. In the first two steps we show that there exist local coordinates (x, y) in which the map H_t has the form:

$$(21) \quad \begin{cases} x_1 = \lambda_t x + a_2 x^2 + \dots + a_{2q+1} x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \dots, \\ y_1 = \nu_t y + x h(x, y), \end{cases}$$

where a_2, \dots, a_{2q+1} are constants. In the third step we show how to eliminate the terms $a_k x^k$ for $2 \leq k \leq 2q + 1$, k not congruent to 1 modulo q , and obtain equation (19).

(1) Reduction to $a_1(y) = \lambda_t$. Consider as in [Ha] and [U] a coordinate transformation

$$\begin{cases} X = u(y)x \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X/u(Y), \\ y = Y, \end{cases}$$

where u is a germ of analytic functions from $(\mathbb{C}, 0)$ to \mathbb{C} with $u(0) = \lambda_t$. We need to find u such that

$$\begin{aligned} X_1 &= u(y_1)x_1 = u(\nu_t y + x h(x, y)) (a_1(y)x + a_2(y)x^2 + \dots) \\ &= u(\nu_t Y + X/u(Y)h(X/u(Y), Y)) (a_1(Y)X/u(Y) + a_2(Y)(X/u(Y))^2 + \dots) \\ &= \frac{u(\nu_t Y)a_1(Y)}{u(Y)}X + \mathcal{O}(X^2) = \lambda_t X + \mathcal{O}(X^2). \end{aligned}$$

Let $b_1(Y) = a_1(Y)/\lambda_t$. The map u satisfies the equation $u(Y) = u(\nu_t Y)b_1(Y)$. We successively substitute $\nu_t Y$ instead of Y in this equation and obtain the unique solution

$$u(Y) = \prod_{n=0}^{\infty} b_1(\nu_t^n Y).$$

This product converges in a neighborhood of 0 since $|\nu_t| < 1$ and $b_1(Y) = 1 + \mathcal{O}(Y)$.

(2) Reduction to $a_k(y)$ constants for $2 \leq k \leq 2q + 1$. We proceed by induction on k . The base case $k = 1$ was discussed above. Suppose that $k \geq 2$ and that there exist local coordinates (x, y) in which H_t has the form

$$\begin{cases} x_1 = \lambda_t x + a_2 x^2 + \dots + a_{k-1} x^{k-1} + a_k(y) x^k + \dots, \\ y_1 = \nu_t y + x h(x, y), \end{cases}$$

with a_2, \dots, a_{k-1} constant. We would like to find local coordinates so that $a_k(y)$ is also constant. Consider the transformation

$$\begin{cases} X = x + v(y)x^k \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - v(Y)X^k + \dots, \\ y = Y, \end{cases}$$

where v is a germ of analytic functions from $(\mathbb{C}, 0)$ to \mathbb{C} with $v(0) = 0$. Using the coordinates given by this transformation we get

$$\begin{aligned} X_1 &= x_1 + v(y_1)x_1^k \\ &= \lambda_t x + a_2 x^2 + \dots + a_{k-1} x^{k-1} + (a_k(y) + \lambda_t^k v(\nu_t y)) x^k + \mathcal{O}(x^{k+1}) \\ &= \lambda_t X + \dots + a_{k-1} X^{k-1} + (a_k(Y) + \lambda_t^k v(\nu_t Y) - \lambda_t v(Y)) X^k + \mathcal{O}(X^{k+1}). \end{aligned}$$

We need v such that the coefficient of X^k is constant. This gives the functional equation $\lambda_t v(Y) - \lambda_t^k v(\nu_t Y) = a_k(Y) - a_k(0)$. We successively substitute $\nu_t Y$

instead of Y in this equation and obtain

$$\lambda_t v(Y) = \sum_{n=0}^{\infty} (a_k (\nu_t^n Y) - a_k(0)) \lambda_t^{n(k-1)}.$$

The series converges in a neighborhood of 0 if $|\nu_t| |\lambda_t|^{k-1} < 1$. This is clearly achieved when $|\lambda_t| \leq 1$ since $|\nu_t| < 1$. If $|\lambda_t| > 1$, then $|\nu_t| |\lambda_t|^{k-1} \leq |\nu_t| |\lambda_t|^{2q} < 1$ and the latter inequality is true by hypothesis. Therefore H_t can be written as in equation (21).

(3) Suppose $2 \leq k \leq 2q + 1$ is not congruent to 1 modulo q . Assume by induction on k that H_t can be written as

$$\begin{cases} x_1 = \lambda_t x + a_k x^k + \dots + a_{2q+1} x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \dots, \\ y_1 = \nu_t y + x h(x, y). \end{cases}$$

Let $b = \frac{a_k}{\lambda_t - \lambda_t^k}$ and consider the coordinate transformation

$$(22) \quad \begin{cases} X = x + b x^k \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - b X^k + \dots, \\ y = Y. \end{cases}$$

In the new coordinate system, we get

$$\begin{aligned} X_1 &= x_1 + b x_1^k = (\lambda_t x + a_k x^k + \dots) + b(\lambda_t x + a_k x^k + \dots)^k \\ &= \lambda_t x + (a_k + b \lambda_t^k) x^k + \dots \\ &= \lambda_t (X - b X^k + \dots) + (a_k + b \lambda_t^k) (X - b X^k + \dots)^k + \dots \\ &= \lambda_t X + (a_k + b(\lambda_t^k - \lambda_t)) X^k + \dots \end{aligned}$$

By this procedure, the term containing X^k has been eliminated. The first monomial that cannot be eliminated in this way will be $a_{mq+1} X^{mq+1}$ for some integer m . If we assume that the parabolic multiplicity of the semi-parabolic germ is 1, then $a_{q+1} \neq 0$ for $t = 0$ (hence also for small t) and $a_{q+1} X^{q+1}$ will be the first term that we cannot eliminate by the above procedure. We can further reduce the normal form to $a_{q+1} = 1$ by considering a linear transformation of the form $X = Ax, Y = y$, where A is a constant such that $A^q = a_{q+1}$. We can therefore assume that the Hénon map can be written as

$$\begin{cases} x_1 = \lambda_t (x + x^{q+1} + a_k x^k + \dots + a_{2q+1} x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \dots), \\ y_1 = \mu y + x h(x, y). \end{cases}$$

By repeating the coordinate transformations (22), we can eliminate all monomials $a_k x^k$ with $q + 1 < k < 2q + 1$. By abuse of notation we still denote by a_k the term $\lambda_t a_k$. In the new coordinate system, we get

$$\begin{aligned} X_1 &= x_1 + b x_1^k = (\lambda_t (x + x^{q+1}) + a_k x^k + \dots) + b(\lambda_t (x + x^{q+1}) + a_k x^k + \dots)^k \\ &= \lambda_t (x + x^{q+1}) + (a_k + b \lambda_t^k) x^k + \dots \\ &= \lambda_t (X - b X^k + (X - b X^k)^{q+1} + \dots) + (a_k + b \lambda_t^k) (X - b X^k + \dots)^k + \dots \\ &= \lambda_t (X + X^{q+1}) + (a_k + b(\lambda_t^k - \lambda_t)) X^k + \dots \end{aligned}$$

therefore the term containing X^k has been eliminated. □

This following theorem is a generalization of [RT, Theorem 6.2].

Theorem 3.5. *Let $r > 3$ be a fixed constant. There exist $\delta, \delta' > 0$ such that for any $(c, a) \in \mathcal{P}_{\lambda_t}$ with $|a| < \delta$ and $|t| < \delta'$ there exists a coordinate transformation $\phi_{a,t}$ from a tubular neighborhood $B = \mathbb{D}_{\rho'}(\alpha_t) \times \mathbb{D}_r$ of the local strong stable manifold of the fixed point $\mathbf{q}_{a,t}$*

$$\phi_{a,t} : B \rightarrow \mathbb{D}_\rho \times \mathbb{D}_{r+\mathcal{O}(|a|)}$$

in which $H_{a,t}$ has the form $\tilde{H}_{a,t}(x, y) = (x_1, y_1)$, with

$$(23) \quad \begin{cases} x_1 = \lambda_t(x + x^{q+1} + C_{a,t}x^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots), \\ y_1 = \nu_{a,t}y + xh_{a,t}(x, y), \end{cases}$$

and $C_{a,t}$ is a constant (depending on a and t) and $xh_{a,t}(x, y) = \mathcal{O}(a)$. Moreover the transformations $\phi_{a,t}$ are analytic in a and t , and

$$\lim_{a \rightarrow 0} \phi_{a,t} = (\phi_t(x), y),$$

uniformly with respect to t . The map $\phi_t : \mathbb{D}_{\rho'}(\alpha_t) \rightarrow \mathbb{D}_\rho$ is the change of coordinates from Proposition 2.6 for the polynomial p_t with a fixed point α_t of multiplier λ_t and

$$\phi_t \circ p_t \circ \phi_t^{-1}(x) = \lambda_t(x + x^{q+1} + C_t x^{2q+1} + \mathcal{O}(x^{2q+2})).$$

Proof. We choose δ small enough so that the local strong stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,t})$ has no foldings inside $\mathbb{D}_r \times \mathbb{D}_r$ and it can therefore be straightened using a holomorphic change of coordinates. The first part of the proof follows directly from Proposition 3.4. We need to verify the conditions imposed on the eigenvalues at the fixed point. We have $|\lambda_t| \in (1 - \delta', 1 + \delta')$ and $|\nu_{a,t}| = |a|^2/|\lambda_t| \in [0, \delta^2/(1 - \delta')]$. The bounds δ and δ' are chosen small enough so that $\delta \ll 1 - \delta'$. It follows that $|\nu_{a,t}| \ll |\lambda_t|$ and $|\nu_{a,t}| < 1$. Then $|\lambda_t||\nu_{a,t}|^{2q} < (1 + \delta')\delta^{4q}(1 - \delta')^{-2q} < 1$. This inequality is not so restrictive. For example, it is verified for $\delta' < 1/(2q)$ and $\delta < 1/2$. The convergence of the coordinate transformation $\phi_{a,t}$ as $a \rightarrow 0$ follows immediately by comparing the coordinate transformations done in Proposition 3.4 to those done in Proposition 2.6. □

It is also worth mentioning that the change of coordinates function $\phi_{a,t}$ from Theorem 3.5 maps horizontal curves to horizontal curves, that is,

$$(24) \quad \phi_{a,t}(\mathbb{D}_{\rho'}(\alpha_t) \times \{y_1\}) \subset \mathbb{C} \times \{y_2\},$$

which will be useful later on.

3.2. Attracting and repelling sectors. In this section we continue the analysis of the local dynamics of holomorphic germs of diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-parabolic fixed point and their nearby perturbations. Consider the set

$$\Delta_R = \left\{ x \in \mathbb{C} : \left(\operatorname{Re}(x^q) + \frac{1}{2R} \right)^2 + \left(|\operatorname{Im}(x^q)| - \frac{1}{2R} \right)^2 < \frac{1}{2R^2} \right\}$$

in the complex plane. There are q connected components of Δ_R , which we denote $\Delta_{R,j}$, for $1 \leq j \leq q$. Define $\mathcal{P}_{R,r} = \Delta_R \times \mathbb{D}_r$ and let $\mathcal{P}_{R,r,j} = \Delta_{R,j} \times \mathbb{D}_r$ be the connected components of $\mathcal{P}_{R,r}$.

Proposition 3.6. *For R large enough and r small enough there exists a positive number δ' such that for all $t \in (-\delta', \delta')$*

$$H_{a,t}(\overline{\mathcal{P}}_{R,r,j}) \subset \mathcal{P}_{R,r,j+p} \cup \{0\} \times \mathbb{D}_r \quad \text{for } 1 \leq j \leq q.$$

In particular $H_{a,t}(\overline{\mathcal{P}}_{R,r}) \subset \mathcal{P}_{R,r} \cup \{0\} \times \mathbb{D}_r$.

Proof. Assume that R is large enough and r is small enough so that the map $H_{a,t}$ is well defined and has the expansion from Theorem 3.5:

$$\begin{cases} x_1 = \lambda_t(x + x^{q+1} + C_{a,t}x^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots), \\ y_1 = \nu_{a,t}y + xh(x, y). \end{cases}$$

Define the region $U_R := \{X \in \mathbb{C} \mid R/q - \operatorname{Re}(X) < |\operatorname{Im}(X)|\}$ and set

$$W_{R,r} := U_R \times \mathbb{D}_r.$$

Suppose $(x, y) \in \mathcal{P}_{R,r,j}$. The transformation $X = -1/(qx^q)$, $Y = y$ maps each petal $\mathcal{P}_{R,r,j}$ to $W_{R,r}$. Thus $X \in U_R$ and $|Y| < r$. Let $\hat{H}(X, Y) = (X_1, Y_1)$ be the corresponding map in the new coordinates:

$$\begin{aligned} X_1 &= -\frac{1}{qx_1^q} = \frac{X}{\lambda_t^q(1 + x^q + C_{a,t}x^{2q} + a_{2q+2}(y)x^{2q+1} + \dots)^q} \\ &= \frac{X}{\lambda_t^q} \left(1 - q(x^q + C_{a,t}x^{2q} + \dots) + \frac{q(q+1)}{2}x^{2q} + \dots \right) \\ &= \frac{1}{\lambda_t^q} \left(X + 1 + \frac{A}{X} + \mathcal{O}_Y \left(\frac{1}{|X|^{1+1/q}} \right) \right), \text{ where } A := \frac{1}{q} \left(\frac{q+1}{2} - C_{a,t} \right); \\ Y_1 &= \nu_{a,t}y + xh(x, y) = \nu_{a,t}Y + \mathcal{O}_Y \left(\frac{1}{|X|^{1/q}} \right). \end{aligned}$$

The notation $\mathcal{O}_Y(|X|^\alpha)$ represents a holomorphic function of (X, Y) in $W_{R,r}$ which is bounded by $K|X|^\alpha$ for some constant K .

One can check that $|X| > \frac{R}{q\sqrt{2}}$ throughout the region U_R . Clearly $|\lambda_t|^q > 1/2$ for small $|t|$. There exist constants K', K'' and $K_1 = 2K'q\sqrt{2}$, $K_2 = K''(q\sqrt{2})^{1/q}$ such that

$$\begin{aligned} \left| X_1 - \frac{1}{\lambda_t^q}(X + 1) \right| &\leq \frac{K'}{|\lambda_t|^q|X|} < \frac{K_1}{R}, \\ |Y_1 - \nu_{a,t}Y| &\leq \frac{K''}{|X|^{1/q}} < \frac{K_2}{R^{1/q}}. \end{aligned}$$

Choose R large enough and r small enough so that

$$(25) \quad \begin{cases} \frac{K_1}{R} < \frac{1}{4}, \\ \frac{K_2}{R^{1/q}} < (1 - |\nu_{a,t}|)r. \end{cases}$$

The second condition immediately gives

$$|Y_1| \leq |Y_1 - \nu_{a,t}Y| + |\nu_{a,t}||Y| < \frac{K_2}{R^{1/q}} + |\nu_{a,t}|r < r.$$

The first condition of (25) implies that

$$(26) \quad \left| X_1 - \frac{1}{\lambda_t^q}(X + 1) \right| < \frac{1}{4}.$$

In our case $\lambda_t^q = (1 \pm t)^q$ is a real positive number. Hence inequality (26) yields the following estimates:

$$\begin{aligned} \operatorname{Re}(X_1) &> \frac{1}{(1 \pm t)^q} \operatorname{Re}(X) + \frac{1}{(1 \pm t)^q} - \frac{1}{4}, \\ |\operatorname{Im}(X_1)| &> \frac{1}{(1 \pm t)^q} |\operatorname{Im}(X)| - \frac{1}{4}. \end{aligned}$$

Using these estimates and the fact that $X \in U_R$, we get

$$|Im(X_1)| > \frac{R/q + 1}{(1 \pm t)^q} - \frac{1}{2} - Re(X_1) > R/q - Re(X_1),$$

provided that $|t|$ is small enough so that

$$(27) \quad \frac{R + q}{R + q/2} > (1 \pm t)^q.$$

The constant $\delta' > 0$ is chosen so that this inequality holds for all $0 \leq t < \delta'$. It follows that $\widehat{H}(\overline{W}_{R,r}) \subset W_{R,r}$. □

Since the dynamics are different, we will treat the cases $t < 0$ and $t > 0$ separately. We begin with the latter case, so suppose R, r and $t > 0$ are as in Proposition 3.6. We show that points in $\mathcal{P}_{R,r}$ are attracted by an attractive orbit of period q under iterations by $H_{a,t}$. Each region $\mathcal{P}_{R,r,j}$ contains a point of this orbit. The fixed point 0 is hyperbolic.

Suppose $\rho_t > 0$ is a small enough radius such that $\rho_t \leq \sqrt[q]{t/(2q)}$. The number ρ_t is just a local variable which will be used in the proposition below. Define

$$\mathcal{D}_{R,r,t} = \mathcal{P}_{R,r} - \{(x, y) \in \mathbb{C}^2 : |x|^q \leq \rho_t, |y| < r\}$$

and let $\mathcal{D}_{R,r,t,j}$ with $1 \leq j \leq q$, be the connected components of $\mathcal{D}_{R,r,t}$ (see Figure 3).

Proposition 3.7 (Trapping regions – t positive). *For R large enough and r small enough there exists a positive number δ' such that for all $t \in (0, \delta')$*

$$H_{a,t}(\overline{\mathcal{D}}_{R,r,t,j}) \subset \mathcal{D}_{R,r,t,j+p} \quad \text{for } 1 \leq j \leq q.$$

In particular $H_{a,t}(\overline{\mathcal{D}}_{R,r,t}) \subset \mathcal{D}_{R,r,t}$ and all points of $\mathcal{D}_{R,r,t}$ are attracted to an attractive orbit of period q under iterations by $H_{a,t}$.

Proof. We make a change of variables $X = -1/(qx^q)$, $Y = y$ and analyze the situation at infinity. This transformation maps each component $\mathcal{D}_{R,r,t,j}$ to a region $W_{R,r,t} := U_{R,t} \times \mathbb{D}_r$, where

$$U_{R,t} := \left\{ X \in \mathbb{C} : R/q - Re(X) < |Im(X)| \text{ and } |X| < \frac{1}{q\rho_t^q} \right\}.$$

From equation (26) from the proof of the previous proposition we have

$$|X_1| - \frac{1}{(1+t)^q}|X| - \frac{1}{(1+t)^q} \leq \left| X_1 - \frac{1}{(1+t)^q}(X+1) \right| < \frac{1}{4}$$

which gives

$$|X_1| < \frac{1}{q(1+t)^q\rho_t^q} + \frac{1}{(1+t)^q} + \frac{1}{4} < \frac{1}{q\rho_t^q}.$$

The last inequality holds because

$$q\rho_t^q < \frac{t}{2} < \frac{(1+t)^q - 1}{(1+t)^q/4 + 1},$$

based on our assumption on t and our choice of ρ_t . With this choice we showed that $\widehat{H}(\overline{W}_{R,r,t}) \subset W_{R,r,t}$ and all points of $W_{R,r,t}$ are attracted to an attractive orbit under iterations by \widehat{H} . The existence of this orbit follows immediately since we have a nested intersection of compact sets. □

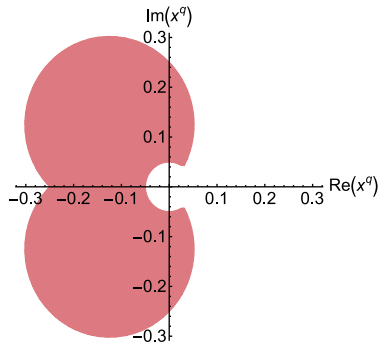


FIGURE 3. (case $t > 0$) The image of $\mathcal{D}_{R,r,t}$ under the map $x \mapsto x^q$ at a height $y = \text{const}$. A small disk of radius ρ_t is removed around the origin.

By choosing a smaller ρ_t as necessary we can show that all points in $\mathcal{P}_{R,r}$ are attracted by the q -periodic attractive orbit under forward iterations by $H_{a,t}$. Moreover, every point that is attracted to this orbit must eventually land in the interior of one of the regions $\mathcal{P}_{R,r,j}$ for $1 \leq j \leq q$.

Let $\epsilon_0 = \tan(2\pi/9)$. To simplify notation, define ρ such that $\rho^q := \frac{1-\epsilon_0}{R\sqrt{1+\epsilon_0^2}}$. The number ρ measures the distance between the origin and one of the points of intersection of the lines $\text{Re}(x^q) = \epsilon_0|\text{Im}(x^q)|$ with the boundary of Δ_R .

Define the *attractive* sectors

$$(28) \quad \Delta^+ := \{x \in \mathbb{C} : \text{Re}(x^q) \leq \epsilon_0|\text{Im}(x^q)| \text{ and } |x^q| < \rho^q\},$$

and the *repelling* sectors

$$(29) \quad \Delta^- := \{x \in \mathbb{C} : \text{Re}(x^q) > \epsilon_0|\text{Im}(x^q)| \text{ and } |x^q| < \rho^q\}.$$

Let $W^+ := \Delta^+ \times \mathbb{D}_r \subset \mathcal{P}_{R,r}$ and $W^- := \Delta^- \times \mathbb{D}_r$.

We will call W^- *repelling* because as we will see, the Hénon map expands horizontally when the Jacobian is small enough. We will call W^+ *attractive* because points in W^+ are attracted to the q -periodic attractive orbit as we have shown above. There are q components of W^\pm which we denote W_j^\pm for $1 \leq j \leq q$.

In the regions Δ^- and W^- we have

$$(30) \quad \text{Re}(x^q) > \epsilon_1|x|^q, \quad \text{where } \epsilon_1 := \frac{\epsilon_0}{\sqrt{1+\epsilon_0^2}} > \frac{3}{5}.$$

The constants ϵ_0 and ϵ_1 are chosen such that the image of Δ^- under $x \mapsto x^q$ has an angle opening of $5\pi/9$ (see Figure 4).

The definition of the sectors W^\pm for $t > 0$ is the same as in the case $t = 0$ [RT, Section 4]. However, when $t < 0$, we need to modify the definition of the repelling sector W^- so that we have a good horizontal expansion for the Hénon map. Suppose therefore that $t < 0$.

Remark 3.8. When $\lambda = 1$ the parametric paths described by $\lambda_t = 1 \pm t$ are in fact the same, so we can assume that $q \geq 2$.

Assumption on t . Suppose t is sufficiently small so that $|t| < \frac{1}{24q+12}$. A restriction on t of this form is needed for the local dynamics, but the choice for this bound

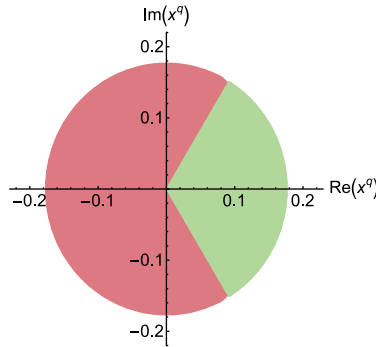


FIGURE 4. (case $t \geq 0$) Repelling and attractive sectors near the semi-parabolic/hyperbolic fixed point. The attracting sector Δ^+ is shown in red and the repelling sector Δ^- is shown in green. The angle of the green sector is $5\pi/9$.

will become clear later on. Let

$$(31) \quad R_t := \frac{|t|}{(q + 1/3)\epsilon_1}$$

be fixed from now on. The constant $\epsilon_1 > 3/5$ is the same as in equation (30). Suppose further that $|t|$ is small enough so that $R_t < 1/(9R)$, where R is as in Proposition 3.6.

Define

$$\mathcal{D}_{r,t} = \{(x, y) \in \mathbb{C}^2 : |x|^q \leq R_t, |y| < r\}.$$

Proposition 3.9 (Attracting region – t negative). *For R large enough and r small enough there exists a positive number δ' such that $H_{a,t}(\overline{\mathcal{D}_{r,t}}) \subset \mathcal{D}_{r,t}$ and all points of $\mathcal{D}_{r,t}$ are attracted to the origin under iterations by $H_{a,t}$ for all $t \in (-\delta', 0)$.*

Proof. We make the change of variables $X = -1/(qx^q)$, $Y = y$ and analyze the situation at infinity. Let \widehat{H} be the map written in these coordinates. This transformation maps $\mathcal{D}_{r,t}$ to the region

$$W_{r,t} := \left\{ X \in \mathbb{C} : |X| \geq \frac{1}{qR_t} \right\} \times \mathbb{D}_r.$$

Let $(X_1, Y_1) = \widehat{H}(X, Y)$ for $(X, Y) \in W_{r,t}$. Note that $\lambda_t = 1 - |t|$, as t is negative. Similar to equation (26) we have that

$$\left| X_1 - \frac{1}{(1 - |t|)^q} X \right| < \frac{1}{4}.$$

This gives

$$\frac{1}{(1 - |t|)^q} |X| - \frac{1}{(1 - |t|)^q} - |X_1| \leq \left| X_1 - \frac{1}{(1 - |t|)^q} (X + 1) \right| < \frac{1}{4}$$

and, after rearranging the terms, we want to obtain the following estimate:

$$|X_1| > \frac{1}{(1 - |t|)^q} |X| - \frac{1}{(1 - |t|)^q} - \frac{1}{4} > |X| + \frac{1}{40}.$$

The last inequality is equivalent to

$$|X| \left(\frac{1}{(1 - |t|)^q} - 1 \right) > \frac{1}{(1 - |t|)^q} + \frac{11}{40}.$$

Note that $\frac{1}{(1 - |t|)^q} - 1 > q|t|$ for small $|t|$. Using the fact that $|X| \geq (qR_t)^{-1}$ and the particular choice of R_t we get that

$$|X| \left(\frac{1}{(1 - |t|)^q} - 1 \right) > (q + 1/3) \epsilon_1 \geq \frac{7}{5} > \frac{1}{(1 - |t|)^q} + \frac{11}{40},$$

which is true whenever $(1 - |t|)^q > 8/9$. This condition is satisfied because, based on our assumption on t , we have

$$(1 - |t|)^q > \left(1 - \frac{1}{24q + 12} \right)^q \geq \left(\frac{59}{60} \right)^2 > \frac{8}{9}.$$

Note that the function $x \mapsto (1 - 1/(24x + 12))^x$ is increasing on $[2, \infty)$ and that is why we can use the middle inequality. We have therefore shown that $|X_1| > |X| + 1/40$. Let $(X_n, Y_n) = \widehat{H}(X_{n-1}, Y_{n-1})$ for some $(X_0, Y_0) \in W_{r,t}$. By induction we get that $|X_n| > |X_0| + n/40$. It follows that all points in $W_{r,t}$ are attracted to $(\infty, 0)$. In order to prove that indeed $Y_n \rightarrow 0$ as $n \rightarrow \infty$ we need to do a similar analysis as in [RT]; we leave the details to the reader. \square

When $t < 0$, we define the *repelling* sectors as follows:

$$(32) \quad \Delta_{R_t}^- := \{x \in \mathbb{C} : \operatorname{Re}(x^q) > \epsilon_0 |\operatorname{Im}(x^q)| \text{ and } R_t < |x^q| < \rho^q\},$$

where R_t is given in (31). The excluded region belongs to the basin of attraction of 0 (see Figure 5). Set as before $W_{R_t}^- := \Delta_{R_t}^- \times \mathbb{D}_r$. The definition of the set W^+ is the same as in the case when t is positive, i.e., $W^+ = \Delta^+ \times \mathbb{D}_r$, where Δ^+ is given in equation (28). Also, $\Delta_{R_t}^-$ is a subset of Δ^- , defined in equation (29). When $t \rightarrow 0^-$, the sets $\Delta_{R_t}^-$ converge to Δ^- , so the definition of W^- when $t = 0$ is the same as in [RT].

By choosing t small enough so that $R_t < \frac{1}{9R} < \rho^q$ we made sure that the excluded region $\{x \in \mathbb{C} \mid |x|^q < R_t\} \cap \Delta_R$ is contained in Δ^+ .

Let $B = \mathbb{D}_{\rho'}(\alpha_t) \times \mathbb{D}_r$ be the polydisk from Theorem 3.5 and $\phi_{a,t}$ the coordinate transformation defined on B . We define attractive and repelling sectors relative to B .

Definition 3.10. Let $W_B^+ := \phi_{a,t}^{-1}(W^+)$ for $t \geq 0$ and $W_B^+ := \phi_{a,t}^{-1}(\mathbb{D}_\rho \times \mathbb{D}_r - W_{R_t}^-)$ for $t < 0$ be the attractive sectors inside B . Similarly, let $W_B^- := \phi_{a,t}^{-1}(W^-)$ for $t \geq 0$ and $W_B^- := \phi_{a,t}^{-1}(W_{R_t}^-)$ for $t < 0$ be the repelling sectors inside B .

Proposition 3.11 (Local dynamics).

a) If $t \geq 0$, then the compact region W_B^+ satisfies

$$H_{a,t}(W_B^+) \subset \operatorname{int}(K_{a,t}^+) \cup W_{loc}^{ss}(\mathbf{q}_{a,t}).$$

b) If $t < 0$, then the compact region W_B^+ lies in the interior of $K_{a,t}^+$.

Proof. Using Definition 3.10, the proof follows directly from Propositions 3.6 and 3.9. \square

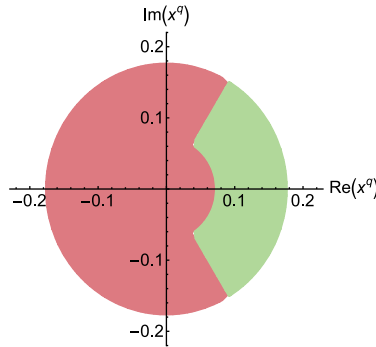


FIGURE 5. (case $t < 0$) The repelling sector $\Delta_{R_t}^-$ is shown in green. The angle of the green sector is $5\pi/9$. The red region belongs to the basin of attraction of 0.

3.3. Deforming the local semi-parabolic structure into a hyperbolic structure. In the parabolic case we have shown in [RT, Propositions 6.8, 9.2] that in the repelling sectors W^- near a semi-parabolic fixed point, the Hénon map is weakly expanding in horizontal cones and strongly contracting in vertical cones, with respect to the Euclidean metric. We will reuse these cones and show that when t is non-zero, the Hénon map is strongly expanding in horizontal cones and strongly contracting in vertical cones, and therefore has a local hyperbolic structure. In this section we only use the local normal form of the map, so all results are applicable to holomorphic germs of diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-parabolic fixed point at 0.

Definition 3.12. Define the **vertical cone** at a point (x, y) from the set $\mathbb{D}_\rho \times \mathbb{D}_r$ as

$$\mathcal{C}_{(x,y)}^v = \{(\xi, \eta) \in T_{(x,y)}\mathbb{D}_\rho \times \mathbb{D}_r, |\xi| \leq |x|^{2q}|\eta|\}.$$

Define the **horizontal cone** at a point (x, y) from the set $\mathbb{D}_\rho \times \mathbb{D}_r$ to be

$$\mathcal{C}_{(x,y)}^h = \{(\xi, \eta) \in T_{(x,y)}\mathbb{D}_\rho \times \mathbb{D}_r, |\xi| \geq |\eta|\}.$$

We consider the interior of a cone to be its topological interior together with the origin.

Consider the Hénon map $\tilde{H}_{a,t} : \mathbb{D}_\rho \times \mathbb{D}_r \rightarrow \mathbb{C}^2$ written in the normal form given in equation (23). We write $\tilde{H}_{a,t}$ whenever we want to stress the dependency on the parameters a and t , but otherwise we simply write \tilde{H} . We have

$$\tilde{H}_{a,t}(x, y) = (\lambda_t(x + x^{q+1} + g_{a,t}(x, y)), \nu_{a,t}y + xh_{a,t}(x, y)),$$

where

$$\begin{aligned} g_{a,t}(x, y) &= C_{a,t}x^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots, \\ h_{a,t}(x, y) &= b_1(y) + \dots + b_k(y)x^k + \dots, \end{aligned}$$

and $g_{a,t}(x, y) = g_{0,t}(x) + \mathcal{O}(a)$ and $h_{a,t}(x, y) = \mathcal{O}(a)$. Here the term $\mathcal{O}(a)$ is in fact a holomorphic function in both a and t .

When $a = 0$, $\tilde{H}_{0,t}(x, y) = (\tilde{p}_t(x), 0)$, where $\tilde{p}_t(x) = \lambda_t(x + x^{q+1} + g_{0,t}(x))$ and

$$g_{0,t}(x) = C_{0,t}x^{2q+1} + a_{2q+1}x^{2q+2} + \dots$$

The function $g_{0,t}$ depends only on x and t , hence $\partial_y g_{0,0}(x, y) \equiv 0$. For $|a| < \delta$ and $|t| < \delta'$ we can assume that there exists a constant $M_{a,t}$ with $0 < M_{a,t} < 1$ such that

$$(33) \quad |\partial_y g_{a,t}(x, y)| < M_{a,t}|x|^{2q+2}.$$

When $a = 0$ we also know that $xh_{0,t}(x, y) \equiv 0$. Moreover, by the construction of the normalizing coordinates, we have $xh_{a,t}(x, y) = \mathcal{O}(a)$. There exists a constant $N_{a,t}$, depending on a , with $0 < N_{a,t} < 1$ such that when $|a| < \delta$ the following bounds hold:

$$(34) \quad |\partial_x(xh_{a,t})(x, y)| < N_{a,t} \quad \text{and} \quad |\partial_y(xh_{a,t})(x, y)| < N_{a,t}.$$

Let $\partial_x g_{a,t}(x, y) = x^{2q}\beta_{a,t}(x, y)$, for some function $\beta_{a,t}$. As usual, ∂_x denotes the partial derivative with respect to the variable x . Denote by m the supremum of $|\beta_{a,t}(x, y)|$ on the set W^- , where the supremum is taken after all $|a| < \delta$ and $|t| < \delta'$. Thus

$$(35) \quad m := \sup_{(x,y) \in W^-, |a| < \delta, |t| < \delta'} |\beta_{a,t}(x, y)|$$

and so $|\partial_x g_{a,t}(x, y)| < m|x|^{2q}$ for all $(x, y) \in W^-$. The repelling sectors Δ^- and $W^- = \Delta^- \times \mathbb{D}_r$ are defined in equation (29).

By eventually reducing the radius $\rho > 0$ from the definition of the set Δ^- , we can assume that

$$(36) \quad |1 + (q + 1)x^q| - m|x|^{2q} > 1 + (q + 2/3)\epsilon_1|x|^q > 1 \quad \text{for all } x \in \Delta^-,$$

where ϵ_1 is given in equation (30). Consider the polynomial \tilde{p}_t as in Proposition 2.6 with its corresponding repelling sector $\Delta_{R_t}^-$ (see equations (29) and (32)). The estimate above allows us to show that $|\tilde{p}'_t(x)| > |\lambda_t|(1 + (q + 2/3)\epsilon_1|x|^q)$ for all $x \in \Delta_{R_t}^-$. The polynomial \tilde{p}'_t is clearly expanding if t is non-negative since $|\lambda_t| = 1 + t \geq 1$, but Lemma 3.15 shows that it is also expanding on $\Delta_{R_t}^-$ for negative t .

Proposition 3.13 (Vertical cones). *Consider (x, y) and (x_1, y_1) in the repelling sectors $W^- \subset \mathbb{D}_\rho \times \mathbb{D}_r$ (respectively in $W_{R_t}^-$ for $t < 0$) such that $\tilde{H}(x, y) = (x_1, y_1)$. Then*

$$D\tilde{H}_{(x_1, y_1)}^{-1} \left(\mathcal{C}_{(x_1, y_1)}^v \right) \subset \text{Int } \mathcal{C}_{(x, y)}^v$$

and $\|D\tilde{H}_{(x_1, y_1)}^{-1}(\xi', \eta')\| \geq (|\nu_{a,t}| + 3/2N_{a,t})^{-1} \|(\xi', \eta')\|$ for $(\xi', \eta') \in \mathcal{C}_{(x_1, y_1)}^v$.

Proof. Let $(\xi', \eta') \in \mathcal{C}_{(x_1, y_1)}^v$ with $(\xi', \eta') \neq (0, 0)$, and set $(\xi, \eta) = D\tilde{H}_{(x, y)}^{-1}(\xi', \eta')$. We need to show that $(\xi, \eta) \in \mathcal{C}_{(x, y)}^v$. A direct computation gives

$$D\tilde{H}_{(x, y)} = \begin{pmatrix} \lambda_t(1 + (q + 1)x^q + \partial_x g_{a,t}(x, y)) & \lambda_t \partial_y g_{a,t}(x, y) \\ \partial_x(xh_{a,t})(x, y) & \nu_{a,t} + x \partial_y h_{a,t}(x, y) \end{pmatrix}$$

and so

$$(37) \quad \xi' = \lambda_t(1 + (q + 1)x^q + \partial_x g_{a,t}(x, y))\xi + \lambda_t \partial_y g_{a,t}(x, y)\eta,$$

$$(38) \quad \eta' = \partial_x(xh_{a,t})(x, y)\xi + (\nu_{a,t} + \partial_y(xh_{a,t})(x, y))\eta.$$

Using the bounds from equations (34) and (33) we get

$$(39) \quad |\xi'| \geq |\lambda_t| \left(|1 + (q + 1)x^q| - m|x|^{2q} \right) |\xi| - |\lambda_t| M_{a,t} |x|^{2q+2} |\eta|,$$

$$(40) \quad |\eta'| \leq N_{a,t} |\xi| + (|\nu_{a,t}| + N_{a,t}) |\eta|.$$

Since (ξ', η') belongs to the vertical cone at (x_1, y_1) , we also know that

$$\begin{aligned} |\xi'| \leq |x_1|^{2q} |\eta'| &\leq |\lambda_t|^{2q} |x|^{2q} |1 + x^q + g_{a,t}(x, y)/x|^{2q} |\eta'| \\ &\leq |\lambda_t|^{2q} |x|^{2q} M_1^{2q} |\eta'|, \end{aligned}$$

where M_1 is the supremum of $|1 + x^q + g_{a,t}(x, y)/x|$ on the repelling sectors W^- of the tubular neighborhood $\mathbb{D}_\rho \times \mathbb{D}_r$, that is,

$$M_1 := \sup_{\substack{(x,y) \in W^-, \\ |a| < \delta, |t| < \delta'}} |1 + x^q + g_{a,t}(x, y)/x|.$$

Since $Re(x^q) > \epsilon_1 |x|^q$ on W^- we can take $M_1 > 1$, but any constant $M_1 > 0$ would suffice. We have assumed that $|t| < 1/(2q)$, so $|\lambda_t|^{2q} < 3|\lambda_t|$ and

$$(41) \quad |\xi'| < 3|\lambda_t| |x|^{2q} M_1^{2q} |\eta'|.$$

By combining estimates (39), (40), and (41) we get

$$\begin{aligned} |\lambda_t| (|1 + (q + 1)x^q| - m|x|^{2q}) |\xi| - |\lambda_t| M_{a,t} |x|^{2q+2} |\eta| &\leq |\xi'| \\ < 3|\lambda_t| |x|^{2q} M_1^{2q} |\eta'| &\leq 3|\lambda_t| M_1^{2q} N_{a,t} |x|^{2q} |\xi| + 3|\lambda_t| M_1^{2q} (|\nu_{a,t}| + N_{a,t}) |x|^{2q} |\eta|. \end{aligned}$$

After regrouping the terms, we write

$$|\xi| < \frac{A_2}{A_1} |x|^{2q} |\eta|,$$

where A_1 and A_2 are defined as follows:

$$\begin{aligned} A_1 &:= |1 + (q + 1)x^q| - (m + 3M_1^{2q} N_{a,t}) |x|^{2q}, \\ A_2 &:= 3M_1^{2q} (|\nu_{a,t}| + N_{a,t}) + M_{a,t} |x|^2. \end{aligned}$$

Since x is chosen from the repelling sectors we have $|1 + (q + 1)x^q| - m|x|^{2q} > 1$. The quantities $N_{a,t}$, $M_{a,t}$ and $\nu_{a,t} = -a^2/\lambda_t$ depend on a and on t , and they tend to 0 as $a \rightarrow 0$, uniformly with respect to t . For $|a|$ and $|t|$ small we can therefore assume that $A_1 > 2/3$ and $A_2 < 1/3$. Hence $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$, so

$$D\tilde{H}_{(x_1,y_1)}^{-1} \left(\mathcal{C}_{(x_1,y_1)}^v \right) \subset \text{Int } \mathcal{C}_{(x,y)}^v.$$

We now show that inside the vertical cones the derivative $D\tilde{H}^{-1}$ is expanding with respect to the Euclidean metric. We have

$$\begin{aligned} |\eta'| &\leq N_{a,t} |\xi| + (|\nu_{a,t}| + N_{a,t}) |\eta| < N_{a,t} \frac{A_2}{A_1} |x|^{2q} |\eta| + (|\nu_{a,t}| + N_{a,t}) |\eta| \\ &< \left(\frac{1}{2} N_{a,t} |x|^{2q} + |\nu_{a,t}| + N_{a,t} \right) |\eta| < \left(|\nu_{a,t}| + \frac{3}{2} N_{a,t} \right) |\eta|, \end{aligned}$$

provided that $|x| < 1$ (which is already assumed since $\rho < 1$). By definition, as both (ξ, η) and (ξ', η') are taken from the vertical cones, we have

$$\|(\xi, \eta)\| = \max(|\xi|, |\eta|) = |\eta| \quad \text{and} \quad \|(\xi', \eta')\| = \max(|\xi'|, |\eta'|) = |\eta'|.$$

We obtain $\|(\xi, \eta)\| > (|\nu_{a,t}| + 3/2 N_{a,t})^{-1} \|(\xi', \eta')\|$. □

For $|a|$ and $|t|$ sufficiently small the expansion factor $(|\nu_{a,t}| + 3/2 N_{a,t})^{-1}$ can be easily made larger than 1. Hence $D\tilde{H}^{-1}$ expands in the vertical cones with a factor strictly greater than 1.

Proposition 3.14 (Horizontal cones). *Consider two points (x, y) and (x_1, y_1) in the repelling sectors $W^- \subset \mathbb{D}_\rho \times \mathbb{D}_r$ (respectively in $W_{R_t}^-$ for $t < 0$) such that $\tilde{H}(x, y) = (x_1, y_1)$. Then*

$$D\tilde{H}_{(x,y)} \left(\mathcal{C}_{(x,y)}^h \right) \subset \text{Int } \mathcal{C}_{(x_1,y_1)}^h$$

and $\|D\tilde{H}_{(x,y)}(\xi, \eta)\| \geq |\lambda_t| (1 + (q + 1/2)\epsilon_1|x|^q) \|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$.

Proof. Consider $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$, $(\xi, \eta) \neq (0, 0)$, and let $(\xi', \eta') = D\tilde{H}_{(x,y)}(\xi, \eta)$. We first need to show that $(\xi', \eta') \in \mathcal{C}_{(x_1,y_1)}^h$. Consider ξ' and η' written as in equations (37) and (38), from the proof of the previous proposition. Since (ξ, η) belongs to the horizontal cone at (x, y) , we know that $|\xi| \geq |\eta|$. As before, by using equations (33), (34), and (35), we get the following estimates:

$$\begin{aligned} |\xi'| &\geq |\lambda_t| (|1 + (q + 1)x^q| - m|x|^{2q}) |\xi| - |\lambda_t| M_{a,t} |x|^{2q+2} |\eta| \\ (42) \quad &\geq |\lambda_t| (|1 + (q + 1)x^q| - m|x|^{2q} - M_{a,t} |x|^{2q+2}) |\xi|, \\ |\eta'| &\leq N_{a,t} |\xi| + (|\nu_{a,t}| + N_{a,t}) |\eta| \leq (2N_{a,t} + |\nu_{a,t}|) |\xi|. \end{aligned}$$

In the final analysis we obtain

$$|\eta'| \leq \frac{B_2}{B_1} |\xi'|,$$

where B_1 and B_2 are defined in the obvious way

$$\begin{aligned} B_2 &:= 2N_{a,t} + |\nu_{a,t}|, \\ B_1 &:= |\lambda_t| (|1 + (q + 1)x^q| - m|x|^{2q} - M_{a,t} |x|^{2q+2}). \end{aligned}$$

The bounds $N_{a,t}$, $M_{a,t}$ and $|\nu_{a,t}|$ tend to 0 as $a \rightarrow 0$, uniformly with respect to t , so one can assume that for $|a|$ and $|t|$ small enough we have $B_2 < 1/2$. Moreover, using the bound from equation (36), we can assume that

$$(43) \quad B_1 > |\lambda_t| (1 + (q + 1/2)\epsilon_1|x|^q).$$

Clearly, B_1 is bounded below by $|\lambda_t| > 1 - 1/(2q)$. In conclusion, we get

$$|\eta'| < \frac{q}{2q - 1} |\xi'|,$$

which implies that $(\xi', \eta') \in \text{Int } \mathcal{C}_{(x',y')}^h$. The norm of the two vectors from the horizontal cones are $\|(\xi', \eta')\| = \max(|\xi'|, |\eta'|) = |\xi'|$ and $\|(\xi, \eta)\| = \max(|\xi|, |\eta|) = |\xi|$. We have already shown in equation (42) that $|\xi'| \geq B_1|\xi|$. Together with the lower bound on B_1 from equation (43) this yields

$$\|(\xi', \eta')\| > |\lambda_t| ((1 + (q + 1/2)\epsilon_1)|x|^q) |\xi| = |\lambda_t| ((1 + (q + 1/2)\epsilon_1)|x|^q) \|(\xi, \eta)\|,$$

which is what we needed to prove. □

We now analyze the expansion factor $|\lambda_t| (1 + (q + 1/2)\epsilon_1|x|^q)$ in the horizontal cones from Proposition 3.14. If $t = 0$, then $|\lambda_t| = 1$ and the expansion factor reduces to $1 + (q + 1/2)\epsilon_1|x|^q$. In this case $D\tilde{H}$ expands strictly, but not strongly in the horizontal cones. The expansion factor goes to 1 when $x \rightarrow 0$, i.e., when we approach the local strong stable manifold of the semi-parabolic fixed point.

If t is positive, then $|\lambda_t| > 1$ and $D\tilde{H}$ expands strongly in the horizontal cones, by a factor of $(1+t) (1 + (q + 1/2)\epsilon_1|x|^q) \geq (1+t)$. If t is negative, then $|\lambda_t| < 1$ and we need to use the definition of the repelling sector $W_{R_t}^-$ to get a good expansion.

The repelling sectors were carefully defined in equation (32), from the previous section. We use the fact that $|x|^q > R_t$ for the choice of R_t from equation (31) to make the product $|\lambda_t| \cdot (1 + (q + 1/2)\epsilon_1|x|^q)$ strictly greater than 1 throughout $\Delta_{R_t}^-$. So we use the particular choice of R_t to make the second term dominate $|\lambda_t|$, which is in fact smaller than 1. The following technical lemma deals with this situation.

Lemma 3.15 (Expansion estimate). *If $t \in (-\delta', 0)$, then*

$$|\lambda_t| (1 + (q + 1/2)\epsilon_1|x|^q) > (1 + \epsilon_2|t|) \left(1 + \frac{\epsilon_1}{16}|x|^q\right),$$

for all $x \in \Delta_{R_t}^-$, where $\epsilon_2 := \frac{1}{16(q+1)}$. The inequality is also true for $t \in [0, \delta')$ and $x \in \Delta^-$.

Proof. The proof is straightforward if t is non-negative. Suppose that t is negative. Note that $|\lambda_t| = 1 - |t|$. We first show that for all $x \in \Delta_{R_t}^-$

$$|\lambda_t| (1 + (q + 1/2)\epsilon_1|x|^q) = (1 - |t|) (1 + (q + 1/2)\epsilon_1|x|^q) > 1 + \frac{\epsilon_1}{8}|x|^q,$$

which is equivalent to showing that $|x|^q ((1 - |t|)(q + 1/2)\epsilon_1 - \epsilon_1/8) > |t|$. On $\Delta_{R_t}^-$ we have that $|x|^q > R_t$ for $R_t = \frac{|t|}{(q+1/3)\epsilon_1}$, so

$$|x|^q ((1 - |t|)(q + 1/2)\epsilon_1 - \epsilon_1/8) > \frac{|t|}{(q + 1/3)} ((1 - |t|)(q + 1/2) - 1/8) > |t|.$$

This is verified for $|t| < \frac{1}{24q+12}$, which is one of the bounds already imposed on t .

We then show by direct computation that

$$1 + \frac{\epsilon_1}{8}|x|^q > (1 + \epsilon_2|t|) \left(1 + \frac{\epsilon_1}{16}|x|^q\right),$$

for all $x \in \Delta_{R_t}^-$ and some constant ϵ_2 . We take $\epsilon_2 = \frac{1}{16(q+1)}$, but the choice is not optimal. The computational details are left to the reader. □

3.4. Global analysis of the Julia set. We would first like to show that the corresponding Hénon map is hyperbolic on its Julia set $J_{a,t}$. We must show that the derivative of the Hénon map has appropriate contraction and expansion in a family of vertical, respectively horizontal cones. We have already shown this to be true locally around the fixed point $\mathbf{q}_{a,t}$ in Section 3.3.

If we look at the form of the Hénon map, $H_{a,t}(x, y) = (p_t(x) + a^2w + ay, ax)$ given in equation (17), we notice that the presence of the multiplicative factor a in the second coordinate implies that the derivative of the Hénon map is strongly contracting in the “vertical direction”, or equivalently, DH^{-1} is expanding in the “vertical direction”. If we analyze the first coordinate, we notice that the expanding properties of DH in the horizontal direction are closely related to the expanding properties of the polynomial p_t on a neighborhood of its Julia set. We will construct a neighborhood V of $J_{a,t}^+$ for the Hénon map $H_{a,t}$ inside a polydisk $\mathbb{D}_r \times \mathbb{D}_r$, and put a metric on it with respect to which the derivative of the Hénon map is expanding in the “horizontal direction”.

For the Hénon map $H_{a,t}$, the construction of the neighborhood V will be similar to the construction of the neighborhood U_t' in the polynomial case in Section 2.4 (see also [RT, Section 7] for the construction of the neighborhood in the semi-parabolic case).

Let α_t be a fixed point for the polynomial p_t . For $|a| < \delta$ consider the normalizing coordinates of the Hénon map $H_{a,t}$ on the tubular neighborhood $B = \mathbb{D}_{\rho'}(\alpha_t) \times \mathbb{D}_r$ as defined in Theorem 3.5. Let $\mathbf{q}_{a,t}$ denote the hyperbolic/semi-parabolic/attracting fixed point. Let W_B^+ and W_B^- be the attractive, respectively repelling sectors inside B from Definition 3.10. By Proposition 3.11, the set W_B^+ belongs to $\text{int}(K_{a,t}^+) \cup W_{loc}^{ss}(\mathbf{q}_{a,t})$. The set $H_{a,t}^{-1}(B) \cap \mathbb{D}_r \times \mathbb{D}_r$ has two connected components, so let us denote by

$$(44) \quad B' := (H_{a,t}^{-1}(B) - B) \cap \mathbb{D}_r \times \mathbb{D}_r$$

the component disjoint from B . Let $W_{B'}^+$ be the preimage of the attractive sectors W_B^+ in B' , that is, $W_{B'}^+ := H_{a,t}^{-1}(W_B^+) \cap B'$.

We start by defining a box neighborhood $U'_t \times \mathbb{D}_r$, where U'_t is constructed as in the one-dimensional case (see equation (6)). Recall that the set U'_t was defined as $U'_t = p_t^{-1}(U_t)$, where

$$U_t := \mathbb{C} - \overline{p_t^{-\circ N}(S_{att})} - \{z \in \mathbb{C} - K_{p_t} : |\Psi_{p_t}^{-1}(z)| \geq R\}.$$

We choose $R > 2$ large enough so that the outer boundary of U_t is in the set V^+ defined in (15) (and implicitly in the escaping set U^+).

The only difference will be that instead of removing the attractive sectors $p_t^{-\circ N}(\Delta_t^+)$, we want to remove a bit less. Construct attractive sectors $S_{att} \subset \Delta_t^+$ associated with the polynomial p_t in $\mathbb{D}_{\rho'}(\alpha_t)$, thin enough along the *attractive axes* so that

$$\left(p_t^{-\circ(N+1)}(S_{att}) \cap A \right) \times \mathbb{D}_r \subset W_B^+.$$

We denoted by A the annulus between the disk of radius ρ' and the disk of radius $\rho'' < \rho'/2$ centered at α_t , so $A = \mathbb{D}_{\rho'}(\alpha_t) - \mathbb{D}_{\rho''}(\alpha_t)$. Otherwise said, in the annular region A , we want the small attractive sectors $p_t^{-\circ(N+1)}(S_{att})$ of the polynomial p_t to be compactly contained in the attractive sectors W_B^+ of the Hénon map. As in Section 2.4, when writing $p_t^{-\circ N}(S_{att})$ we do not take into account all preimages of S_{att} , but rather only the preimage of S_{att} that is contained in the immediate Fatou components of the fixed point α_t and has α_t in the boundary.

Inside the tubes B and B' , we forget all together about the polynomial dynamics. So we take out the tubes completely and put back in only the repelling sectors W_B^- and $W_{B'}^-$. We can now finally define the set V as

$$(45) \quad V := (U'_t \times \mathbb{D}_r - (B \cup B')) \cup (W_B^- \cup W_{B'}^-).$$

Remark 3.16. Note that the set B' (and consequently $W_{B'}^-$) is contained in the larger set $U_t \times \mathbb{D}_r$ and its projection on the first coordinate is compactly contained in U_t and bounded away from the critical point 0 of the polynomial p_t . Denote by B'' the polydisk $\mathbb{D}_{\rho''}(\alpha_t) \times \mathbb{D}_r$. When $|a|$ is small, the set B'' should be thought of as a small neighborhood of the local stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,t})$ of the fixed point $\mathbf{q}_{a,t}$. By the construction above, the set difference $W_B^- - B''$ is contained in $U'_t \times \mathbb{D}_r$. Hence $V - B''$ is a subset of $U_t \times \mathbb{D}_r$.

For $t \geq 0$, let \bar{V} denote the set V together with the local stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,t})$ and together with $H^{-1}(W_{loc}^{ss}(\mathbf{q}_{a,t})) \cap B'$. When $t < 0$ there is no need to add the two stable manifolds as they belong to the interior of $K_{a,t}^+$. However, to preserve notation, we set $\bar{V} = V$ in this case.

Lemma 3.17. $J_{a,t}^+ \cap \bar{V} = J_{a,t}^+ \cap \mathbb{D}_r \times \mathbb{D}_r$. Moreover, the Julia set $J_{a,t}$ is contained in \bar{V} and

$$J_{a,t} = \bigcap_{n \geq 0} H^{\circ n}(J_{a,t}^+ \cap \bar{V}).$$

Proof. The outer boundary of the set V is an equipotential of the polynomial p_t cross \mathbb{D}_r , which belongs to the escaping set $U_{a,t}^+$. From the tubular neighborhood B of the local stable manifold we have removed only the attractive sectors W_B^+ , which are contained in the interior of $K_{a,t}^+$ when $t < 0$ and in the interior of $K_{a,t}^+$ union the local stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,t})$ when $t \geq 0$. From B' we only removed the attractive sectors $W_{B'}^+$, which are contained in the interior of $K_{a,t}^+$ when $t < 0$, and respectively in the interior of $K_{a,t}^+$ union a preimage of the local stable manifold $H^{-1}(W_{loc}^{ss}(\mathbf{q}_{a,t})) \cap B'$ when $t \geq 0$. Outside of $B \cup B'$, we have removed a vertical tube $p_t^{-\circ(N+1)}(S_{att}) \times \mathbb{D}_r$ which belongs to the interior of $K_{a,t}^+$. Therefore, when $t < 0$, the set $J_{a,t}^+ \cap (\mathbb{D}_r \times \mathbb{D}_r)$ is contained in the set V . When $t \geq 0$, in our construction process of the neighborhood V , we have lost from $J_{a,t}^+ \cap (\mathbb{D}_r \times \mathbb{D}_r)$ only two local stable manifolds. These local stable manifolds are no longer in V , but they lie in the larger set \bar{V} ,

$$J_{a,t}^+ \cap (\mathbb{D}_r \times \mathbb{D}_r) \subset \bar{V} \text{ when } t \geq 0.$$

Any point in $J_{a,t}^+ \cap \bar{V}$ remains in \bar{V} under forward iterates of the Hénon map, so the Julia set $J_{a,t}$ is contained in \bar{V} . □

3.5. Vertical and horizontal cones in the product metric. We construct an invariant family of horizontal and vertical cones on the set V defined in (45), such that the derivative of the Hénon map expands in the horizontal cones, and contracts in the vertical cones.

In Section 3.3 we have already constructed such an invariant family of cones in the repelling sectors of the fixed point $\mathbf{q}_{a,t}$. These cones live only in a small neighborhood of $\mathbf{q}_{a,t}$, where the map is conjugate to the normal form (3.4).

In this section we define a family of cones on the set V away from a small neighborhood of $\mathbf{q}_{a,t}$. At the end of this section, we show how to patch together these two types of cones, from Sections 3.3 and 3.5, to get an invariant family on the entire set V .

The set $U_t \times \mathbb{D}_r$ comes equipped with the product metric $\mu_{U_t} \times \mu_E$ of the Poincaré metric μ_{U_t} of the set U_t and the regular Euclidean metric μ_E on the vertical disk \mathbb{D}_r . Tangent vectors (ξ, η) from $T_{(x,y)}\mathbb{C}^2$ will be measured with respect to the product metric

$$\|(\xi, \eta)\| := \max(\mu_{U_t}(x, \xi), |\eta|),$$

where $|\eta|$ is the absolute value of the complex number η .

By Remark 3.16, the set $V - B''$ is a subset of $U_t \times \mathbb{D}_r$, and we can endow $V - B''$ with the product metric that we have just constructed on the set $U_t \times \mathbb{D}_r$. Denote by U_t'' the projection of V on the first coordinate, which is equal to $U_t' \cup pr_1(W_{B'})$. The set $U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$ is compactly contained in U_t , so the Poincaré metric μ_{U_t} is bounded above and below by the Euclidean metric on the set $U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$, that is, there exist two positive constants m_1 and m_2 such that

$$(46) \quad m_1 < \rho_{U_t}(x) < m_2,$$

for any $t \in [-\delta', \delta']$ and any $x \in U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$. Therefore, the product metric is bounded on the set $V - B''$. If we let ρ_{U_t} be the density function of the metric μ_{U_t} ,

$$\mu_{U_t}(x, \xi) = \rho_{U_t}(x)|\xi|,$$

then ρ_{U_t} is positive and C^∞ -smooth on $U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$.

The sets U_t'' , $|t| < \delta'$, avoid a neighborhood of fixed size of the critical point of the polynomial p_t . Hence there exists a lower bound $r_1 > 0$ such that

$$(47) \quad r_1 < |p_t'(x)| \text{ for any } x \in U_t''.$$

Definition 3.18. Let $\tau < 1$. Let the **vertical cone** at a point (x, y) from $V - B''$ be

$$\mathcal{C}_{(x,y)}^v = \{(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2, \mu_{U_t}(x, \xi) \leq \tau \cdot |\eta|\}.$$

Define the **horizontal cone** at a point (x, y) from the set $U_t' \times \mathbb{D}_r$ to be

$$\mathcal{C}_{(x,y)}^h = \{(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2, \mu_{U_t}(x, \xi) \geq |\eta|\}.$$

We will show that the vertical cones are invariant under $DH_{a,t}^{-1}$ and that the horizontal cones are invariant under $DH_{a,t}$.

Proposition 3.19 (Vertical cones). Consider (x, y) and (x', y') in $V - B''$ such that $H(x', y') = (x, y)$. Then

$$DH_{(x,y)}^{-1}(\mathcal{C}_{(x,y)}^v) \subset \text{Int } \mathcal{C}_{(x',y')}^v$$

and $\|DH_{(x,y)}^{-1}(\xi, \eta)\| \geq |a|^{-1}\|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$.

Proof. Let $(\xi, \eta) \in \mathcal{C}_{(x,y)}^v$ and $(\xi', \eta') = DH_{(x,y)}^{-1}(\xi, \eta)$. From the formula of the inverse of the Hénon map

$$(x', y') = \left(\frac{y}{a}, \frac{x - p_t(y/a) - a^2w}{a}\right)$$

we find $\xi' = \frac{1}{a}\eta$ and $\eta' = \frac{1}{a}\left(\xi - \frac{2x'}{a}\eta\right)$. Assume $(\xi, \eta) \neq (0, 0)$, otherwise the proof is trivial. The vector (ξ, η) belongs to the vertical cone, so $\mu_{U_t}(x, \xi) = \rho_{U_t}(x)|\xi| \leq \tau|\eta|$. This implies that

$$(48) \quad |\xi| < \frac{\tau}{m_1}|\eta|.$$

We can evaluate

$$(49) \quad \mu_{U_t}(x', \xi') = \rho_{U_t}(x')\frac{|\eta|}{|a|} \leq \frac{m_2}{|a|}|\eta|.$$

Next, by using inequality (48), we compute

$$(50) \quad |\eta'| = \frac{1}{|a|}\left|\xi - \frac{2x'}{a}\eta\right| > \frac{1}{|a|}\left(\frac{|2x'|}{|a|} - \frac{\tau}{m_1}\right)|\eta|.$$

The point x' belongs to U_t' , so $|2x'| > r_1$ by equation (47). Choose $|a|$ small so that $\frac{r_1}{|a|} - \frac{\tau}{m_1} > \max\left(\frac{2m_2}{\tau}, 1\right)$. Combining equations (49) and (50) gives $\mu_{U_t}(x', \xi') < \frac{\tau}{2}|\eta|$. Therefore $DH_{(x,y)}^{-1}(\mathcal{C}_{(x,y)}^v) \subset \text{Int } \mathcal{C}_{(x',y')}^v$, which proves the cone invariance.

Inequality (50) shows that DH^{-1} expands in the vertical cone as $|\eta'| > |a|^{-1}|\eta|$. By definition, since both (ξ, η) and (ξ', η') belong to vertical cones, we have

$$\|(\xi, \eta)\| = \max(\mu_{U_t}(x, \xi), |\eta|) = |\eta| \quad \text{and} \quad \|(\xi', \eta')\| = \max(\mu_{U_t}(x', \xi'), |\eta'|) = |\eta'|.$$

We therefore obtain $\|(\xi, \eta)\| > |a|^{-1}\|(\xi', \eta')\|$, as claimed. □

Remark 3.20. When dealing with vertical cones, it is not really necessary to measure the horizontal component of vectors with respect to the Poincaré metric. Any bounded metric in the horizontal direction would work, because we can always choose $|a|$ small to get the invariance of the vertical cone field and the strong expansion of DH^{-1} in the vertical cones. The choice of the Poincaré metric is essential however to show expansion of DH in the horizontal cones.

The scalar $0 < \tau < 1$ in the definition of the vertical cone will typically be chosen less than $(\rho/2)^{2q}$, so that on a neighborhood of the boundary of B , the vertical cones $\mathcal{C}_{(x,y)}^v$ from Definition 3.18 are contained in the pull-back by $D\phi_{a,t}$ of the vertical cones from Definition 3.12 defined in the normalized coordinates.

Proposition 3.21 (Horizontal cones). *Let (x, y) and (x', y') in $V - B''$ such that $H(x, y) = (x', y')$. Then we have*

$$DH_{(x,y)}\left(\mathcal{C}_{(x,y)}^h\right) \subset \text{Int } \mathcal{C}_{(x',y')}^h$$

and $\|DH_{(x,y)}(\xi, \eta)\| \geq k\|(\xi, \eta)\|$ for $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$.

Proof. Let $(\xi, \eta) \in \mathcal{C}_{(x,y)}^h$ with $(\xi, \eta) \neq (0, 0)$ and let $(\xi', \eta') = DH_{(x,y)}(\xi, \eta)$. We first need to show that $(\xi', \eta') \in \text{Int } \mathcal{C}_{(x',y')}^h$. Since $DH_{(x,y)}(\xi, \eta) = (2x\xi + a\eta, a\xi)$, we find $\xi' = 2x\xi + a\eta$ and $\eta' = a\xi$. The vector (ξ, η) belongs to the horizontal cone at (x, y) , so

$$(51) \quad |\eta| \leq \mu_{U_t}(x, \xi) = \rho_{U_t}(x)|\xi| < m_2|\xi|.$$

Using $(x', y') = (p_t(x) + a^2w + ay, ax)$ we evaluate

$$(52) \quad \mu_{U_t}(x', \xi') = \rho_{U_t}(p_t(x) + a^2w + ay) |2x\xi + a\eta|.$$

Since ρ_{U_t} is C^∞ -smooth, its derivative ρ'_{U_t} is also bounded on $U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$. There exists a constant $c > 0$ (which is just a local variable) such that

$$(53) \quad \frac{|\rho_{U_t}(p_t(x) + a^2w + ay) - \rho_{U_t}(p_t(x))|}{|a|} \leq |aw + y| \cdot \sup \rho'_{U_t} \cdot \frac{\rho_{U_t}(p_t(x))}{\inf \rho_{U_t}} < c \cdot \rho_{U_t}(p_t(x)).$$

The polynomial p_t is expanding with respect to the Poincaré metric μ_{U_t} . As in Lemma (2.9) part (a), there exists $\kappa_t > 1$ with $\inf_{t \in [-\delta', \delta']} \kappa_t > 1$ such that

$$(54) \quad \rho_{U_t}(p_t(x)) |p'_t(x)\xi| > \kappa_t \cdot \rho_{U_t}(x)|\xi|,$$

whenever $x, p_t(x) \in U_t'' - \mathbb{D}_{\rho''}(\alpha_t)$. We now turn back to relation (52). Using (53), (54), (51) and (47) one gets

$$(55) \quad \begin{aligned} \mu_{U_t}(x', \xi') &> (1 - c|a|) \cdot \rho_{U_t}(p_t(x)) |2x\xi| \cdot \frac{|2x\xi + a\eta|}{|2x\xi|} \\ &> (1 - c|a|) \cdot \kappa_t \cdot \rho_{U_t}(x)|\xi| \cdot \left(1 - |a| \frac{|\eta|}{|2x||\xi|}\right) \\ &> \kappa_t \cdot (1 - c|a|) \cdot \left(1 - |a| \frac{m_2}{r_1}\right) \cdot \rho_{U_t}(x)|\xi|. \end{aligned}$$

The constant κ_t is larger than 1 for all $t \in [-\delta', \delta']$. We write the dependence on t to preserve notation from equation (7), but we could drop the dependence on t by

working with $\inf_{t \in [-\delta', \delta']} \kappa_t$ which is also strictly larger than 1. The factors $1 - c|a|$ and $1 - |a| \frac{m_2}{r_1}$ are independent of t , and they can be made arbitrarily close to 1 by reducing $|a|$. In conclusion, for $|a|$ sufficiently small we can assume that

$$(56) \quad k := \inf_{t \in [-\delta', \delta']} \kappa_t \cdot (1 - c|a|) \cdot \left(1 - |a| \frac{m_2}{r_1}\right) > 1.$$

From relation (55), we obtain

$$\mu_{U_t}(x', \xi') > k \cdot \rho_{U_t}(x) |\xi| = k \cdot \mu_{U_t}(x, \xi),$$

which shows that DH expands in the horizontal cones. Also from (55) we infer that

$$|a| \cdot \mu_{U_t}(x', \xi') > k \cdot \rho_{U_t}(x) |a\xi| > k \cdot m_1 \cdot |\eta'|,$$

which proves that $DH_{(x,y)}(\mathcal{C}_{(x,y)}^h) \subset \text{Int } \mathcal{C}_{(x',y')}^h$, so the horizontal cones are invariant. □

On the set $V - B''$ we have one family of horizontal/vertical cones, $\mathcal{C}_{(x,y)}^h$ and $\mathcal{C}_{(x,y)}^v$, defined in Definition 3.18. On W_B^- we have another family of horizontal/vertical cones

$$D\phi_{a,t}^{-1}|_{\phi_{a,t}(x,y)}(\mathcal{C}_{\phi_{a,t}(x,y)}^h) \text{ and } D\phi_{a,t}^{-1}|_{\phi_{a,t}(x,y)}(\mathcal{C}_{\phi_{a,t}(x,y)}^v)$$

defined in Definition 3.12 with respect to the Euclidean metric in the normalized coordinates given by $\phi_{a,t}$. For those points $(x, y) \in W_B^-$ where both types of cones are defined, we take the horizontal/vertical cone to be their intersection.

3.6. Combining infinitesimal metrics. On the neighborhood V defined in (45) we have given two infinitesimal metrics. On the set $V - B''$, where $B'' \subset B$ was defined in Remark 3.16, we put the product of the Poincaré metric μ_{U_t} with the Euclidean metric on \mathbb{D}_r ,

$$(57) \quad \mu_P((x, y), (\xi, \eta)) := \max(\mu_{U_t}(x, \xi), |\eta|),$$

where $(x, y) \in V - B''$ and $(\xi, \eta) \in T_{(x,y)}V - B''$.

In the repelling sectors W_B^- of the tubular neighborhood B of the local strong stable manifold of the hyperbolic/semi-parabolic/attractive fixed point (see Definition 3.10), we have the pull-back Euclidean metric from the normalizing coordinates $\phi_{a,t} : W_B^- \rightarrow W^- \subset \mathbb{D}_\rho \times \mathbb{D}_r$. Let

$$\mu_B((x, y), (\xi, \eta)) := \max(|\tilde{\xi}|, |\tilde{\eta}|),$$

where $(\tilde{\xi}, \tilde{\eta}) = D\phi_{a,t}|_{(x,y)}(\xi, \eta)$ and $\phi_{a,t} : B \rightarrow \mathbb{D}_\rho \times \mathbb{D}_r$ is the change of coordinate function from Theorem 3.5.

Just like in the polynomial case, we can define an infinitesimal pseudo-norm on the set V ,

$$(58) \quad \mu := \inf(M\mu_B, \mu_P),$$

where M is a positive real number, chosen so that the derivative of the Hénon map is still expanding in the horizontal cones when we map from the repelling sectors

$W_{B'}^-$ of B' (see (44)) into the repelling sectors W_B^- of B . We take M so that

$$(59) \quad M > \sup_{\substack{(x,y) \in W_{B'}^- \\ (\xi,\eta) \in \mathcal{C}_{(x,y)}^h - \{(0,0)\}}} \frac{2\mu_P((x,y),(\xi,\eta))}{\mu_B(H(x,y),DH_{(x,y)}(\xi,\eta))}.$$

The supremum from equation (59) is bounded, because $\mu_P((x,y),(\xi,\eta)) = \mu_{U_t}(x,\xi)$ for $(x,y) \in W_{B'}^-$, $(\xi,\eta) \in \mathcal{C}_{(x,y)}^h$, and the Poincaré metric on B' is bounded.

Let (x,y) be any point in V . Let $(\xi,\eta), (\xi',\eta')$ be any vectors from the two dimensional tangent space $T_{(x,y)}V$.

Notice that μ is homogeneous, that is, $\mu((x,y),\alpha(\xi,\eta)) = |\alpha|\mu((x,y),(\xi,\eta))$, for all complex numbers α , as both μ_P and μ_B are homogeneous metrics. It also satisfies the relation $\mu((x,y),(\xi,\eta)) \geq 0$, with equality if and only if $(\xi,\eta) = (0,0)$. However, μ does not necessarily satisfy the triangle inequality

$$\mu((x,y),(\xi + \xi',\eta + \eta')) \leq \mu((x,y),(\xi,\eta)) + \mu((x,y),(\xi',\eta')).$$

Nonetheless, μ induces a regular path metric on horizontal curves between points in V (see (8) and (9)) by integration. If $g : [0,1] \rightarrow V$ is a horizontal rectifiable path $g(s) = (g_1(s),y)$, then its length with respect to μ is given by the formula $\ell_\mu(g) = \int_0^1 \mu(g(s),g'_1(s),0) ds$. The distance between two points (x,y) and (x',y) from V with respect to the induced metric μ is

$$(60) \quad d_\mu((x,y),(x',y)) = \inf \ell_\mu(g),$$

where the infimum is taken after all horizontal rectifiable paths $g : [0,1] \rightarrow V$ with $pr_2(g) = y, g(0) = (x,y)$ and $g(1) = (x',y)$.

Another way to combine the two metrics is by defining a true product metric, where the second coordinate is measured with respect with the Euclidean metric, and the first coordinate is an infimum of two metrics. Choose $(x,y) \in U'_t \times \mathbb{D}_r \cap W_B^-$ and non-zero $(\xi,\eta) \in T_{(x,y)}U'_t \times \mathbb{D}_r \cap W_B^-$ as before. Using relation (24), let $(\tilde{\xi},0) = D\phi_{a,t}|_{(x,y)}(\xi,0)$ and define

$$\mu'((x,y),(\xi,\eta)) := \max \left(\inf \left(\mu_{U_t}(x,\xi), M|\tilde{\xi}| \right), |\eta| \right).$$

The constant M is greater than $\sup \left(2\mu_{U_t}(x,\xi) / |\tilde{\xi}_1| \right)$, where

$$(\xi_1,\eta_1) = DH_{(x,y)}(\xi,\eta)$$

and the supremum is taken after all points $(x,y) \in W_{B'}^-$ and non-zero vectors $(\xi,\eta) \in \mathcal{C}_{(x,y)}^h$.

If we let $\phi_{a,t} = (\phi_1,\phi_2)$, then $\tilde{\xi} = \partial_x \phi_1(x,y)\xi$. Also $\mu_{U_t}(x,\xi) = \rho_{U_t}(x)\xi$, where ρ_{U_t} is the density function of the Poincaré metric of U_t . In conclusion, if we define $m(x,y) := \inf(\rho_{U_t}(x), M|\partial_x \phi_1(x,y)|)$ we get

$$\mu'((x,y),(\xi,\eta)) := \max(m(x,y)|\xi|, |\eta|).$$

With this definition it is easy to see that the triangle inequality is satisfied and μ' is an infinitesimal metric, i.e., a norm. Notice also that μ and μ' coincide when restricted to horizontal curves, and they induce the same horizontal path metric.

Lemma 3.22. *Let $t \in [-\delta', \delta']$ and $|a| < \delta$. Let k be chosen as in Proposition 3.21 and the constant ϵ_2 as in Lemma 3.15. There exists a constant*

$$k_t \geq \min(1 + \epsilon_2|t|, k) > 1,$$

independent of a , such that

$$(61) \quad \mu \left(H_{a,t}(x, y), DH_{a,t}|_{(x,y)}(\xi, \eta) \right) > k_t \cdot \mu((x, y), (\xi, \eta)),$$

for any $(x, y) \in V$ and any non-zero tangent vector (ξ, η) in the horizontal cone at (x, y) .

If $t = 0$ and $|a| < \delta$, there exists $k_0(x, y) > 1$ such that

$$(62) \quad \mu \left(H_{a,t}(x, y), DH_{a,t}|_{(x,y)}(\xi, \eta) \right) > k_0(x, y) \cdot \mu((x, y), (\xi, \eta)),$$

and $k_0(x, y)$ goes to 1 precisely when (x, y) tends to the local stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,0})$ of the semi-parabolic fixed point.

Moreover, the inequalities (61) and (62) hold true for μ' instead of μ .

Proof. The proof is identical to the proof of Lemma 2.9. We use the estimates in the horizontal cones from Propositions 3.14 and 3.21. The choice of the constant M in (59) is useful when dealing with the analogue of case (d)(i), from Lemma 2.9. The case $t = 0$ is given by [RT, Theorem 8.7]. □

A larger region of hyperbolicity. The following theorem is a classical result on dominated splitting.

Theorem 3.23 ([KH]). *A compact f -invariant set Λ is hyperbolic if there exists $\beta_0 < 1 < \beta_1$ such that for every $x \in \Lambda$ there is a decomposition $T_x M = S_x \oplus T_x$, a family of horizontal cones $\mathcal{C}_x^h \supset S_x$, and a family of vertical cones $\mathcal{C}_x^v \supset T_x$ associated with that decomposition such that*

$$Df_x \mathcal{C}_x^h \subset \text{Int } \mathcal{C}_{f(x)}^h, \quad Df_x^{-1} \mathcal{C}_{f(x)}^v \subset \text{Int } \mathcal{C}_x^v,$$

$$\|Df_x \xi\| \geq \beta_1 \|\xi\| \text{ for } \xi \in \mathcal{C}_x^h \text{ and } \|Df_x^{-1} \xi\| \geq \beta_0^{-1} \|\xi\| \text{ for } \xi \in \mathcal{C}_x^v.$$

We now have all the ingredients to prove the hyperbolicity part of Theorem 1.2.

Theorem 3.24 (Hyperbolicity). *There exist $\delta, \delta' > 0$ such that in the parametric region*

$$\mathcal{HR}_{\delta, \delta'} = \{(c, a) \in \mathcal{P}_{\lambda_t} : 0 < |a| < \delta \text{ and } -\delta' < t < \delta', t \neq 0\}$$

the Julia set $J_{c,a}$ is connected and the Hénon map $H_{c,a}$ is hyperbolic.

Proof. In Section 3.5, we built a family of horizontal and vertical cones, invariant under DH , respectively under DH^{-1} , such that DH expands with a factor of $\beta_1 > 1$ inside the horizontal cones, and DH^{-1} expands with a factor of $1/\beta_0 > 1$ inside the vertical cones. The expansion is measured with respect to the metric μ' from Lemma 3.22. The proof follows from Propositions 3.13, 3.14, 3.19 and 3.21 and Lemma 3.15 by taking $\beta_0 = \max(|a|, |\nu_{a,t}| + 3/2N_{a,t}) \ll 1$ and $\beta_1 = k_t > 1$ for $t \neq 0$. The constant k_t is given in Lemma 3.22. We then apply Theorem 3.23 for the set $\Lambda := \overline{V}$, which includes $J^+ \cap \mathbb{D}_r \times \mathbb{D}_r$, by Lemma 3.17. The set V was constructed in Section 3.4.

The fact that this hyperbolic region is inside a component of the connectedness locus follows from Corollary 3.31.1. □

It is worth mentioning that $J = J^*$ (the closure of the saddle periodic points) throughout the parametric region defined by $(c, a) \in \mathcal{P}_{\lambda_t}$, $|a| < \delta$ and $-\delta' < t < \delta'$. This follows from Theorem 3.24 and [BS1] for $t \neq 0$ and [RT] for $t = 0$.

Period doubling. Let $\mathcal{P}_{-1}^{2^n}$ be the set of parameters $(c, a) \in \mathbb{C}^2$ for which the Hénon map $H_{c,a}$ has a cycle of period 2^n with one multiplier $\lambda = -1$. Theorem 3.24 can be generalized to show that there are regions of hyperbolicity to the left and to the right of the real curve $\mathcal{P}_{-1}^{2^n} \cap \mathbb{R}^2$ (see Figure 1). Moreover, for each n , there is a region of hyperbolicity connecting $\mathcal{P}_{-1}^{2^n} \cap \mathbb{R}^2$ to $\mathcal{P}_{-1}^{2^{n+1}} \cap \mathbb{R}^2$ of “vertical” size (the size of the parameter a) $\delta_n > 0$. Presumably $\delta_n \rightarrow 0$ as we approach the Feigenbaum parameter.

3.7. The function space \mathcal{F} . We will do the same construction as in Section 2.5. Let R be fixed as in equation (6). Recall that

$$\gamma_{t,0} : \mathbb{S}^1 \rightarrow U'_t, \quad \gamma_{t,0}(s) = \Psi_{p_t} \left(R^{1/2} e^{2\pi i s} \right)$$

is the equipotential of p_t that defines the outer boundary of the neighborhood U'_t constructed in Section 2.4. Define $f_0 : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow V$ as

$$f_0(s, z) = (\gamma_{t,0}(s), z).$$

The image of f_0 is thus a solid torus contained in the escaping set U^+ that represents the outer boundary of the set V .

Definition 3.25. Consider the space of functions:

$$\mathcal{F}_{a,t} = \{f_n : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow V : f_0(s, z) = (\gamma_{t,0}(s), z), f_n(s, z) = F_{a,t} \circ f_{n-1}(s, z) \text{ for } n \geq 1\},$$

Where the graph transform $F_{a,t} : \mathcal{F}_{a,t} \rightarrow \mathcal{F}_{a,t}$ is defined as $F_{a,t}(f) = \tilde{f}$, where the map \tilde{f} is continuous with respect to s , holomorphic with respect to z , and $\tilde{f}|_{s \times \mathbb{D}_r}$ is the reparametrization $\tilde{f}(s, z) = (\varphi_s(z), z)$ of one of the two vertical-like disk components of

$$H_{a,t}^{-1}(f(2s \times \mathbb{D}_r)) \cap V$$

as a graph of a function over the second coordinate, via the Inverse Function Theorem.

Remark 3.26. The maps f_n are essentially reparametrizations of the backward iterates of f_0 (inside V) under the Hénon map. The picture to keep in mind is the following: The image of the map $f_n \in \mathcal{F}_{a,t}$, $n \geq 0$, is a solid torus T_n contained in the escaping set U^+ . In the s -coordinate, the Hénon map behaves like angle doubling, whereas in the vertical z -coordinate, it behaves like a strong contraction. Therefore, the Hénon map maps T_{n+1} to another solid torus, wrapped around two times inside T_n .

We say that a complex disk is vertical-like if any tangent vector to it belongs to the vertical cones (Definitions 3.18 and 3.12) if both of them are defined, or to the one that is defined. The invariance of vertical cones (Propositions 3.19 and 3.13) and the fact that the Hénon map has degree 2 imply that the preimage of a vertical-like complex disk contained in the set $V \cap U^+$ consists of two vertical-like complex disks. Assume by induction on $n \geq 0$, that we have $f_n(s, z) = (\varphi_s^n(z), z)$, where f_n is injective, continuous with respect to s and holomorphic with respect to

z , and for any $s \in \mathbb{S}^1$, $L = f_n(2s \times \mathbb{D}_r)$ is a vertical-like disk in the escaping set U^+ . Let us show how to construct f_{n+1} . The projection Δ of L on the first coordinate is almost constant and bounded away from 0, the critical point of p_t , so the preimage $H_{a,t}^{-1}(L) \cap V$ is a disjoint union of two vertical-like disks that we would like to first label as s and $s + 1/2$ and then parametrize as graphs over the second coordinate. As in the polynomial case, there are exactly two possible choices of labelings that would make the function f_{n+1} continuous with respect to $s \in \mathbb{S}^1$. There are two holomorphic branches of the backward iterate of the polynomial p_t defined on Δ . Let now $(\varphi_{2s}^n(z), z)$ be any point of L such that $H_{a,t}^{-1}(\varphi_{2s}^n(z), z) \in V$. In particular, by analyzing the second coordinate of $H_{a,t}^{-1}(\varphi_{2s}^n(z), z)$, we see that the condition $|(\varphi_{2s}^n(z) - p(z/a) - a^2w)/a| < r$ must be satisfied, which means exactly that the first coordinate z/a is $\mathcal{O}(a)$ close to one of the two preimages of $\varphi_{2s}^n(z)$ under the polynomial p_t . The curves $f_{n+1}(s \times \mathbb{D}_r)$ and $f_{n+1}((s + 1/2) \times \mathbb{D}_r)$ correspond to different choices of the branch of p_t^{-1} (see also Section 2.5, equation (10)). The Inverse Function Theorem can be used to write the two vertical-like disks as graphs of functions over the second coordinate. Thus $f_{n+1}(s, z) = (\varphi_s^{n+1}(z), z)$ where φ_s^{n+1} is a holomorphic function, continuous with respect to s . The map f_{n+1} is injective.

Remark 3.27. This procedure can be used to define external rays for the Hénon map. External rays are very useful tools, because they give combinatorial models for the Julia set. In [BS6] and [BS7] it was shown that external rays for polynomial diffeomorphisms of \mathbb{C}^2 can be defined when J is connected. A priori we do not know that our family has connected J , but this will be shown to be true as a result of our construction, in Corollary 3.31.1. The construction of the space \mathcal{F} and of the operator F for $t = 0$ is given in [RT, Section 11] and is identical for t real and small.

On the set V we use the modified metric d_μ from (58). On the function space $\mathcal{F}_{a,t}$ we consider the metric

$$(63) \quad d(f, g) = \sup_{s \in \mathbb{S}^1} \sup_{z \in \mathbb{D}_r} d_\mu(f(s, z), g(s, z)),$$

where $d_\mu(f(s, z), g(s, z))$ is defined in (60) as the infimum of the length of horizontal rectifiable paths $\gamma : [0, 1] \rightarrow V$ with $\gamma(0) = f(s, z)$ and $\gamma(1) = g(s, z)$. The length is measured with respect to the metric μ .

Theorem 3.28. *Suppose that $t \in [-\delta', \delta']$ and $|a| < \delta$. If $t \neq 0$, then the operator $F_{a,t} : \mathcal{F}_{a,t} \rightarrow \mathcal{F}_{a,t}$ is a strong contraction, i.e., there exists a constant $K_t > 1$, which depends on t , such that*

$$d(F_{a,t}(f), F_{a,t}(g)) < \frac{1}{K_t} d(f, g) \text{ for any } f, g \in \mathcal{F}_{a,t}.$$

Proof. We can use the expanding properties of the infinitesimal pseudo-metric μ constructed in (58) to show that the operator $F_{a,t}$ contracts distances between vertical-like disks with respect to the induced metric d_μ . We show that there exists $K_t > 1$ such that

$$(64) \quad d(F_{a,t} \circ f(s \times \mathbb{D}_r), F_{a,t} \circ g(s \times \mathbb{D}_r)) < \frac{1}{K_t} d(f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r))$$

for all $f, g \in \mathcal{F}_{a,t}$ and $s \in \mathbb{S}^1$.

We first discuss the strategy in the semi-parabolic case $t = 0$, which is harder and treated in [RT]. When $t = 0$ we showed a similar inequality in [RT, Proposition 11.9]: for $f, g \in \mathcal{F}_{a,0}$ and $s \in \mathbb{S}^1$ there exists a constant $0 \leq C(f, g, s) < 1$ such that

$$d(F_{a,0} \circ f(s \times \mathbb{D}_r), F_{a,0} \circ g(s \times \mathbb{D}_r)) < C(f, g, s)d(f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r)).$$

The contraction factor $C(f, g, s)$ depends only on the distance from the fibers $f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r)$ to $W_{loc}^{ss}(\mathbf{q}_{a,0})$ and goes to 1 precisely when these fibers approach $W_{loc}^{ss}(\mathbf{q}_{a,0})$, the local stable manifold of the semi-parabolic fixed point $\mathbf{q}_{a,0}$.

We briefly explain how the factor $C(f, g, s)$ is obtained when $t = 0$. When the disks $f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r)$ are close to $W_{loc}^{ss}(\mathbf{q}_{a,0})$, they become almost vertical (the vertical cones have angle opening of $\approx |x|^{2q}$, where $|x|$ measures the distance to the stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,0})$ and is close to 0). Meanwhile, by [RT, Proposition 6.8], the expansion factor of the derivative of the Hénon map in the horizontal direction is at least $(1 + \frac{\epsilon_1}{16}|x|^q) > 1$. By using the fact that $|x|^q$ dominates $|x|^{2q}$ when $|x|$ is small, we showed in [RT, Theorem 10.2] that the factor $C(f, g, s)$ is strictly smaller than 1 and goes to 1 precisely when $x \rightarrow 0$. When the disks $f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r)$ do not belong to a small neighborhood of $W_{loc}^{ss}(\mathbf{q}_{a,0})$ we proved in [RT, Proposition 11.9] that in fact we have a strong contraction factor $C(f, g, s) < 1/(k + \mathcal{O}(|a|)) < 1$. The constant $k > 1$ is defined in equation (56).

We now return to the proof of inequality (64). When $t \neq 0$, the same proof as outlined above works, but the computations are greatly simplified, because the expansion factor from Lemma 3.22 is at least $\min(1 + \epsilon_2|t|, k)$, hence strictly greater than 1.

In a small neighborhood of $W_{loc}^{ss}(\mathbf{q}_{a,t})$, the disks $f(2s \times \mathbb{D}_r), g(2s \times \mathbb{D}_r)$ are almost vertical. Indeed, by Definition 3.12 and the invariance of vertical cones from Proposition 3.13, the vertical cones in the repelling sectors W_B^- have a narrow angle opening $\approx |x|^{2q}$ when x is close to 0. By Proposition 3.14 and Lemma 3.15, the derivative $DH_{a,t}$ expands horizontally by a factor of $(1 + \epsilon_2|t|)(1 + \frac{\epsilon_1}{16}|x|^q)$, so the operator $F_{a,t}$ contracts the distance between vertical-like disks by a factor of $(1 + \epsilon_2|t|)C(f, g, s)$. Away from the local stable manifold $W_{loc}^{ss}(\mathbf{q}_{a,t})$, we have the same strong contraction factor $1/(k + \mathcal{O}(|a|))$ as in the case $t = 0$. Let K_t be $\min(1 + \epsilon_2|t|, k + \mathcal{O}(|a|))$. In conclusion, when $t \neq 0$, we have a strong contraction factor $1/K_t$, strictly less than 1. \square

As in the polynomial case, we can reduce the hyperbolic case $t \in [-\delta', \delta']$ to the semi-parabolic case by considering

$$h : [0, \infty) \rightarrow [0, \infty), \quad h(s) := \sup_{t \in [-\delta', \delta']} h_t(s),$$

where

$$h_t(s) := \sup_{|a| \leq \delta} \sup \{d(F_{a,t} \circ f(\theta \times \mathbb{D}_r), F_{a,t} \circ g(\theta \times \mathbb{D}_r)) : f, g \in \mathcal{F}_{a,t} \text{ and } \theta \in \mathbb{S}^1 \\ \text{and } d(f(2\theta \times \mathbb{D}_r), g(2\theta \times \mathbb{D}_r)) \leq s\}.$$

By definition, for each t , the function $h_t : [0, \infty) \rightarrow [0, \infty)$ is increasing and satisfies

$$d(F_{a,t}(f), F_{a,t}(g)) < h_t(d(f, g)) \text{ for any } f, g \in \mathcal{F}_{a,t}.$$

By Theorem 3.28 we know that for $t \neq 0$, $h_t(s) < \frac{1}{K_t}s < s$ for all $s > 0$. When $t = 0$, we know a bit more: by [RT, Theorem 11.10], $h_0(s+) < s$ for all $s > 0$. We

can therefore apply Lemma 2.14 and conclude that the function $h^+ : s \mapsto h(s+)$ is a Browder function that works for all $t \in [-\delta', \delta']$. This proves that the construction can be done uniformly with respect to t and a . Uniformity with respect to $|a| < \delta$ was already shown in the semi-parabolic case $t = 0$ [RT]. Now Browder’s Theorem 2.10 proves the existence of a unique fixed point $f_{a,t}^* : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \overline{V}$ of the operator $F_{a,t}$. We summarize below some basic properties of the fixed point $f_{a,t}^*$, which are direct consequences of our construction.

Proposition 3.29. *The operator $F_{a,t}$ has a unique fixed point*

$$f_{a,t}^* : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow J_{a,t}^+ \cap \overline{V}, \quad f_{a,t}^*(s, z) = (\varphi_{t,s}(z), z).$$

The map $f_{a,t}^$ is surjective, continuous with respect to t and s , and holomorphic with respect to a and z .*

3.8. Stability and continuity of J and J^+ . The Julia sets J and J^+ depend lower-semicontinuously on the parameters, and discontinuities can occur at a parameter for which the Hénon map has a semi-parabolic fixed point [BSU17]. In Theorem 3.32 we prove that in our family of complex Hénon maps $H_{a,t}$ the sets J and J^+ depend continuously on the parameters as $t \rightarrow 0$.

We begin by analyzing the properties of the fixed point $f_{a,t}^*$ in more detail. The analysis is similar to [RT, Section 12], but the role of the parameter t is different. By Proposition 3.29, $f_{a,t}^*(s, z) = (\varphi_{t,s}(z), z)$, where $\varphi_{t,s}(z)$ is continuous with respect to $s \in \mathbb{S}^1$ and analytic with respect to $z \in \mathbb{D}_r$. The map $\varphi_{t,s}$ depends analytically on the parameter a as well, but we choose to disregard this to simplify notation. We will point out the dependency on a when needed. For each t , let $\sigma_{a,t} : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$ be defined by

$$(65) \quad \sigma_{a,t}(s, z) = (2s, a\varphi_{t,s}(z)).$$

By Proposition 3.30 below, for sufficiently small $|t|$ and $|a| \neq 0$, the map $\sigma_{a,t}$ is well-defined, open, and injective.

As in [RT, Lemma 12.2], for each t , the map $\varphi_{t,s}$ has the following expansion:

$$(66) \quad \varphi_{t,s}(z) = \gamma_t(s) - \frac{az}{2\gamma_t(s)} + a^2\beta_t(s, z, a),$$

where $\gamma_t : \mathbb{S}^1 \rightarrow J_{p_t}$ is the Carathéodory loop associated to the polynomial p_t . Recall that γ_t is continuous, surjective, and does not vanish on \mathbb{S}^1 since the critical point of the polynomial p_t does not belong to the Julia set J_{p_t} . The tail $\beta_t(s, z, a)$ is bounded with respect to a and t .

Proposition 3.30. *For sufficiently small $|t| < \delta'$ and $0 < |a| < \delta$ the map $\sigma_{a,t}$ is open and injective. Moreover $\sigma_{a,t}(\mathbb{S}^1 \times \mathbb{D}_r) \subset \mathbb{S}^1 \times \mathbb{D}_{|a|r'}$, for some $r' < r$.*

Proof. Since the critical point of p_t is far away from the Julia set J_{p_t} , there exists $\epsilon > 0$ such that $|\gamma_t(s) - \gamma_t(s+1/2)| > \epsilon$ for all $s \in \mathbb{S}^1$ and $t \in [-\delta', \delta']$. The expansion from (66) shows that there exist constants $M_t > 0$ such that $|\varphi_{t,s}(z) - \gamma_t(s)| < |a|M_t$ for all $s \in \mathbb{S}^1$ and $z \in \mathbb{D}_r$. The constant $M := \sup_{|t| < \delta'} M_t$ does not depend on t . Then for $|a| < \frac{\epsilon}{2M}$ the map σ_t is injective. It is open because locally it is a homeomorphism. □

The following theorem is a direct consequence of our construction so far.

Theorem 3.31. *Let $\lambda = e^{2\pi ip/q}$ and $\lambda_t = (1+t)\lambda$. There exists $\delta, \delta' > 0$ such that*

- for all $-\delta' < t < \delta'$ and
- for all parameters $(c, a) \in \mathcal{P}_{\lambda_t}$ with $0 < |a| < \delta$

the diagram commutes:

$$\begin{CD} \mathbb{S}^1 \times \mathbb{D}_r @>f_{a,t}^*>> J^+ \cap \mathbb{D}_r \times \mathbb{D}_r \\ @V\sigma_{a,t}VV @VVH_{c,a}V \\ \mathbb{S}^1 \times \mathbb{D}_r @>f_{a,t}^*>> J^+ \cap \mathbb{D}_r \times \mathbb{D}_r \end{CD}$$

Proof. The existence of the fixed point $f_{a,t}^*$ has already been established in the previous Section 3.7. By construction, we have that $H \circ f_{a,t}^*(s \times \mathbb{D}_r)$ is compactly contained in $f_{a,t}^*(2s \times \mathbb{D}_r)$. Thus we can write

$$\begin{aligned} H \circ f_{a,t}^*(s, z) &= (p_t(\varphi_{t,s}(z)) + a^2w + az, a\varphi_{t,s}(z)) \\ &= (\varphi_{t,2s}(a\varphi_{t,s}(z)), a\varphi_{t,s}(z)) = f_{a,t}^* \circ \sigma_{a,t}(s, z). \end{aligned}$$

The last equality follows from the fact that

$$f_{a,t}^* \circ \sigma_{a,t}(s, z) = f_{a,t}^*(2s, a\varphi_{t,s}(z)) = (\varphi_{t,2s}(a\varphi_{t,s}(z)), a\varphi_{t,s}(z)).$$

Therefore $f_{a,t}^*$ semi-conjugates H on $J^+ \cap \bar{V}$ to $\sigma_{a,t}$ on $\mathbb{S}^1 \times \mathbb{D}_r$, as claimed. The fact that $J^+ \cap \bar{V} = J^+ \cap \mathbb{D}_r \times \mathbb{D}_r$ follows from Lemma 3.17. □

Corollary 3.31.1. *The Julia set J is connected.*

Proof. By Theorem 3.31 and Lemma 3.17, we get

$$(67) \quad J = f_{a,t}^* \left(\bigcap_{n \geq 0} \sigma_{a,t}^{on}(\mathbb{S}^1 \times \mathbb{D}_r) \right).$$

By Proposition 3.30, the intersection above is a nested intersection of connected, relatively compact sets, hence connected. Then J is connected, since $f_{a,t}^*$ is continuous. See Figure 6 for parameter space pictures of the connectivity region. □

The Hénon map $H_{a,t}$ has a fixed point $\mathbf{q}_{a,t}$ with eigenvalues λ_t and ν_t . The product of the eigenvalues equals the Jacobian of the map, so $|\lambda_t||\nu_t| = |a|^2$. We write $J_{(\lambda_t, \nu_t)}$ and $J_{(\lambda_t, \nu_t)}^+$ to denote the dependency of the Julia sets J and J^+ on the eigenvalues, rather than on the parameters a and t .

Theorem 3.32 (Continuity). *There exists $\delta > 0$ such that if $|\nu_t| < \delta$ and $\nu_t \rightarrow \nu$ as $t \rightarrow 0$, then the Julia sets J and J^+ depend continuously on the parameters, i.e.,*

$$J_{(\lambda_t, \nu_t)}^+ \rightarrow J_{(\lambda, \nu)}^+ \quad \text{and} \quad J_{(\lambda_t, \nu_t)} \rightarrow J_{(\lambda, \nu)}$$

in the Hausdorff topology.

Proof. By Theorem 3.31, Lemma 3.17, and Proposition 3.29 we know that

$$(68) \quad J_{(\lambda_t, \nu_t)}^+ \cap \mathbb{D}_r \times \mathbb{D}_r = f_{a,t}^*(\mathbb{S}^1 \times \mathbb{D}_r)$$

and $f_{a,t}^*$ is continuous with respect to t and holomorphic in a . Therefore

$$J_{(\lambda_t, \nu_t)}^+ \cap \mathbb{D}_r \times \mathbb{D}_r \rightarrow J_{(\lambda, \nu)}^+ \cap \mathbb{D}_r \times \mathbb{D}_r$$

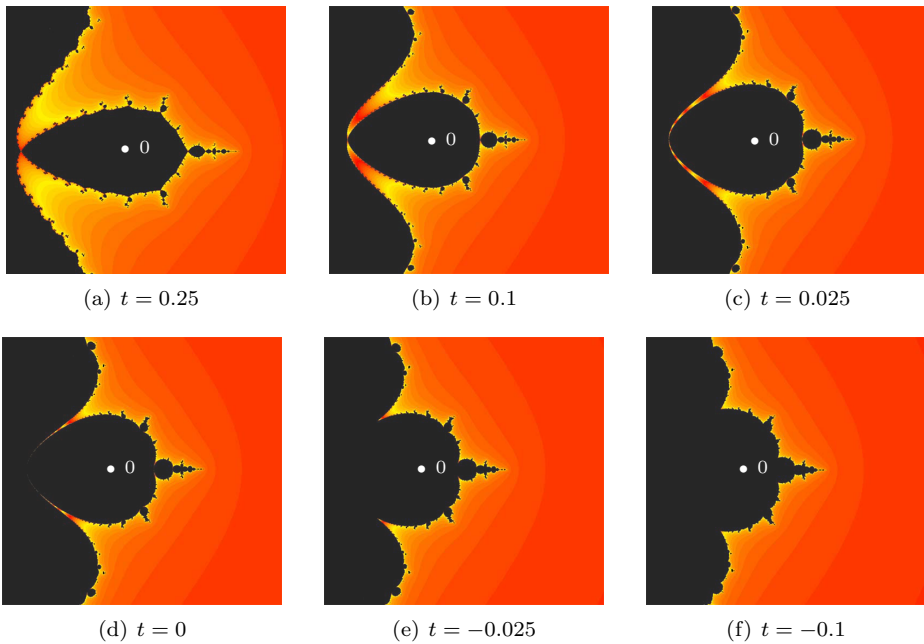


FIGURE 6. Parameter plots inside the curves $\mathcal{P}_{(1+t)\lambda}$ for $\lambda = -1$ and several values of t . In each picture, the large region in the center contains the disk $|a| < \delta$. The black region represents a rough approximation of the set of parameters $(c, a) \in \mathcal{P}_{(1+t)\lambda}$ for which the Julia set $J_{c,a}$ is connected. Here the Hénon map is written in the standard form $H_{c,a}(x, y) = (x^2 + c - ay, x)$. The pictures were generated using FractalStream.

in the Hausdorff topology, as $t \rightarrow 0$. Let $H = H_{(\lambda_t, \nu_t)}$ be the Hénon map corresponding to a pair of eigenvalues (λ_t, ν_t) . Clearly H^{-1} is continuous with respect to t . Let n be a positive integer. Taking H^{-on} in equation (68) gives

$$H^{-on} \left(J_{(\lambda_t, \nu_t)}^+ \cap \mathbb{D}_r \times \mathbb{D}_r \right) = H^{-on} f_t^* (\mathbb{S}^1 \times \mathbb{D}_r),$$

which converge in the Hausdorff topology, as $t \rightarrow 0$. We have accounted for all of J^+ , because globally the set J^+ is $\bigcup_{n \geq 0} H^{-on}(J^+ \cap \mathbb{D}_r \times \mathbb{D}_r)$.

The Julia set $J_{(\lambda_t, \nu_t)}$ can be written as in equation (67). The maps $f_{a,t}^*$ and $\sigma_{a,t}$ are continuous in a and t , so $J_{(\lambda_t, \nu_t)}$ converges to $J_{(\lambda, \nu)}$ in the Hausdorff topology. □

Remark 3.33. We have established a continuity result for real values of t , but the situation is much more general, similar to the one-dimensional case. If t is real, then the local attractive/repelling sectors from Section 3.2 are “straight”, as in the semi-parabolic case $t = 0$. If we allow t to be complex, then we need to adapt the computations from Sections 3.2 and 3.5 to construct “spiralling petals” for the Hénon map.

Suppose t is fixed. For each $s \in \mathbb{S}^1$, $f_{a,t}^*(s \times \mathbb{D}_r)$ is a vertical-like holomorphic disk. Any two such disks corresponding to distinct angles s_1 and s_2 are either disjoint or coincide (since they were obtained as a uniform limit of disjoint holomorphic disks $f_n(s_1 \times \mathbb{D}_r)$ and $f_n(s_2 \times \mathbb{D}_r)$). In the latter case, their parametrizing maps coincide, i.e.,

$$f_{a,t}^*(s_1, z) = (\varphi_{s_1}(z), z) = (\varphi_{s_2}(z), z) = f_{a,t}^*(s_2, z)$$

for all $z \in \mathbb{D}_r$. The fixed point $f_{a,t}^*$ is holomorphic with respect to a , so we can determine the equivalence classes of $f_{a,t}^*$ by letting $a \rightarrow 0$. When $a = 0$, we have $J^+ \cap \bar{V} = J_{p_t} \times \mathbb{D}_r$ so all the identifications are given by the polynomial p_t . An application of Hurwitz’s theorem (see [RT, Propositions 12.4-12.6]) gives

$$f_{a,t}^*(s_1, z_1) = f_{a,t}^*(s_2, z_2) \text{ if and only if } \gamma_t(s_1) = \gamma_t(s_2) \text{ and } z_1 = z_2.$$

Definition 3.34. We define an equivalence relation \sim on $\mathbb{S}^1 \times \mathbb{D}_r$ as follows:

$$(s_1, z) \sim (s_2, z) \text{ whenever } \gamma_t(s_1) = \gamma_t(s_2).$$

We obtain $\varphi_{t,s_1}(z) = \varphi_{t,s_2}(z)$ iff $\gamma_t(s_1) = \gamma_t(s_2)$. By equation (66) this also gives $\beta_t(s_1, z, a, t) = \beta_t(s_2, z, a, t)$ whenever $\gamma_t(s_1) = \gamma_t(s_2)$. The relation \sim is clearly closed. Moreover, since all polynomials p_t have the same Thurston lamination for $t \in [0, \delta')$ hence the same combinatorial model (see [Th]), the equivalence relation \sim does not depend on t when $t \in [0, \delta')$. Thus, in 1-D, the polynomial p_t acting on J_{p_t} is conjugate to the parabolic polynomial p_0 on J_{p_0} , for all $t \in [0, \delta')$. Note that this is not true for all $t \in (-\delta', \delta')$. For example, if $\lambda = -1$, the Julia set of p_{1-t} is a quasicircle and the associated Thurston lamination is empty. However, the Julia set of p_{1+t} is homeomorphic to the Julia set of $z \mapsto z^2 - 3/4$ (the “fat Basilica”) and the corresponding lamination is non-empty. The same situation is true in 2-D as we will show below.

Theorem 3.35 (Stability). *The family of complex Hénon maps $\mathcal{P}_{\lambda_t} \ni (c, a) \rightarrow H_{c,a}$ is a structurally stable family on J and J^+ for $0 < |a| < \delta$ and $0 \leq t < \delta'$.*

Proof. In view of equation (66), the map $\sigma_{a,t} : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$ has the form

$$(69) \quad \sigma_{a,t}(s, z) = \left(2s, a\gamma_t(s) - \frac{a^2 z}{2\gamma_t(s)} + \mathcal{O}(a^3) \right).$$

By Theorem 3.31, the map $H_{a,t}$ on $J_{a,t}^+ \cap \mathbb{D}_r \times \mathbb{D}_r$ is semi-conjugate to $\sigma_{a,t}$ on $\mathbb{S}^1 \times \mathbb{D}_r$. For $0 < |a| < \delta$ and $t \in [0, \delta')$ small enough, the maps $\sigma_{a,t}$ are conjugate to each other. The proof of this fact is the same as that of [RT, Lemmas 12.7, 12.8] stated below.

Lemma 3.36 ([RT]). *Suppose $0 < |a| < \delta$ and $t = 0$.*

- a) *The map $\sigma_{a,0} : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$ is conjugate to $\sigma'_{a,0} : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$, defined by $\sigma'_{a,0}(s, z) = \left(2s, a\gamma_0(s) - \frac{a^2 z}{2\gamma_0(s)} \right)$.*
- b) *The maps $\sigma'_{a,0}$ are conjugate to $\sigma'_{\epsilon,0}$ for some $\epsilon > 0$ independent of a .*

It is important to note that in each fiber $\{s\} \times \mathbb{D}_r$ the image of $\sigma_{a,t}$ consists of two disjoint disks. This follows from Proposition 3.30 as $\gamma_t(s)$ and $\gamma_t(s + 1/2)$ are at least ϵ -apart, for some $\epsilon > 0$ independent of a and t .

Suppose $0 < |a| < \delta$ and $t \in [0, \delta')$. The equivalence classes of $f_{a,t}^*$ are exactly the ones given by the equivalence relation \sim , in the sense that $(s_1, z) \sim (s_2, z)$ iff $f_{a,t}^*(s_1, z) = f_{a,t}^*(s_2, z)$. Moreover, by Definition 3.34 and the discussion following

it, if $(s_1, z) \sim (s_2, z)$, then $\sigma_{a,t}(s_1, z) \sim \sigma_{a,t}(s_2, z)$. Hence, $f_{a,t}^*$ and $\sigma_{a,t}$ are well defined on the quotient $\mathbb{S}^1 \times \mathbb{D}_r / \sim$ and $f_{a,t}^* : \mathbb{S}^1 \times \mathbb{D}_r / \sim \rightarrow J_{a,t}^+ \cap \mathbb{D}_r \times \mathbb{D}_r$ is bijective. The equivalence relation does not depend on the parameters t or a . We obtain that $(H_{a,t}, J_{a,t}^+ \cap \mathbb{D}_r \times \mathbb{D}_r)$ is conjugate to $(\sigma_{a,t}, \mathbb{S}^1 \times \mathbb{D}_r / \sim)$, which are conjugate to each other and to $(\sigma'_{\epsilon,0}, \mathbb{S}^1 \times \mathbb{D}_r / \sim)$, for all $0 < |a| < \delta$ and $t \in [0, \delta']$. Stability on J and J^+ follows from these observations. \square

Using the same arguments as in the previous theorem, we also have that the family $\mathcal{P}_{\lambda_t} \ni (c, a) \rightarrow H_{c,a}$ is a structurally stable family on J and J^+ for $0 < |a| < \delta$ and $t \in (-\delta', 0)$. However, this is not so surprising: by Theorem 3.24 this family of maps is hyperbolic and has connected Julia set J , so the family belongs to the same hyperbolic component of the Hénon connectedness locus. As in the polynomial case presented earlier, stability does not hold for all parameters $t \in (-\delta', \delta')$.

Using the equivalence relation from Definition 3.34 we can identify the quotient space $\mathbb{S}^1 \times \mathbb{D}_r / \sim$ with $J_{p_t} \times \mathbb{D}_r$ and the map $\sigma_{a,t} : \mathbb{S}^1 \times \mathbb{D}_r \rightarrow \mathbb{S}^1 \times \mathbb{D}_r$ defined in equation (65) with a similar map $\psi_{a,t} : J_{p_t} \times \mathbb{D}_r \rightarrow J_{p_t} \times \mathbb{D}_r$ of the form

$$(70) \quad \psi_{a,t}(\zeta, z) = \left(p_t(\zeta), a\zeta - \frac{a^2 z}{2\zeta} + \mathcal{O}(a^3) \right).$$

The following theorem is a direct consequence of the construction above and provides concrete model maps for the Hénon family. The corollaries following the theorem are immediate consequences.

Theorem 3.37. *Let $\lambda = e^{2\pi i p/q}$ and $\lambda_t = (1 + t)\lambda$. Suppose $p_t(x) = x^2 + c_t$ is a polynomial with a fixed point of multiplier λ_t . There exists $\delta, \delta' > 0$ such that*

- for all $t \in (-\delta', \delta')$ and
- for all parameters $(c, a) \in \mathcal{P}_{\lambda_t}$ with $0 < |a| < \delta$

there exists a homeomorphism $\Phi_{a,t} : J_{p_t} \times \mathbb{D}_r \rightarrow J^+ \cap \mathbb{D}_r \times \mathbb{D}_r$ which makes the diagram

$$\begin{array}{ccc} J_{p_t} \times \mathbb{D}_r & \xrightarrow{\Phi_{a,t}} & J^+ \cap \mathbb{D}_r \times \mathbb{D}_r \\ \psi_t \downarrow & & \downarrow H_{c,a} \\ J_{p_t} \times \mathbb{D}_r & \xrightarrow{\Phi_{a,t}} & J^+ \cap \mathbb{D}_r \times \mathbb{D}_r \end{array}$$

commute, where

$$(71) \quad \psi_t(\zeta, z) = \left(p_t(\zeta), \epsilon\zeta - \frac{\epsilon^2 z}{2\zeta} \right)$$

for some $\epsilon > 0$ independent of a and t .

Proof. Most of the work has already been done. The idea of the proof is the same as in [RT, Theorem 1.1]. As in [RT, Lemma 12.7] we can construct a homeomorphism $h_{a,t} : J_{p_t} \times \mathbb{D}_r \rightarrow J_{p_t} \times \mathbb{D}_r$ conjugating the map $\psi_{a,t}$ from equation (70) to the map ψ_t from equation (71), for some $\epsilon > 0$, independent of a and t . The map $\Phi_{a,t}$ is just a composition between the homeomorphism $h_{a,t}$ and the map $f_{a,t}^*$ from Theorem 3.31. \square

Corollary 3.37.1. *The Julia set J is homeomorphic to $\bigcap_{n \geq 0} \psi_t^{*n}(J_{p_t} \times \mathbb{D}_r)$.*

Corollary 3.37.2. *Passing to the inductive limit we obtain a global model for the Julia set J^+ . The map $\Phi_{a,t}$ extends naturally to a homeomorphism $\check{\Phi}_{a,t}$ which makes the following diagram:*

$$\begin{array}{ccc} \varinjlim (J_{p_t} \times \mathbb{D}_r, \psi_t) & \xrightarrow{\check{\Phi}_{a,t}} & J^+ \\ \check{\psi}_t \downarrow & & \downarrow H_{c,a} \\ \varinjlim (J_{p_t} \times \mathbb{D}_r, \psi_t) & \xrightarrow{\check{\Phi}_{a,t}} & J^+ \end{array}$$

commute.

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REFERENCES

- [A] Zin Arai, *On hyperbolic plateaus of the Hénon map*, Experiment. Math. **16** (2007), no. 2, 181–188. MR2339274
- [AI] Z. Arai and Y. Ishii, *On parameter loci of the Hénon family*, arXiv:1501.01368v2
- [B] Eric Bedford, *Complex Hénon maps with semi-parabolic fixed points*, J. Difference Equ. Appl. **16** (2010), no. 5-6, 425–426, DOI 10.1080/10236190903203838. MR2642457
- [BD] P. Berger and R. Dujardin, *On stability and hyperbolicity for polynomial automorphisms of \mathbb{C}^2* , arXiv:1409.4449, to appear in Ann. École Norm. Sup.
- [BH] X. Buff and J. H. Hubbard, *Dynamics in One Complex Variable*, to appear.
- [Br] Felix E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math. **30** (1968), 27–35. MR0230180
- [BS] Eric Bedford and John Smillie, *Real polynomial diffeomorphisms with maximal entropy: Tangencies*, Ann. of Math. (2) **160** (2004), no. 1, 1–26, DOI 10.4007/annals.2004.160.1. MR2119716
- [BS1] Eric Bedford and John Smillie, *Polynomial diffeomorphisms of \mathbb{C}^2 : currents, equilibrium measure and hyperbolicity*, Invent. Math. **103** (1991), no. 1, 69–99, DOI 10.1007/BF01239509. MR1079840
- [BS6] Eric Bedford and John Smillie, *Polynomial diffeomorphisms of \mathbb{C}^2 . VI. Connectivity of J* , Ann. of Math. (2) **148** (1998), no. 2, 695–735, DOI 10.2307/121006. MR1668567
- [BS7] Eric Bedford and John Smillie, *Polynomial diffeomorphisms of \mathbb{C}^2 . VII. Hyperbolicity and external rays* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 4, 455–497, DOI 10.1016/S0012-9593(99)80020-2. MR1693587
- [BSU17] Eric Bedford, John Smillie, and Tetsuo Ueda, *Semi-parabolic bifurcations in complex dimension two*, Comm. Math. Phys. **350** (2017), no. 1, 1–29. MR3606468
- [D] Adrien Douady, *Does a Julia set depend continuously on the polynomial?*, Complex dynamical systems (Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 91–138, DOI 10.1090/psapm/049/1315535. MR1315535
- [DH] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes. Partie I* (French), Publications Mathématiques d’Orsay [Mathematical Publications of Orsay], vol. 84, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984. MR762431
- [DL] Romain Dujardin and Mikhail Lyubich, *Stability and bifurcations for dissipative polynomial automorphisms of \mathbb{C}^2* , Invent. Math. **200** (2015), no. 2, 439–511, DOI 10.1007/s00222-014-0535-y. MR3338008
- [FM] Shmuel Friedland and John Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99, DOI 10.1017/S014338570000482X. MR991490

- [FS] John Erik Fornæss and Nessim Sibony, *Complex Hénon mappings in \mathbf{C}^2 and Fatou-Bieberbach domains*, Duke Math. J. **65** (1992), no. 2, 345–380, DOI 10.1215/S0012-7094-92-06515-X. MR1150591
- [H] J. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics. Volume 2: Surface Homeomorphisms and Rational Functions*. Matrix Editions, Ithaca, NY (2016). MR3675959
- [Ha] Monique Hakim, *Attracting domains for semi-attractive transformations of \mathbf{C}^p* , Publ. Mat. **38** (1994), no. 2, 479–499, DOI 10.5565/PUBLMAT.38294.16. MR1316642
- [Hai] P. Haïssinsky, *Applications de la chirurgie holomorphe aux systèmes dynamiques, notamment aux points paraboliques*, Thèse de l’Université de Paris-Sud, Orsay, 1998.
- [HOV1] John H. Hubbard and Ralph W. Oberste-Vorth, *Hénon mappings in the complex domain. I. The global topology of dynamical space*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 5–46. MR1307296
- [HOV2] John H. Hubbard and Ralph W. Oberste-Vorth, *Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials*, Real and complex dynamical systems (Hillerød, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 464, Kluwer Acad. Publ., Dordrecht, 1995, pp. 89–132. MR1351520
- [KH] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza. MR1326374
- [KS] William A. Kirk and Brailey Sims (eds.), *Handbook of metric fixed point theory*, Kluwer Academic Publishers, Dordrecht, 2001. MR1904271
- [Kw] Tomoki Kawahira, *Semiconjugacies between the Julia sets of geometrically finite rational maps*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1125–1152, DOI 10.1017/S0143385702001682. MR1997970
- [L] P. Lavaurs, *Systèmes dynamiques holomorphiques: explosion de points périodiques*, Thèse, Université Paris-Sud, 1989.
- [Mc] Curtis T. McMullen, *Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps*, Comment. Math. Helv. **75** (2000), no. 4, 535–593, DOI 10.1007/s000140050140. MR1789177
- [Mi] John Milnor, *Dynamics in one complex variable*, 3rd ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR2193309
- [MNTU] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, vol. 66, Cambridge University Press, Cambridge, 2000. Translated from the 1995 Japanese original and revised by the authors. MR1747010
- [R] Remus Andrei Radu, *Topological models for hyperbolic and semi-parabolic complex Hénon maps*. Pro-Quest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Cornell University. MR3193181
- [RT] Remus Radu and Raluca Tanase, *A structure theorem for semi-parabolic Hénon maps*, arXiv:1411.3824v1
- [S] Michael Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987. With the collaboration of Albert Fathi and Rémi Langevin; Translated from the French by Joseph Christy. MR869255
- [T] Raluca Elena Tanase, *Hénon maps, discrete groups and continuity of Julia sets*, Pro-Quest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Cornell University. MR3193182
- [Th] William P. Thurston, *On the geometry and dynamics of iterated rational maps*, Complex dynamics, A K Peters, Wellesley, MA, 2009, pp. 3–137, DOI 10.1201/b10617-3. Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher. MR2508255
- [U] Tetsuo Ueda, *Local structure of analytic transformations of two complex variables. I*, J. Math. Kyoto Univ. **26** (1986), no. 2, 233–261, DOI 10.1215/kjm/1250520921. MR849219

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