

## ON EXISTENCE OF GENERIC CUSP FORMS ON SEMISIMPLE ALGEBRAIC GROUPS

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ABSTRACT. In this paper we discuss the existence of certain classes of cuspidal automorphic representations having non-zero Fourier coefficients for a general semisimple algebraic group  $G$  defined over a number field  $k$  such that its Archimedean group  $G_\infty$  is not compact. When  $G$  is quasi-split over  $k$ , we obtain a result on existence of generic cuspidal automorphic representations which generalize results of Vignéras, Henniart, and Shahidi. We also discuss: (i) the existence of cuspidal automorphic forms with non-zero Fourier coefficients for congruence of subgroups of  $G_\infty$ , and (ii) applications related to the work of Bushnell and Henniart on generalized Whittaker models.

### 1. INTRODUCTION

Possibly degenerate Fourier coefficients of automorphic cuspidal forms are important for the theory of automorphic  $L$ -functions ([30], [12], [31], [17]). Recent classification of discrete global spectrum for classical groups due to Arthur [1] cannot be used directly to study Fourier coefficients of cuspidal automorphic forms. The goal of the present paper is to adjust methods of compactly supported Poincaré series as developed in [24] in order to show existence of various types of cuspidal automorphic forms with non-zero Fourier coefficients for a general semisimple algebraic group  $G$  over a number field  $k$ . We warn the reader that compactly supported Poincaré series are of a quite different nature than more classical Poincaré series considered in [7], [8], [2], where the Archimedean group  $G_\infty$  must possess representations in discrete series (see [16], [22], [23], [25], [26]).

Now, we explain the results of the present paper. We let  $G$  be a semisimple algebraic group defined over a number field  $k$ . We write  $V_f$  (resp.,  $V_\infty$ ) for the set of finite (resp., Archimedean) places. For  $v \in V := V_\infty \cup V_f$ , we write  $k_v$  for the completion of  $k$  at  $v$ . If  $v \in V_f$ , we let  $\mathcal{O}_v$  denote the ring of integers of  $k_v$ . Let  $\mathbb{A}$  be the ring of adèles of  $k$ . For almost all places of  $k$ ,  $G$  is a group scheme over  $\mathcal{O}_v$ , and  $G(\mathcal{O}_v)$  is a hyperspecial maximal compact subgroup of  $G(k_v)$  ([33, 3.9.1]); we say  $G$  is unramified over  $k_v$ . The group of adelic points  $G(\mathbb{A}) = \prod'_v G(k_v)$  is a restricted product over all places of  $k$  of the groups  $G(k_v)$ . The group  $G(\mathbb{A})$  is a locally compact group and  $G(k)$  is embedded diagonally as a discrete subgroup. The group  $G_\infty = \prod_{v \in V_\infty} G(k_v)$  is a semisimple Lie group with finite center but possibly disconnected. We assume that  $G_\infty$  is not compact. We denote by  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  a unitary representation of  $G(\mathbb{A})$  on the space of all

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cuspidal  $L^2$ -functions on  $G(k) \backslash G(\mathbb{A})$  (see Section 2 for details). It decomposes into a direct sum of irreducible unitary representations of  $G(\mathbb{A})$  called cuspidal automorphic representations. As opposed to [26], where we deal with underlying Fréchet spaces, in this paper we mostly deal with  $L^2$  spaces.

Let  $U$  be a unipotent  $k$ -subgroup of  $G$ . Let  $\psi : U(k) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. We warn the reader that  $\psi$  might be trivial. In Section 3 we define a  $(\psi, U)$ -Fourier coefficient of  $\varphi \in L^2(G(k) \backslash G(\mathbb{A}))$  by the integral

$$(1.1) \quad \mathcal{F}_{(\psi,U)}(\varphi)(g) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du,$$

which converges almost everywhere for  $g \in G(\mathbb{A})$ . We say that  $\varphi$  is  $(\psi, U)$ -generic if  $\mathcal{F}_{(\psi,U)}(\varphi) \neq 0$  (a.e.) for  $g \in G$ . According to [29], if  $G$  is quasi-split over  $k$ ,  $U$  is the unipotent radical of a Borel subgroup of  $G$  defined over  $k$ , and  $\psi$  is non-degenerate in the appropriate sense, then we use the term  $\psi$ -generic instead of  $(\psi, U)$ -generic. We refer to these settings as the ordinary generic case.

In Section 3 we adapt the arguments of ([24, §4, Theorem 4.2] to construct compactly supported Poincaré series with non-zero  $(\psi, U)$ -Fourier coefficients. We give some details. As input we take a finite set of places  $S$ , containing  $V_\infty$ , large enough such that  $G$ ,  $U$ , and  $\psi$  are unramified for  $v \notin S$ , and for each  $v \in V_f$  we have  $f_v \in C_c^\infty(G(k_v))$  and an open compact subgroup  $L_v \subset G(k_v)$  satisfying the following conditions:

- (I-a)  $f_v(1) \neq 0$ , for all  $v \in V_f$ ,
- (I-b)  $f_v = 1_{G(\mathcal{O}_v)}$  and  $L_v = G(\mathcal{O}_v)$  for all  $v \notin S$ ,
- (I-c) for  $v \in S - V_\infty$ , we have  $\int_{U(k_v)} f_v(u_v)\overline{\psi_v(u_v)}du_v \neq 0$ ,
- (I-d) and, for each  $v \in S - V_\infty$ , we require that  $f_v$  is right-invariant under  $L_v$ .

Then, as output, we find that  $f_\infty \in C_c^\infty(G_\infty)$ , so that if we let  $f = f_\infty \otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$ , then the compactly supported Poincaré series

$$(1.2) \quad P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma g), \quad g \in G(\mathbb{A})$$

satisfies:

- (I-i)  $\mathcal{F}_{(\psi,U)}(P(f))(1) \neq 0$ . In particular,  $P(f)$  is a non-zero element of  $L^2(G(k) \backslash G(\mathbb{A}))^L$ , where the open compact subgroup  $L$  is defined by  $L = \prod_{v \in V_f} L_v$ , and  $P(f)$  is  $(\psi, U)$ -generic.
- (I-ii)  $P(f)|_{G_\infty} \neq 0$  and is an element of  $L^2(\Gamma_L \backslash G_\infty)$ , where  $\Gamma_L$  is a congruence subgroup which corresponds to  $L$  from (i) (see (2.2)).
- (I-iii)  $\int_{\Gamma_L \cap U_\infty \backslash U_\infty} P(f)(u_\infty)\overline{\psi_\infty(u_\infty)}du_\infty \neq 0$ , where  $U_\infty = \prod_{v \in V_\infty} U(k_v)$ .

The reader may observe that among conditions (I-a)–(I-d), only the conditions (I-c) and (I-d) are delicate. First, we explain how to assure (I-c) and what the consequences are of (I-i). Later we explain how to deal with (I-d) and what the consequences are of (I-ii) and (I-iii).

In Section 4, we fix  $v \in V_f$  and consider local  $(\psi_v, U(k_v))$ -generic representations. Using Bernstein theory [3], we show how to construct functions  $f_v$  satisfying the conditions (I-c) while at the same time we control the smooth module generated by  $f_v$  under right translations. Lemma 4.6 contains the result regarding the relation between non-vanishing of Fourier coefficients and theory of Bernstein classes (it generalizes ([24, Lemma 5.2])). We end Section 4 with an application (see

Theorem 4.9) about the decomposition of the compactly induced representation  $\text{c-Ind}_{U(k_v)}^{G(k_v)}(\psi_v)$  (its smooth contragredient is  $\text{Ind}_{U(k_v)}^{G(k_v)}(\overline{\psi}_v)$ ) according to Bernstein classes in the ordinary generic case (see above). The proof of Theorem 4.9 is a simple consequence of our very general global results (see Lemma 5.2 and Corollary 5.8) and sufficient understanding of local theory in the ordinary generic case to be able to globalize the local situation (see the proof of Theorem 4.9). The reader should note that the result is the most important case of a result of Bushnell and Kutzko [10], but the proof here is simpler and in our opinion more natural than the one in [10].

In Section 8, we improve on the result of [10], considering a more complicated case of the decomposition into Bernstein classes of  $\text{Ind}_{U(k_v)}^{G(k_v)}(\overline{\psi}_v)$  (see Theorem 8.7). This section is local, and we work in the settings of [10].

In Section 5, we prove the main global results. Before describing them, we introduce some notation. In Section 5, we define the notion of a  $(\psi, U)$ -generic  $G(\mathbb{A})$ -irreducible closed subspace of  $L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A}))$  as follows. First, we define a closed subrepresentation

$$L_{\text{cusp}, (\psi, U)\text{-degenerate}}^2(G(k) \backslash G(\mathbb{A})) = \{ \varphi \in L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A})); \\ \varphi \text{ is not } (\psi, U)\text{-generic} \}.$$

Then, an irreducible closed subrepresentation  $\mathfrak{U}$  of  $L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A}))$  is  $(\psi, U)$ -generic if

$$\mathfrak{U} \not\subset L_{\text{cusp}, (\psi, U)\text{-degenerate}}^2(G(k) \backslash G(\mathbb{A})).$$

The reader might be surprised with this definition, but passing to  $K$ -finite vectors  $\mathfrak{U}_K$  ( $K$  is a maximal compact subgroup of  $G(\mathbb{A})$ ) we obtain the usual definition [29]. In particular, if we decompose  $\mathfrak{U}_K$  into a restricted tensor product of local representations  $\mathfrak{U}_K = \pi_\infty \otimes_{v \in V_f} \pi_v$ , then all local representations  $\pi_v$  ( $v \in V_f$ ) are  $(\psi_v, U(k_v))$ -generic in the usual sense (see Lemma 5.1). Introducing the notion of  $(\psi, U)$ -generic representation in this way makes it possible to detect the existence of  $(\psi, U)$ -generic representations contributing to the spectral decomposition of the Poincaré series  $P(f)$  (defined by (1.2)).

We remark here—and this is crucial for considerations in Section 5—that by combining local results of Section 4 with ([24, Proposition 5.3]) we may control local components in (I-c) not only to assure that the Poincaré series  $P(f)$  has a non-zero Fourier coefficient (see (I-i) above) but also that  $P(f) \in L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A}))$ . Finally, after all of these preparations, the main result of the present paper is the following theorem (see Theorem 5.9):

**Theorem 1.3.** *Assume that  $G$  is a semisimple algebraic group defined over a number field  $k$ . Let  $U$  be a unipotent  $k$ -subgroup. Let  $\psi : U(k) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. Let  $S$  be a finite set of places, containing  $V_\infty$ , large enough such that  $G$  and  $\psi$  are unramified for  $v \notin S$  (in particular,  $\psi_v$  is trivial on  $U(\mathcal{O}_v)$ ). For each finite place  $v \in S$ , let  $\mathfrak{M}_v$  be a  $(\psi_v, U(k_v))$ -generic Bernstein's class (i.e., there is a  $(\psi_v, U(k_v))$ -generic irreducible representation which belongs to that class; see Definition 4.1) such that the following holds: if  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  for all finite  $v$  in  $S$ , then  $P = G$ . Then, there exists an irreducible sub-*

space in  $L_{cusp}^2(G(k) \backslash G(\mathbb{A}))$  which is  $(\psi, U)$ -generic such that its  $K$ -finite vectors  $\pi_\infty \otimes_{v \in V_f} \pi_v$  satisfy the following:

- (i)  $\pi_v$  is unramified for  $v \notin S$ .
- (ii)  $\pi_v$  belongs to the class  $\mathfrak{M}_v$  for all finite  $v \in S$ .
- (iii)  $\pi_v$  is  $(\psi_v, U(k_v))$ -generic for all finite  $v$ .

In the ordinary generic case, the local results of Rodier ([27], [28]) are used to reformulate the requirement that the classes  $\mathfrak{M}_v$  are  $(\psi_v, U(k_v))$ -generic in its standard form (see Lemma 4.8). In this particular case, the theorem is a vast generalization of similar results of Henniart, Shahidi, and Vignéras ([15], [32], [30, Proposition 5.1]) about existence of cuspidal automorphic representations with supercuspidal local components. (See Corollary 5.10 for details.) This is because our assumption

*If  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  for all finite  $v$  in  $S$ , then  $P = G$ .*

is satisfied if one of the classes is supercuspidal. In general, none of the classes needs to be supercuspidal (see [24] for examples).

A final remark regarding the theorem is about the case in which  $U$  is the unipotent radical of a proper  $k$ -parabolic subgroup of  $G$ , and  $\psi$  is trivial. In this case, the assumptions of the theorem taken together do not hold (see the text after Lemma 5.2 for explanation). Therefore, the theorem cannot be applied to this case. Of course, this is expected since constant terms along proper  $k$ -parabolic subgroups of cuspidal automorphic forms vanish (they are Fourier coefficients in this particular case).

In Section 6 we deal with (I-d). For  $v \in S - V_\infty$ , we construct very specific matrix coefficients  $f_v$  of generic local supercuspidal representations of  $G(k_v)$  and open compact subgroups  $L_v \subset G(k_v)$  such that (I-c) and (I-d) hold (see Proposition 6.11). We use the results of ([20], [21]). In Theorem 7.3 of Section 7 we use these results along with the methods of [19] to prove the existence of certain  $(\psi_\infty, U_\infty)$ -generic cuspidal automorphic representations on  $L_{cusp}^2(\Gamma_L \backslash G_\infty)$ . We use (I-ii) and (I-iii).

## 2. PRELIMINARIES

We let  $G$  be a semisimple algebraic group defined over a number field  $k$ . We write  $V_f$  (resp.,  $V_\infty$ ) for the set of finite (resp., Archimedean) places. For  $v \in V := V_\infty \cup V_f$ , we write  $k_v$  for the completion of  $k$  at  $v$ . If  $v \in V_f$ , we let  $\mathcal{O}_v$  denote the ring of integers of  $k_v$ . Let  $\mathbb{A}$  be the ring of adèles of  $k$ . For almost all places of  $k$ ,  $G$  is a group scheme over  $\mathcal{O}_v$ , and  $G(\mathcal{O}_v)$  is a hyperspecial maximal compact subgroup of  $G(k_v)$  ([33, 3.9.1]); we say  $G$  is unramified over  $k_v$ . The group of adelic points  $G(\mathbb{A}) = \prod'_v G(k_v)$  is a restricted product over all places of  $k$  of the groups  $G(k_v)$ :  $g = (g_v)_{v \in V} \in G(\mathbb{A})$  if and only if  $g_v \in G(\mathcal{O}_v)$  for almost all  $v$ .  $G(\mathbb{A})$  is a locally compact group, and  $G(k)$  is embedded diagonally as a discrete subgroup of  $G(\mathbb{A})$ .

For a finite subset  $S \subset V$ , we let

$$G_S = \prod_{v \in S} G(k_v).$$

In addition, if  $S$  contains all Archimedean places  $V_\infty$ , we let  $G^S = \prod'_{v \notin S} G(k_v)$ . Then

$$(2.1) \quad G(\mathbb{A}) = G_S \times G^S.$$

We let  $G_\infty = G_{V_\infty}$  and  $G(\mathbb{A}_f) = G^{V_\infty}$ .

Let  $S \subset V$  be a finite set of places containing  $V_\infty$  such that  $G$  is unramified over  $k_v$ . For each  $v \in V_f$  we select an open-compact subgroup  $L_v$  such that  $L_v = G(\mathcal{O}_v)$  for all  $v \notin S$ . We define an open compact subgroup  $L \subset G(\mathbb{A}_f)$  as follows:

$$L = \prod_{v \in V_f} L_v.$$

We consider  $G(k)$  embedded diagonally in  $G^S$  and define

$$\Gamma_S = \left( \prod_{v \notin S} G(\mathcal{O}_v) \right) \cap G(k).$$

This can be considered as a subgroup of  $G_S$  using the diagonal embedding of  $G(k)$  into the product (2.1) and then the projection to the first component. Since  $G(k)$  is a discrete subgroup of  $G(\mathbb{A})$ , it follows that  $\Gamma_S$  is a discrete subgroup of  $G_S$ . In particular for  $S = V_\infty$ , considering  $G(k)$  embedded diagonally in  $G(\mathbb{A}_f)$ , we define

$$(2.2) \quad \Gamma_L = L \cap G(k),$$

where  $L$  is any open-compact subgroup of  $G(\mathbb{A}_f)$ . We obtain a discrete subgroup of  $G_\infty$  called a congruence subgroup.

The topological space  $G(k) \backslash G(\mathbb{A})$  has a finite volume  $G(\mathbb{A})$ -invariant measure:

$$(2.3) \quad \int_{G(k) \backslash G(\mathbb{A})} P(f)(g) dg \stackrel{def}{=} \int_{G(\mathbb{A})} f(g) dg, \quad f \in C_c^\infty(G(\mathbb{A})),$$

where the adelic compactly supported Poincaré series  $P(f)$  is defined as follows:

$$(2.4) \quad P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g) \in C_c^\infty(G(k) \backslash G(\mathbb{A})).$$

We remark that the space  $C_c^\infty(G(k) \backslash G(\mathbb{A}))$  is a subspace of  $C^\infty(G(\mathbb{A}))$  consisting of all functions which are  $G(k)$ -invariant on the left and which are compactly supported modulo  $G(k)$ .

The measure introduced in (2.3) enables us to introduce the Hilbert space  $L^2(G(k) \backslash G(\mathbb{A}))$ , where the inner product is the usual Petersson inner product

$$\langle \varphi, \psi \rangle = \int_{G(k) \backslash G(\mathbb{A})} \varphi(g) \overline{\psi(g)} dg.$$

It is a unitary representation of  $G(\mathbb{A})$  under right translations. Next, we define a closed subrepresentation  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  consisting of all cuspidal functions. We recall the definition of  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  and its basic properties.

Since  $G(k) \backslash G(\mathbb{A})$  has a finite volume, Hölder inequality implies that  $L^2(G(k) \backslash G(\mathbb{A}))$  is a subset of  $L^1(G(k) \backslash G(\mathbb{A}))$ . Every function  $\varphi \in L^1(G(k) \backslash G(\mathbb{A}))$  is locally integrable on  $G(\mathbb{A})$ . This means that for every compact set  $C \subset G(\mathbb{A})$  we have  $\int_C |\varphi(g)| dg < \infty$ . Next, if  $U$  is a  $k$ -unipotent subgroup of  $G$ , then  $U(k) \backslash U(\mathbb{A})$

is compact. Thus, there exists a compact neighborhood  $D$  of identity of  $U(\mathbb{A})$  such that  $U(\mathbb{A}) = U(k)D$ . Then, for every compact set  $C \subset G(\mathbb{A})$  we have

$$\begin{aligned} \int_C |\varphi(g)| dg &= \int_{U(\mathbb{A}) \backslash U(\mathbb{A})C} \left( \int_{U(k) \backslash U(\mathbb{A})} |\varphi(ug)| \sum_{\gamma \in U(k)} 1_C(\gamma ug) du \right) dg \\ &\geq \int_{U(\mathbb{A}) \backslash U(\mathbb{A})C} \left( \int_{U(k) \backslash U(\mathbb{A})} |\varphi(ug)| du \right) dg. \end{aligned}$$

Letting  $C$  vary, this implies that

$$\int_{U(k) \backslash U(\mathbb{A})} |\varphi(ug)| du < \infty, \quad (\text{a.e.}) \text{ for } g \in G(\mathbb{A}).$$

If  $P$  is a  $k$ -parabolic subgroup of  $G$ , then we denote by  $U_P$  the unipotent radical of  $P$ . For  $\varphi \in L^1(G(k) \backslash G(\mathbb{A}))$ , the constant term is a function

$$\varphi_P(g) = \int_{U_P(k) \backslash U_P(\mathbb{A})} \varphi(ug) du$$

defined almost everywhere on  $G(\mathbb{A})$ . We say that  $\varphi$  is a cuspidal function if  $\varphi_P = 0$  almost everywhere on  $G(\mathbb{A})$  for all proper  $k$ -parabolic subgroups of  $G$ . Later in the paper we need compactly supported Poincaré series which are cuspidal functions. Their construction is rather delicate. Using theory of Bernstein classes [3] and smooth representation theory of  $p$ -adic groups we describe fairly general construction of such functions in ([24, Proposition 5.3]). We use this construction later in the proofs of our main results. A different construction of such functions which are spherical has been done by Lindenstrauss and Venkatesh [18]. They rely on Satake isomorphism.

We continue with the description of  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$ . The space  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  consists of all cuspidal functions in  $L^2(G(k) \backslash G(\mathbb{A}))$ . Obviously, it is  $G(\mathbb{A})$ -invariant. It is closed since it is exactly the subspace of  $L^2(G(k) \backslash G(\mathbb{A}))$  orthogonal to all pseudo-Eisenstein series

$$E(\eta, P)(g) = \sum_{U_P(k) \backslash G(k)} \eta(\gamma g), \quad g \in G(\mathbb{A}),$$

where  $P$  ranges over all proper  $k$ -parabolic subgroups of  $G$ , and  $\eta \in C_c(U_P(\mathbb{A}) \backslash G(\mathbb{A}))$ . This follows immediately from the integration formula

$$\langle \varphi, E(\eta, P) \rangle = \int_{U_P(\mathbb{A}) \backslash G(\mathbb{A})} \varphi_P(g) \overline{\eta(g)} dg.$$

We remark that since  $U_P(k) \backslash U_P(\mathbb{A})$  is compact, we have that  $\eta$  is compactly supported modulo  $U(k)$ . Consequently, we have  $E(\eta, P) \in L^2(G(k) \backslash G(\mathbb{A}))$ .

We have the following result from representation theory:

**Theorem 2.5.** *The space  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  can be decomposed into a direct sum of irreducible unitary representations of  $G(\mathbb{A})$  each occurring with a finite multiplicity.*

3. FOURIER COEFFICIENTS AND NON-VANISHING OF POINCARÉ SERIES

We begin the section with the following standard definition (see [29, §3, for the generic case]. Let  $U$  be a unipotent  $k$ -subgroup of  $G$ . Let  $\psi : U(k) \setminus U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. We warn the reader that  $\psi$  might be trivial. As with the constant term recalled in Section 2, for  $\varphi \in L^2(G(k) \setminus G(\mathbb{A}))$ , the integral

$$(3.1) \quad \mathcal{F}_{(\psi,U)}(\varphi)(g) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du$$

converges almost everywhere for  $g \in G(\mathbb{A})$ . We say that  $\varphi$  is  $(\psi, U)$ -generic if  $\mathcal{F}_{(\psi,U)}(\varphi) \neq 0$  (a.e.) for  $g \in G(\mathbb{A})$ .

It follows from (3.1) that

$$(3.2) \quad \mathcal{F}_{(\psi,U)}(\varphi)(ug) = \psi(u)\mathcal{F}_{(\psi,U)}(\varphi)(g), \quad u \in U(\mathbb{A}), \text{ (a.e.) for } g \in G(\mathbb{A}).$$

The space defined by

$$L^2_{(\psi,U)\text{-degenerate}}(G(k) \setminus G(\mathbb{A})) = \{ \varphi \in L^2(G(k) \setminus G(\mathbb{A})); \varphi \text{ is not } (\psi, U)\text{-generic} \}$$

is closed and  $G(\mathbb{A})$ -invariant. The later is obvious, while the former follows as in Section 2, where we discussed  $L^2_{cusp}(G(k) \setminus G(\mathbb{A}))$ . Indeed, we let

$$(3.3) \quad E(\eta)(g) = \sum_{U(k)\backslash G(k)} \eta(\gamma g), \quad g \in G(\mathbb{A}),$$

where  $\eta \in C^\infty(G(\mathbb{A}))$  satisfies the following conditions:

- $\eta(ug) = \psi(u)\eta(g), \quad u \in U(\mathbb{A}), g \in G(\mathbb{A}),$
- there exists a compact subset  $C \subset G(\mathbb{A})$  (depending on  $\eta$ ) such that  $\text{supp}(\eta) \subset U(\mathbb{A}) \cdot C.$

Since  $U(k) \setminus U(\mathbb{A})$  is compact, we have that  $\eta$  is compactly supported modulo  $U(k)$ . Consequently, we have  $E(\eta) \in L^2(G(k) \setminus G(\mathbb{A}))$ . Finally,  $\varphi$  is not  $(\psi, U)$ -generic if and only if it is orthogonal to all  $E(\eta)$ . This follows immediately from the integration formula

$$(3.4) \quad \langle \varphi, E(\eta) \rangle = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \mathcal{F}_{(\psi,U)}(\varphi)(g)\overline{\eta(g)}dg,$$

whose simple proof we leave to the reader as an exercise.

After these preliminary claims, we turn our attention to construction of compactly supported Poincaré series having non-zero  $(\psi, U)$ -Fourier coefficients. We need them in Sections 5 and 7 for the proof of our main results.

**Lemma 3.5.** *Let  $G$  be a semisimple group defined over  $k$ . Let  $U$  be a unipotent  $k$ -subgroup. Let  $\psi : U(k) \setminus U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. Let  $S$  be a finite set of places, containing  $V_\infty$ , large enough such that  $G, U,$  and  $\psi$  are unramified for  $v \notin S$  (in particular,  $\psi_v$  is trivial on  $U(\mathcal{O}_v)$ ). Assume that for each  $v \in V_f$  we have  $f_v \in C_c^\infty(G(k_v))$  and an open compact subgroup  $L_v$  such that*

- (a)  $f_v = 1_{G(\mathcal{O}_v)}$  and  $L_v = G(\mathcal{O}_v)$  for all  $v \notin S,$
- (b) for  $v \in S - V_\infty,$  we have  $\int_{U(k_v)} f_v(u_v)\overline{\psi_v(u_v)}du_v \neq 0,$
- (c) and, for each  $v \in S - V_\infty,$  we require that  $f_v$  is right-invariant under  $L_v.$

Then, we can find  $f_\infty \in C_c^\infty(G_\infty)$  such that when we let  $f = f_\infty \otimes_{v \in V_f} f_v$  the following hold:

- (i)  $\mathcal{F}_{(\psi,U)}(P(f))(1) \neq 0$ . In particular,  $P(f)$  is a non-zero element of  $L^2(G(k) \backslash G(\mathbb{A}))^L$ , where the open compact subgroup  $L$  is defined by  $L = \prod_{v \in V_f} L_v$ .
- (ii)  $P(f)|_{G_\infty} \neq 0$  and is an element of  $L^2(\Gamma_L \backslash G_\infty)$ , where  $\Gamma_L$  is a congruence subgroup which corresponds to  $L$  from (i) (see (2.2)).
- (iii)  $\int_{\Gamma_L \cap U_\infty \backslash U_\infty} P(f)(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty \neq 0$ .

*Proof.* Since  $U(k) \backslash U(\mathbb{A})$  is compact, there exists a compact set  $C \subset U(\mathbb{A})$  such that  $U(\mathbb{A}) = U(k)C$ . We explain how we can choose this set more precisely. First, by strong approximation, we have

$$U(\mathbb{A}_f) = U(k) (L \cap U(\mathbb{A}_f)).$$

We consider the decomposition

$$U(\mathbb{A}) = U_\infty \times U(\mathbb{A}_f),$$

with  $U(k)$  diagonally embedded. Then, we define the continuous map

$$U_\infty \times (L \cap U(\mathbb{A}_f)) \longrightarrow U(k) \backslash U(\mathbb{A})$$

given by

$$(u_\infty, l) \mapsto U(k)(u_\infty, l).$$

By strong approximation, this map is surjective and it induces a homeomorphism of topological spaces

$$\Gamma_L \cap U_\infty \backslash U_\infty \times (L \cap U(\mathbb{A}_f)) \longrightarrow U(k) \backslash U(\mathbb{A}).$$

This implies that  $\Gamma_L \cap U_\infty \backslash U_\infty$  is compact. In particular, we can select a compact set  $C_\infty \subset U_\infty$  such that

$$U_\infty = U(k)C_\infty.$$

Hence, this implies that we can select a compact set

$$(3.6) \quad C = C_\infty \times (L \cap U(\mathbb{A}_f))$$

in order to obtain  $U(\mathbb{A}) = U(k)C$ .

Since  $G(k)$  is discrete in  $G(\mathbb{A})$  and the set (see (3.6))

$$(3.7) \quad D \stackrel{\text{def}}{=} C_\infty^{-1} \times \prod_{v \in S - V_\infty} \text{supp}(f_v) \times \prod_{v \notin S} G(\mathcal{O}_v)$$

is compact, we have that the set  $G(k) \cap D$  is finite. We claim that

$$(3.8) \quad G(k) \cap D \subset U(k).$$

Indeed, considering the projection to the first factor in (3.7), we find that

$$G(k) \cap D \subset C_\infty^{-1}$$

when we consider  $G(k)$  as a subgroup of  $G_\infty$ . But  $C_\infty^{-1} \subset U_\infty$ , so that

$$G(k) \cap D \subset C_\infty^{-1} \cap G(k) \subset U_\infty \cap G(k) = U(k).$$

This proves (3.8). Next, we can find an open set  $V'_\infty$  in  $G_\infty$  containing  $C_\infty^{-1}$  such that

$$G(k) \cap \left( V'_\infty \times \prod_{v \in S - V_\infty} \text{supp}(f_v) \times \prod_{v \notin S} G(\mathcal{O}_K) \right) = G(k) \cap D.$$



We select an open neighborhood  $V_\infty$  of identity in  $G_\infty$  such that  $V_\infty \cdot C_\infty^{-1} \subset V'_\infty$ . In particular,

$$(3.9) \quad G(k) \cap \left( V_\infty \cdot C_\infty^{-1} \times \prod_{v \in S - V_\infty} \text{supp}(f_v) \times \prod_{v \notin S} G(\mathcal{O}_K) \right) = G(k) \cap D \subset U(k).$$

Next, we select  $f_\infty \in C_c^\infty(G_\infty)$  such that  $\text{supp}(f_\infty) \subset V_\infty$  and

$$(3.10) \quad \int_{U_\infty} f_\infty(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty \neq 0.$$

This can be achieved by requiring that support of  $f_\infty$  is small enough so that it is contained in the image of the restriction of  $\exp$  to a small neighborhood of  $0 \in \mathfrak{g}_\infty$  where that restriction is a diffeomorphism onto its image. Then, we can transfer statement (3.10) to the Lie algebra by writing the Haar measure on  $U_\infty$  in local coordinates (as the differential form of top degree which never vanish). The obtained claim is easy to verify directly.

Now, we are ready to prove (i). We compute

$$\mathcal{F}_{(\psi,U)}(P(f))(1) = \int_{U(k) \setminus U(\mathbb{A})} \sum_{\gamma \in G(k)} f(\gamma \cdot u) \overline{\psi(u)} du.$$

We reduce the above expression using the following observation:

$$(3.11) \quad f(\gamma \cdot u) \neq 0, \text{ for some } u \in U(\mathbb{A}) \text{ and } \gamma \in G(k), \text{ implies that } \gamma \in U(k).$$

Let us prove (3.11). Using  $U(\mathbb{A}) = U(k)C$ , we can write  $u = \delta c$  where  $\delta \in U(k)$  and  $c \in C$ . Since  $f(\gamma \cdot u) \neq 0$ , we obtain

$$\gamma \delta \in G(k) \cap \text{supp}(f) \cdot C^{-1}.$$

The key observation is that the assumptions (a) and (c) from the statement of the lemma as well as (3.6) imply that

$$\text{supp}(f) \cdot C^{-1} = \text{supp}(f_\infty) \cdot C_\infty^{-1} \times \prod_{v \in S - V_\infty} \text{supp}(f_v) \times \prod_{v \notin S} G(\mathcal{O}_v).$$

Using this and  $\text{supp}(f_\infty) \subset V_\infty$ , (3.9) implies that

$$G(k) \cap \text{supp}(f) \cdot C^{-1} \subset U(k),$$

which shows that  $\gamma \delta \in U(k)$ . Hence  $\gamma \in U(k)$ . This proves (3.11).

Using (3.11), (b) from the statement of the lemma, and (3.10), the above integral becomes

$$\begin{aligned} \mathcal{F}_{(\psi,U)}(P(f))(1) &= \int_{U(k) \setminus U(\mathbb{A})} \sum_{\gamma \in U(k)} f(\gamma \cdot u) \overline{\psi(u)} du \\ &= \int_{U(\mathbb{A})} f(u) \overline{\psi(u)} du \\ &= \left( \int_{U(k_\infty)} f_\infty(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty \right) \prod_{v \in S - V_\infty} \int_{U(k_v)} f_v(u_v) \overline{\psi_v(u_v)} du_v \neq 0. \end{aligned}$$

This implies (i) in view of the assumption (c).

To prove (ii) and (iii), we recall that in ([24, Proposition 3.2], we proved that  $P(f)|_{G_\infty}$  is a compactly supported Poincaré series on  $G_\infty$  for  $\Gamma_L$ . This shows that it belongs to  $L^2(\Gamma_L \backslash G_\infty)$ . In order to complete the proofs of (ii) and (iii), we observe that (b) implies that each  $\psi_v$  is invariant under  $L_v \cap U(k_v)$ . This means that  $P(f)|_{U(\mathbb{A}_f)} \left( \bigotimes_{v \in V_f} \overline{\psi_v} \right)$  is right invariant under  $L \cap U(\mathbb{A}_f)$ . This enables us to apply ([24, Lemma 3.3])

$$\mathcal{F}_{(\psi,U)}(P(f))(1) = \text{vol}_{U(\mathbb{A}_f)}(L \cap U(\mathbb{A}_f)) \cdot \int_{\Gamma_L \cap U_\infty \backslash U_\infty} P(f)(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty.$$

In view of (i), this proves (ii) and (iii). □

#### 4. LOCAL GENERIC REPRESENTATIONS

In this section we discuss local generic representation. We drop index  $v$  and let  $k$  be a non-Archimedean local field. We assume that  $G$  is a semisimple group defined over  $k$ . We write  $G$  for  $G(k)$  in order to simplify notation. Similarly, we do the same for subgroups of  $G$ . The goal of this section is to explain how to construct functions satisfying Lemma 3.5(b) using the theory of Bernstein [3]. The reader may also want to consult ([24, §5]). In the present section we refine some of the results proved there for our particular application.

We introduce some notation following standard references [5] and [6]. We consider the category of all smooth complex representations of  $G$ . For a smooth representation  $\pi$ , we denote by  $\tilde{\pi}$  the smooth dual of  $\pi$ . We call it a contragredient representation.

Let  $P$  be a parabolic subgroup of  $G$  given by a Levi decomposition  $P = MU_P$ , where  $M$  is a Levi factor and  $U_P$  is the unipotent radical of  $P$ . If  $\sigma$  is a smooth representation of  $M$  extended trivially across  $U_P$  to a representation of  $P$ , then we denote the normalized induction by  $\text{Ind}_P^G(\sigma)$ . If  $\pi$  is a smooth representation of  $G$ , then we denote by  $\text{Jacq}_G^P(\pi)$  a normalized Jacquet module of  $\pi$  with respect to  $P$ . When restricted to  $U_P$ ,  $\text{Jacq}_G^P(\pi)$  is a direct sum of (possibly infinitely many) copies of a trivial representation. Therefore, when  $M$  is fixed, we write  $\text{Jacq}_G^M(\pi) = \text{Jacq}_G^P(\pi)$ . Let  $|\cdot|$  be an absolute value on  $k$ . Let  $M^0$  be the subgroup of  $M$  given as the intersection of the kernels of all characters  $m \mapsto |\chi(m)|$ , where  $\chi$  ranges over the group of all  $k$ -rational algebraic characters  $M \rightarrow k^\times$ . We say that a character  $\chi : M \rightarrow \mathbb{C}^\times$  is unramified if it is trivial on  $M^0$ . We say that an irreducible representation  $\rho$  of  $M$  is supercuspidal if  $\text{Jacq}_M^Q(\rho) = 0$  for all proper parabolic subgroups  $Q$  of  $M$ .

We recall Bernstein’s decomposition of the category of smooth complex representations of  $G$  [3]. On the set of pairs  $(M, \rho)$ , where  $M$  is a Levi subgroup of  $G$  and  $\rho$  is a smooth irreducible supercuspidal representation of  $M$ , we introduce the relation of equivalence as follows:  $(M, \rho)$  and  $(M', \rho')$  are equivalent if we can find  $g \in G$  and an unramified character  $\chi$  of  $M'$  such that  $M' = gMg^{-1}$  and  $\rho' \simeq \chi\rho^g$ , i.e.,

$$\rho^g(m') = \chi(m')\rho(g^{-1}m'g), \quad m' \in M'.$$

We write  $[M, \rho]$  for Bernstein’s equivalence class associated to a pair  $(M, \rho)$ . We say that a class  $[M, \rho]$  is supercuspidal if  $M = G$ . The contragredient Bernstein’s class  $\widetilde{\mathfrak{M}}$  of the class  $\mathfrak{M} = [M, \rho]$  is the class  $[M, \tilde{\rho}]$ .

Let  $V$  be a smooth complex representation of  $G$ . Let

$$V([M, \rho])$$

be the largest smooth submodule of  $V$  such that every irreducible subquotient of  $V$  is a subquotient of  $\text{Ind}_P^G(\chi\rho)$ , for some unramified character  $\chi$  of  $M$ . Here  $P$  is an arbitrary parabolic subgroup of  $G$  containing  $M$  as a Levi subgroup. The fundamental result of Bernstein is the following decomposition:

$$V = \bigoplus_{\mathfrak{M}} V(\mathfrak{M}),$$

where  $\mathfrak{M}$  ranges over all Bernstein equivalence classes.

We say that a smooth representation  $\pi$  of  $G$  belongs to the class  $[M, \rho]$  if the following holds:

$$\pi([M, \rho]) = \pi.$$

It is obvious that any non-zero subquotient of  $\pi$  belongs to the same class. It is well-known that each irreducible representation  $\pi$  belongs to a unique Bernstein's class.

Now, we apply this theory to study generic representations. We consider the following very general set-up. Later in the section we give examples. Let  $U$  be any unipotent  $k$ -subgroup of  $G$  and let  $\chi : U \rightarrow \mathbb{C}^\times$  be a character. Since  $U$  is a union of open compact subgroups,  $\chi$  is unitary. For the same reason,  $U$  is unimodular. We form the two types of induced representations (see ([5, 6]):

(1)  $\text{Ind}_U^G(\chi)$  on the space of all functions  $f : G \rightarrow \mathbb{C}$  satisfying  $f(ug) = \chi(u)f(g)$ , for all  $g \in G, u \in U$ , and there exists an open-compact subgroup  $L$  such that  $f(gl) = f(g)$ , for all  $g \in G, l \in L$ .

(2)  $\text{c-Ind}_U^G(\chi)$  on the space of all functions  $f \in \text{Ind}_U^G(\chi)$  which are compactly supported modulo  $U$ .

The contragredient of the representation  $\text{c-Ind}_U^G(\bar{\chi})$  is  $\text{Ind}_U^G(\chi)$ . The canonical pairing

$$\text{c-Ind}_U^G(\bar{\chi}) \times \text{Ind}_U^G(\chi) \rightarrow \mathbb{C}$$

is given by

$$\langle f, F \rangle = \int_{U \backslash G} f(g)F(g)dg.$$

Let  $\pi$  be a smooth representation of  $G$ . Let  $V$  be the space on which  $\pi$  acts. Let  $V(U, \chi)$  be the span of all vectors  $\pi(u)v - \chi(u)v, v \in V$ . Put  $r_{U, \chi}(V) = V/V(U, \chi)$ . It is the largest quotient of  $V$  on which  $U$  acts as  $\chi$ . The assignment  $V \mapsto r_{U, \chi}(V)$  can be considered as a functor from the category of smooth  $G$ -representations to the category of smooth  $U$ -representations. Since  $U$  is the union of open compact subgroups, the functor is exact ([5, Proposition 2.3.5]). The following definition is standard. Let  $\pi$  be a smooth representation of  $G$ . We say that  $\pi$  is  $(\chi, U)$ -generic if

$$\text{Hom}_G(\pi, \text{Ind}_U^G(\chi)) \neq 0.$$

By Frobenius reciprocity, this is equivalent to

$$r_{U, \chi}(\pi) \neq 0.$$

**Definition 4.1.** Let  $\mathfrak{M}$  be a Bernstein’s class. We say that  $\mathfrak{M}$  is  $(\chi, U)$ -generic if there exists an irreducible representation in this class which is  $(\chi, U)$ -generic.

In the settings of Definition 4.1, we have the following simple result:

**Lemma 4.2.** *Let  $\mathfrak{M}$  be a Bernstein’s class. Then we have the following:*

- (i) *If  $\text{c-Ind}_U^G(\chi)(\mathfrak{M}) \neq 0$ , then  $\text{Ind}_U^G(\chi)(\mathfrak{M}) \neq 0$ .*
- (ii) *If the class  $\mathfrak{M}$  is  $(\chi, U)$ -generic, then  $\text{Ind}_U^G(\chi)(\mathfrak{M}) \neq 0$ .*
- (iii) *The class  $\mathfrak{M}$  is  $(\chi, U)$ -generic if and only if  $\text{Ind}_U^G(\chi)(\mathfrak{M})$  has an irreducible subrepresentation or, equivalently,  $\text{c-Ind}_U^G(\bar{\chi})(\mathfrak{M})$  has an irreducible quotient.*
- (iv) *If a supercuspidal representation  $\rho$  is  $(\chi, U)$ -generic, then  $\text{c-Ind}_U^G(\chi)([G, \rho]) \neq 0$ , and  $\text{c-Ind}_U^G(\bar{\chi})([G, \tilde{\rho}]) \neq 0$ .*
- (v) *Conversely, if  $\text{Ind}_U^G(\chi)([G, \rho]) \neq 0$ , where  $\rho$  is a supercuspidal representation, then  $\rho$  is  $(\chi, U)$ -generic.*

*Proof.* The claims (i) and (ii) and the first claim in (iii) are obvious. For the second claim in (iii), we note that

$$\left(\text{c-Ind}_U^G(\bar{\chi})\right)^\sim \simeq \text{Ind}_U^G(\chi).$$

Hence,

$$\text{Hom}_G\left(\pi, \text{Ind}_U^G(\chi)\right) \simeq \text{Hom}_G\left(\pi, \left(\text{c-Ind}_U^G(\bar{\chi})\right)^\sim\right) \simeq \text{Hom}_G\left(\text{c-Ind}_U^G(\bar{\chi}), \tilde{\pi}\right).$$

From this observation the second claim easily follows. Let us prove (iv). By our assumption

$$\text{Hom}_G\left(\rho, \text{Ind}_U^G(\chi)\right) \neq 0.$$

By the computation in (iii), this implies that

$$(4.3) \quad \text{Hom}_G\left(\text{c-Ind}_U^G(\bar{\chi}), \tilde{\rho}\right) \neq 0.$$

Since the group  $G$  has finite center (being semisimple),  $\rho$  is a projective object in the category of all smooth representations of  $G$ . Thus, (4.3) implies that

$$(4.4) \quad \text{Hom}_G\left(\tilde{\rho}, \text{c-Ind}_U^G(\bar{\chi})\right) \neq 0.$$

This implies that

$$\text{c-Ind}_U^G(\bar{\chi})([G, \tilde{\rho}]) \neq 0.$$

Next, obviously (4.4) implies that

$$(4.5) \quad \text{Hom}_G\left(\tilde{\rho}, \text{Ind}_U^G(\bar{\chi})\right) \neq 0.$$

So again, by the proof of (iii), we obtain from (4.5) the following:

$$\text{Hom}_G\left(\text{c-Ind}_U^G(\chi), \rho\right) \simeq \text{Hom}_G\left(\text{c-Ind}_U^G(\chi), \tilde{\tilde{\rho}}\right) \neq 0.$$

Thus, finally applying the projectivity argument one more time, we obtain

$$\text{Hom}_G\left(\rho, \text{Ind}_U^G(\chi)\right) \neq 0.$$

This completes the proof of (iv). The claim (v) follows from the fact that  $\rho$  is a projective object in the category of smooth representations of  $G$ . □

In the case of usual  $\chi$ -generic representations (see the text after the proof of Lemma 4.6 below), part (iv) was proved earlier by Casselman and Shalika ([11, Corollary 6.5]).

Now, we use Bernstein’s theory to show existence of certain types of functions with non-vanishing Fourier coefficients. This will be crucial in Section 5 for global applications.

Let  $f \in C_c^\infty(G)$ . Then, we define a Fourier coefficient of  $f$  along  $U$  with respect to  $\chi$  as follows:

$$\mathcal{F}_{(\chi,U)}(f)(g) = \int_U f(ug)\overline{\chi(u)}du, \quad g \in G.$$

Clearly

$$\mathcal{F}_{(\chi,U)}(f) \in \text{c-Ind}_U^G(\chi).$$

**Lemma 4.6.** *Let  $\mathfrak{N}$  be a Bernstein’s class which satisfies  $\text{c-Ind}_U^G(\chi)(\mathfrak{N}) \neq 0$ . (By Bernstein theory there exists at least one such class.) Then, considering  $C_c^\infty(G)$  as a smooth module under right translations, there exists  $f \in C_c^\infty(G)(\mathfrak{N})$  such that the Fourier coefficient  $\mathcal{F}_{(\chi,U)}(f)$  is not identically equal to zero.*

*Proof.* We observe the following simple fact. If  $V$  and  $W$  are smooth representations such that  $W$  is a quotient of  $V$ , then, for any Bernstein’s class  $\mathfrak{N}$ ,  $W(\mathfrak{N})$  is a quotient of  $V(\mathfrak{N})$ . This follows immediately from the facts that  $V$  and  $W$  can be decomposed into a direct sum of modules  $V(\mathfrak{N})$  and  $W(\mathfrak{N})$ , and  $V(\mathfrak{N})$  is mapped into  $W(\mathfrak{N})$ .

Next, it is a standard fact that the map  $C_c^\infty(G) \rightarrow \text{c-Ind}_U^G(\chi)$  given by  $f \mapsto \mathcal{F}_{(\chi,U)}(f)$  is a surjective intertwining operator of right regular representations. This clearly implies the surjectivite map

$$C_c^\infty(G)(\mathfrak{N}) \rightarrow \text{c-Ind}_U^G(\chi)(\mathfrak{N})$$

for any Bernstein’s class  $\mathfrak{N}$ . The lemma follows. □

Now, we list some examples for the above theory. First of all, there are various trivial cases such as the case  $U = \{1\}$  and  $\chi$  is trivial or the case  $U = U_P$  and  $\chi$  is trivial, for some proper parabolic subgroup of  $G$ . The reader may want to compute generic representations in both cases as an easy exercise.

For global applications (see [30]), one instance of Lemma 4.6 is of the greatest importance. Assume that  $G$  is quasi-split over  $k$ . Let  $B = TU_B$  be a Borel subgroup defined over  $k$  given by its Levi decomposition, let  $T$  be a torus, and let  $U_B$  be a unipotent radical, both defined over  $k$ . We let  $U = U_B$  and assume that  $\chi$  is generic in the sense that  $\chi$  is not trivial when restricted to any root subgroup  $U_\alpha$ , where  $\alpha$  is a simple root corresponding to the choice of  $B$ . It is a fundamental result of Rodier [27, 28] that  $\dim r_{U,\chi}(\pi) \leq 1$ . Moreover, if  $P = MU_P$  is a standard parabolic subgroup of  $G$  (i.e.,  $B \subset P, T \subset M$  a standard choice for Levi subgroup; the details can be found in ([11, p. 208]) and  $\sigma$  is an admissible representation of  $M$ , then we have an isomorphism of vector spaces [11, 27, 28]

$$r_{U,\chi}\left(\text{Ind}_P^G(\sigma)\right) \simeq r_{U \cap M, \chi'}(\sigma),$$

where  $\chi'$  is again a generic character defined by

$$(4.7) \quad \chi'(u) = \chi(w^{-1}uw), \quad u \in U \cap M.$$

The element  $w$  is any element of  $N_G(A)$ , where  $A$  is a split component in the center of  $M$  which satisfies that the quotient  $P \setminus PwB$  is a unique open double coset in

$P \setminus G$ . As is more usual, in this case we speak of  $\chi$ -generic representations and  $\chi$ -generic Bernstein classes. In this case, the above discussion implies the following standard lemma, whose proof we leave to the reader as an exercise.

**Lemma 4.8.** *Assume that  $G$  is quasi-split over  $k$ . The class  $\mathfrak{M}$  is  $\chi$ -generic if and only if for a representative  $(M, \rho)$  of  $\mathfrak{M}$  which is taken among the set of standard Levi subgroups we have that  $\rho$  is  $\chi'$ -generic.*

We end this section with the following local result, which we prove using global methods from the next section. In the theorem below we give a simple and natural global proof of one of the main results of [10] in the most important case.

**Theorem 4.9.** *Assume that  $G$  is quasi-split over  $k$ . Let  $\chi$  be a generic character of  $U = U_B$ . Let  $\mathfrak{M}$  be any Bernstein's class such that  $\text{c-Ind}_U^G(\chi)(\mathfrak{M}) \neq 0$ . Then,  $\text{c-Ind}_U^G(\chi)(\mathfrak{M})$  has an irreducible quotient (i.e., the class  $\widetilde{\mathfrak{M}}$  is  $\bar{\chi}$ -generic).*

*Proof.* Let us make some preliminary reductions to the proof. Let us fix a generic character  $\chi_0$  of  $U = U_B$ . Let  $\bar{k}$  be the algebraic closure of  $k$ . Then, as indicated in ([30, §3]), for each generic character  $\chi$  of  $U$  there exists an element  $a \in A(\bar{k})$ , where  $A$  is a maximal split  $k$ -torus in  $T$ , such that the following hold:

- the map  $g \mapsto a^{-1}ga$  is a continuous automorphism of  $G = G(k)$ ,
- $a^{-1}Ua = U$ ,
- $\chi(u) = \chi_0(a^{-1}ua)$ , for all  $u \in U$ ,
- it fixes the set of standard parabolic subgroups of  $G$  and their standard Levi subgroups (with respect to the choice of  $B$  and  $A$ ),
- it permutes the set of supercuspidal representations and the set of unramified characters of each standard Levi subgroup  $M$ :  $\rho^a(m) = \rho(a^{-1}ma)$  and  $\chi^a(m) = \chi(a^{-1}ma)$ ,  $m \in M$ ,
- the map  $\pi \mapsto \pi^a$  permutes irreducible representations,
- if  $\pi$  is  $\chi_0$ -generic, then  $\pi^a$  is  $\chi$ -generic,
- if  $\pi$  is a subquotient of  $\text{Ind}_P^G(\chi\rho)$ , then  $\pi^a$  is a subquotient of  $\text{Ind}_P^G(\chi^a\rho^a)$ .

These facts show that it is enough to establish the theorem for some convenient character  $\chi$ . We complete the proof using Corollary 5.8, Lemma 4.10, and the fact that when  $G$  is split there exist generic supercuspidal representations of  $G$  (see Proposition 6.11 for the case of simple groups). □

**Lemma 4.10.** *Let  $H$  be a reductive group defined over a number field  $K$ . Then there exist infinitely many places  $v$  of  $K$  such that  $H$  is split over  $K_v$ .*

*Proof.* There exists a finite Galois extension  $K \subset L$  such that  $H$  splits over  $L$ ; i.e.,  $H$  has a maximal torus defined over  $L$  and split over  $L$ . On the other hand, by the Chebotarev density theorem, there exists a set of finite primes  $v$  of  $K$  of positive density which are split in the sense of algebraic number theory with respect to the extension  $K \subset L$ . For such  $v$  and a finite place  $w|v$  of  $L$ , we have  $K_v = L_w$ . Since  $H$  is obviously split over  $L_w$  (being split over  $L$ ),  $H$  is split over  $K_v$ . □

### 5. MAIN GLOBAL THEOREMS

In this section we return to the global settings of Section 3. Let  $K = K_\infty \times \prod_{v \in V_f} K_v$  be a maximal compact subgroup of  $G(\mathbb{A})$ , where  $K_v = G(\mathcal{O}_v)$  for almost all  $v$ . By Theorem 2.5,  $L^2_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$  can be decomposed into a Hilbert direct sum of irreducible unitary representations of  $G(\mathbb{A})$  each occurring with finite

multiplicity. Then, the same is true for any closed subrepresentation of  $L^2_{cusp}(G(k)\backslash G(\mathbb{A}))$ .

Let  $\mathfrak{U}$  be an irreducible subrepresentation of  $L^2_{cusp}(G(k)\backslash G(\mathbb{A}))$ . On the space  $\mathfrak{U}_K$  of  $K$ -finite vectors we have an irreducible representation  $\pi$  of  $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$ , where  $\mathfrak{g}_\infty$  is a real Lie algebra of  $G_\infty$ . In fact,  $\pi$  is an irreducible subspace of the space of all cuspidal automorphic forms  $\mathcal{A}_{cusp}(G(k)\backslash G(\mathbb{A}))$  and is dense in  $\mathfrak{U}$  (see [9]). The representation  $\pi$  is a restricted tensor product of local representations:  $\pi \simeq \pi_\infty \otimes_{v \in V_f} \pi_v$ , where for almost all  $v \in V_f$  the representation  $\pi_v$  is unramified.

Let  $U$  be a unipotent  $k$ -subgroup of  $G$ . Let  $\psi : U(k)\backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. We define a closed subrepresentation (see Section 3)

$$L^2_{cusp, (\psi, U)\text{-degenerate}}(G(k)\backslash G(\mathbb{A})) = L^2_{cusp}(G(k)\backslash G(\mathbb{A})) \cap L^2_{(\psi, U)\text{-degenerate}}(G(k)\backslash G(\mathbb{A})).$$

Let  $\mathfrak{U}$  be an irreducible closed subrepresentation of  $L^2_{cusp}(G(k)\backslash G(\mathbb{A}))$  such that

$$\mathfrak{U} \not\subset L^2_{cusp, (\psi, U)\text{-degenerate}}(G(k)\backslash G(\mathbb{A})).$$

Then we say that  $\mathfrak{U}$  is  $(\psi, U)$ -generic.

We have the following standard result:

**Lemma 5.1.** *Let  $\mathfrak{U}$  be an irreducible subrepresentation of  $L^2_{cusp}(G(k)\backslash G(\mathbb{A}))$  which is  $(\psi, U)$ -generic. Then, for every  $v \in V_f$ , the representation  $\pi_v$  is  $(\psi_v, U(k_v))$ -generic.*

*Proof.* Let  $\Lambda : \mathfrak{U}_K \rightarrow \mathbb{C}$  be a linear functional defined by

$$\varphi \mapsto \mathcal{F}_{(\psi, U)}(\varphi)(1) = \int_{U(k)\backslash U(\mathbb{A})} \varphi(u) \overline{\psi(u)} du.$$

We show that  $\Lambda$  is non-zero. Assuming this for a moment, we complete the proof of the lemma. Let us fix a finite place  $v$ . Then, for  $u \in U(k_v)$  and  $\varphi \in \mathfrak{U}_K$ , we have the following:

$$\Lambda(\pi(u_v)\varphi) = \int_{U(k)\backslash U(\mathbb{A})} \pi(u_v)\varphi(u) \overline{\psi(u)} du = \int_{U(k)\backslash U(\mathbb{A})} \varphi(uu_v) \overline{\psi(u)} du = \psi_v(u_v)\Lambda(\varphi).$$

This means that  $\mathfrak{U}_K$  is  $(\psi_v, U(k_v))$ -generic considered as a smooth  $G(k_v)$ -representation. But this representation is a direct sum of possibly infinitely many copies of  $\pi_v$ . This means that  $\pi_v$  is  $(\psi_v, U(k_v))$ -generic.

It remains to show that  $\Lambda \neq 0$ . If not, we have

$$\mathcal{F}_{(\psi, U)}(\varphi)(1) = 0$$

for all  $\varphi \in \mathfrak{U}_K$ . Since  $\mathfrak{U}_K$  is  $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$ -invariant, writing

$$G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f),$$

we conclude that

$$\mathcal{F}_{(\psi, U)}(\varphi)(k_\infty \exp(X), g_f) = \sum_{n=0}^\infty \frac{1}{n!} X^n \cdot \mathcal{F}_{(\psi, U)}(\varphi)(k_\infty, g_f) = 0,$$

for any  $g \in G(\mathbb{A}_f)$ ,  $k_\infty \in K_\infty$ , and for  $X$  in a neighborhood of 0 (depending on  $k_\infty$ ) in  $\mathfrak{g}_\infty$ . This means that there exists an open set  $V \subset G_\infty$  which meets all connected components (in usual metric topology) of  $G_\infty$  such that

$$\mathcal{F}_{(\psi, U)}(\varphi) = 0 \text{ on } V \times G(\mathbb{A}_f).$$

This implies that

$$\mathcal{F}_{(\psi,U)}(\varphi) = 0 \text{ on } G(\mathbb{A})$$

since  $\mathcal{F}_{(\psi,U)}(\varphi)$  is real-analytic in the first variable being an integral over a compact set of  $\varphi$  which is obviously a real analytic function in the first variable.

Thus, we conclude that  $\mathcal{F}_{(\psi,U)} = 0$  on the dense subset  $\mathfrak{U}_K$  of  $\mathfrak{U}$ . Now let  $\varphi \in \mathfrak{U}$ . Then, using the discussion at the beginning of Section 3 (see (3.4)), we conclude that  $\langle \varphi, \eta \rangle = 0$  for all  $\eta$  described there. From this, applying again (3.4), we conclude that  $\mathcal{F}_{(\psi,U)}(\varphi) = 0$ . Since  $\varphi \in \mathfrak{U}$  is arbitrary, we conclude that  $\mathfrak{U}$  is not  $(\psi, U)$ -generic.  $\square$

Now, we state and prove the main technical result of the present section.

**Lemma 5.2.** *Assume that  $G$  is a semisimple algebraic group defined over a number field  $k$ . Let  $U$  be a unipotent  $k$ -subgroup of  $G$ . Let  $\psi : U(k) \setminus U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. Let  $S$  be a finite set of places, containing  $V_\infty$ , large enough such that  $G$  and  $\psi$  are unramified for  $v \notin S$  (in particular,  $\psi_v$  is trivial on  $U(\mathcal{O}_v)$ ). For each finite place  $v \in S$ , let  $\mathfrak{M}_v$  be a Bernstein’s class such that  $\text{c-Ind}_{U(k_v)}^{G(k_v)}(\psi_v)(\mathfrak{M}_v) \neq 0$ . Assume the following property: if  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  for all finite  $v$  in  $S$ , then  $P = G$ . Then, there exists an irreducible subspace in  $L_{\text{cusp}}^2(G(k) \setminus G(\mathbb{A}))$  which is  $(\psi, U)$ -generic such that its  $K$ -finite vectors  $\pi_\infty \otimes_{v \in V_f} \pi_v$  satisfy the following:*

- (i)  $\pi_v$  is unramified for  $v \notin S$ .
- (ii)  $\pi_v$  belongs to the class  $\mathfrak{M}_v$  for all finite  $v \in S$ .
- (iii)  $\pi_v$  is  $(\psi_v, U(k_v))$ -generic for all finite  $v$ .

*In particular, for each finite  $v \in S$ , the class  $\mathfrak{M}_v$  is  $(\psi_v, U(k_v))$ -generic.*

Before we start the proof we make some preliminary remarks. If  $U = \{1\}$  and  $\chi = 1$ , then Lemma 5.2 is just ([24], Theorem 1.1). On the other hand, assuming that  $\chi$  is trivial and  $U$  is a unipotent radical of a proper  $k$ -parabolic subgroup  $Q$  of  $G$ , our assumptions on  $\mathfrak{M}_v$  (for finite  $v \in S$ ) mean that there exists a non-zero function  $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$  such that

$$\int_{U(k_v)} f_v(u_v g_v) du_v \neq 0$$

for some  $g_v$ . Then, ([24, Lemma 5.1]) implies that a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  is contained in a Levi subgroup of  $Q(k_v)$ . Since this holds for all  $v \in S$ , we would get  $Q = G$ , which is not possible. So, in this case, as it should be, the theorem does not give anything.

*Proof of Lemma 5.2.* As in Lemma 3.5, we let  $f_v = 1_{G(\mathcal{O}_v)}$  for all  $v \notin S$ . For finite  $v \in S$ , applying Lemma 4.6, we select  $f \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$  such that

$$\int_{U(k_v)} f_v(u_v) \overline{\psi_v(u_v)} du_v \neq 0.$$

We select open compact subgroups  $L_v$  ( $v \in V_f$ ) as required in Lemma 3.5. Then, by Lemma 3.5, there exists  $f_\infty \in C_c^\infty(G_\infty)$  such that letting  $f = f_\infty \otimes_{v \in V_f} f_v$  we have

$$\mathcal{F}_{(\psi,U)}(P(f)) \neq 0.$$



Thus,  $P(f)$  is a non-zero element of  $L^2(G(k) \backslash G(\mathbb{A}))$ . To show its cuspidality we use our assumption: if  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  for all finite  $v$  in  $S$ , then  $P = G$ , and then apply ([24, Proposition 5.3]). Thus, we obtain

$$P(f) \in L^2_{cusp}(G(k) \backslash G(\mathbb{A})).$$

Let  $\mathfrak{V}$  be a closed subspace of  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  generated by  $P(f)$ . It can be decomposed into a direct sum of irreducible unitary representations of  $G(\mathbb{A})$  each occurring with finite multiplicity:

$$\mathfrak{V} = \widehat{\bigoplus}_j \mathfrak{U}_j, \text{ each } \mathfrak{U}_j \text{ is closed and irreducible.}$$

Let us write according to this decomposition

$$(5.3) \quad P(f) = \sum_j \psi_j, \quad \psi_j \in \mathfrak{U}_j.$$

Since  $P(f)$  generates  $\mathfrak{V}$ , we must have

$$\psi_j \neq 0, \text{ for all } j.$$

Also, since

$$P(f) \notin L^2_{cusp, (\psi, U)\text{-degenerate}}(G(k) \backslash G(\mathbb{A})),$$

there exists an index  $i$  such that we have

$$\mathfrak{U}_i \not\subset L^2_{cusp, (\psi, U)\text{-degenerate}}(G(k) \backslash G(\mathbb{A})).$$

From now on, we use arguments similar to those used in the proof of ([24, Theorem 7.2]). We just outline the argument. It follows from (5.3) that the following inner product is not zero:

$$(5.4) \quad \int_{G(k) \backslash G(\mathbb{A})} P(f)(g) \overline{\psi_i(g)} dg = \int_{G(k) \backslash G(\mathbb{A})} |\psi_i(g)|^2 dg > 0.$$

Since the space of cusp forms is dense in  $\mathfrak{U}_i$  we can assume that  $\psi_i$  is a cusp form in the above inequality. In particular, this means that

$$(5.5) \quad \psi_i \in C^\infty(G(k) \backslash G(\mathbb{A})).$$

The integral on the left-hand side in (5.4) can be written as follows:

$$(5.6) \quad \int_{G(\mathbb{A})} f(g) \overline{\psi_i(g)} dg = \int_{G(k) \backslash G(\mathbb{A})} P(f)(g) \overline{\psi_i(g)} dg > 0.$$

Next, as is well-known in unitary theory, the space  $\overline{\mathfrak{U}}_i$  consisting of all  $\overline{\psi}$ ,  $\psi \in \mathfrak{U}_j$ , is a contragredient representation of  $\mathfrak{U}_i$ . Next, (5.5) and (5.6) tell us that  $f$  acts non-trivially on  $\overline{\mathfrak{U}}_i$ . If we write

$$(\mathfrak{U}_i)_K = \pi_\infty^i \bigotimes_{v \in V_f} \pi_v^i,$$

then

$$(\overline{\mathfrak{U}}_i)_K = \tilde{\pi}_\infty^i \bigotimes_{v \in V_f} \tilde{\pi}_v^i,$$

and

$$\tilde{\pi}_\infty^i(f_\infty) \bigotimes_{v \in V_f} \tilde{\pi}_v^i(f_v) \neq 0.$$

In particular, for each finite place  $v$ , we have

$$(5.7) \quad \tilde{\pi}_v^i(f_v) \neq 0.$$

Since,  $f_v = 1_{G(\mathcal{O}_v)}$ , for all  $v \notin S$ , (5.7) implies that  $\tilde{\pi}_v^i$  and hence  $\pi_v^i$  are unramified. Also, since for finite  $v \in S$ ,  $f \in C_{\mathbb{C}}^\infty(G(k_v))(\mathfrak{M}_v)$ , (5.7) and ([24, Lemma 5.2(ii)]) imply that  $\tilde{\pi}_v^i$  belongs to the class  $\widetilde{\mathfrak{M}}_v$ . Hence,  $\pi_v^i$  belongs to the class  $\mathfrak{M}_v$ . Thus, if we let  $\mathfrak{U} = \mathfrak{U}_i$ , then (i) and (ii) hold. Finally, (iii) holds by Lemma 5.1.  $\square$

We need the following result in the proof of Theorem 4.9.

**Corollary 5.8.** *Assume that  $G$  is a semisimple quasi-split algebraic group defined over a number field  $k$ . Let  $U$  be the unipotent radical of a Borel subgroup defined over  $k$ . Let  $\psi : U(k) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a non-degenerate character. Assume that  $v_0$  is a finite place of  $k$  such that  $G$  is unramified over  $k_{v_0}$  and such that there exists a  $\psi_{v_0}$ -generic supercuspidal representation of  $G(k_{v_0})$ . Then, for any other finite place  $v$ , any Bernstein's class which satisfies  $\text{c-Ind}_{U(k_v)}^{G(k_v)}(\psi_v)(\mathfrak{M}_v) \neq 0$  is  $\psi_v$ -generic.*

*Proof.* This corollary is a direct consequence of Lemma 5.2. We just need to select  $S$  large enough such that it contains both  $v$  and  $v_0$ . For each finite place  $w \in S$ ,  $w \neq v, v_0$ , let  $\mathfrak{M}_w$  be a Bernstein's class such that  $\text{c-Ind}_{U(k_w)}^{G(k_w)}(\psi_w)(\mathfrak{M}_w) \neq 0$  (at least one such class exists by Bernstein's theory since  $\text{c-Ind}_{U(k_w)}^{G(k_w)}(\psi_w) \neq 0$ ).  $\square$

The following theorem is the main result of the present section and the paper:

**Theorem 5.9.** *Assume that  $G$  is a semisimple algebraic group defined over a number field  $k$ . Let  $U$  be a unipotent  $k$ -subgroup. Let  $\psi : U(k) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a (unitary) character. Let  $S$  be a finite set of places, containing  $V_\infty$ , large enough such that  $G$  and  $\psi$  are unramified for  $v \notin S$  (in particular,  $\psi_v$  is trivial on  $U(\mathcal{O}_v)$ ). For each finite place  $v \in S$ , let  $\mathfrak{M}_v$  be a  $(\psi_v, U(k_v))$ -generic Bernstein's class such that the following holds: if  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of a Levi subgroup defining  $\mathfrak{M}_v$  for all finite  $v$  in  $S$ , then  $P = G$ . Then, there exists an irreducible subspace in  $L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A}))$  which is  $(\psi, U)$ -generic such that its  $K$ -finite vectors  $\pi_\infty \otimes_{v \in V_f} \pi_v$  satisfy the following:*

- (i)  $\pi_v$  is unramified for  $v \notin S$ .
- (ii)  $\pi_v$  belongs to the class  $\mathfrak{M}_v$  for all finite  $v \in S$ .
- (iii)  $\pi_v$  is  $(\psi_v, U(k_v))$ -generic for all finite  $v$ .

*Proof.* By Lemma 4.2(iii), for each finite  $v \in S$ , the class  $\widetilde{\mathfrak{M}}_v$  satisfies

$$\text{c-Ind}_{U(k_v)}^{G(k_v)}(\overline{\psi}_v)(\widetilde{\mathfrak{M}}_v) \neq 0.$$

Thus, by Lemma 5.2, there exists an irreducible subspace  $\mathfrak{U}$  in  $L_{\text{cusp}}^2(G(k) \backslash G(\mathbb{A}))$  which is  $(\overline{\psi}, U)$ -generic such that its  $K$ -finite vectors  $\rho_\infty \otimes_{v \in V_f} \rho_v$  satisfy the following:

- (a)  $\rho_v$  is unramified for  $v \notin S$ .
- (b)  $\rho_v$  belongs to the class  $\mathfrak{M}_v$  for all finite  $v \in S$ .
- (c)  $\rho_v$  is  $(\overline{\psi}_v, U(k_v))$ -generic for all finite  $v$ .

The contragredient representation of  $\mathfrak{U}$  can be realized on the space of all functions  $\overline{\varphi}$  where  $\varphi$  ranges over  $\mathfrak{U}$ . Then, by conjugating the Fourier coefficient of  $\mathfrak{U}$ , we see that the contragredient is  $(\psi, U)$ -generic. Thus, if we let  $\pi_\infty = \widehat{\rho}_\infty$  and  $\pi_v = \widehat{\rho}_v$ ,

for  $v \in V_f$ , then we get (i) and (ii) from (a) and (b), respectively. Finally, (iii) follows from Lemma 5.1 since the contragredient is  $(\psi, U)$ -generic.  $\square$

The following corollary of Theorem 5.9 is a generalization of similar results of Henniart, Shahidi, and Vignéras ([15], [32], [30, Proposition 5.1]). They considered the case of generic cusp forms having only supercuspidal representations as ramified local components. Those forms have non-trivial Fourier coefficients with respect to  $(\psi, U)$  where  $B = TU$  is a Borel subgroup defined over  $k$  ( $T$  is a maximal torus,  $U$  is the unipotent radical, both defined over  $k$ ) of  $G$  assumed to be quasi-split, and  $\psi$  is generic in the sense that it is not trivial when restricted to any root subgroup  $U_\alpha(\mathbb{A})$ , where  $\alpha$  is a simple root corresponding to the choice of  $B$ . As usual we call such cuspidal forms  $\psi$ -generic cuspidal forms.

**Corollary 5.10.** *Assume that  $G$  is a semisimple quasi-split algebraic group defined over a number field  $k$ . Let  $U$  be the unipotent radical of a Borel subgroup defined over  $k$ . Let  $\psi : U(k) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$  be a non-degenerate character. Let  $S$  be a finite set of places, containing  $V_\infty$ , large enough such that  $G$  and  $\psi$  are unramified for  $v \notin S$  (in particular,  $\psi_v$  is trivial on  $U(\mathcal{O}_v)$ ). For each finite place  $v \in S$ , let  $[M_v, \rho_v]$  be a Bernstein's class such that  $M_v$  is a standard Levi subgroup of  $G(k_v)$  and  $\rho_v$  is a  $\psi'_v$ -generic supercuspidal representation of  $M_v$  (see the paragraph containing (4.7) in Section 4 for notation). Assume that the following holds: if  $P$  is a  $k$ -parabolic subgroup of  $G$  such that a Levi subgroup of  $P(k_v)$  contains a conjugate of  $M_v$  for all finite  $v \in S$ , then  $P = G$ . Then, there exists an irreducible subspace in  $L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$  which is  $\psi$ -generic such that its  $K$ -finite vectors  $\pi_\infty \otimes_{v \in V_f} \pi_v$  satisfy the following:*

- (i)  $\pi_v$  is unramified for  $v \notin S$ .
- (ii)  $\pi_v$  belongs to the class  $[M_v, \rho_v]$  for all finite  $v \in S$ .
- (iii)  $\pi_v$  is  $\psi_v$ -generic for all finite  $v$ .

### 6. GENERICNESS OF THE REPRESENTATIONS OF [19]

Suppose  $k_v$  is a  $p$ -adic field with ring of integers  $\mathcal{R}_v$ . Let  $G$  be a split simple algebraic group defined over  $\mathcal{R}_v$ . As in ([19, §3.2]), set

$$\mathcal{G} := G(k_v) \text{ and } \mathcal{K} := G(\mathcal{R}_v) \text{ a maximal compact subgroup of } \mathcal{G}.$$

If  $L \subset G$  is a subgroup defined over  $\mathcal{R}_v$ , let  $L_v = L(k_v)$  be the group of  $k_v$ -rational points. Let  $B$  be a Borel subgroup defined over  $\mathcal{R}_v$ .

Let  $\mathcal{B}(\mathcal{G})$  be the Bruhat-Tits building of  $\mathcal{G}$ . Let  $x_{\mathcal{K}} \in \mathcal{B}(\mathcal{G})$  be the point fixed by  $\mathcal{K}$ . The Borel subgroup  $B$  then determines an Iwahori subgroup  $\mathcal{I} \subset \mathcal{K}$ . Let  $C = \mathcal{B}(\mathcal{G})^{\mathcal{I}}$  be the fixed points of the Iwahori subgroup  $\mathcal{I}$ . It is an alcove in  $\mathcal{B}(\mathcal{G})$ .

Take a maximally split torus  $A \subset B$  defined over  $\mathcal{R}_v$  so that  $C$  is contained in the apartment  $\mathcal{A}(A_v)$  associated to  $A_v$ . Let  $\Phi = \Phi(G, A)$  and  $\Phi^+ = \Phi^+(B, A)$  be the root system of  $A$  and positive root system with respect to  $G$  and  $B$ .

For  $\alpha \in \Phi$ , let  $U_\alpha \subset G$  denote the corresponding root group. We have

$$(6.1) \quad U(k_v) = \prod_{\alpha \in \Phi^+} U_\alpha(k_v).$$

Let  $\Gamma = \mathbb{Z}\gamma_0 \subset \mathbb{Q}$  be the additive subgroup so that the affine roots have the form  $\alpha + \eta$  with  $\alpha \in \Phi$  and  $\eta \in \Gamma$ . Let  $U_{\alpha+\eta}$  be the subgroup of  $U_\alpha(k_v)$  associated to the affine root  $\alpha + \eta$ .

Let  $\Delta$  and  $\Delta^{\text{aff}}$  be the simple roots and simple affine roots of  $\mathcal{A}(A_v)$  with respect to the Borel and Iwahori subgroups  $B$  and  $\mathcal{I}$  respectively. We recall that every  $\alpha \in \Delta$  is the gradient part of a unique root  $\psi \in \Delta^{\text{aff}}$ . In this way, we view  $\Delta$  as a subset of  $\Delta^{\text{aff}}$ .

Let  $\beta \in \Phi^+$  be the highest root, and let  $-\beta + \gamma_0$  ( $\gamma_0 > 0$ ) be the simple affine root. Let  $\ell$  be the height of  $\beta$  and take  $x_0 \in C$  to be the point satisfying

$$(6.2) \quad \forall \alpha \in \Delta \subset \Delta^{\text{aff}} : \quad \alpha(x_0) = -\beta(x_0) + \gamma_0 = \frac{\gamma_0}{\ell + 1}.$$

For  $j \geq 0$  an integer, set

$$(6.3) \quad j' := j + \left(\frac{\gamma_0}{\ell + 1}\right).$$

So,

$$(6.4) \quad \forall \alpha \in \Delta : \quad (\alpha + j)(x_0) = j + \frac{\gamma_0}{\ell + 1},$$

$$\text{and } (-\beta + \gamma_0 + j)(x_0) = j + \frac{\gamma_0}{\ell + 1}.$$

Let  $\Phi^{\text{aff}}$  denote the affine roots. We consider the Moy-Prasad groups

$$(6.5) \quad \mathcal{G}_{x_0, j'} = (A_v)_j \prod_{\substack{\psi \in \Phi^{\text{aff}} \\ \psi(x_0) \geq j'}} U_\psi \quad \text{and} \quad \mathcal{G}_{x_0, (j')^+} = (A_v)_j \prod_{\substack{\psi \in \Phi^{\text{aff}} \\ \psi(x_0) > j'}} U_\psi$$

defined in [20, 21], and

$$(6.6) \quad \text{the quotient group } \mathcal{G}_{x_0, j'} / \mathcal{G}_{x_0, (j')^+} \text{ is canonically } \prod_{\psi \in \Delta^{\text{aff}}} U_{(\psi+j)} / U_{(\psi+j^+)}.$$

As in ([19, §3.2]), let  $\chi$  be a character of the quotient  $\mathcal{G}_{x_0, j'} / \mathcal{G}_{x_0, (j')^+}$  which is non-degenerate in the sense that under the canonical isomorphism of (6.6),  $\chi$  is non-trivial on each of the groups  $U_{(\psi+j)} / U_{(\psi+j^+)}$ . Then, the proof of Lemma 3-19 in ([19, §3.2]) generalizes to show the following lemma:

**Lemma 6.7.** *Let  $\chi$  be a non-degenerate character of  $\mathcal{G}_{x_0, j'} / \mathcal{G}_{x_0, j^+}$ . Then:*

- (i) *The inflation of  $\chi$  to  $\mathcal{G}_{x_0, j'}$ , when extended to  $\mathcal{G}$  by zero outside  $\mathcal{G}_{x_0, j'}$ , is a cusp form of  $\mathcal{G}$ .*
- (ii) *For each  $j \geq 0$ , there exists an irreducible supercuspidal representation  $(\rho, W)$  which has a non-zero  $\mathcal{G}_{x_0, (j')^+}$ -invariant vector but no non-zero  $\mathcal{G}_{x_0, j'}$ -invariant vector.*

We show that the irreducible supercuspidal representations arising from the cusp form  $\chi$  are generic for a suitable (non-degenerate) character of the unipotent radical  $U(k_v)$  of  $B(k_v)$ .

Recall that the cusp form  $\chi$  satisfies the following: For  $\alpha \in \Delta$  (positive simple roots), the restriction of the character  $\chi$  to  $U_{\alpha+j}$  factors to a non-trivial character of  $U_{\alpha+j} / U_{\alpha+j^+}$ . Let  $\xi$  be a character of  $U(k_v)$  so that

$$(6.8) \quad \xi|_{U_{\alpha+j}} \text{ equals } \chi|_{U_{\alpha+j}}.$$

Clearly,  $\xi$  is a non-degenerate character of the unipotent group  $U(k_v)$ .

Recall for  $f \in C_c^\infty(G(k_v))$  that the Fourier coefficient of  $f$  along  $U(k_v)$  with respect to  $\xi$  is the function  $\mathcal{F}_{(\xi,U(k_v))}(f)$  on  $G$  defined as

$$(6.9) \quad \mathcal{F}_{(\xi,U(k_v))}(f)(g) := \int_{U(k_v)} f(ug) \overline{\xi(u)} du .$$

The coefficient  $\mathcal{F}_{(\xi,U(k_v))}(f)$  is in the induced representation

$$(6.10) \quad \text{c-Ind}_{U(k_v)}^{G(k_v)}(\xi) .$$

**Proposition 6.11.** *Consider the cusp form  $\chi$  defined in Lemma 6.7 and  $\xi$  a character of  $U(k_v)$  satisfying (6.8). Then, the Fourier coefficient  $\mathcal{F}_{(\xi,U(k_v))}(\chi)$  satisfies that  $\mathcal{F}_{(\xi,U(k_v))}(\chi)(1)$  is non-zero.*

*Proof.*

$$(6.12) \quad \begin{aligned} \mathcal{F}_{(\xi,U(k_v))}(\chi)(1) &= \int_{U \cap \mathfrak{g}_{x_0,j'}} \chi(u) \overline{\xi(u)} du = \int_{U \cap \mathfrak{g}_{x_0,j'}} 1 du \\ &= \text{meas}(U \cap \mathfrak{g}_{x_0,j'}) . \end{aligned}$$

In particular, the Fourier coefficient function  $\mathcal{F}_{(\xi,U(k_v))}(\chi)$  is a non-zero function. □

Let  $V_\chi$  be the  $G(k_v)$ -subrepresentation of  $C_c^\infty(G(k_v))$  consisting of the right translates of  $\chi$ . It is a finite length supercuspidal representation of  $G(k_v)$ , and

$$(6.13) \quad \mathcal{F}_{(\xi,U(k_v))} : V_\chi \longrightarrow \text{c-Ind}_{U(k_v)}^{G(k_v)}(\xi)$$

is a  $G(k_v)$ -map. Let  $\mathfrak{B}$  be the (finite) set of Bernstein components which appear in the representation  $V_\chi$ . Similarly, let  $\text{c-Ind}_U^G(\xi)(\mathfrak{B})$  be the Bernstein projection of  $\text{c-Ind}_U^G(\xi)(\mathfrak{B})$  to the  $\mathfrak{B}$  components. Then

$$(6.14) \quad \mathcal{F}_{(\xi,U(k_v))} : V_\chi \longrightarrow \text{c-Ind}_{U(k_v)}^{G(k_v)}(\xi)(\mathfrak{B}) ,$$

and the non-zero Fourier coefficient function  $\mathcal{F}_{(\xi,U(k_v))}(\chi)$  belongs to  $\text{c-Ind}_{U(k_v)}^{G(k_v)}(\xi)(\mathfrak{B})$ .

### 7. A RELATION TO [19]

In this section we combine the results of the current paper with the results of our previous paper [19] in order to prove the existence of generic cuspidal forms on a simply connected absolutely almost simple algebraic group  $G$  defined over  $\mathbb{Q}$  such that  $G_\infty = G(\mathbb{R})$  is not compact. We remind the reader that these are the assumptions of [19]. Examples of such groups are split Chevalley groups such as  $SL(n)$ ,  $Sp(n)$ , or split  $G_2$ . In this section we let  $k = \mathbb{Q}$ .

For each prime  $p$ , let  $\mathbb{Z}_p$  denote the  $p$ -adic integers inside  $\mathbb{Q}_p$ . Recall that for almost all primes  $p$ , the group  $G$  is unramified over  $\mathbb{Q}_p$ . Thus,  $G$  is a group scheme over  $\mathbb{Z}_p$ , and  $G(\mathbb{Z}_p)$  is a hyperspecial maximal compact subgroup of  $G(\mathbb{Q}_p)$  ([33, 3.9.1]).

As in Section 3, we let  $U$  be a unipotent  $\mathbb{Q}$ -subgroup of  $G$ . Let  $\psi : U(\mathbb{Q}) \setminus U(\mathbb{A}) \longrightarrow \mathbb{C}^\times$  be a (unitary) character.

As in ([19, Assumptions 1-3]) we consider a finite family of open compact subgroups but which satisfies more restrictive properties. We consider a finite family of open compact subgroups

$$(7.1) \quad \mathcal{F} = \{L\}$$

satisfying the following assumptions:

**Assumptions 7.2.**

- (i) Under the partial ordering of inclusion there exists a subgroup  $L_{\min} \in \mathcal{F}$  that is a subgroup of all the others.
- (ii) The groups  $L \in \mathcal{F}$  are factorizable, i.e.,  $L = \prod_p L_p$ , and for all but finitely many  $p$ 's, the group  $L_p$  is the maximal compact subgroup  $G(\mathbb{Z}_p)$ .
- (iii) There exists a non-empty finite set of primes  $T$  such that for  $p \in T$  the group  $G(\mathbb{Q}_p)$  has a local cusp form  $f_p \in C_c^\infty(G(\mathbb{Q}_p))$  which satisfies the following conditions:
  - (a)  $f_p$  is  $L_{\min,p}$ -invariant on the right, and
  - (b)  $\int_{U(\mathbb{Q}_p)} f_p(u_p) \overline{\psi_p(u_p)} du_p \neq 0$ .
 Moreover, we assume that for  $L \neq L_{\min}$  there exists  $p \in T$  such that the integral  $\int_{L_p} f_p(g_p l_p) dl_p = 0$  for all  $g_p \in G(\mathbb{Q}_p)$ .
- (iv)  $\psi_p$  is trivial on  $U(\mathbb{Q}_p) \cap L_{\min,p}$  for all  $p \notin T$ .

The reader may want to compare these assumptions with ([19, Assumptions 1-3]). We remark that using results of Section 6 we can write examples of families  $\mathcal{F}$  satisfying Assumptions 7.2 in the ordinary generic case (see the Introduction) by globalizing non-degenerate characters from that section. But this is very technical, and we do not write details here. An analogous result can be found in [19].

Let  $L \subset G(\mathbb{A}_f)$  be an open compact subgroup. We define a congruence subgroup  $\Gamma_L$  of  $G_\infty$  using (2.2). We define  $L^2_{cusp}(\Gamma_L \backslash G_\infty)$  to be the subset of  $L^2(\Gamma_L \backslash G_\infty)$  consisting of all measurable functions  $\varphi \in L^2(\Gamma_L \backslash G_\infty)$  such that

$$\int_{U_P(\mathbb{R}) \cap \Gamma_L \backslash U_P(\mathbb{R})} \varphi(ug) = 0, \quad (\text{a.e.}) \text{ for } g \in G_\infty,$$

where  $U_P$  is the unipotent radical of any proper  $\mathbb{Q}$ -parabolic subgroup  $P$ .

Further, assume that  $L$  is factorizable  $L = \prod_p L_p$  and that  $\psi_p$  is trivial on  $L_p \cap U(\mathbb{Q}_p)$  for all  $p$ . Then,  $\psi_\infty$  is trivial on  $U_\infty \cap \Gamma_L$ . We remind the reader that in the proof of Lemma 3.5 we proved that  $U_\infty \cap \Gamma_L \backslash U_\infty$  is compact. The basic considerations similar to those given at the beginning of Section 3 can be carried without difficulties. So, as in Section 3, for  $\varphi \in L^2(\Gamma_L \backslash G_\infty)$ , we define the  $(\psi_\infty, U_\infty)$ -Fourier coefficient

$$\mathcal{F}_{(\psi_\infty, U_\infty)}(\varphi)(g_\infty) = \int_{U_\infty \cap \Gamma_L \backslash U_\infty} \varphi(u_\infty g_\infty) \overline{\psi_\infty(u_\infty)} du_\infty, \quad \text{a.e. for } g_\infty \in G_\infty.$$

We say that  $\varphi$  is  $(\psi_\infty, U_\infty)$ -generic if  $\mathcal{F}_{(\psi_\infty, U_\infty)}(\varphi) \neq 0$  (a.e.). We define the closed  $G_\infty$ -invariant subspace  $L^2_{(\psi_\infty, U_\infty)\text{-degenerate}}(\Gamma_L \backslash G_\infty)$  as in Section 3. As in Section 5, we define

$$L^2_{cusp, (\psi_\infty, U_\infty)\text{-degenerate}}(\Gamma_L \backslash G_\infty) = L^2_{(\psi_\infty, U_\infty)\text{-degenerate}}(\Gamma_L \backslash G_\infty) \cap L^2_{cusp}(\Gamma_L \backslash G_\infty).$$

As before, we say that an irreducible closed subrepresentation in  $L^2_{cusp}(\Gamma_L \backslash G_\infty)$  is  $(\psi_\infty, U_\infty)$ -generic if

$$\mathfrak{U} \not\subset L^2_{cusp, (\psi_\infty, U_\infty)\text{-degenerate}}(\Gamma_L \backslash G_\infty).$$

As in the proof of Lemma 5.1, we see that the functional from the space of cuspidal automorphic forms (i.e., the space of  $K_\infty$ -finite vectors) in  $\mathfrak{U}$  given by

$$\varphi \longmapsto \int_{U_\infty \cap \Gamma_L \backslash U_\infty} \varphi(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty$$

is not zero.

After these preliminaries, we are ready to state and prove the main result of the present section. It is analogous to the main result of [19].

**Theorem 7.3.** *Suppose  $G$  is a simply connected, absolutely almost simple algebraic group defined over  $\mathbb{Q}$ , such that  $G_\infty$  is non-compact and  $\mathcal{F} = \{L\}$  is a finite set of open compact subgroups of  $G(\mathbb{A}_f)$  satisfying Assumptions (7.2). Then, the orthogonal complement of*

$$\sum_{\substack{L \in \mathcal{F} \\ L_{\min} \subsetneq L}} L^2_{cusp}(\Gamma_L \backslash G_\infty)$$

*in  $L^2_{cusp}(\Gamma_{L_{\min}} \backslash G_\infty)$  contains an  $(\psi_\infty, U_\infty)$ -generic irreducible (closed) subrepresentation.*

*Proof.* The proof of this theorem is similar to the proof of ([19, Theorem 1-4]), but instead of ([24, Theorem 4-2]), we use Lemma 3.5. For  $p \notin T$ , we let  $f_p = 1_{L_{\min, p}}$ . For  $p \in T$ , we use the cusp form  $f_p$  given by Assumptions 7.2(iii).

Now, in view of our Assumptions 7.2, we see that all assumptions (a)–(c) of Lemma 3.5 hold. As a consequence, Lemma 3.5 asserts that there exists  $f_\infty \in C_c^\infty(G_\infty)$ ,  $f_\infty \neq 0$ , such that if we let  $f = f_\infty \otimes_p f_p$ , then the following holds:

$$(7.4) \quad \int_{U_\infty \cap \Gamma_{L_{\min}} \backslash U_\infty} P(f)(u_\infty) \overline{\psi_\infty(u_\infty)} du_\infty \neq 0.$$

Next, as in the proof of Lemma 5.2, we see that  $P(f)$  is cuspidal. Hence, ([24, Proposition 3.2]) implies that  $P(f)|_{G_\infty}$  is  $\Gamma_L$ -cuspidal. Thus, (7.4) implies that  $P(f)|_{G_\infty}$  is a non-zero element of  $L^2_{cusp}(\Gamma_{L_{\min}} \backslash G_\infty)$ .

Next, as in ([19, Lemmas 2-18, 2-19]), we show that  $P(f)|_{G_\infty}$  is orthogonal to  $L^2_{cusp}(\Gamma_L \backslash G_\infty)$  in  $L^2_{cusp}(\Gamma_{L_{\min}} \backslash G_\infty)$  for all  $L \in \mathcal{F}$ ,  $L \neq L_{\min}$ . Thus, the closed  $G_\infty$ -invariant subspace  $\mathcal{U}$  in  $L^2_{cusp}(\Gamma_{L_{\min}} \backslash G_\infty)$  generated by  $P(f)|_{G_\infty}$  is non-trivial by (7.4), and consequently it is a direct sum of irreducible unitary representations each appearing with finite multiplicity [13]. Finally, using (7.4) and arguing as in the proof of Lemma 5.2, we see that some of those representations must be  $(\psi_\infty, U_\infty)$ -generic. □

### 8. DUALS AND BERNSTEIN PROJECTORS

In this section we use notation of Section 4. We relate the decomposition into Bernstein classes of a smooth representation and its contragredient. We apply obtained results to complete our investigation of [10].

Let  $Z(G)$  denote the Bernstein center of  $G$  (see [4]). If  $(\pi, V)$  is a smooth representation of  $G$ , let  $\pi : Z(G) \rightarrow \text{End}_G(V)$  be the canonical extension of  $\pi$

to  $Z(G)$ . If  $\Omega$  is a Bernstein class, let  $e_\Omega$  denote the projector distribution of  $\Omega$ , and set

$$V(\Omega) = \pi(e_\Omega)V.$$

We note the distribution  $e_\Omega$  can be represented by a local  $L^1$  function on  $G$ . For convenience we identify the distribution with its local  $L^1$  function. Define the function and distribution  $\check{e}_\Omega$  as

$$\check{e}_\Omega(x) := e_\Omega(x^{-1}).$$

**Proposition 8.1.** *Suppose  $e_\Omega$  is the projector distribution to the Bernstein class  $\Omega$ . Suppose also that  $(\pi_X, X)$  and  $(\pi_Y, Y)$  are smooth representations of  $G$  and  $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$  is a  $G$ -equivariant pairing, i.e.,*

$$\forall x \in X, y \in Y, g \in G : \quad \langle \pi_X(g)x, y \rangle = \langle x, \pi_X(g^{-1})y \rangle.$$

Then for any open compact subgroup  $J$ , we have

$$(8.2) \quad \forall x \in X, y \in Y : \quad \langle \pi_X(e_\Omega \star 1_J)x, y \rangle = \langle x, \pi_Y(\check{e}_\Omega \star 1_J)y \rangle.$$

*Proof.* We have  $(e_\Omega \star 1_J)(g) = \int_J e_\Omega(gk) dk$ . For  $x \in X$ , we have

$$\pi_X(e_\Omega \star 1_J)x = \int_G \int_J e_\Omega(gk) dk \pi_X(g)x dg.$$

If  $x \in X$  and  $y \in Y$ , then:

$$\begin{aligned} \langle \pi_X(e_\Omega \star 1_J)x, y \rangle &= \left\langle \int_G \int_J e_\Omega(gk) dk \pi_X(g)x dg, y \right\rangle \\ &= \int_G \int_J \langle e_\Omega(gk)\pi_X(g)x, y \rangle dk dg = \int_G \int_J \langle x, e_\Omega(gk)\pi_Y(g^{-1})y \rangle dk dg \\ &= \int_G \int_J \langle x, \check{e}_\Omega(k^{-1}g^{-1})\pi_Y(g^{-1})y \rangle dk dg \\ &= \int_G \langle x, \left( \int_J \check{e}_\Omega(k^{-1}h) dk \right) \pi_Y(h)y \rangle dh \\ &= \langle x, \int_G \left( \int_J \check{e}_\Omega(hk^{-1}) dk \right) \pi_Y(h)y \rangle dh = \langle x, \pi_Y(\check{e}_\Omega \star 1_J)y \rangle. \end{aligned}$$

□

**Corollary 8.3.** *In the setting of the proposition, the distribution  $\check{e}_\Omega$  is the projector to the Bernstein class  $\tilde{\Omega}$ , and we have for any  $x \in X, y \in Y$ ,*

$$(8.4) \quad \langle \pi_X(e_\Omega)x, y \rangle = \langle x, \pi_Y(\check{e}_\Omega)y \rangle.$$

*Proof.* We use the formula (8.2) in the proposition and note that for any  $x \in X, y \in Y$ , if  $J$  is a sufficiently small open compact subgroup, then  $\pi_X(e_\Omega)x = \pi_X(e_\Omega \star \frac{1}{\text{meas}}(J)1_J)x$ , and  $\pi_Y(e_\Omega)y = \pi_Y(e_\Omega \star \frac{1}{\text{meas}}(J)1_J)y$ . To prove  $\check{e}_\Omega$  is the projector to the Bernstein class  $\tilde{\Omega}$ , we apply the pairing formula (8.4) to the situation of the evaluation pairing between an irreducible  $X$  and its smooth dual  $Y = X'$ . □

**Corollary 8.5.** *Assume that  $(\pi, V)$  is a smooth representation of  $G$ . Then, for any Bernstein class  $\Omega$ , we have  $[V(\Omega)]' = V'(\tilde{\Omega})$ .*



*Proof.* Let  $\langle \cdot, \cdot \rangle : V \times V' \rightarrow \mathbb{C}$  be the evaluation pairing. Then, for any  $x \in V$ ,  $y \in V'$ , we have  $\langle \pi(e_\Omega)x, y \rangle = \langle x, \pi(e_{\tilde{\Omega}})y \rangle$ . The  $G$ -subspace  $V(\Omega)$  is a direct  $G$ -summand of  $V$ . This means that  $[V(\Omega)]'$  is a  $G$ -summand of  $V'$ . If we take  $x \in V(\Omega)$  and  $y \in [V(\Omega)]'$ , we deduce from the pairing formula that  $[V(\Omega)]'$  is  $V'(\tilde{\Omega})$ .  $\square$

Finally, we have the following:

**Corollary 8.6.** *Assume that  $(\pi, V)$  is a smooth representation of  $G$ . Then, for any Bernstein class  $\Omega$ , if  $V'(\Omega) \neq 0$ , then  $V(\tilde{\Omega}) \neq 0$ .*

*Proof.* Apply Corollary 8.5 to the class  $\tilde{\Omega}$ .  $\square$

Now, we apply our results to improve the results of Bushnell and Henniart [10]. We use notation and results of our Section 4. The reader may wish to review Section 4. Assume that  $U$  is the unipotent radical of a minimal  $k$ -parabolic subgroup of  $G$  and that  $\chi$  is a non-degenerate character of  $U$  (see §1 of [10]). In the classical Whittaker case described in the second half of Section 4, the notion coincides with the one introduced there. One of the main results of [10] says that if the Bernstein class  $\Omega$  contributes to the decomposition of  $\text{c-Ind}_U^G(\chi)$  i.e.,  $\text{c-Ind}_U^G(\chi)(\Omega) \neq 0$ , then  $\text{c-Ind}_U^G(\chi)(\Omega)$  has an irreducible quotient. Equivalently,  $\text{Ind}_U^G(\bar{\chi})(\tilde{\Omega})$  has an irreducible subrepresentation (see Lemma 4.2). In view of Definition 4.1, we say that the class  $\tilde{\Omega}$  is  $(\bar{\chi}, U)$ -generic. In the most important case of usual generic representations, the result of Bushnell and Henniart is a simple and natural consequence of our main global results (see Theorem 4.9).

**Theorem 8.7.** *In the settings of [10] mentioned above, let us decompose into Bernstein classes:*

$$\text{Ind}_U^G(\bar{\chi}) = \bigoplus_{\Omega} \text{Ind}_U^G(\bar{\chi})(\Omega).$$

*Then, if  $\text{Ind}_U^G(\bar{\chi})(\Omega) \neq 0$ , the Bernstein class  $\Omega$  is  $(\bar{\chi}, U)$ -generic.*

*Proof.* We note that  $\text{Ind}_U^G(\bar{\chi})$  is a contragredient representation of  $\text{c-Ind}_U^G(\chi)$ . Now, the theorem is a direct consequence of Corollary 8.6 and the above described result of Bushnell and Henniart.  $\square$

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