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CONTINUED FRACTIONS WITH $SL(2,Z)$-BRANCHES: COMBINATORICS AND ENTROPY

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ABSTRACT. We study the dynamics of a family K_α of discontinuous interval maps whose (infinitely many) branches are Möbius transformations in $SL(2,\mathbb{Z})$ and which arise as the critical-line case of the family of (a,b)-*continued fractions*.

We provide an explicit construction of the bifurcation locus $\mathcal{E}_{KU}$ for this family, showing it is parametrized by Farey words and it has Hausdorff dimension zero. As a consequence, we prove that the metric entropy of K_α is analytic outside the bifurcation set but not differentiable at points of $\mathcal{E}_{KU}$ and that the entropy is monotone as a function of the parameter.

Finally, we prove that the bifurcation set is combinatorially isomorphic to the main cardioid in the Mandelbrot set, providing one more entry to the dictionary developed by the authors between continued fractions and complex dynamics.

1. INTRODUCTION

It is well-known that the usual continued fraction algorithm is encoded by the dynamics of the Gauss map $G(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$; moreover, the Gauss map is known to be related, via a Poincaré section, to the geodesic flow on the modular surface $\mathbb{H}^2/SL(2,\mathbb{Z})$. In greater generality, the modular group $SL(2,\mathbb{Z})$ is generated by the transformations $\mathbf{S}x := -1/x$ and $\mathbf{T}x := x+1$, and several different continued fraction algorithms have been constructed by applying the generators according to different rules (see e.g. [22]). As in the Gauss case, to any such algorithm is associated an interval map whose branches are Möbius transformations.

Examples of such algorithms are the *continued fraction to the nearest integer* going back to Hurwitz [21], as well as the *backward continued fraction*, which is related to the reduction theory of quadratic forms [23, 38].

It turns out that the maps generating these algorithms can be seen as members of a continuous, one-parameter family of interval maps K_α. In this paper we shall be interested in describing this family K_α from a dynamical point of view: in particular, we shall identify explicitly the set of bifurcation parameters in terms of the usual continued fraction expansion and study the metric entropy as a function of the parameter.

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For each $\alpha \in [0,1]$, the map K_α is defined by fixing a "fundamental interval" $[\alpha-1,\alpha)$ and at each step applying the inversion $\mathbf{S}$ followed by as many translations $\mathbf{T}$ as are needed to come back to the fundamental domain. In particular, $K_{1/2}$ generates the nearest-integer continued fraction, while K_1 generates the backward continued fraction.

In symbols, for each $\alpha \in [0,1]$, the map $K_\alpha : [\alpha-1,\alpha] \to [\alpha-1,\alpha]$ is defined by $K_\alpha(0) = 0$ and

$$K_\alpha(x) = -\frac{1}{x} - c_\alpha(x),$$

where $c_\alpha(x) \in \mathbb{Z}$ is chosen so that the result lies in $[\alpha-1,\alpha)$. For each x, the orbit of x under K_α generates a continued fraction expansion of type

$$x = -\cfrac{1}{c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \dots}}}$$

with coefficients $c_n := c_\alpha(K_\alpha^{n-1}(x))$. In recent years, S. Katok and I. Ugarcovici, following a suggestion of D. Zagier, defined the two-dimensional family $f_{a,b}$ of (a,b)-*continued fraction transformations* and studied their dynamics and natural extensions [24, 25]. The maps K_α are the first return maps of $f_{\alpha-1,\alpha}$ on the interval $[\alpha-1,\alpha)$ and, as will be explained, they capture all the essential dynamical features. Similarly to the Gauss map, each K_α has infinitely many expanding branches and a unique absolutely continuous invariant measure μ_α.

The definition of K_α is very similar to the definition of the α-*continued fraction transformations* T_α introduced by Nakada [34] and subsequently studied by several authors [1, 6, 10, 12, 27, 28, 31, 35], the main difference being that all branches of K_α are orientation-preserving, while this is not true for T_α. In this paper we shall use techniques similar to the ones in [10] to study the K_α: as we shall see in greater detail, this will also highlight the substantial differences in the combinatorial structures of the respective bifurcation sets. In particular, we shall see that the bifurcation set of the K_α is canonically isomorphic to the set of external rays landing on the main cardioid of the Mandelbrot set (while the bifurcation set for the α-continued fractions T_α was shown to be isomorphic to the real slice of the Mandelbrot set [6]).

From a dynamical systems perspective, we shall be interested in studying the variation of the dynamics of K_α as a function of the parameter. As we shall see, there exist infinitely many islands of "stability", and each of them corresponds to a *Farey word* (see section 2). Namely, to each Farey word w we shall associate an open interval $J_w \subseteq [0,1]$ called *quadratic maximal interval*, or *quamterval* for short (see section 4.1); the *bifurcation set* $\mathcal{E}_{KU}$ is defined as the complement of all such intervals:

$$\mathcal{E}_{KU} := [0,1] \setminus \bigcup_{w \in FW} J_w.$$

The set $\mathcal{E}_{KU}$ is a Cantor set of Hausdorff dimension zero (Proposition 4.4). We shall prove (Theorem 5.1) that on each J_w we have the following matching between the orbits of α and $\alpha-1$: namely, there exist integers m_0 and m_1 (which depend only on J_w) such that

$$K_\alpha^{m_0+1}(\alpha-1) = K_\alpha^{m_1+1}(\alpha) \tag{1}$$

for all $\alpha \in J_w$.

One way to study the bifurcations of the family K_α is by considering its entropy, in the spirit of [31]. Indeed, let us define $h(\alpha)$ to be the metric entropy of the map K_α with respect to the measure μ_α (see Figure 1).

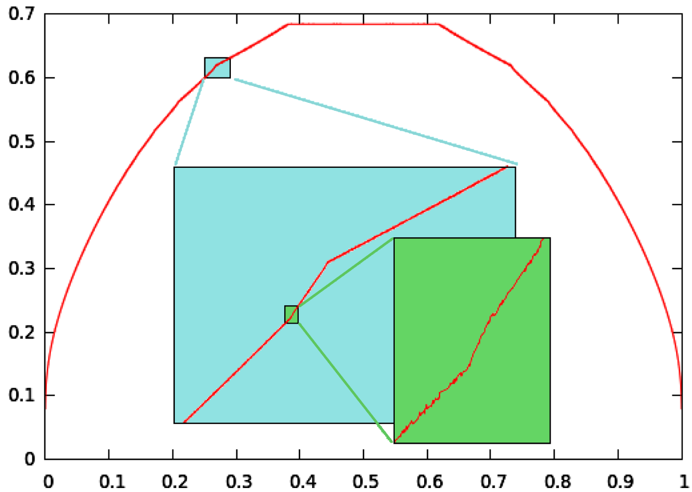

FIGURE 1. The entropy of K_α as a function of α, and a sequence of zooms around a parameter in the bifurcation set $\mathcal{E}_{KU}$. Note that the slope is increasing in each zoom, due to the fact that the entropy is not locally Lipschitz at points of $\mathcal{E}_{KU}$ (Theorem 1.1). However, the entropy is globally monotone on $[0, \frac{3-\sqrt{5}}{2}]$, as stated in Theorem 1.2.

We shall prove that the set $\mathcal{E}_{KU}$ is precisely the set of parameters for which the entropy function is not smooth.

Theorem 1.1. *The entropy function* $\alpha \mapsto h(\alpha)$

(1) *is analytic on* $[0,1] \setminus \mathcal{E}_{KU}$*;*
(2) *is not differentiable* (*and not locally Lipschitz*) *at any* $\alpha \in \mathcal{E}_{KU}$*.*

Thus, as the parameter α varies, the dynamics of K_α goes through infinitely many stable regimes, one for each connected component of the complement of $\mathcal{E}_{KU}$. We shall prove, however, that the entropy function is globally monotone across the (Cantor set of) bifurcations. In order to state the theorem, let us note that the graph of the entropy function is symmetric with respect to the transformation $\alpha \mapsto 1-\alpha$, because K_α and $K_{1-\alpha}$ are measurably conjugate (see equation (28)). Moreover, it is not hard to see by an explicit computation that the entropy is constant (and equal to $\frac{\pi^2}{6\log(1+g)}$) on the interval $[g^2, g]$, where $g := \frac{\sqrt{5}-1}{2}$ is the golden mean (so $g^2 = 1 - g = \frac{3-\sqrt{5}}{2}$).

The main theorem is the following monotonicity result for the entropy h.

Theorem 1.2. *The function* $\alpha \mapsto h(\alpha)$ *is strictly monotone increasing on* $[0, g^2]$*, constant on* $[g^2, g]$*, and strictly monotone decreasing on* $[g, 1]$*.*

Note that Theorem 1.2 highlights a major difference with the α-continued fraction case, where the entropy is not monotone [35] in any neighbourhood of $\alpha = 0$, and actually the set of parameters where the entropy is locally non-monotone has Hausdorff dimension 1 [12]. For the K_α, the study of the metric entropy was introduced by Katok and Ugarcovici in [24], [25], who gave an algorithm to produce the natural extension for any given element in the complement of $\mathcal{E}_{KU}$; as a consequence, they computed the entropy in some particular cases. The present work gives a global approach which makes it possible to study the entropy as a function of the parameter. The condition in equation (1) was introduced in [24], where it is called the *cycle property*, and it is also completely analogous to the *matching condition* used by Nakada and Natsui [35] to study the family (T_α).

Finally, we shall prove (Proposition 7.3) that the entropy tends to 0 as $\alpha \to 0^+$, and there its modulus of continuity is of order $\frac{1}{|\log \alpha|}$ (which is the same behaviour as in the case of α-continued fractions).

1.1. **Connection with the main cardioid in the Mandelbrot set.** The fact that each connected component of the complement of $\mathcal{E}_{KU}$ is naturally labelled by a Farey word can be used to draw an unexpected connection between the combinatorial structure of $\mathcal{E}_{KU}$ and the Mandelbrot set.

Recall that the *main cardioid* of the Mandelbrot set is the set of parameters $c \in \mathbb{C}$ for which the map $f_c(z) := z^2 + c$ has an attractive or indifferent fixed point. The exterior of the Mandelbrot set admits a canonical uniformization map, and to each angle $\theta \in \mathbb{R}/\mathbb{Z}$ there corresponds an associated *external ray* $R(\theta)$. Let us denote Ω to be the set of angles θ for which the ray $R(\theta)$ lands on the main cardioid.

Recall that Minkowski's *question mark function* $Q : [0,1] \to [0,1]$ is a homeomorphism of the interval which is defined by converting the continued fraction expansion of a number into a binary expansion. More precisely, if $x = [0; a_1, a_2, a_3, \dots]$ is the usual continued fraction expansion of x, then we define

$$Q(x) := 0.\underbrace{0\dots0}_{a_1-1}\underbrace{1\dots1}_{a_2}\underbrace{0\dots0}_{a_3}\dots. \tag{2}$$

We shall prove that Minkowski's function induces the following correspondence.

Theorem 1.3. *Minkowski's question mark function $Q(x)$ maps homeomorphically the bifurcation set $\mathcal{E}_{KU}$ onto the set Ω of external angles of rays landing on the main cardioid of the Mandelbrot set. In formulas, we have*

$$Q(\mathcal{E}_{KU}) = \Omega.$$

The connection may seem incidental, but it is an instance of a more general correspondence discovered by the authors in recent years. Indeed, the Minkowski map provides an explicit dictionary between sets of numbers defined using continued fractions and sets of external angles for certain fractals arising in complex dynamics. More precisely, the question mark function:

(1) maps homeomorphically the *bifurcation set for α-continued fractions* onto the set of external rays landing on the *real slice* of the boundary of the Mandelbrot set (see [6] and [37], Theorem 1.1);
(2) maps the sets of *numbers of generalized bounded type* defined in [11] to the sets of external rays landing on the real slice of the boundary of Julia sets for real quadratic polynomials ([37], Theorem 1.4);
(3) conjugates the tuning operators defined by Douady and Hubbard for the real quadratic family to tuning operators corresponding to renormalization schemes for the α-continued fractions [12].

For an introduction and more details about such correspondence we refer to one of the authors' thesis [37]. The dictionary proves to be especially useful to derive results about families of continued fractions using the large body of information known about the combinatorics of the quadratic family; moreover, it can also be used to obtain new results about the quadratic family and the Mandelbrot set from the combinatorics of continued fractions (e.g. [37], Theorem 1.6).

Being intimately connected to the structure of $\mathbb{Q}$, Farey words play a distinguished role in several other dynamical, combinatorial, or algebraic problems. To list just a few, we mention: kneading sequences for Lorentz maps [20, 29], the coding of cutting sequences on the flat torus [18] as well as on the hyperbolic one-punctured torus ([26], pp. 726-727); the Markov spectrum (in particular the Cohn tree, see [5], p. 201); primitive elements in rank two free groups [16]; the Burrows-Wheeler transform [32]; digital convexity [7]. For more information we also refer to the survey [2] or the books [15] and [3].

1.2. **Behaviour of (a, b)-continued fractions on the critical line.** We conclude the introduction by explaining in more detail the results of [24, 25] and how they relate to the present paper. For further details, see also section 7.

In [24], Katok and Ugarcovici consider the two-parameter family of continued fraction algorithms induced by the maps

$$f_{a,b}(x) := \begin{cases} x+1 & \text{if } x < a, \\ -1/x & \text{if } a \leq x < b, \\ x-1 & \text{if } b \leq x, \end{cases} \tag{3}$$

where the parameters (a, b) range in a closed subset $\mathcal{P}$ of the plane. The segment

$$C := \{(a, b) \ : \ b - a = 1, \ b \in [0, 1]\}$$

is a piece of the boundary of $\mathcal{P}$, and the first return map of $f_{b-1,b}$ on the interval $[b-1, b)$ coincides with the map K_b we are going to study (these maps are also mentioned in [24] under the name "Gauss-like maps" and denoted $\hat{f}_{b-1,b}$).

Katok and Ugarcovici also consider a closely related family $(F_{a,b})_{(a,b)\in\mathcal{P}}$ of maps of the plane: each $F_{a,b}$ has an attractor $D_{a,b} \subset \mathbb{R}^2$ such that $F_{a,b}$ restricted to $D_{a,b}$ is invertible and it is a geometric realization of the natural extension of $f_{a,b}$. They also show that for most parameters in $\mathcal{P}$ the attractor $D_{a,b}$ has *finite rectangular structure*, meaning that it is a finite union of rectangles. Moreover, all exceptions to this property belong to a Cantor set $\widetilde{\mathcal{E}}$ which is contained in the critical line C and whose 1-dimensional Lebesgue measure is zero.

It turns out that the set $\mathcal{E}_{KU}$ we are considering is just the projection of the set $\widetilde{\mathcal{E}}$ onto the second coordinate, up to a countable set. Making explicit the structure of $\mathcal{E}_{KU}$ allows us to prove that it is not just a zero measure set, but it also has zero Hausdorff dimension.

Structure of the paper. In section 2 we provide background material on Farey words in order to establish the properties which are needed to describe the combinatorial dynamics of the (a, b)-continued fractions. Then in section 3 we recall basic facts about continued fractions and define the *runlength map* RL which passes from binary expansions to continued fraction expansions. In section 4 we define *quamtervals* and the bifurcation set $\mathcal{E}_{KU}$. Then, we apply all these properties to the case of (a, b)-continued fractions; in section 5 we determine the combinatorial dynamics of the orbits of α and $\alpha - 1$, thus proving that the matching condition holds on each quamterval. In section 6 we draw consequences for the entropy function, proving Theorem 1.2: indeed, we prove that the Cantor set $\mathcal{E}_{KU}$ has Hausdorff dimension zero (Proposition 4.4) and h is Hölder continuous, so it can be extended to a monotone function across the Cantor set. Then, in section 7 we combine the previous properties with the construction of the attractors given in [24] to prove Theorem 1.1. Finally, in section 8 we establish the connection between the bifurcation set and the main cardioid in the Mandelbrot set, proving Theorem 1.3. For the sake of readability, the proofs of some technical lemmas will be postponed to the appendix.

2. Farey words and dynamics

We shall start by constructing the set of Farey words and establishing the properties which are needed in the rest of the paper. Many of these results appear in various sources, for instance in the books [2, 3, 15, 30]. For the convenience of the reader and in order to set up the notation for the rest of the paper, we shall give a fairly self-contained treatment.

2.1. **Alphabets and orderings.**

An *alphabet* $\mathcal{A}$ will be a finite set of symbols, which we shall call *digits*. Given an alphabet $\mathcal{A}$, we shall denote by $\mathcal{A}^n$ the set of words of length n, by $\mathcal{A}^{\mathbb{N}}$ the set of infinite words, and by $\mathcal{A}^\star := \bigcup_{n\geq 0} \mathcal{A}^n$ the set of *finite words* of arbitrary length. If w is a finite word, then the symbol $\overline{w}$ will denote the infinite word given by infinite repetition of the word w.

If $w = (\epsilon_1, \ldots, \epsilon_\ell) \in \mathcal{A}^*$, we shall denote as $|w|$ the *length* of the word w, i.e., the number of digits. Moreover, if we fix a digit $a \in \mathcal{A}$, the symbol $|w|_a$ will denote the number of digits in the word which are equal to a. Moreover, given a word $w = (\epsilon_1, \ldots, \epsilon_\ell) \in \mathcal{A}^*$, we define its *transpose* to be the word ${}^t w$ with

$$ {}^t w := (\epsilon_\ell, \ldots, \epsilon_1); $$

a word which is equal to its transpose is called a *palindrome*. Moreover, we define the *cyclic permutation* operator τ to act on the word $w = (\epsilon_1, \ldots, \epsilon_\ell)$ as

$$ \tau w := (\epsilon_2, \ldots, \epsilon_\ell, \epsilon_1). $$

A total order $<$ on the alphabet $\mathcal{A}$ induces, for each n, a total order on the set $\mathcal{A}^n$ of words of length n by using the lexicographic order, and similarly it induces a total order on the set $\mathcal{A}^{\mathbb{N}}$ of infinite words. We shall extend this order to a (partial) order on the set $\mathcal{A}^*$ of finite words by defining that

$$ u < v \quad \text{if} \quad uv < vu. $$

Note that it is not difficult to check that if $u, v \in \mathcal{A}^*$, then the inequality $u < v$ is equivalent to $\bar{u} < \bar{v}$ (this fact also proves that $<$ is an order relation).

Finally, we shall also define the stronger partial order relation $\ll$ on the set $\mathcal{A}^*$ of finite strings by saying that

$$ u \ll v $$

if there exist a prefix u_1 of u and a prefix v_1 of v with $|u_1| = |v_1|$ and such that $u_1 < v_1$. Note that $u \ll v$ implies $u < v$, and moreover that any infinite word beginning with u is smaller than any infinite word beginning with v.

In the following we will mainly be interested in the binary alphabet $\mathcal{A} := \{0, 1\}$, with the natural order $0 < 1$. For $\epsilon \in \{0, 1\}$, we also define the *negation* operator $\check{\epsilon} := 1 - \epsilon$, which can be extended digit-wise to binary words: if $w = (\epsilon_1, \ldots, \epsilon_\ell) \in \{0, 1\}^*$, we define $\check{w} := (\check{\epsilon}_1, \ldots, \check{\epsilon}_\ell)$.

Every infinite word $w = (\epsilon_1, \epsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$ also corresponds to the unique real value in $[0,1]$ which has w as its binary expansion, which will be denoted by $.w := \sum_{k=1}^{\infty} \epsilon_k 2^{-k}$; the same is true for finite binary words in $\{0,1\}^*$, which correspond to dyadic rationals.

2.2. **Farey words.** We are now ready to define one of the main ingredients of the paper, namely the set of *Farey words*. As we shall see, several equivalent definitions can be given; we shall start with a recursive definition.

For each integer $n \geq 0$, we shall construct a list F_n of finite words in the alphabet $\{0,1\}$, called a *Farey list* of level n. Let us start with $F_0 := (0,1)$, the list consisting of the two one-digit words. For each n, the next list F_{n+1} is obtained by inserting between two consecutive words v, w in the list F_n the concatenation vw. In formulas, if $F_n = (w_1, \dots, w_k)$ with each w_i a finite word, then the next list is $F_{n+1} = (v_1, \dots, v_{2k-1})$ with

$$\begin{aligned} v_{2i-1} &:= w_i && \text{for } 1 \leq i \leq k, \\ v_{2i} &:= w_i w_{i+1} && \text{for } 1 \leq i \leq k-1. \end{aligned}$$

Definition 2.1. The set of Farey words FW is the union of all Farey lists:

$$FW := \bigcup_{n \geq 0} F_n.$$

As an example, the first few Farey lists[1] are

$$\begin{aligned} F_0 &= (0,1), \\ F_1 &= (0,01,1), \\ F_2 &= (0,001,01,011,1), \\ F_3 &= (0,0001,001,00101,01,01011,011,0111,1), \end{aligned}$$

and all their elements are Farey words. Note that each F_n contains $2^n + 1$ elements, and its elements are in a strictly increasing order. A Farey word will be called *non-degenerate* if it has more than one digit: we shall denote the set of non-degenerate Farey words as $FW^\star = FW \setminus \{0,1\}$. These words are also sometimes called *Christoffel words*, as in the book [3], or *standard words* as in [32].

Note moreover that each Farey word is naturally equipped with a *standard factorization*; indeed, if w is a Farey word, let n be the smallest integer for which w belongs to F_n; by definition, the word w is generated in the iterative construction as a concatenation $w = w_1 w_2$, where w_1 and w_2 belong to the level F_{n-1}. Thus, the decomposition $w = w_1 w_2$ will be called the standard factorization of w. One has the following characterization ([3], section 3.1):

Proposition 2.2. *Given $w \in FW$, let us consider a decomposition $w = w'w''$ where w', w'' are non-empty words. Then the following conditions are equivalent:*

1. *w' and w'' are Farey words;*
2. *$w = w'w''$ is the standard factorization of w.*

We shall now construct a natural correspondence between the set of Farey words and the set of rational numbers between 0 and 1. Given a word $w \in \{0,1\}^*$, let us

[1] Let us remark that, despite its similarity in name, the Farey lists we defined are different from the also classical *Farey sequences*, which are defined as the set of rational numbers with a bound on the denominator. In fact, if one converts a Farey list to a set of rationals using the map of Proposition 2.3, the resulting set of rationals is in general not a Farey sequence.

define the rational number $\rho(w)$ to be the ratio between the number of occurrences of the digit 1 and the total length of the word:

$$\rho(w) := \frac{|w|_1}{|w|}.$$

Clearly, $0 \leq \rho(w) \leq 1$. Moreover, we have the following correspondence.

Proposition 2.3. *The map $\rho : FW \to \mathbb{Q} \cap [0,1]$ is a bijection between the set of Farey words and the set of rational numbers between* 0 *and* 1*.*

In the rest of this section we shall prove Proposition 2.3 and meanwhile establish more properties of Farey words. In particular, we shall see how to construct an inverse of ρ, i.e., to produce a Farey word given a rational number.

Let $r := \frac{p}{q} \in [0,1]$ be a rational number, with $(p,q) = 1$, and consider the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$, with the marked point $x_0 = 0$. Let $C_q := \{x \in \mathbb{R}/\mathbb{Z} : qx \not\equiv 0 \mod 1\}$. For each $x \in C_q$, we shall define the binary word $\Phi_r(x) \in \{0,1\}^q$ using the dynamics of the circle rotation

$$R_r(x) := x + r \mod 1.$$

The word $\Phi_r(x)$ will be constructed as follows: starting at x, we successively apply the rotation $R = R_r$ and each time we write down 1 if we cross the x_0 mark, and 0 otherwise. More precisely, we define $\Phi_r(x) := (\epsilon_1, \dots, \epsilon_q)$, where, for each k between 1 and q, the k^{th} digit ϵ_k is given by

$$\epsilon_k := \begin{cases} 0 & \text{if } x_0 \notin (R^{k-1}(x), R^k(x)], \\ 1 & \text{if } x_0 \in (R^{k-1}(x), R^k(x)]. \end{cases}$$

It is immediate to check that one can also write the formula

$$\epsilon_k = \lfloor x + kr \rfloor - \lfloor x + (k-1)r \rfloor \qquad \text{for } 1 \leq k \leq q. \tag{4}$$

An equivalent way to describe Farey words is as *cutting sequences* of straight lines with respect to a square grid: see Figure 2 and section 2.3.

Note that the map Φ_r intertwines the rotation with the cyclic permutation τ, i.e.,

$$\Phi_r \circ R_r = \tau \circ \Phi_r.$$

The map $\Phi_r(x)$ is (weakly) increasing for $0 \leq x \leq 1$ and is constant on connected components of C_q; we will be particularly interested in the word W_r defined as

$$W_r := \Phi_r(0^+) = \lim_{x \to 0^+} \Phi_r(x).$$

Lemma 2.4. *The map $W : \mathbb{Q} \cap [0,1] \to \{0,1\}^\star$ is a right inverse of ρ; that is, for each $r \in \mathbb{Q} \cap [0,1]$ we have*

$$\rho(W_r) = r.$$

Proof. Since all digits of W_r are either 0 or 1, the number of 1 digits of W_r is just the sum of the digits, so by using equation (4) we get the telescoping sum:

$$|W_r|_1 = \epsilon_1 + \dots + \epsilon_q = \sum_{k=1}^{q} (\lfloor kr \rfloor - \lfloor (k-1)r \rfloor) = \lfloor qr \rfloor = p,$$

so $\rho(W_r) = |W_r|_1 / |W_r| = p/q = r$. $\square$

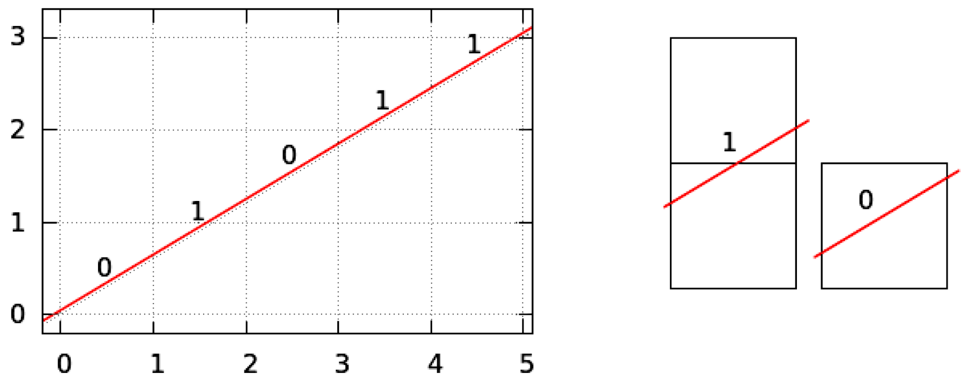

FIGURE 2. The word $\Phi_r(x)$ can also be interpreted as the period of the cutting sequence determined by a straight line of slope r passing through $(0, x)$, where the digit 1 corresponds to the case when this line cuts some horizontal boundary of the unitary tiling, while the digit 0 corresponds to the case when the line cuts the vertical sides of a square tile (see right figure for a legend). Thus, W_r is the cutting sequence of a line of slope r which crosses the y-axis just above 0. The same can also be considered for a straight line with irrational slope: in this case the cutting sequence generated is not periodic but has low complexity and is called a *Sturmian sequence* (see section 2.3).

A pair (r, r') of rational numbers $r := \frac{p}{q}$ and $r' := \frac{p'}{q'}$ with $\gcd(p, q) = \gcd(p', q') = 1$ and $pq' - p'q = 1$ is called a *Farey pair*; the *Farey sum* of a Farey pair is defined as

$$r \oplus r' := \frac{p + p'}{q + q'}.$$

It is easy to check that $r \oplus r'$ lies in between r and r'; that is, if $r < r'$ we have

$$r < r \oplus r' < r'. \tag{5}$$

Moreover, if we let r, r' be a Farey pair with $r < r'$, then we have the identity

$$W_{r \oplus r'} = W_r W_{r'}, \tag{6}$$

where on the right-hand side we mean the concatenation of W_r and $W_{r'}$. In fact, the map ρ is a bijection between the tree of Farey words and the tree of Farey fractions (see Figure 3).

Proof of Proposition 2.3. By Lemma 2.4, the function ρ is surjective, and moreover its restriction to the set

$$\operatorname{Im} W := \{W_r \ : \ r \in \mathbb{Q} \cap [0, 1]\}$$

is a bijection between $\operatorname{Im} W$ and $\mathbb{Q} \cap [0, 1]$. Therefore, we just need to show that the set $\operatorname{Im} W$ coincides with the set FW of all Farey words. Now, since $W_0 = 0$ and $W_1 = 1$, the elements of the Farey list $F_0 = (0, 1)$ belong to $\operatorname{Im} W$, and note that $(0, 1)$ is a Farey pair. Thus, by induction using identity (6), for each n the elements of the list F_n belong to $\operatorname{Im} W$, so all Farey words belong to $\operatorname{Im} W$. Since it is well-known that every rational number can be obtained from 0 and 1 by taking

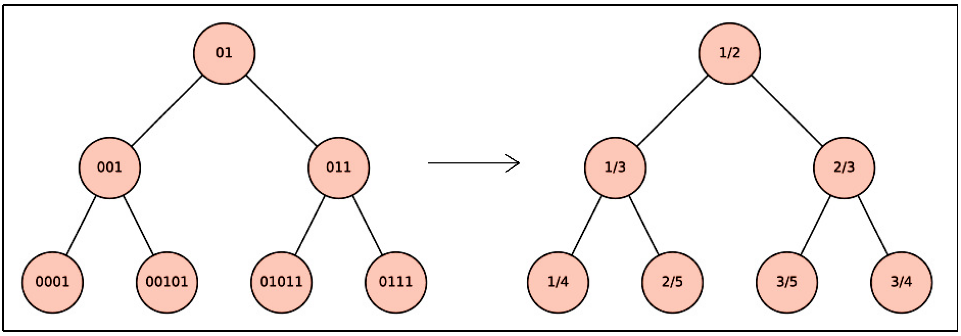

FIGURE 3. The tree structure of Farey words and its corresponding tree of rational numbers, which is known as the Farey tree.

successive Farey sums of Farey pairs, then W_r is a Farey word for any rational numbers $r \in [0,1]$, and the claim is proven. $\square$

For $w = (\epsilon_1, \dots, \epsilon_\ell) \in \{0,1\}^*$ we set

$$^\vee w := (\check{\epsilon}_1, \epsilon_2, \dots, \epsilon_\ell), \quad w^\vee := (\epsilon_1, \dots, \epsilon_{\ell-1}, \check{\epsilon}_\ell).$$

We shall now see Farey words have many symmetries, arising from the symmetries of the dynamical system R_r.

Proposition 2.5. *If $w = W_r$ is a Farey word, then:*

(a) *the word $^t\check{w}$ is still a Farey word: in particular,*
$$W_{1-r} = {}^t\check{w};$$
(b) *moreover, we have the identity*
$$\Phi_{1-r}(0^-) = \check{w}$$
(c) *and*
$$\Phi_r(0^-) = {}^\vee w^\vee = {}^t w;$$
(d) *both $^\vee w$ and $w^\vee$ are palindromes;*
(e) *finally, we have*
$$^t w < {}^\vee w.$$

As an example, let us pick $w = W_{2/5} = 00101$. One can check that $^\vee w = 10101$ and $w^\vee = 00100$ are both palindromes, and $^\vee w^\vee = 10100$ equals the transpose of w. Finally, the word $^t\check{w} = 01011$ is also a Farey word ($= W_{3/5}$).

Proof. (a) Let us note that considering the rotation R_{1-r} instead of R_r is equivalent to inverting the direction (clockwise or counterclockwise) of the rotation. Thus, for each $x \in C_q$, the first $q+1$ elements of the orbit of x under R_r are the same as the first $q+1$ elements of the orbit of x under R_{1-r}, but the order of visit is reversed (in symbols, $R_r^k(x) \equiv R_{1-r}^{q-k}(x) \mod 1$ for $0 \le k \le q$), which proves the claim.

(b) This identity relies on the fact that the circle is symmetric under reflection $\sigma(x) := -x \mod 1$; indeed, for each x the orbit of x under R_r is the reflection of the orbit of $1-x$ under R_{1-r} (in symbols, $R_r^k(x) \equiv -R_{1-r}^k(-x) \mod 1$), while the marked point $x_0 = 0$ is fixed by σ.

(c) The first equality follows by noting that the iterates $R_r^k(0)$ encounter a discontinuity of the function $\lfloor \cdot \rfloor$ if and only if $k \equiv 0 \mod q$; thus, changing the starting point x from 0^+ to 0^- only affects the first and last digits of $\Phi_r(x)$. For the second equality, denote $v := \Phi_{1-r}(0^+) = {}^t\check{w}$; we have by (b) and then (a)

$$\Phi_r(0^-) = \check{v} = {}^t w.$$

(d) follows immediately from (c): indeed we have

$${}^t({}^\vee w) = ({}^t w)^\vee = ({}^\vee w^\vee)^\vee = {}^\vee w,$$

where the first and third equalities are elementary, and the second one uses (c); a completely analogous proof works for $w^\vee$.

(e) Applying (c) and using the fact that the last digit of each (non-zero) Farey word is 1, we get

$${}^t w = {}^\vee w^\vee < {}^\vee w.$$

□

It will be crucial in the following to study the ordering of the set of cyclic permutations of a given Farey word. The essential properties are contained in the following lemma.

Lemma 2.6. *Let $w = W_r$ be a Farey word, and consider the set*

$$\Sigma(w) := \{\tau^k w \ : \ k \in \mathbb{N}\}$$

of its cyclic permutations. Moreover, let $w = w_1 w_2$ be the standard factorization of w, and let $q_1 := |w_1|$, $q_2 := |w_2|$. Then the following are true:

(1) *the smallest cyclic permutation of w is w itself* (*i.e.*, $\min \Sigma(w) = w$);
(2) *the second smallest cyclic permutation of w is*

$$\tau^{q_1} w = w_2 w_1;$$

(3) *the largest cyclic permutation of w is*

$$\tau^{q_2} w = {}^t w.$$

Proof. Let us start by noting that if $w = \Phi_r(x)$, then the set of cyclic permutations of w is given by

$$\{\tau^k w \ : \ 0 \le k < q\} = \{\Phi_r(R_r^k(x)) \ : \ 0 \le k < q\};$$

moreover, since the map Φ_r is increasing, the order in the above set is the same as the order in the set

$$S_r(x) := \{\{x + kr\} \ : \ 0 \le k < q\}.$$

Thus, the smallest cyclic permutation of $w = W_r$ corresponds to the smallest possible value of $\{kr\}$, which is attained for $k = 0$, hence by $w = \Phi_r(0^+)$ itself, proving (1).

Moreover, let $w = w_1 w_2$ be the standard factorization of w. Then by definition we have $w = W_r$, while $w_1 = W_{r_1}$ and $w_2 = W_{r_2}$, in such a way that (r_1, r_2) is a Farey pair, with $r_1 < r_2$ and $r := r_1 \oplus r_2$. Note now that, writing $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$, we have $p_2 q_1 - p_1 q_2 = 1$ by the definition of Farey pair; hence we can write

$$q_1(p_1 + p_2) \equiv 1 \mod (q_1 + q_2). \tag{7}$$

Thus, the second smallest element of $S_r(0)$ is attained for $k = q_1$; hence the second smallest element of $\Sigma(w)$ is $\tau^{q_1} w = w_2 w_1$, proving (2).

Finally, the largest element of the set $\Sigma(w)$ is $\tau^k w$, where k is such that $\{kr\} = 1 - \frac{1}{q}$; thus, the corresponding word is $\Phi_r(0^-)$, which equals ${}^t w$ by Proposition 2.5(c). Moreover, from equation (7) one also gets

$$q_2(p_1 + p_2) \equiv -1 \mod (q_1 + q_2);$$

hence $\{q_2 r\} = 1 - \frac{1}{q}$, so the largest element of the set $\Sigma(w)$ is $\tau^{q_2} w$. $\square$

Let us now state one more consequence of the previous lemma, in terms of ordering of subsets of the circle. Recall the *doubling map* $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is defined as $D(x) := 2x \mod 1$. We say that a finite set $X \subseteq S^1$ has *rotation number* $r = \frac{p}{q} \in \mathbb{Q}$ if it is invariant for the doubling map, and the restriction of D to X is conjugate to the circle rotation R_r via an orientation-preserving homeomorphism of S^1. More concretely, this means that if we write the elements of X in cyclic order as $X = (\theta_0, \theta_1, \dots, \theta_{q-1})$ with $0 \leq \theta_0 < \theta_1 < \dots < \theta_{q-1} < 1$, then we have for each index i,

$$D(\theta_i) = \theta_{i+p},$$

where the index $i + p$ is taken modulo q. The proof of the previous lemma also yields the following (uniqueness follows from [17], Corollary 8):

Lemma 2.7. *Let $w = W_r$ be a Farey word. Then the set*

$$C(w) = \{0.\overline{\tau^k w} \; : \; 0 \leq k \leq q-1\} \subseteq S^1$$

is the unique subset of S^1 which has rotation number r for the doubling map.

For an example, if $w = 00101$, then $C(w) = (\frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{18}{31}, \frac{20}{31})$ (see also Figure 10).

Recall that a word $w \in \{0,1\}^*$ which is minimal (with respect to lexicographic order) among all its cyclic permutations is also called a *Lyndon word*; hence property (1) of Lemma 2.6 can be paraphrased as saying that every Farey word is a Lyndon word (but not vice versa: e.g. 0011 is a Lyndon word but not a Farey word). Let us recall that all Lyndon words of length greater than 1 begin with the digit 0 and end with the digit 1; moreover one has the following (see [30]):

Proposition 2.8. *If $w = ps$ is a Lyndon word (in particular, if w is a Farey word), then $w \ll s$.*

2.3. **Infinite cutting sequences and Sturmian sequences.** Let us now extend the construction of W_r to irrational values of r. Given a number $r \in (0,1)$, which we interpret as the slope of a straight line (see Figure 2), we define its *upper cutting sequence* as the infinite binary sequence

$$W_r^+ := \lim_{s \in \mathbb{Q}, s \to r^+} W_s.$$

It turns out that W_r^+ can also be obtained by the formula

$$W_r^+ = (\lfloor (n+1)r \rfloor - \lfloor nr \rfloor)_{n \geq 0}.$$

Note that the above definition also makes sense for $r \in \mathbb{Q}$, and in that case it produces the infinite repetition of the finite word W_r, i.e., $W_r^+ = \overline{W_r}$. Moreover, we define the *lower cutting sequence* of slope r as the infinite word

$$W_r^- := \lim_{s \in \mathbb{Q}, s \to r^-} W_s.$$

The two maps $W^+, W^- : (0,1) \to \{0,1\}^{\mathbb{N}}$ are both strictly increasing and continuous on the irrationals; moreover, $W_r^- < W_r^+$ if $r \in \mathbb{Q}$, while $W_r^- = W_r^+$ if

$r \notin \mathbb{Q}$. Note moreover that if r is rational, then by Proposition 2.5(c) one has $W_r^- = {}^\vee(\overline{{}^tW_r})$, where ${}^\vee w$ denotes the operation of switching the first digit of the word w. As an example, if $r = \frac{2}{5}$, then $W_r^+ = \overline{00101}$, while $W_r^- = 0\overline{01001}$.

We will be especially interested in the set

$$\mathcal{C} := \overline{W^+((0,1))},$$

i.e., the closure of the image of W^+, which we call the set of *cutting sequences.* By the above properties one gets the decomposition

$$\mathcal{C} = \mathcal{S}_0 \setminus \bigcup_{r\in\mathbb{Q}\cap(0,1)} \mathcal{A}_r, \tag{8}$$

where $\mathcal{S}_0$ is the set of infinite binary sequences beginning with 0, while $\mathcal{A}_r := \{\sigma \in \mathcal{S}_0 \ : \ W_r^- < \sigma < W_r^+\}$ and the union is disjoint.

The set $\mathcal{C}$ is closely connected to the set of *Sturmian sequences* widely studied in the literature (see e.g. [30], Chapter 2). In fact, the set $\mathcal{C}_{irr} := W^+((0,1)\setminus\mathbb{Q})$ of cutting sequences of irrational slopes is characterized by the property

$$\mathcal{C}_{irr} = \{0S \ : \ S \text{ is a characteristic Sturmian sequence}\}.$$

2.4. **Substitutions.** Another way to generate Farey words is by substitutions. Given a pair of words $U = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \in \{0,1\}^* \times \{0,1\}^*$ we can define the *substitution operator* associated to U to be the operator acting on $\{0,1\}^*$ (or on $\{0,1\}^{\mathbb{N}}$) as

$$w = (\epsilon_1, \epsilon_2, \dots) \mapsto (u_{\epsilon_1}, u_{\epsilon_2}, \dots);$$

the action of U on w will be denoted by $w \star U$. Let us note that if $u_0 < u_1$, then the operator is order-preserving, while if $u_0 > u_1$ it is order-reversing. Moreover, the negation operator can be obtained as the substitution associated to $V := \begin{bmatrix} (1) \\ (0) \end{bmatrix}$. We can also extend the substitution operator to pairs of words: if $U = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$ and $W = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$ let us define $U \star W := \begin{bmatrix} u_0 \star W \\ u_1 \star W \end{bmatrix}$; in this way we get the following associativity property, that for each word w we have

$$(w \star U) \star W = w \star (U \star W). \tag{9}$$

Finally, the substitution and transposition operators are compatible, in the sense that

$${}^t(w \star U) = {}^tw \star {}^tU \quad \text{where} \quad {}^tU := \begin{bmatrix} {}^tu_0 \\ {}^tu_1 \end{bmatrix}. \tag{10}$$

It turns out that one can produce all (non-degenerate) Farey words by successive iteration of two substitution operators, starting with the word $w_0 = (01)$. Namely, let us define the two substitution operators

$$U_0 : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01, \end{cases} \qquad U_1 : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1. \end{cases}$$

It is not difficult to realize that the action of U_0 and U_1 preserves the set of Farey words. More precisely, let us set $F_n^* := F_n \setminus \{(0),(1)\}$,

$$FW_0 := \{w \in FW^* : |w|_0 > |w|_1\}, \quad FW_1 := \{w \in FW^* : |w|_0 < |w|_1\},$$

and $F_n^\epsilon := F_n \cap FW_\epsilon$ (note that, by Proposition 2.3, for each $n \geq 2$ one has $F_n^\star = F_n^0 \cup \{(01)\} \cup F_n^1$). The following properties easily follow by induction.

Proposition 2.9. *For each $\epsilon = 0, 1$, the operator $U_\epsilon : F_n^* \to F_{n+1}^\epsilon$ is a bijection. Moreover, for all $n \geq 1$, the following characterization holds:*

$$F_n^* = \{(01) \star U_{\epsilon_1} \star \cdots \star U_{\epsilon_\ell} \ : \ \epsilon_k \in \{0,1\}, \quad 0 \leq \ell < n\}.$$

3. Regular continued fraction expansions

Let us first fix some notation regarding the classical continued fractions expansions. Any irrational number admits a unique infinite continued fraction expansion, which will be denoted as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \dots]$$

with $a_k \in \mathbb{Z}\ \forall k$ and $a_k \geq 1\ \forall k \geq 1$. Moreover, any rational value r admits exactly two finite expansions; indeed, we can write

$$r = [a_0; a_1, \dots, a_n] = [a_0; a_1, \dots, a_n - 1, 1]$$

with $a_n \geq 2$. Any non-empty string of positive integers $S = (a_1, \dots, a_n)$ defines a rational value $r = [0; a_1, \dots, a_n] \in (0, 1]$, which we will sometimes denote as $r = [0; S]$.

We then define the *right conjugate* of S to be the only string S' which defines the same rational value as S, i.e., such that $[0; S'] = [0; S]$. For instance $(3, 1, 3)' = (3, 1, 2, 1)$ and vice versa (conjugation is involutive and affects only the last one or two digits). We also define the *left conjugate* $'S$ of a (finite or infinite) string S in a similar way, just acting on the leftmost digits: that is, if $S = (a_1, a_2, \dots)$ we define

$$'S := \begin{cases} (1, a_1 - 1, a_2, \dots) & \text{if } a_1 \geq 2, \\ (1 + a_2, a_3, \dots) & \text{if } a_1 = 1. \end{cases}$$

Thus, the left conjugate of $(3, 1, 3)$ will be $'(3, 1, 3) = (1, 2, 1, 3)$. It is not difficult to check that this manipulation on strings translates into the map $\sigma : [0, 1] \to [0, 1]$ defined as $\sigma(x) := 1 - x$ on the side of continued fraction expansions; namely, for any string of positive integers we have

$$\sigma([0; S]) = [0; 'S]. \tag{11}$$

Another operation on strings we shall often use in the following is the operator ∂ defined on (finite or infinite) strings as

$$\partial(a_1, a_2, \dots) := \begin{cases} (a_1 - 1, a_2, \dots) & \text{if } a_1 > 1, \\ (a_2, \dots) & \text{if } a_1 = 1. \end{cases}$$

We shall sometimes also use the transposition: the transpose string of $S = (a_1, \dots, a_\ell)$ is the string ${}^tS = (a_\ell, \dots, a_1)$. Finally, if S is a finite string of positive integers we will denote by $q(S)$ the denominator of the rational number whose c.f. expansion is S, i.e., such that $\frac{p(S)}{q(S)} = [0; S]$ with $(p(S), q(S)) = 1$, $q(S) > 0$.

Let us also recall the well-known estimate

$$q(S)q(T) \leq q(ST) \leq 2q(S)q(T). \tag{12}$$

Moreover, we define the map $f_S : x \mapsto S \cdot x$, which corresponds to appending the string S at the beginning of the continued fraction expansion of x. That is, if $S = (a_1, \dots, a_n)$ we can write, by identifying matrices with Möbius transformations,

$$S \cdot x := \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \cdot x. \tag{13}$$

It is easy to realize that concatenation of strings corresponds to composition, namely $(ST) \cdot x = S \cdot (T \cdot x)$. Moreover, the map f_S is increasing if $|S|$ is even, decreasing if $|S|$ is odd. The image of f_S is a *cylinder set*

$$I(S) := \{x = S \cdot y, \ y \in [0,1]\},$$

which is a closed interval with endpoints $[0; a_1, \dots, a_n]$ and $[0; a_1, \dots, a_n + 1]$. The map f_S is a contraction of the unit interval, and it is easy to see that

$$\frac{1}{4q(S)^2} \leq |f'_S(x)| \leq \frac{1}{q(S)^2} \qquad \forall x \in [0,1] \tag{14}$$

and that the length of $I(S)$ is bounded by

$$\frac{1}{2q(S)^2} \leq |I(S)| \leq \frac{1}{q(S)^2}. \tag{15}$$

Given two strings of positive integers $S = (a_1, \dots, a_n)$ and $T = (b_1, \dots, b_n)$ of equal length, let us define the *alternate lexicographic order* as

$$S < T \text{ if } \exists k \leq n \text{ s.t. } a_i = b_i \ \forall 1 \leq i \leq k-1 \text{ and } \begin{cases} a_n < b_n & \text{if } n \text{ even,} \\ a_n > b_n & \text{if } n \text{ odd.} \end{cases}$$

The importance of such order lies in the fact that given two strings of equal length $S < T$ iff $[0; S] < [0; T]$. In order to compare quadratic irrationals with periodic expansion, the following *string lemma* ([10], Lemma 2.12) is useful: for any pair of strings S, T of positive integers, we have the equivalence

$$ST < TS \Leftrightarrow [0; \overline{S}] < [0; \overline{T}]. \tag{16}$$

The order $<$ is a total order on the strings of positive integers of fixed length; to be able to compare strings of different lengths we define the partial order

$$S \ll T \quad \text{if } \exists i \leq \min\{|S|, |T|\} \text{ s.t. } S_1^i < T_1^i,$$

where $S_1^i = (a_1, \dots, a_i)$ denotes the truncation of S to the first i characters. Let us note the following basic properties:

(1) if $|S| = |T|$, then $S < T$ iff $S \ll T$;
(2) if S, T, U are any strings, $S \ll T \Rightarrow SU \ll T, S \ll TU$;
(3) if $S \ll T$, then $S \cdot z < T \cdot w$ for any $z, w \in (0,1)$.

3.1. **Farey legacy.** We shall now see how to construct, using continued fractions, an irrational number given a binary word; this way, starting from the set of Farey words we shall define the fractal subset $\mathcal{E}_{KU}$ of the interval and establish its properties from the properties of Farey words we obtained in the previous sections.

Indeed, let us define the *runlength map* RL to be the map which associates to a (finite or infinite) binary word w the string of positive integers which records the size of blocks of consecutive equal digits: namely, if

$$w = \underbrace{0 \dots 0}_{a_1} \underbrace{1 \dots 1}_{a_2} \dots$$

we set

$$RL(w) := (a_1, a_2, \dots).$$

For instance, $RL(0001001001) = RL(1110110110) = (3,1,2,1,2,1)$. Note that RL is a two-to-one map ($RL(w) = RL(\check{w})$), but it is strictly increasing when restricted to words beginning with the digit 0. If $S = RL(w)$ for some $|w| > 1$, then

$$RL({}^{\vee}w) = {}'S, \qquad RL(w^{\vee}) = S'. \tag{17}$$

Note also that if $w = w_0 w_1$ and the last digit of w_0 is different from the first of w_1, then one has

$$RL(w) = RL(w_0)RL(w_1)$$

(note this is always the case when w_i are non-degenerate Farey words). For the runlength string of Farey words some more nice properties hold.

Lemma 3.1. *Let $w \in FW^*$ be a Farey word and let $S := RL(w)$. Then the length $|S|$ is even and*

(i) *there exists an integer $a \geq 1$ and a Farey word $f = (\epsilon_1, \dots, \epsilon_n)$ such that one can write*

$$S = B_{\epsilon_1} \dots B_{\epsilon_n}$$

with

$$B_0 = (a+1, 1), \quad B_1 = (a, 1) \quad \text{ if } w \in FW_0$$

or

$$B_0 = (1, a), \quad B_1 = (1, a+1) \quad \text{ if } w \in FW_1.$$

The Farey word f is unique as long as $w \neq (01)$; since it plays a central role in the following, it will be referred to as the Farey structure *of the string S.*

(ii) *The runlength of the Farey word ${}^t\check{w}$ is*

$$RL({}^t\check{w}) = {}'S' = {}^tS;$$

(iii) *if $S = (a_1, \dots, a_\ell)$ and $1 \leq k < \ell/2$, we set $P_k := (a_1, \dots, a_{2k})$, $S_k := (a_{2k+1}, \dots, a_\ell)$ (so that $S = P_k S_k$), then*

$$S \ll S_k; \tag{18}$$

(iv) *using the same notation as above, if $w \in FW_0$, then*

$$S_k P_k \ll \partial S. \tag{19}$$

Proof. (i) Recall that $FW^* = FW_0 \cup (01) \cup FW_1$. Clearly, for $w = (01)$ we have $S = RL(w) = (1,1)$, so $a = 1$, and we can choose $f = (0)$ or $f = (1)$. Let us now assume $w \in FW_0$. Then by Proposition 2.9 we can write

$$w = f \star U_1 \star U_0^a \tag{20}$$

for some $f = (\epsilon_1, \dots, \epsilon_\ell) \in FW$ and $a \geq 1$. On the other hand $U_1 \star U_0^a = \begin{bmatrix} (0^{a+1}1) \\ (0^a 1) \end{bmatrix}$ and $RL(0^a 1) = (a, 1)$, so, calling $B_0 := (a+1, 1)$ and $B_1 := (a, 1)$ we get that S is the concatenation $B_{\epsilon_1} \dots B_{\epsilon_\ell}$. Note that since the image of U_0 is contained in FW_0 and the image of U_1 is contained in FW_1 (Proposition 2.9), the factorization of equation (20) is unique; hence also the Farey structure of S is unique. The case $w \in FW_1$ is analogous.

(ii) The second claim is an immediate consequence of equation (17) and the fact that ${}^{\vee}w^{\vee} = {}^t w$, together with the fact that $RL(\check{w}) = RL(w)$.

(iii) It follows from Proposition 2.8 and the fact that the runlength map preserves the strong order $\ll$ when restricted to words which begin with 0.

(iv) By unwinding the definitions it is not hard to see that if $w \in FW_0$, we can write the identities

$$\begin{aligned} S &= RL(f \star U_1 \star U_0^a), \\ \partial S &= RL((^\vee f) \star U_1 \star U_0^a), \\ S_k P_k &= RL((\tau^k f) \star U_1 \star U_0^a). \end{aligned}$$

Moreover, by Proposition 2.5(e) and Lemma 2.6(3) we have

$$^\vee f \gg {}^t f \geq \tau^k f \quad \forall k;$$

hence the claim follows from the fact that both the substitution operator $f \mapsto f \star U_1 \star U_0^a$ and the runlength map (when restricted to words beginning with 0) are order-preserving. $\square$

For $x \in [0, 1/2]$ we shall consider the map $\phi : [0, 1/2] \to [0, 1]$ induced by runlength as follows: if $x = \sum_{j \geq 1} \epsilon_j 2^{-j}$ is the binary expansion of x, with $\epsilon_j \in \{0, 1\}$, then we define $\phi(x)$ to be the number with continued fraction

$$\phi(x) := [0; RL(\epsilon)],$$

where $RL(\epsilon)$ is the runlength of the sequence $(\epsilon_j)_{j \geq 1}$. This map is certainly well-defined for those values of x which admit a unique (and infinite) binary expansion; in fact, it also extends continuously to dyadic rationals, since the two binary expansions of a dyadic rational are mapped onto two continued fraction expansions of the same rational. For instance, if $x = \frac{3}{8} = 0.011$, then $\phi(x) = [0; 1, 2] = \frac{2}{3}$; on the other hand, we can write $\frac{3}{8} = 0.010\overline{1}$, which maps to $[0; 1, 1, 1, \infty] = \frac{2}{3}$. It is not difficult to check that this map is a homeomorphism between $[0, 1/2]$ and $[0, 1]$. The inverse of ϕ is essentially *Minkowski's question mark* function Q defined in (2) in the introduction. In fact, one has for each $x \in [0, 1/2]$,

$$Q(\phi(x)) = 2x. \tag{21}$$

4. The bifurcation set $\mathcal{E}_{KU}$

4.1. **Quamtervals.** We now associate to any Farey word w an interval J_w by choosing as its endpoints the points whose coding is the upper and lower cutting sequence. As the endpoints are quadratic irrationals, such an interval will be called a *quadratic maximal interval*, or *quamterval*.

Definition 4.1. Let $w = W_r$ be a Farey word, with $r \in (0, 1)$. We define the *quamterval of label* w to be the interval J_w with endpoints

$$J_w = (\alpha^-, \alpha^+) \quad \text{with} \quad \begin{array}{l} \alpha^+ := \phi(.W_r^+), \\ \alpha^- := \phi(.W_r^-). \end{array} \tag{22}$$

In terms of regular continued fractions, by unraveling the definition and using equation (17) and Lemma 3.1, we get that if $S = RL(w)$, then

$$\begin{aligned} \alpha^+ &= [0; \overline{S}], \\ \alpha^- &= [0; S'\overline{{}^tS}]. \end{aligned}$$

As an example, the Farey word $w = 001$ yields $S = (2, 1)$; hence $\alpha^+ = [0; \overline{2, 1}] = \frac{\sqrt{3}-1}{2}$ and $\alpha^- = [0; 3, \overline{1, 2}] = 2 - \sqrt{3}$. The rational value $s := [0; S]$ is the (unique!) rational value in J_w with least denominator and will be called the *pseudocenter* of

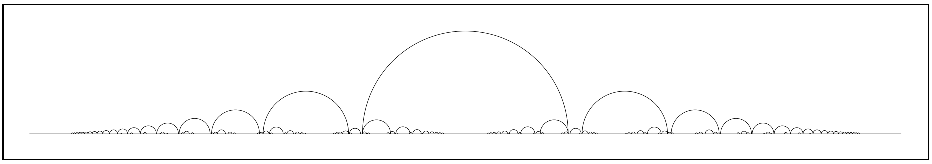

FIGURE 4. The quadratic intervals $\widetilde{J}_s$. Each interval is represented by a half-circle with the same endpoints. The intervals which are maximal with respect to inclusion are precisely the connected components of the complement of the bifurcation set $\mathcal{E}_{KU}$.

J_w (see [10] for more general properties of the pseudocenter of an interval). Note that by using equation (11) and Lemma 3.1(ii), the left endpoint α^- can also be described by the property

$$1 - \alpha^- = [0; \overline{{}^tS}]. \tag{23}$$

We now define the *bifurcation set* $\mathcal{E}_{KU}$ as the complement of the quamtervals:

$$\mathcal{E}_{KU} := [0,1] \setminus \bigcup_{w \in FW^*} J_w. \tag{24}$$

By comparing with equation (8), one gets that $\mathcal{E}_{KU}$ is the set of points whose continued fraction expansion equals the runlength of a cutting sequence.

Lemma 4.2. *The set $\mathcal{E}_{KU}$ has measure zero (hence, it has empty interior).*

Proof. Let $\sigma \in \mathcal{C}$ be a cutting sequence. Then σ is either periodic or Sturmian: in either case, the maximum length of a block of consecutive equal digits in σ is bounded. By definition of runlength, this implies that every element of $\mathcal{E}_{KU}$ has bounded continued fraction coefficients. The claim follows by recalling that the set of numbers with bounded continued fraction expansion has measure 0. ☐

4.2. **Thickening $\mathbb{Q}$.** We shall now perform an alternative construction of $\mathcal{E}_{KU}$ which is not essential for the main results of this paper, but it is useful for a comparison with the results in [10]. Given any rational value $s \in (0,1)$, let us consider its continued fraction expansion of even length $s = [0; S]$; then set $\beta(s) := [0; \overline{S}]$ and

$$\widetilde{J}_s := (\sigma\beta(\sigma s), \beta(s)).$$

Since $\beta(s) > s$ and σ is order-reversing, we can easily see that the $\widetilde{J}_s$ is an open interval containing s, and in fact s is the pseudocenter of $\widetilde{J}_s$. For all $w \in FW^*$ we have that $J_w = \widetilde{J}_s$ for $s = \phi(.w)$. Indeed quamtervals have the following maximality property (which will be proven in the appendix).

Proposition 4.3. *For any $s' \in \mathbb{Q} \cap (0,1)$ there is a Farey word $w \in FW^*$ such that $\widetilde{J}_{s'} \subset J_w$.*

As a consequence of the above proposition one gets the identity

$$\mathcal{E}_{KU} = [0,1] \setminus \bigcup_{s \in \mathbb{Q} \cap (0,1)} \widetilde{J}_s. \tag{25}$$

Let us now compute the dimension of $\mathcal{E}_{KU}$.

Proposition 4.4. *The Hausdorff dimension of $\mathcal{E}_{KU}$ is zero:*

$$\text{H.dim } \mathcal{E}_{KU} = 0.$$

Proof. We shall actually prove the stronger statement that for each $N \geq 2$ the set $\mathcal{E}_{KU} \cap [\frac{1}{N+1}, \frac{1}{N}]$ has zero box-counting dimension. The claim then follows since the box-counting dimension is an upper bound for the Hausdorff dimension. Fix $N \geq 2$, set

$$C_N := \left\{ w \in FW_0 \ : J_w \cap \left[\frac{1}{N+1}, \frac{1}{N} \right] \neq \emptyset \right\},$$

and consider the geometric ζ-function defined by

$$\zeta_N(t) := \sum_{w \in C_N} |J_w|^t.$$

Since the abscissa of convergence of the series ζ_N coincides with the upper box dimension of $\mathcal{E}_{KU} \cap [\frac{1}{N+1}, \frac{1}{N}]$ (see [14], p. 54), it is enough to prove that the above series converges for any $t > 0$. Now, it is not hard to prove that, for all $N \geq 2$, one has

$$|J_w| < 2\, b^{|w|} \quad \forall w \in C_N, \quad \text{where } b := N^{-\frac{2}{N+1}}. \tag{26}$$

Indeed, it is easy to check that

$$|s - \beta(s)| = |S \cdot 0 - S \cdot \beta(s)| \leq \sup |f'_S| \beta(s) \leq \frac{1}{q(S)^2},$$

where the last inequality is a consequence of equation (14). Since an analogous estimate holds for the distance between s and the left endpoint, one gets

$$|J_w| < \frac{2}{q(S)^2}. \tag{27}$$

On the other hand, if $w \in C_N$, then $S = RL(w)$ is a concatenation of n blocks of the type $B_0 := (N, 1)$ or $B_1 := (N-1, 1)$, where $n(N+1) < |w|$. Thus we get that

$$q(S) = q(B_{\epsilon_1} \cdots B_{\epsilon_n}) \geq q(B_1^n) \geq q(B_1)^n,$$

and since $q(B_1) \geq N$ we get $q(S) \geq N^{\frac{|w|}{N+1}}$. Thus (26) follows from this last estimate and equation (27). Since $\#\{w \in FW \ : \ |w| = k\} \leq k$, the estimate (26) implies that ζ_N is dominated by the sum $2^t \sum_1^\infty k b^{tk}$; therefore it converges for all $t > 0$, proving the claim. □

Finally, the following lemma will be needed in section 7.2.

Lemma 4.5. *Let w_∞ be a Sturmian sequence which begins with* 00*. Then*

$$\liminf_{w \in FW^*, w \to w_\infty} (|w|_0 - |w|_1) = +\infty.$$

If $x \in \mathcal{E}_{KU} \cap [0, 1-g)$, then the following limit is infinite:

$$\lim_{\delta \to 0} \inf\{|w|_0 - |w|_1 \ : \ J_w \subset [x - \delta, x + \delta]\} = +\infty.$$

Proof. Indeed, note that setting $\rho := \rho(w)$ we can rewrite $|w|_0 - |w|_1 = |w|(1 - 2\rho)$. Thus, if w_n is a sequence of Farey words such that $w_n \to w_\infty$, then $\rho(w_n)$ tends to a finite number which is $< \frac{1}{2}$ if w starts with the digits 00, while $|w|$ tends to infinity. Hence the liminf of the product is infinite.

For the second statement it is enough to recall that $\mathcal{E}_{KU}$ is the set of points whose continued fraction expansion equals the runlength of a Sturmian sequence.

So if $x \in [0, 1-g) \cap \mathcal{E}_{KU}$, then x corresponds to a Sturmian sequence w_∞ starting with 00, and the claim easily follows from the previous point. □

5. Matching intervals for continued fractions with $SL(2,\mathbb{Z})$-branches

Let us return to the maps $K_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ which are defined by $K_\alpha(0) = 0$ and

$$K_\alpha(x) = -\frac{1}{x} - c_\alpha(x), \qquad c_\alpha(x) := \left\lfloor -\frac{1}{x} + 1 - \alpha \right\rfloor \in \mathbb{Z}.$$

The goal of this section is to prove that a matching condition between the orbits of the endpoints α and $\alpha - 1$ is achieved for any parameter which belongs to some quamterval (Theorem 5.1 and Corollary 5.2). In order to formulate the result precisely, we need some notation.

Recall that the group $PSL(2,\mathbb{Z})$ acts on the real projective line by Möbius transformations. Indeed, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $x \in \mathbb{R} \cup \{\infty\}$, then we shall write $\mathbf{A}x := \frac{ax+b}{cx+d}$. The group $PSL(2,\mathbb{Z})$ is generated by the two elements $\mathbf{S}$ and $\mathbf{T}$, which are represented by the matrices

$$\mathbf{S} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{T} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and act respectively as the inversion $\mathbf{S}x := -1/x$ and the translation $\mathbf{T}x := x + 1$. For any fixed $\alpha \in (0, 1)$, the map K_α is just given by the inversion followed by an integer power of the translation which brings the point back to the interval $[\alpha - 1, \alpha]$. Thus, each branch of $K_\alpha(x)$ is represented by the map $x \mapsto \mathbf{T}^{-c_\alpha(x)}\mathbf{S}x$. Now, in order to keep track of the inverse branches of the powers of K_α, we shall now use the notation $c_{j,\alpha}(x) := c_\alpha(K_\alpha^{j-1}(x))$ for each positive integer j and define the matrices

$$\mathbf{M}_{\alpha,x,\ell} := \begin{pmatrix} 0 & -1 \\ 1 & c_{1,\alpha}(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & c_{\ell,\alpha}(x) \end{pmatrix}.$$

Note that these matrices represent the inverses of K_α, in the sense that

$$\mathbf{M}_{\alpha,x,\ell}(K_\alpha^\ell(x)) = x$$

for each $\alpha \in [0, 1], x \in [\alpha - 1, \alpha], \ell \in \mathbb{N}$. Finally, note that the family K_α possesses the following fundamental symmetry: the maps K_α and $K_{1-\alpha}$ are measurably conjugate; namely, one has

$$K_\alpha(x) = -K_{1-\alpha}(-x) \tag{28}$$

for all $x \in [\alpha - 1, \alpha] \setminus \bigcup_{k\in\mathbb{Z}} \frac{1}{k-\alpha}$ (the countable set of exceptions is due to the convention about the floor function). As a consequence, it is sufficient to study the dynamics for $\alpha \in [0, 1/2]$.

We are now ready to formulate the main result of this section.

Theorem 5.1. *Let $w \in FW^*$ be a Farey word, let $m_0 := |w|_0, m_1 := |w|_1$, and let J_w be the corresponding quamterval. Moreover let $S := RL(w)$ denote the runlength of S and let $s := [0; S]$ be the pseudocenter of J_w. Then there exist two elements $\mathbf{M}, \mathbf{M}' \in PSL(2,\mathbb{Z})$ such that, for all $\alpha \in J_w$, we have the equalities*

$$\begin{aligned} \mathbf{M}_{\alpha,\alpha-1,m_0} &= \mathbf{M}, \\ \mathbf{M}_{\alpha,\alpha,m_1} &= \mathbf{M}'. \end{aligned} \tag{29}$$

Moreover, the following matching condition *holds:*

$$\mathbf{TM} = \mathbf{M}'\mathbf{S}\mathbf{T}^{-1}\mathbf{S}. \tag{30}$$

The matching condition (30) implies the following identification between the orbits of the two endpoints α and $\alpha - 1$.

Corollary 5.2. *For each parameter $\alpha \in J_w$ we have the identity*

$$K_\alpha^{m_0+1}(\alpha - 1) = K_\alpha^{m_1+1}(\alpha). \tag{31}$$

Note that (29) implies that the first m_0 steps of the (symbolic) itinerary of $\alpha - 1$ is constant for all α in the same quamterval, and the same is true for the first m_1 steps of the orbit of α. A condition of this kind is called a **strong cycle condition** in [24]; see section 7 for a more detailed comparison.

As an illustration of Theorem 5.1, let us consider the case $\alpha \in J_w$ with $w = 01$. It turns out that for every $\alpha \in J_{01} = (g^2, g)$ the following identity holds:

$$K_\alpha^2(\alpha) = K_\alpha^2(\alpha - 1) \quad \forall \alpha \in (g^2, g). \tag{32}$$

Indeed, this is due to the fact that the analytic expression of K_α at the endpoints does not change as $\alpha \in J_{01}$. In this simple case in fact we can work out the explicit form of K_α, and we get

$$K_\alpha(\alpha) = \mathbf{T}^2\mathbf{S}\alpha = \frac{2\alpha - 1}{\alpha}, \qquad K_\alpha(\alpha - 1) = \mathbf{T}^{-2}\mathbf{S}(\alpha - 1) = \frac{2\alpha - 1}{1 - \alpha},$$

whence we have $\mathbf{M} = \mathbf{S}\mathbf{T}^2$ and $\mathbf{M}' = \mathbf{S}\mathbf{T}^{-2}$, and we can check that

$$\mathbf{TM} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \mathbf{M}'\mathbf{S}\mathbf{T}^{-1}\mathbf{S},$$

which is an instance of equation (30). Note that the essential point is that the matrices $\mathbf{M}$ and $\mathbf{M}'$ do not depend on the particular α as long as α belongs to J_w; thus, the matching condition is just an identity between elements of the group $PSL(2,\mathbb{Z})$. However, to different matching intervals J_w there correspond different identities in the group.

The proof of Theorem 5.1 follows from an explicit description of the symbolic orbits of α and $\alpha - 1$ in terms of the regular continued fraction expansion of the pseudocenter of J_w, as stated in the following proposition.

Proposition 5.3. *Let $w \in FW_0$ (hence $J_w \subset (0, 1/2)$), denote $m := |w|_0$ and $n := |w|_1$, and let $RL(w) = (a_1, 1, \ldots, a_n, 1)$ be the associated string of positive integers. Then for each $\alpha \in J_w$ we have the identities*

$$\begin{aligned} \mathbf{M}_{\alpha,\alpha-1,m} &= \mathbf{M}, \\ \mathbf{M}_{\alpha,\alpha,n} &= \mathbf{M}', \end{aligned} \tag{33}$$

where the above matrices are constructed as

$$\begin{aligned} \mathbf{M} &:= (\mathbf{S}\mathbf{T}^2)^{a_1}\mathbf{T}(\mathbf{S}\mathbf{T}^2)^{a_2}\mathbf{T}\cdots(\mathbf{S}\mathbf{T}^2)^{a_{n-1}}\mathbf{T}(\mathbf{S}\mathbf{T}^2)^{a_n}, \\ \mathbf{M}' &:= \mathbf{S}\mathbf{T}^{-a_1-1}\mathbf{S}\mathbf{T}^{-a_2-2}\cdots\mathbf{T}^{-a_{n-1}-2}\mathbf{S}\mathbf{T}^{-a_n-2}. \end{aligned} \tag{34}$$

Let us point out that in the special case $n = 1$ we have $RL(w) = (N, 1)$ for some $N \geq 1$, and the above equations must be interpreted as yielding $\mathbf{M} = (\mathbf{S}\mathbf{T}^2)^N$, $\mathbf{M}' = \mathbf{S}\mathbf{T}^{-N-1}$. Moreover, as a consequence of the proposition, the matrices determining matching conditions behave well under concatenation. Indeed, if the Farey word w is the concatenation of two Farey words w' and w'', then the left-hand side of the matching condition on J_w is the concatenation of the left-hand sides of the matching conditions of $J_{w'}$ and $J_{w''}$.

Before delving into the core of the proofs of Proposition 5.3 and Theorem 5.1 let us make some elementary observations and define some more notation. The action of both $\mathbf{S}$ and $\mathbf{T}$ can be easily expressed in terms of regular continued fraction expansion; thus the action of K_α on the regular continued fraction expansion of x follows some simple rules. Namely, if $x = [-1; a_1, a_2, a_3, \dots] \in [\alpha - 1, 0)$, then one gets the formulas

$$(35) \quad K_\alpha(x) = \begin{cases} \mathbf{T}^{-2}\mathbf{S}x = [-1; a_1 - 1, a_2, a_3, a_4 \dots] & a_1 > 1, \\ \mathbf{T}^{-(a_2+1-\epsilon)}\mathbf{S}x = [\epsilon; a_3, a_4, \dots] & a_1 = 1, \end{cases} \qquad \epsilon \in \{-1, 0\},$$

where to decide whether ϵ is -1 or 0 one has to check which of these choices returns an element in $[\alpha - 1, \alpha]$. On the other hand, if $x = [0; a_1, a_2, a_3, \dots] \in (0, \alpha]$, then

$$(36) \quad K_\alpha(x) = \mathbf{T}^{a_1+1+\epsilon}\mathbf{S}x = \begin{cases} [\epsilon; 1, a_2 - 1, a_3, \dots] & a_2 > 1, \\ [\epsilon; 1 + a_3, a_4, \dots] & a_2 = 1, \end{cases} \qquad \epsilon \in \{-1, 0\}.$$

Again, the choice between $\epsilon = -1$ and $\epsilon = 0$ is forced by the condition that the range of K_α must be $[\alpha - 1, \alpha]$. To write some of the above branches of K_α in compact form we shall also use the following fractional transformations:

$$\partial^- := \mathbf{S}\mathbf{T}^{-1}\mathbf{S}, \qquad \partial := \mathbf{S}\mathbf{T}\mathbf{S}.$$

Note that if $0 < x < 1$, one has $\partial^- x < x$; in fact in terms of regular continued fractions we get $\partial^-([0; a_1, a_2, a_3, \dots]) = [0; 1 + a_1, a_2, a_3, \dots]$, while ∂ is the inverse of ∂^- (and is consistent with the previous definition in section 3). Finally, if $S = (a_1, \dots, a_n)$ and $x \in [0, 1]$, we shall use the *string action* notation $S \cdot x$ to denote the number whose continued fraction expansion is obtained by appending S at the beginning of the continued fraction expansion of x; in terms of $\mathbf{S}$ and $\mathbf{T}$, this can be defined as

$$(37) \qquad S \cdot x := \mathbf{S}\mathbf{T}^{-a_1}\mathbf{S}\mathbf{T}^{a_2} \cdots \mathbf{S}\mathbf{T}^{-a_{2n-1}}\mathbf{S}\mathbf{T}^{a_{2n}} x.$$

Proof of Proposition 5.3. Let $w \in FW_0$, and let $\alpha \in J_w$. Let us denote $S := RL(w)$ as the runlength of S and $s := [0; S]$ as the pseudocenter of J_w. Now, by Lemma 3.1 we have that S is of the form

$$S = (a_1, 1, a_2, 1, \dots, a_m, 1)$$

with $a_j \in \{a, a+1\}$, $m_0 = \sum_{j=1}^m a_j$, and $m_1 = m$. Moreover, for $1 \le k \le m$ we define the even prefixes and suffixes of S as

$$P_k := (a_1, 1, a_2, 1, \dots, a_k, 1), \qquad S_k := (a_{k+1}, 1, a_{k+2}, 1, \dots, a_m, 1).$$

Recall also that by definition the endpoints of $J_w = (\alpha^-, \alpha^+)$ are

$$\alpha^- = [0; S'\overline{{}^tS}], \qquad \alpha^+ = [0; \overline{S}].$$

Case A. Let us first take into account the case $\alpha \in [r, \alpha^+)$. Then we can write $\alpha := S \cdot y$ for some $y \in [0, \alpha^+)$. In this case we claim that the orbits of the endpoints α and $\alpha - 1$ under K_α eventually match, and before getting to the matching point the orbits (and symbolic orbits, on the right column) of $\alpha - 1$ and α are given by Table 1.

One can go from one line to the following just using the rules (35) (for the upper part) or (36) (for the lower part). So we only have to check that at each stage we actually get a value which lies in the interval $[\alpha - 1, \alpha]$.

TABLE 1. Orbits of α and $\alpha - 1$, for $\alpha \in [r, \alpha^+)$.

	$\alpha - 1$	$=$	$-1 + S \cdot y$	
	$K_\alpha(\alpha - 1)$	$=$	$-1 + \partial S \cdot y$	$c_{1,\alpha} = 2$
	$K_\alpha^2(\alpha - 1)$	$=$	$-1 + \partial^2 S \cdot y$	$c_{2,\alpha} = 2$
	$\dots$			$\dots$
	$K_\alpha^{a_1-1}(\alpha - 1)$	$=$	$-1 + \partial^{a_1-1} S \cdot y$	$c_{a_1-1,\alpha} = 2$
$\maltese$	$K_\alpha^{a_1}(\alpha - 1)$	$=$	$-1 + S_1 \cdot y$	$c_{a_1,\alpha} = 3$
	$K_\alpha^{a_1+1}(\alpha - 1)$	$=$	$-1 + \partial S_1 \cdot y$	$c_{a_1+1,\alpha} = 2$
	$\dots$			$\dots$
$\maltese$	$K_\alpha^{a_1+a_2}(\alpha - 1)$	$=$	$-1 + S_2 \cdot y$	$c_{a_1+a_2,\alpha} = 3$
	$\dots$			$\dots$
	$\dots$			$\dots$
$\maltese$	$K_\alpha^{a_1+\cdots+a_{m-1}}(\alpha - 1)$	$=$	$-1 + S_{m-1} \cdot y$	$c_{a_1+\cdots+a_{m-1},\alpha} = 3$
	$\dots$			$\dots$
	$K_\alpha^{a_1+\cdots+a_m-1}(\alpha - 1)$	$=$	$-1 + \partial^{a_m-1} S_{m-1} \cdot y$	$c_{a_1+\cdots+a_m-1,\alpha} = 2$
$\maltese$	$K_\alpha^{a_1+\cdots+a_m}(\alpha - 1)$	$=$	y	$c_{a_1+\cdots+a_m,\alpha} = 2$
	α	$=$	$S \cdot y$	
	$K_\alpha(\alpha)$	$=$	$\partial^- S_1 \cdot y$	$c_{1,\alpha} = -a_1 - 1$
	$K_\alpha^2(\alpha)$	$=$	$\partial^- S_2 \cdot y$	$c_{2,\alpha} = -a_2 - 2$
	$\dots$			$\dots$
	$K_\alpha^{m-1}(\alpha)$	$=$	$\partial^- S_{m-1} \cdot y$	$c_{m-1,\alpha} = -a_{m-1} - 2$
	$K_\alpha^m(\alpha)$	$=$	$\partial^- y = \frac{y}{y+1}$	$c_{m,\alpha} = -a_m - 2$

As far as the orbit of $\alpha - 1$ is concerned, all items in the list except for the last one are negative, so we just have to check that we never drop below $\alpha - 1$. Since the operator ∂ increases the value of its argument (i.e., $\partial x \geq x$), it is sufficient to check the iterates of K_α of order $a_1 + \cdots + a_k$ (with $k \in \{1, \cdots\}$): the corresponding lines are marked by the symbol $\maltese$. That is, we need to check the following:

(1) $-1 + S_k \cdot y \geq \alpha - 1 \quad$ for all $k \in \{1, \dots, m-1\}$,

(2) $y \leq \alpha$.

(1) is true by Lemma 3.1(iii); indeed, we have the inequality $S_k \gg S$, from which it follows that

$$-1 + S_k \cdot y \geq -1 + S \cdot y = -1 + \alpha$$

as needed. Now, since by construction $S \cdot y = \alpha < \alpha^+$ and the map $x \mapsto S \cdot x$ is increasing with a fixed point at α^+, we have that $y \leq S \cdot y = \alpha$, which proves (2).

Checking that the values in the lower part of Table 1 are actually in $[\alpha - 1, \alpha]$ is slightly more tricky. We need to prove that $\partial^- S_k \cdot y \leq S \cdot y = \alpha$ for $k \in \{1, \dots, m-1\}$; as a matter of fact by Lemma 3.1(iv) one has

$$S_k P_k \ll \partial S,$$

which implies, since P_k is a prefix of the continued fraction expansion of α^+, that

$$S_k \cdot \alpha^+ \leq \partial S \cdot y.$$

Since the map $x \mapsto S_k \cdot x$ is increasing we then get $S_k \cdot y \leq S_k \cdot \alpha^+ < \partial S \cdot y$, which implies, by applying ∂^- to both sides of the equation, $\partial^- S_k \cdot y < S \cdot y = \alpha$. Since $0 < \partial^- y < y < S \cdot y = \alpha$, we get the last step for free.

TABLE 2. Orbits of α and $\alpha - 1$, for $\alpha \in (\alpha^-, r]$.

$\alpha - 1 = -1 + S' \cdot y$	
$K_\alpha(\alpha - 1) = -1 + \partial S' \cdot y$	$c_{1,\alpha} = 2$
$K_\alpha^2(\alpha - 1) = -1 + \partial^2 S' \cdot y$	$c_{2,\alpha} = 2$
$\dots$	$\dots$
$K_\alpha^{a_1 - 1}(\alpha - 1) = -1 + \partial^{a_1 - 1} S' \cdot y$	$c_{a_1 - 1,\alpha} = 2$
✠ $K_\alpha^{a_1}(\alpha - 1) = -1 + S'_1 \cdot y$	$c_{a_1,\alpha} = 3$
$K_\alpha^{a_1 + 1}(\alpha - 1) = -1 + \partial S'_1 \cdot y$	$c_{a_1 + 1,\alpha} = 2$
$\dots$	$\dots$
✠ $K_\alpha^{a_1 + a_2}(\alpha - 1) = -1 + S'_2 \cdot y$	$c_{a_1 + \dots + a_2,\alpha} = 3$
$\dots$	$\dots$
$\dots$	$\dots$
✠ $K_\alpha^{a_1 + \dots + a_{m-1}}(\alpha - 1) = -1 + S'_{m-1} \cdot y$	$c_{a_1 + \dots + a_{m-1},\alpha} = 3$
$\dots$	$\dots$
$K_\alpha^{a_1 + \dots + a_m - 1}(\alpha - 1) = -1 + \partial^{a_m - 1} S'_{m-1} \cdot y$	$c_{a_1 + \dots + a_m - 1,\alpha} = 2$
✠ $K_\alpha^{a_1 + \dots + a_m}(\alpha - 1) = -y/(y+1)$	$c_{a_1 + \dots + a_m,\alpha} = 2$
$\alpha = S' \cdot y$	
$K_\alpha(\alpha) = \partial^- S'_1 \cdot y$	$c_{1,\alpha} = -a_1 - 1$
$K_\alpha^2(\alpha) = \partial^- S'_2 \cdot y$	$c_{2,\alpha} = -a_2 - 2$
$\dots$	$\dots$
$K_\alpha^{m-1}(\alpha) = \partial^- S'_{m-1} \cdot y$	$c_{m-1,\alpha} = -a_{m-1} - 2$
$K_\alpha^m(\alpha) = -y$	$c_{m,\alpha} = -a_m - 2$

Case B. We must now settle the case $\alpha \in (\alpha^-, r]$. Let us recall that $'S' = {}^tS$, so that we must have $\alpha = S' \cdot y$ for $0 \le y \le [0; \overline{{}^tS}]$ or, which is equivalent, $\sigma\alpha = {}^tS \cdot y$, $0 \le y \le [0; \overline{{}^tS}]$. In this case we claim that the orbits of the endpoints, before reaching the matching point, are given by Table 2.

Again, one can go from one line to the following just using the rules (35) for the upper list or (36) for the lower; we just have to check that at each stage we actually get values inside the interval $[\alpha - 1, \alpha]$.

As far as the orbit of $\alpha - 1$ is concerned, all items of the list are negative, and we just have to check that we never drop below $\alpha - 1$. Therefore the only steps which need some comment are those corresponding to iterates of K_α of order $a_1 + \dots + a_k$ (marked by ✠ in the table). Let us observe that by Lemma 3.1(iii) we have $S_k \gg S$. Then we have two cases: either $S'_k \gg S'$ and we are done, or we can write $S' = S'_k Z$, with Z a suffix of S' (hence also a suffix of $'S' = {}^tS$), and the length of Z is even. In the latter case, by applying Lemma 3.1 to ${}^tS$, one gets $Z \gg {}^tS$; hence since $y \le [0; \overline{{}^tS}] \le [0; \overline{Z}]$ and the length of S'_k is odd, we have

$$y \le Z \cdot y \Rightarrow S'_k \cdot y \ge S'_k Z \cdot y \Rightarrow -1 + S'_k \cdot y \ge S' \cdot y,$$

proving the required inequality. The last step is immediate, since $-y/(y+1) > -1/2 > \alpha - 1$ (note that $\alpha < 1/2$ since $w \in FW_0$).

To check that the values in the lower part of the list of Table 2 are actually in $[\alpha - 1, \alpha]$ we need to prove that $\partial^- S'_k \cdot y < S' \cdot y$ for $k \in \{1, \dots, m-1\}$. Indeed, since σ is order-reversing and $'S' = {}^tS$, we get

$$\partial^- S'_k \cdot y < S' \cdot y \iff \sigma(\partial^- S'_k \cdot y) > \sigma(S' \cdot y) \iff {}'(\partial^- S'_k) \cdot y = ({}^tS)_k \cdot y > {}^tS \cdot y,$$

and the last inequality holds because we can apply Lemma 3.1(iii) to ${}^tS$, yielding $({}^tS)_k \gg {}^tS$.

Finally, to check that $K^m(\alpha) = -y$ we have to show that $-y > \alpha - 1$: indeed, since $y < [0; \overline{{}^tS}]$ we get ${}^tS \cdot y > y$ and $1 - \alpha = \ 'S' \cdot y = {}^tS \cdot y > y$.

If we now keep track of the symbolic orbit of $\alpha - 1$ and α as we described (see the right column of the Tables 1, 2), we realize that the values of the coefficients $c_{j,\alpha}(\alpha)$ and $c_{j,\alpha}(\alpha-1)$ are

$$\begin{aligned} (c_{j,\alpha}(\alpha-1))_{1\le j\le m_0} &= (\underbrace{2,\dots,2}_{a_1-1},3,\underbrace{2,\dots,2}_{a_2-1},3,\dots,\underbrace{2,\dots,2}_{a_{m-1}-1},3,\underbrace{2,\dots,2}_{a_m}), \\ (c_{j,\alpha}(\alpha))_{1\le j\le m_1} &= (-a_1-1,-a_2-2,\dots,-a_m-2). \end{aligned}$$

Thus we can compute the matrices $\mathbf{M}_{\alpha,\alpha-1,m_0}$ and $\mathbf{M}_{\alpha,\alpha,m_1}$, recovering formula (34). ☐

Proof of Theorem 5.1. By symmetry (28), we need only to check what happens for $\alpha \le \frac{1}{2}$. Now, the previous proposition gives us formulas for $\mathbf{M}$ and $\mathbf{M}'$, so we just need to check that equation (30) holds given these formulas; this is a simple algebraic manipulation as follows. Let us first prove the case $n = 1$, for which equation (30) becomes

$$\mathbf{T}(\mathbf{S}\mathbf{T}^2)^N = \mathbf{S}\mathbf{T}^{-N-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{S} \tag{38}$$

(with $N = a_1$). It is well-known and easy to check that $(\mathbf{ST})^3$ is the identity, from which it follows that

$$\mathbf{TST} = \mathbf{S}\mathbf{T}^{-1}\mathbf{S}.$$

Thus, by writing $\mathbf{TST} = \mathbf{T}(\mathbf{S}\mathbf{T}^2)\mathbf{T}^{-1}$ and raising both sides to the N^{th} power, we have

$$\mathbf{T}(\mathbf{S}\mathbf{T}^2)^N\mathbf{T}^{-1} = \mathbf{S}\mathbf{T}^{-N}\mathbf{S},$$

from which $\mathbf{T}(\mathbf{S}\mathbf{T}^2)^N = \mathbf{S}\mathbf{T}^{-N}\mathbf{S}\mathbf{T} = \mathbf{S}\mathbf{T}^{-N-1}\mathbf{TST} = \mathbf{S}\mathbf{T}^{-N-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{S}$, proving (38). Thus, in general for each $1 \le k \le n$ we have the identity $\mathbf{T}(\mathbf{S}\mathbf{T}^2)^{a_k} = \mathbf{S}\mathbf{T}^{-a_k-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{S}$, and concatenating all pieces we get precisely equation (30). ☐

Proof of Corollary 5.2. By taking the inverses of both sides of equation (30) and acting on α we get the equality

$$\mathbf{M}_{\alpha,\alpha-1,m_0}\mathbf{T}^{-1}\alpha = \mathbf{STS}\mathbf{M}^{-1}_{\alpha,\alpha,m_1}(\alpha).$$

Hence using that $K_\alpha^{m_1}(\alpha) = \mathbf{M}^{-1}_{\alpha,\alpha,m_0}(\alpha)$ and $K_\alpha^{m_0}(x) = \mathbf{M}^{-1}_{\alpha,\alpha-1,m_0}(\alpha-1)$ we get that

$$-\frac{1}{K_\alpha^{m_1}(\alpha)} + 1 = -\frac{1}{K_\alpha^{m_0}(\alpha-1)}.$$

Hence for any $k \in \mathbb{Z}$ we have that

$$-\frac{1}{K_\alpha^{m_1}(\alpha)} - k \in [\alpha-1,\alpha] \Leftrightarrow -\frac{1}{K_\alpha^{m_0}(\alpha-1)} - (k+1) \in [\alpha-1,\alpha].$$

Thus $c_{m_0+1,\alpha}(\alpha-1) = c_{m_1,\alpha}(\alpha) + 1$, and the claim follows. ☐

Let us conclude this section by studying the ordering between the iterates of K_α, which will be needed in the last section. As we shall see, this also follows from the combinatorics of the underlying Farey words: in particular, the ordering is the same as the ordering between the cyclic permutations of their Farey structure.

Lemma 5.4. *Let $w \in FW_0$. Then for any $\alpha \in J_w$ the ordering of the set*

$$\{K_\alpha^j(\alpha - 1) \ : \ 0 \leq j \leq m_0\}$$

of the first $m_0 + 1$ iterates of $\alpha - 1$ under K_α is independent of α. Similarly, the ordering of the set of the first $m_1 + 1$ iterates of α is also independent of α.

Proof. The first part of the claim can be rephrased as saying that for each $j, j' \in \{0, \dots, m_0\}$ and for each $\alpha, \alpha' \in J_w$,

$$K_\alpha^i(\alpha - 1) < K_\alpha^j(\alpha - 1) \quad \text{if and only if} \quad K_\alpha^i(\alpha - 1) < K_\alpha^j(\alpha - 1).$$

Now let $0 \leq j < j' \leq m_0$ be fixed. To check that $K_\alpha^j(\alpha - 1)$ and $K_\alpha^{j'}(\alpha - 1)$ are ordered in the same way for all $\alpha \in J_w$ it is enough to prove that they are always different, and to prove this latter statement it is enough to prove that $K_\alpha^j(\alpha - 1) < K_\alpha^{m_0}(\alpha - 1)$ for all $\alpha \in J_w$ and all $j \in \{0, \dots, m_0 - 1\}$. This follows from the explicit description of the orbits given in the proof of Proposition 5.3, of which we will keep the notation (see Tables 1 and 2). Indeed, in Case A the claim holds because of the inequality

$$K_\alpha^j(\alpha - 1) < 0 \leq y = K_\alpha^{m_0}(\alpha - 1).$$

In Case B we have

$$K_\alpha^{m_0}(\alpha - 1) = \frac{-y}{y + 1} = -1 + \frac{1}{1 + y} = [-1; 1, a, \dots] \qquad \text{for some } a > 1,$$

while the largest of the previous iterates has a continued fraction expansion beginning with $[-1; 1, 1, \dots]$.

The corresponding claim about iterates of α is proven in the same way. Indeed, it is enough to check that if $j < m_1$, then $K_\alpha^{m_1}(\alpha) < K_\alpha^j(\alpha)$; this is obvious in Case B, while in Case A we just have to check that $\partial^- y < \partial^- S_j \cdot y$. Now, since $y < S \cdot y$ (see also proof of Proposition 5.3, Case A) and $S \ll S_j$ (Lemma 3.1(iii)), one gets $\partial^- y < \partial^- S \cdot y \leq \partial^- S_j \cdot y$ as claimed. □

Proposition 5.5. *Let $w \in FW_0$ be a Farey word, and let $w = w'w''$ be its standard factorization, and denote $j_0 := |w'|_0$ and $j_1 = |w''|_1$. Then for each $\alpha \in J_w$ the following hold:*

$$K_\alpha^{j_0}(\alpha - 1) = \min\{K_\alpha^j(\alpha - 1) \ : 1 \leq j \leq m_0\},$$
$$K_\alpha^{j_1}(\alpha) = \max\{K_\alpha^j(\alpha) \ : 1 \leq j \leq m_1\}.$$

Proof. By Lemma 5.4 and since all the maps $\alpha \to K_\alpha^j(\alpha - 1)$ for $0 \leq j \leq m_0$ are continuous on the closure of J_w (they are given by equation (34)), it is sufficient to verify the statement for $\alpha = \alpha^+$, the right endpoint of J_w. In this case, by Proposition 5.3, the first iterates of $\alpha - 1$ are given by

$$K_\alpha^{a_1 + \dots + a_k + h}(\alpha - 1) = -1 + \partial^h S_k \cdot \alpha^+$$

with $0 \leq h \leq a_{k+1} - 1$, $0 \leq k \leq m - 1$. Note that the homeomorphism ϕ defined in section 4.1 semiconjugates the shift in the binary expansion with the map ∂; hence for each prefix v of w we have

$$\phi(0.\overline{\tau^l w}) = K_\alpha^{l_0}(\alpha - 1) + 1, \tag{39}$$

where $l := |v|$ and $l_0 := |v|_0$. Now, in order to find the smallest non-trivial iterate, recall that by Lemma 2.6, the smallest cyclic permutation of w is w itself, while the second smallest is $\tau^{|w'|}w$, i.e.,

$$w < \tau^{|w'|}w \le \tau^k w \qquad \text{for all } 1 \le k \le m_0 + m_1.$$

Thus, since the homeomorphism ϕ is increasing on the interval $[0, 1/2]$, we get by equation (39) that for each $j \in \{1, \dots, m_0\}$,

$$K_\alpha^{j_0}(\alpha - 1) \le K_\alpha^j(\alpha - 1),$$

where $j_0 = |w'|_0$ as claimed. Similarly, for the orbit of α, we know by Proposition 5.3 that the iterates in case $\alpha = \alpha^+$ are given by

$$K_\alpha^j(\alpha) = \partial^- S_j \cdot \alpha^+, \qquad 1 \le j \le m_1.$$

Once again from Lemma 2.6, the largest cyclic permutation of w is $\tau^l w$ with $l = |w''|$. Hence the largest value of $S_j \cdot \alpha^+$ is attained for $j = |w''|_1 = j_1$, and the claim follows. □

6. Entropy

We shall now use the combinatorial description of the orbits of K_α we have obtained in the previous section to derive consequences about the entropy of the maps, proving Theorem 1.2.

For any fixed $\alpha \in (0, 1)$, the map K_α is a uniformly expanding map of the interval, and many general facts about its measurable dynamics are known (see [25]). Indeed, each K_α has a unique absolutely continuous invariant probability measure (a.c.i.p. for short), which we will denote $d\mu_\alpha = \rho_\alpha(x)dx$, and the dynamical system (K_α, μ_α) is ergodic. In fact, it is even exact and isomorphic to a Bernoulli shift; moreover, its ergodic properties can also be derived from the properties of the geodesic flow on the modular surface.

Let $h(\alpha)$ be the metric entropy of K_α with respect to the measure μ_α: we shall be interested in studying the properties of the function $\alpha \mapsto h(\alpha)$. Recall that for an expanding map of the interval the entropy can also be given by Rohlin's formula

$$h(\alpha) = \int_{\alpha-1}^{\alpha} \log |K'_\alpha| \; d\mu_\alpha.$$

Moreover, for maps generating continued fraction algorithms such as K_α, the entropy is also related to the growth rate of denominators of convergents to a "typical" point. More precisely, we can define the α-convergents to x to be the sequence $(p_{n,\alpha}(x)/q_{n,\alpha}(x))_{n\in\mathbb{N}}$ where

$$\begin{pmatrix} p_{n,\alpha}(x) \\ q_{n,\alpha}(x) \end{pmatrix} := \mathbf{M}_{\alpha,x,n} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For each x, the sequence $p_{n,\alpha}(x)/q_{n,\alpha}(x)$ tends to x. Then, for μ_α-almost every $x \in [\alpha - 1, \alpha]$ we have

$$h(\alpha) = 2 \lim_{x\to+\infty} \frac{1}{n} \log |q_{n,\alpha}(x)|.$$

As far as the global regularity of the entropy function is concerned, one can easily adapt the strategy of [36] to prove that $h(\alpha)$ is Hölder continuous in α.

Theorem 6.1. *For any $a \in (0, 1/2]$ and any $\eta \in (0, 1/2]$, the function $\alpha \mapsto h(\alpha)$ is Hölder continuous of exponent η on $[a, 1 - a]$.*

On the complement of $\mathcal{E}_{KU}$ we can exploit the rigidity due to the matching to gain much more regularity. The key tool will be the following proposition.

Proposition 6.2. *Let $\alpha, \alpha' \in J_w$ be nearby points which both lie on the same side with respect to the pseudocenter, with $\alpha' < \alpha$. Then the following formulas hold:*

$$h(\alpha) = [1 + (|w|_0 - |w|_1)\mu_\alpha([\alpha', \alpha])]h(\alpha'), \tag{40}$$

$$h(\alpha') = [1 - (|w|_0 - |w|_1)\mu_{\alpha'}([\alpha' - 1, \alpha - 1])]h(\alpha). \tag{41}$$

The proof of this proposition follows very closely the proof of the corresponding statement for α-continued fractions in ([35], Theorem 2); a sketch of the argument is included in the appendix. As a first straightforward consequence of Proposition 6.2 we get the local monotonicity of h on the complement of $\mathcal{E}_{KU}$.

Corollary 6.3. *The entropy is locally monotone on $[0,1] \setminus \mathcal{E}_{KU}$. More precisely:*

(1) *the entropy is strictly increasing on J_w if $w \in FW_0$;*
(2) *the entropy is constant on J_w if $w = (01)$;*
(3) *the entropy is strictly decreasing on J_w if $w \in FW_1$.*

Let us point out that Corollary 6.3 alone is still not enough to deduce the monotonicity on $[0, g^2]$ stated in Theorem 1.2. Indeed, h is strictly increasing on each open interval J_w for all $w \in FW_0$, and the union of all such opens is dense in $[0, g^2]$, but to conclude that h is monotone on $[0, g^2]$ we must exclude that h displays pathological behaviour like the "devil's staircase" function. We shall take care of this issue proving that h is absolutely continuous.

Let us consider the decomposition $h(t) = h_r(t) + h_s(t)$ where

$$h_r(t) := \sum_{w \in FW} \operatorname*{Var}_{J_w \cap [0,t]} h, \qquad h_s(t) := h(t) - h_r(t). \tag{42}$$

Recall that the notation $\mathrm{Var}_I\, f$ means the total variation of the function f on the interval I. Intuitively, h_r is the "regular part" which takes into account the behaviour of h on the (open and dense) union of the J_w, while h_s is the remaining "singular part", which we will actually prove to be zero.

Lemma 6.4. *The function h_s is locally constant on $[0, g^2] \setminus \mathcal{E}_{KU}$.*

Proof. If $w \in FW_0$ and $t_1, t_2 \in J_w$, $t_1 < t_2$, then the monotonicity of $h_{|J_w}$ implies that

$$h(t_2) - h(t_1) = \operatorname*{Var}_{J_w \cap [t_1, t_2]} h = h_r(t_2) - h_r(t_1),$$

which implies that $h_s(t_2) - h_s(t_1) = 0$, whence the claim. □

We shall now need the following lemma in fractal geometry, whose proof we postpone to the appendix.

Lemma 6.5. *Let $I_1, I_2, \dots$ be a countable family of disjoint subintervals of some close interval $I \subseteq \mathbb{R}$ and denote*

$$\mathcal{G} := I \setminus \bigcup_{i=1}^{\infty} I_i.$$

Let the upper box-dimension of $\mathcal{G}$ be δ_0. Given any η and δ with $\eta > \delta > \delta_0$, there exists a constant C such that for any choice of a subsequence $J_1, J_2, \dots$ of the *family* $\{I_i\}$*, one gets the following inequality:*

$$\sum_{i=1}^{\infty} |J_i|^{\eta} \le C \left(\sum_{i=1}^{\infty} |J_i| \right)^{\eta-\delta}.$$

Lemma 6.6. *For every* a *and* η *in* $(0,1/2)$ *both functions* h_s *and* h_r *are Hölder continuous of exponent* η *on the interval* $[a,1/2]$*.*

Proof. By Theorem 6.1 there is $C = C(a,\eta)$ such that

$$|h(t) - h(t')| \le C|t-t'|^{\eta} \quad \forall t,t' \in [a,1/2]. \tag{43}$$

Note that in order to prove Hölder continuity of h_r on the whole interval $[a,1/2]$ it is sufficient to show that h_r is Hölder continuous on $[a,1/2] \cap \mathcal{E}_{KU}$. If $\beta,\beta' \in \mathcal{E}_{KU} \cap [a,1/2]$, $\beta < \beta'$, then

$$h_r(\beta') - h_r(\beta) = \sum_{J_w \subset [\beta,\beta']} \operatorname*{Var}_{J_w} h.$$

By equation (43), for each J_w we have that $\mathrm{Var}_{J_w} h \le C|J_w|^{\eta}$; then we get

$$\begin{aligned}|h_r(\beta') - h_r(\beta)| &= \textstyle\sum_{J_w \subset [\beta,\beta']} \mathrm{Var}_{J_w} h \le C \sum_{J_w \subset [\beta,\beta']} |J_w|^{\eta} \\ &\le CC' \Big(\textstyle\sum_{J_w \subset [\beta,\beta']} |J_w| \Big)^{\eta-\delta} = CC' |\beta'-\beta|^{\eta-\delta}\end{aligned}$$

for any $\delta \in (0,\eta)$. The last line is a direct consequence of Lemma 6.5 and the fact that the box-dimension of $\mathcal{E}_{KU}$ is 0 (Proposition 4.4). Finally h_s is Hölder continuous since it is a difference of two Hölder continuous functions (note the domain has unit length). □

Lemma 6.7. *Let* $f : I \to I$ *be a map of a closed real interval which is Hölder continuous of exponent* $\eta > 0$*. Suppose* $E \subseteq I$ *is a closed, measurable subset of Hausdorff dimension less than* η *and that* f *is locally constant on the complement on* E*. Then* f *is constant on all* I*.*

Proof. If f is Hölder continuous of exponent η, it is easy to check using the definition of Hausdorff dimension that for any measurable subset E of the interval one has the estimate

$$\mathrm{H.dim}\ f(E) \le \frac{1}{\eta}\ \mathrm{H.dim}\ E. \tag{44}$$

Now, if f is locally constant on $I \setminus E$, then the image of any connected component of the complement of E is a point; hence we have

$$\mathrm{H.dim}\ f(I) = \mathrm{H.dim}\ f(E).$$

Now, if f is not constant, then by continuity the image $f(I)$ is a non-degenerate interval; hence it has dimension 1. Thus one has by using equation (44)

$$\mathrm{H.dim}\ E \ge \eta\ \mathrm{H.dim}\ f(I) = \eta,$$

which is a contradiction. □

Proof of Theorem 1.2. The second statement follows from Corollary 6.3(2). Moreover, by virtue of the symmetry of h, the third statement is a consequence of the first one, so we just need to prove that the function h is strictly increasing on $[0,g^2]$.

Since the function h is strictly increasing on each J_w which intersects $[0,g^2]$ (Corollary 6.3(1)) and the union of the J_w is dense, then the function h_r is strictly

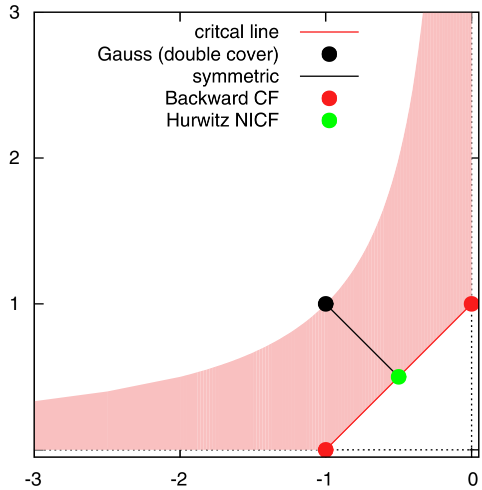

FIGURE 5. The parameter space of (a,b)-continued fractions. We only consider the critical line case $b-a=1$.

increasing on $[0,g^2]$. The claim then follows if we prove that the function h_s is identically zero, so $h=h_r$. Now, by Lemma 6.4 h_s is locally constant on the complement of $\mathcal{E}_{KU}$ and it is η-Hölder continuous for any positive $\eta<1/2$ by Lemma 6.6, so since H.dim $\mathcal{E}_{KU}=0<\eta$ the function h_s is globally constant by Lemma 6.7. $\square$

7. Natural extension and regularity properties of the entropy

Finally, in this section we shall analyze the regularity properties of the entropy function $h(\alpha)$, proving Theorem 1.1. In order to do so, we need some results from the theory of (a,b)-continued fractions due to Katok and Ugarcovici. Therefore we will outline here some of the results contained in [24, 25]; meanwhile, we shall also explain to the reader how our constructions relate to the work of Katok and Ugarcovici and translate between the different notations.

The starting point comprises the "slow" maps $f_{a,b}:\mathbb{R}\cup\{\infty\}\to\mathbb{R}\cup\{\infty\}$ defined as

$$f_{a,b}(y):=\begin{cases}\mathbf{T}y & \text{if } y<a,\\ \mathbf{S}y & \text{if } a\le y<b,\\ \mathbf{T}^{-1}y & \text{if } b\le y,\end{cases} \tag{45}$$

where the parameters (a,b) range in the closed region

$$\mathcal{P}:=\{(a,b)\in\mathbb{R}^2\ :\ a\le 0\le b,\ b-a\ge 1,\ -ab\le 1\}, \tag{46}$$

which is plotted in Figure 5.

An essential role in the theory is played by a condition called *cycle property*, which we recall briefly. If f is a real map and x is a point, we call the *upper orbit* (resp. *lower orbit*) of x the countable set of elements $f_+^k(x):=\lim_{t\to x^+}f^k(t)$ (resp. $f_-^k(x):=\lim_{t\to x^-}f^k(t)$), with $k\in\mathbb{N}$. We say that the map $f_{a,b}$ satisfies the cycle property at the discontinuity points a,b if the upper and lower orbits of a eventually collide, and the same is true for b. As a matter of fact for most parameters the cycle property is *strong*, meaning that it is the consequence of an identity in $SL(2,\mathbb{Z})$, which is stable on an open set.

To build a geometrical realization of the natural extension one defines the family of maps of the plane

$$F_{a,b}(x,y) := \begin{cases} (\mathbf{T}x, \mathbf{T}y) & \text{if } y < a, \\ (\mathbf{S}x, \mathbf{S}y) & \text{if } a \le y < b, \\ (\mathbf{T}^{-1}x, \mathbf{T}^{-1}y) & \text{if } b \le y. \end{cases} \tag{47}$$

Katok and Ugarcovici prove that each $F_{a,b}$ has an attractor $D_{a,b} \subset \mathbb{R}^2$ such that $F_{a,b}$ restricted to $D_{a,b}$ is invertible and it is a geometric realization of the natural extension of $f_{a,b}$. In fact, for most values of (a,b) the attractor $D_{a,b}$ has a simple structure:

Theorem 7.1 ([24]). *There exists an uncountable set $\widetilde{\mathcal{E}}$ of 1-dimensional Lebesgue measure zero that lies on the diagonal boundary $b-a=1$ of $\mathcal{P}$ such that:*

1. *for all $(a,b) \in \mathcal{P} \setminus \widetilde{\mathcal{E}}$ the map $F_{a,b}$ has an attractor $D_{a,b}$ (which is disjoint from the diagonal $x=y$) on which $F_{a,b}$ is essentially bijective;*
2. *the set $D_{a,b}$ consists of two (or one, in the "degenerate" case $ab=0$) connected components each having* finite rectangular structure*, i.e., bounded by non-decreasing step functions with a finite number of steps;*
3. *almost every point (x,y) off the diagonal $x=y$ is mapped to $D_{a,b}$ after finitely many iterations of $F_{a,b}$.*

The above result shows that exceptions to the finiteness condition dwell on the *critical line* $b-a=1$. For this reason, we only consider these cases. With a slight abuse of notation we shall always write $f_\alpha, F_\alpha, D_\alpha$ rather than $f_{\alpha-1,\alpha}, F_{\alpha-1,\alpha}$, $D_{\alpha-1,\alpha}$. Note that if $b-a=1$, the map K_b is precisely the first return map of $f_{b-1,b}$ on the interval $[b-1,b)$.

Note also that in the symmetric case $a+b=0$ (with $b \in [1/2,1]$) the system determined by the first return on $[-b,b]$ is equivalent to a twofold cover of the α-continued fraction transformation T_b.

Let us note that along the critical line $b-a=1$ the cycle property is a bit easier to state. Indeed, in this case the upper orbit of a coincides with the orbit of a, while the lower orbit of a coincides with the orbit of b; on the other hand the upper orbit of b coincides with the orbit of a, while the lower orbit of b is just the orbit of b. Thus, in this case the *cycle property* for $f_{b-1,b}$ is essentially equivalent to the condition (1) for the fast map K_b. Indeed, the set $\mathcal{E}_{KU}$ that we explicitly described coincides, up to a countable set of points,[2] with the projection on the x-axis of the set $\widetilde{\mathcal{E}}$ mentioned in Theorem 7.1.

Now, for $\alpha \in J_w$ the map K_α satisfies the algebraic matching condition (30); hence f_α satisfies the *strong cycle property* (see [24]), and by Theorem 7.1 the extension F_α has an attractor D_α with *finite rectangular structure*; see Figure 6. We may also consider the "first return map" $\hat{F}_\alpha : \mathbb{R} \times [\alpha-1,\alpha) \to \mathbb{R} \times [\alpha-1,\alpha)$ defined as

$$\hat{F}_\alpha(x,y) := (\mathbf{T}^{-c_\alpha(y)}\mathbf{S}x, \mathbf{T}^{-c_\alpha(y)}\mathbf{S}y).$$

[2]Precisely, the projection of $\widetilde{\mathcal{E}}$ equals $\mathcal{E}_{KU} \setminus \bigcup_{w \in FW^*} \partial J_w$.

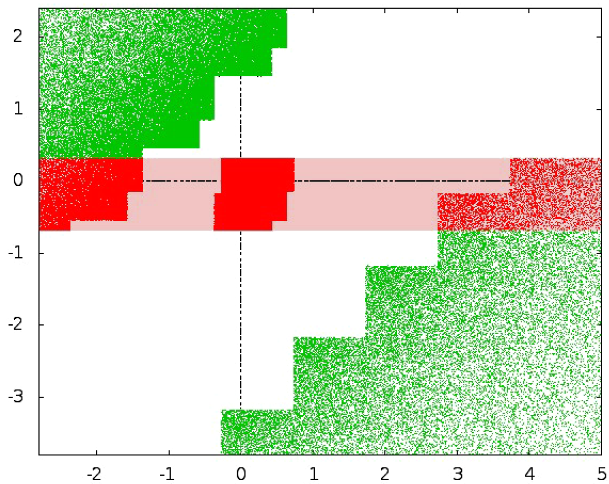

FIGURE 6. The attractor D_α, its intersection with the horizontal strip $\mathbf{T}^{-1} \times [\alpha - 1, \alpha]$, and the compact set Δ_α, which is the attractor for Φ_α.

Note that the map K_α is a factor of $\hat{F}_\alpha$. Obviously the set $D_\alpha \cap \mathbb{R} \times [\alpha - 1, \alpha]$ is an attractor for $\hat{F}_\alpha$. In order to compactify the attractor it is convenient to make the change of coordinates $\xi = \mathbf{S}x$; then, the natural extension map becomes

$$\Phi_\alpha(\xi, y) := (\mathbf{S}\mathbf{T}^{-c_\alpha(y)}\xi, \mathbf{T}^{-c_\alpha(y)}\mathbf{S}y),$$

which is just the map $\hat{F}_\alpha$ in the new coordinates: $\hat{F}_\alpha \circ (\mathbf{S} \times id) = (\mathbf{S} \times id) \circ \Phi_\alpha$. The map Φ_α will have the attractor $\Delta_\alpha := (\mathbf{S} \times id)(D_\alpha \cap \mathbb{R} \times [\alpha - 1, \alpha])$, which is bounded and has finite rectangular structure.

7.1. **Structure of the attractor Δ_α and entropy formula.** In ([24], section 5) the authors prove that the attractor D_α has finite rectangular structure providing an explicit recipe to build it; one can easily translate this recipe in order to obtain the following analogue description for Δ_α for $\alpha \in J_w$.

Let us fix $w \in FW$ a Farey word, with $m_i := |w|_i$, and pick $\alpha \in J_w$. Then the attractor Δ_α is the union of finitely many rectangles with sides parallel to the coordinate axes, and the sides of these rectangles are determined by the dynamics of K_α prior to the matching, as we now describe. See also Figure 7.

The horizontal segments which delimit Δ_α are of precisely two types, corresponding to the orbits of α and $\alpha - 1$, respectively. In particular, the set of levels (ordinates) of the horizontal segments on the "lower-right" part of Δ_α is precisely the set

$$\{\alpha - 1, K_\alpha(\alpha - 1), \dots, K_\alpha^{m_0}(\alpha - 1)\}$$

of iterates of $\alpha - 1$ up to the matching, while the set of levels of the horizontal segments on the "upper-right" side is the set

$$\{\alpha, K_\alpha(\alpha), \dots, K_\alpha^{m_1}(\alpha)\}.$$

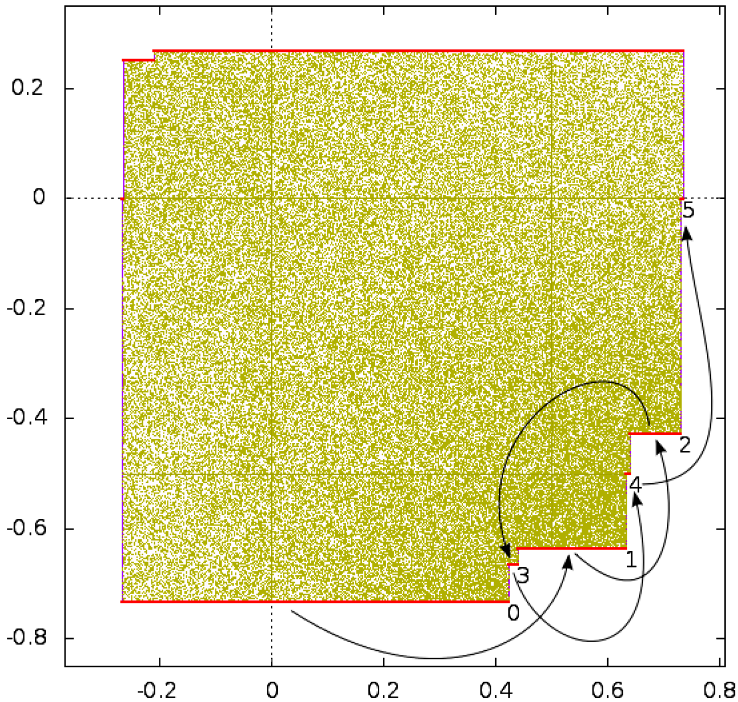

FIGURE 7. The attractor Δ_α for $\alpha = 4/15$: the numbers and arrows indicate the dynamics of the horizontal boundary segments. Note that the vertical ordering of the horizontal segments follows the ordering of the cyclic translates of the corresponding Farey word (in this case, $w = 00101$).

The coordinates of the vertical sides of the boundary of Δ_α (hence the abscissae of its corners) can instead be found in a slightly indirect way, also described in [24]: in order to explain it, let (x, α) and $(y, \alpha - 1)$ denote, respectively, the upper-right and lower-left corners of the attractor Δ_α.

(1) The highest horizontal segment which delimits Δ_α has endpoints $(\frac{y}{y-1}, \alpha)$ and (x, α), while the lowest one is the segment of endpoints $(y, \alpha - 1)$ and $(\frac{x}{x+1}, \alpha - 1)$.

(2) The horizontal segments which form the upper boundary of Δ_α are images under Φ_α of the segment of endpoints $(\frac{y}{y-1}, \alpha)$ and (x, α), and similarly the horizontal segments which bound Δ_α from below are images under Φ_α of the segment of endpoints $(y, \alpha - 1)$ and $(\frac{x}{x+1}, \alpha - 1)$.

(3) The values of x and y are determined by asking that the projection of the horizontal segments bounding Δ_α from above (resp. below) project to adjacent segments. It turns out that it is enough to check this condition on a couple of adjacent levels on the top and on the bottom, and this boils down to an algebraic relation which only depends on the symbolic orbit of α and $\alpha - 1$. In particular, it is enough to ask that the right endpoint of the lowest level matches the left endpoint of the level immediately above it. Similarly, one needs to ask that the left endpoint of the highest level

matches the right endpoint of the level immediately below it. That is, if we let π_1 be the projection on the x-coordinate and let $j_0,\ j_1$ be chosen such that

$$K_\alpha^{j_1}(\alpha) = \max\{K_\alpha^j(\alpha)\ : 1 \le j \le m_1\},\quad K_\alpha^{j_0}(\alpha-1) = \min\{K_\alpha^j(\alpha-1)\ : 1 \le j \le m_0\},$$

then, as a consequence of this discussion, the values x, y are determined by the following system:

$$\begin{cases} \mathbf{STS}y = \pi_1(\Phi_\alpha^{j_1}(x,\alpha)), \\ \mathbf{ST}^{-1}\mathbf{S}x = \pi_1(\Phi_\alpha^{j_0}(y,\alpha-1)). \end{cases} \tag{48}$$

Once x and y are known, the other vertical levels are obtained by iterating Φ_α on x and y. Note that, as a consequence, the abscissae of the vertical segments depend *only on the quamterval* J_w and do not depend on the particular α inside J_w.

We shall now combine this recipe with the results of section 5 and find the following explicit formulas for x and y.

Proposition 7.2. *Let $J_w \subset [0,1/2]$ be a quamterval and let $RL(w) = (a_1,1,\dots,a_n,1)$. Then $x = [0;\overline{1,a_n,\dots,1,a_1}]$ and $-y = [0;\overline{a_1,1,\dots,a_n,1}]$.*

Proof. In general if $(a_1,1,\dots,a_\ell,1)$ and $(a_{\ell+1},1,\dots,a_n,1)$ constitute the splitting of $(a_1,1,\dots,a_n,1)$ which corresponds to the standard factorization of w, then by Proposition 5.5 $j_0 = \sum_{i=1}^{\ell} a_i$, $j_1 = n-\ell$, and (48) becomes

$$\begin{cases} \mathbf{STS}y = \mathbf{ST}^{a_{n-\ell}+2}\ \cdots\ \mathbf{ST}^{a_2+2}\ \mathbf{ST}^{a_1+1}\ x, \\ \mathbf{ST}^{-1}\mathbf{S}x = \mathbf{ST}^{-3}\ (\mathbf{ST}^{-2})^{a_\ell-1}\ \cdots\ \mathbf{ST}^{-3}\ (\mathbf{ST}^{-2})^{a_1-1}\ y. \end{cases} \tag{49}$$

Let us point out that, by Lemma 3.1(ii), $(a_1-1,a_2,\dots,a_n)$ is a palindrome, thus $(a_{n-\ell},a_{n-\ell-1},\dots,a_2,a_1-1) = (a_{\ell+1},\dots,a_n)$. On the other hand, since $(a_{\ell+1},\dots,a_n)$ has Farey structure as well, Lemma 3.1 implies that

$$(a_{n-\ell},a_{n-\ell-1},\dots,a_2,a_1-1) = (a_{\ell+1},\dots,a_n) = (a_n+1,a_{n-1},\dots,a_{\ell+2},a_{\ell+1}-1);$$

therefore the first equation of (49) can be written as

$$\mathbf{STS}y = \mathbf{ST}^{a_n+3}\mathbf{ST}^{a_{n-1}+2}\ \cdots\ \mathbf{ST}^{a_{\ell+2}+2}\ \mathbf{ST}^{a_{\ell+1}+1}x. \tag{50}$$

Note that by applying the equality $\mathbf{TST} = \mathbf{ST}^{-1}\mathbf{S}$ one gets for each k,

$$\mathbf{T}^{k+1}\mathbf{ST} = \mathbf{T}^k\mathbf{TST} = \mathbf{T}^k\mathbf{ST}^{-1}\mathbf{S};$$

hence by applying this identity to each block on the right-hand side of (50) we get

$$y = \mathbf{ST}^{a_n+1}\mathbf{ST}^{-1}\mathbf{ST}^{a_{n-1}}\mathbf{ST}^{-1}\cdots\mathbf{ST}^{-1}\mathbf{ST}^{a_{\ell+1}}x. \tag{51}$$

Similarly, in order to modify the second equation of (49), we note that by leveraging the elementary identity $\mathbf{T}^{-1}\mathbf{ST}^{-1} = \mathbf{STS}$ we get for each k the equality

$$\mathbf{T}^{-1}(\mathbf{ST}^{-2})^k = (\mathbf{T}^{-1}\mathbf{ST}^{-1})^k\mathbf{T}^{-1} = (\mathbf{STS})^k\mathbf{T}^{-1} = \mathbf{ST}^k\mathbf{ST}^{-1}.$$

Now, if we apply it to each block on the right-hand side of the second line of (49), we get the equation

$$x = \mathbf{ST}^{-1}\mathbf{ST}^{a_\ell}\mathbf{ST}^{-1}\cdots\mathbf{ST}^{-1}\mathbf{ST}^{a_1-1}y. \tag{52}$$

We will just prove the claim for y, the other case following in the same way. By putting together (51) and (52), one finds that y satisfies the fixed point equation $y = Gy$ with

$$G = \mathbf{ST}^{a_n+1}\mathbf{ST}^{-1}\mathbf{ST}^{a_{n-1}}\mathbf{ST}^{-1}\cdots\mathbf{ST}^{a_2}\mathbf{ST}^{-1}\mathbf{ST}^{a_1-1}\mathbf{ST}^{-1}.$$

Again, since $(a_1, \dots, a_n)$ has Farey structure, we can use Lemma 3.1(iii) to infer that $(a_n + 1, a_{n-1}, \dots, a_2, a_1 - 1) = (a_1, \dots, a_n)$; hence G can be expressed as

$$G = \mathbf{ST}^{a_1}\ \mathbf{ST}^{-1}\ \mathbf{ST}^{a_2}\ \ \mathbf{ST}^{-1}\ \cdots\ \mathbf{ST}^{a_{n-1}}\ \ \mathbf{ST}^{-1}\ \mathbf{ST}^{a_n}\ \mathbf{ST}^{-1}.$$

Recalling properties (37) we can check that setting

$$\check{G} := \mathbf{ST}^{-a_1}\ \mathbf{ST}\ \mathbf{ST}^{-a_2}\ \ \mathbf{ST}\ \cdots\ \mathbf{ST}^{-a_{n-1}}\ \ \mathbf{ST}\ \mathbf{ST}^{-a_n}\ \mathbf{ST}$$

one gets that $-y = \check{G}(-y)$. On the other hand it is immediate to check that $\check{G}$ coincides with the string action induced by $Z = (a_1, 1, \dots, a_n, 1)$, and since $-y > 0$ we get $-y = [0; \overline{a_1, 1, \dots, a_n, 1}]$. $\square$

For instance, in the case $\alpha = \frac{1}{N+1}$ we get $j_0 = N,\ j_1 = 1$, so (48) reads

$$\mathbf{STS}y = \mathbf{ST}^{N+1}x, \qquad \mathbf{ST}^{-1}\mathbf{S}x = (\mathbf{ST}^{-2})^N y,$$

and a simple computation yields $x = [0; \overline{1, N}]$, $-y = [0; \overline{N, 1}]$.

The map Φ_α admits the invariant density $(1+xy)^{-2}dxdy$, and it is then easy to check that the a.c.i.p. for K_α is $d\mu_\alpha = \rho_\alpha(t)dt$ with invariant density

$$\rho_\alpha(t) := \left(\int_{\Delta_\alpha \cap \{y=t\}} (1+xy)^{-2} dx \right) \Big/ \left(\int_{\Delta_\alpha} (1+xy)^{-2} dx\ dy \right). \tag{53}$$

Moreover, for each α the following formula holds (see [25]):

$$h(\alpha) \int_{\Delta_\alpha} \frac{dx\ dy}{(1+xy)^2} = \frac{\pi^2}{3}. \tag{54}$$

7.2. **Consequences.** Formula (54) says that instead of studying the behaviour of the entropy, we may just study the function

$$\alpha \mapsto A_\alpha := \int_{\Delta_\alpha} \frac{dx\ dy}{(1+xy)^2},$$

and the explicit description of Δ_α provides us with an effective tool to do it.

Proof of Theorem 1.1. The function $\alpha \mapsto A_\alpha$ is smooth on J_w, since the levels of the vertical segments which bound Δ_α are the same for all $\alpha \in J_w$, while the levels of the horizontal segments vary analytically with α. Thus, by equation (54) the function $\alpha \mapsto h(\alpha)$ is smooth as well on each quamterval.

In order to prove the second claim, let us prove that the invariant densities ρ_α are locally bounded from below. In order to do so, let $\alpha \notin \mathcal{E}_{KU}$, and let $J_w = (\alpha^-, \alpha^+)$ be the quamterval to which α belongs. Now, by formula (23) and Proposition 7.2 we have that

$$1 - \alpha^- = [0; \overline{{}^tS}] = [0; \overline{1, a_n, \dots, 1, a_1}] = x.$$

On the other hand, recall that by the discussion in section 7.1 the right endpoint of the lowest horizontal boundary in Δ_α has abscissa

$$x_0 = \frac{x}{x+1} = \frac{1-\alpha^-}{2-\alpha^-} \geq \frac{1}{3}$$

since $\alpha^- \leq \alpha \in [0, 1/2]$; therefore the following inclusion holds:

$$\Delta_\alpha \supset [0, 1/3] \times [\alpha - 1, \alpha]. \tag{55}$$

As a consequence, we can bound the invariant density ρ_α by writing for each $t \in [\alpha-1,\alpha]$,

$$\int_{\Delta_\alpha \cap \{y=t\}} (1+xy)^{-2}dx \geq \int_0^{1/3} (1+xt)^{-2}dx \geq \frac{1}{3}\cdot 2^{-2},$$

from which, using (53) and (54), it immediately follows that $\rho_\alpha(t) \geq \frac{1}{12A_\alpha} = \frac{h(\alpha)}{4\pi^2}$. Now, by Proposition 6.2 the difference quotient of the entropy function $h(\alpha)$ on quamtervals is given in terms of ρ_α and the difference $|w|_0 - |w|_1$:

$$\frac{h(\alpha)-h(\alpha')}{\alpha-\alpha'} = (|w|_0 - |w|_1)h(\alpha')\frac{1}{\alpha-\alpha'}\int_{\alpha'}^{\alpha} \rho_\alpha.$$

Thus, by combining it with the previous lower bound we get for each $\alpha \in J_w$,

$$|h'(\alpha)| \geq C_\alpha ||w|_0 - |w|_1|, \tag{56}$$

where C_α is bounded away from zero as long as α is bounded away from 0 or 1. Now, let us pick $\alpha \in \mathcal{E}_{KU}$, $\alpha \neq 0, 1$: for any α' sufficiently close to α we have

$$|h(\alpha') - h(\alpha)| = \sum_{J_w \subseteq [\alpha,\alpha']} \operatorname*{Var}_{J_w} h \geq \sum_{J_w \subseteq [\alpha,\alpha']} C||w|_0 - |w|_1||J_w|;$$

hence, since $\mathcal{E}_{KU}$ has measure zero,

$$|h(\alpha') - h(\alpha)| \geq C \inf_{J_w \subseteq [\alpha,\alpha']} ||w|_0 - |w|_1||\alpha - \alpha'|.$$

Let us first assume $\alpha \neq g, g^2$. Then by Lemma 4.5 the difference $||w|_0 - |w|_1|$ tends to ∞ as soon as α' tends to some $\alpha \in \mathcal{E}_{KU}$; thus h is not differentiable (and not even Lipschitz continuous) at α.

Suppose instead that $\alpha = g^2$ (the other case is analogous by symmetry). Then we know h is constant to the right of α. On the other hand, by equation (56) and the fact that $||w|_0 - |w|_1| \geq 1$ to the left of α, we get by the same reasoning as before that

$$\liminf_{\alpha' \to \alpha^-} \frac{h(\alpha)-h(\alpha')}{\alpha-\alpha'} \geq C > 0;$$

hence the function h is not differentiable at α. Finally, since g^2 is an accumulation point of parameters in $\mathcal{E}_{KU}$ for which the derivative is unbounded, h is also not locally Lipschitz at $\alpha = g^2$. □

In the same way, one can also use formula (54) to prove the following asymptotic estimate, which is analogous to the result obtained in [35] for the family of Nakada's α-continued fractions.

Proposition 7.3. *The asymptotic behaviour of h at 0 is*

$$h(t) \sim \frac{\pi^2}{3\log(1/t)} \qquad \textit{as} \quad t \to 0^+; \tag{57}$$

hence $\lim_{t\to 0^+} h(t) = 0$, and h is not locally Hölder continuous at 0.

Proof. We shall use formula (54) and prove the asymptotic estimate (57) simply checking that

$$A_\alpha = \log(1/\alpha) + O(1) \qquad \text{as } \alpha \to 0^+. \tag{58}$$

By Theorem 1.2 we know that $\alpha \mapsto A_\alpha$ is decreasing on $[0, g^2]$; therefore it is enough to prove (58) for $\alpha = \frac{1}{N+1}$ with N a positive, even integer.

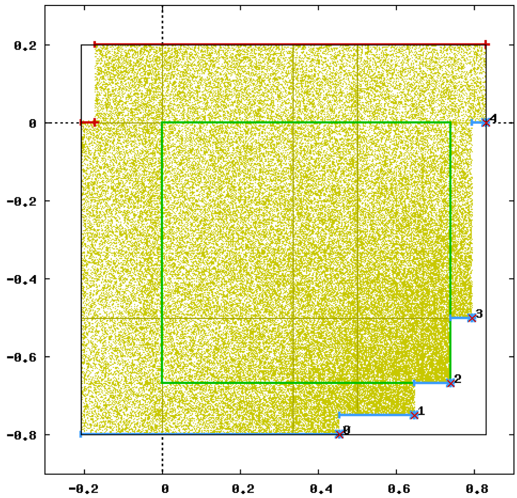

FIGURE 8. The attractor Δ_α for $\alpha = 1/5$: the inner and outer rectangles give the lower and upper bounds for the entropy as in the proof of Propositon 7.3.

Following the recipe of [24] described earlier in this section we see that, for $\alpha = \frac{1}{N+1}$, the attractor Δ_α has a very simple structure, which can be completely described. In particular, it is not difficult to check that the left endpoint of the lowest horizontal boundary of Δ_α has coordinates $([0;2,\overline{N,1}], -\frac{N}{N+1})$; the other lower boundaries are obtained from the lowest applying the function Φ_α, so that the lower-right corners of the attractor are the points $(x_k, y_k) := \Phi_\alpha^k(x_0, y_0)$ with

$$\begin{aligned} x_k &= [0;1,k,\overline{1,N}], \\ y_k &= -\tfrac{N-k}{N-k+1}, \end{aligned}$$

for $1 < k < N$. Now, if we pick an even value $N = 2h$, we get that $-y_h < x_h$; hence the attractor contains the square of coordinates $[0, -y_h] \times [y_h, 0]$ (see Figure 8). Integrating the invariant density $(1+xy)^{-2}dxdy$ on this square we get the lower bound for the measure of the attractor

$$A_{1/(N+1)} \geq \log(N) - \log(4).$$

On the other hand, for the upper bound we note that, using the notation of section 7.1, we have for each $\alpha \notin \mathcal{E}_{KU}$ the inclusion $\Delta_\alpha \subseteq [y,x] \times [\alpha - 1, \alpha]$. Then by taking $\alpha = 1/N$ and using Proposition 7.2 one gets that the attractor $\Delta_{1/N}$ is contained in the rectangle $[-1/(N-1), 1-1/(N+2)] \times [1/N - 1, 1/N]$, which leads to the upper bound $A_{1/N} \leq \log(N) + O(1)$ as $N \to +\infty$. This, together with the previous inequality, proves (58). □

7.3. **Comparison with Nakada's α-continued fractions and open questions.** Let us remark that the study of the entropy h_N in the case of the family (T_α) of α-continued fractions of Nakada is indeed much more complicated than the case examined in this paper (see Figure 9). Actually many statements that we proved before should hold also for the family (T_α), but proofs are missing.

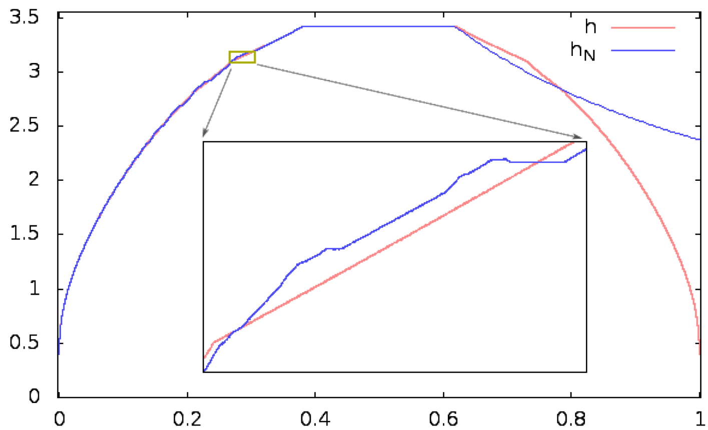

FIGURE 9. The graph of the entropy h_N of Nakada's α-continued fractions (in blue), versus the entropy h of continued fractions with $SL(2,\mathbb{Z})$ branches (in red).

The structure of the matching set for α-continued fractions is quite well understood ([10], [6]), but in this case matching intervals with different monotonic behaviours are mixed up in a complicated way ([12]), so even the fact that the entropy h_N attains its maximum value at $1/2$ is still conjectural.

Another feature which is still unproved is the smoothness of entropy on matching intervals. This is due to the fact that the natural extension has no finite rectangular structure when α ranges in a matching interval (see [28]). We conjecture that, as in the case we examined in this paper, on a matching interval densities are piecewise continuous, with discontinuity points located on the forward images of the endpoints (before matching occurs), while the branches of these densities are fractional transformations which move smoothly with the parameter (see also [9], Conjecture 5.3).

8. Farey words, kneading theory, and external angles

In the last section we will establish the connection between the bifurcation set $\mathcal{E}_{KU}$ and the Mandelbrot set, thus proving Theorem 1.3 in the introduction.

8.1. **A Cantor set defined by Farey words.** Denote as $[a,b]$ the closed interval of the circle from a to b, with positive orientation. Let us define the *binary bifurcation set* $\mathcal{E}_B$ as

$$\mathcal{E}_B := \{x \in [0,1/2] : D^k(x) \in [x, x+1/2] \ \ \forall k \in \mathbb{N}\}.$$

The set $\mathcal{E}_B$ is a closed subset of the interval $[0,1/2]$, and it has no interior as we will see. Let us point out that the only dyadic rationals which belong to $\mathcal{E}_B$ are 0 and $1/2$. Moreover, we shall see that the connected components of the complement of $\mathcal{E}_B$ are canonically labelled by Farey words. Indeed, if $w \in \{0,1\}^*$ is a Farey word we set $I_w := (a^-, a^+)$ with

$$a^+ := 0.\overline{w},$$
$$a^- := 0.\overline{{}^t w} - 1/2.$$

For instance, if $w = 00101$, then $a^+ = 0.\overline{00101} = \frac{5}{31}$ and $a^- = \frac{9}{62}$. We have the following properties (for proofs, see [8]).

Proposition 8.1. *With the notation above we have:*

(1) $a^{\pm} \in \mathcal{E}_B$;
(2) *if* $x \in I_w$, *then* $x \notin \mathcal{E}_B$;
(3) *for each Farey word* w, *the length of* I_w *is*

$$|I_w| = \frac{1}{2(2^n - 1)}$$

with $n = |w|$;
(4) *each* I_w *is a connected component of* $[0, 1/2] \setminus \mathcal{E}_B$; *moreover, we have*

$$[0, 1/2] \setminus \mathcal{E}_B = \bigcup_{w \in FW} I_w;$$

(5) *the Hausdorff dimension of* $\mathcal{E}_B$ *is zero.*

Let us remark that $x = .w \in \mathcal{E}_B$ if and only if $w \in \mathcal{C}$. The set $\mathcal{E}_B$ also appears in the kneading theory for Lorentz maps: indeed, it is the one-dimensional projection of the two-dimensional set of all kneading invariants for Lorentz maps (see [20, 29]).

8.2. **Connection to the main cardioid in the Mandelbrot set.** Let us now highlight a connection between the combinatorics of Farey words and the symbolic coding of rays landing on the main cardioid of the Mandelbrot set. We shall start by recalling a few standard facts in complex dynamics; for an account, we refer to [33] and references therein.

Let us consider the family of quadratic polynomials $f_c(z) := z^2 + c$ with $c \in \mathbb{C}$. Recall that the *filled Julia set* $K(f)$ of a polynomial $f(z)$ is the set of points with bounded orbits:

$$K(f) := \{z \in \mathbb{C} \ : \ \sup_n |f^n(z)| < \infty\}.$$

If $K(f)$ is connected, then its complement in the Riemann sphere is conformally isomorphic to a disk; hence it can be uniformized by a unique map $\Phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K(f)$ such that $\lim_{z\to\infty} |\Phi(z)| = \infty$ and $\lim_{z\to\infty} \Phi(z)/z = 1$. For each $\theta \in \mathbb{R}/\mathbb{Z}$, the *external ray at angle* θ is the set

$$R(\theta) := \{\Phi(\rho e^{2\pi i\theta}) \mid \rho > 1\}.$$

The ray $R(\theta)$ is said to *land* if $\lim_{\rho\to 1^+} \Phi(\rho e^{2\pi i\theta})$ exists (and then it is a point on the boundary of $K(f)$). The *Julia set* $J(f)$ is the topological boundary of $K(f)$. By Carathéodory's theorem, if the Julia set is locally connected, then all rays land. The map f_c has two fixed points (which coalesce if $c = \frac{1}{4}$); we shall call the *β-fixed point* the fixed point where the external ray of angle $\theta = 0$ lands, and the *α-fixed point* the other fixed point. A fixed point z_0 is called *indifferent* when the derivative $f_c'(z_0)$ has modulus 1 (the derivative $f_c'(z_0)$ is usually called the *multiplier* of the fixed point).

In parameter space, let us recall that the *Mandelbrot set* $\mathcal{M}$ is the set of parameters c for which the orbit under f_c of the critical point $z = 0$ is bounded:

$$\mathcal{M} := \{c \in \mathbb{C} \ : \ \sup_n |f_c^n(0)| < \infty\}.$$

The set $\mathcal{M}$ also equals the set of parameters $c \in \mathbb{C}$ for which the Julia set of f_c is connected. Just as the Julia sets, the Mandelbrot set admits a unique uniformizing map $\Phi_M : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$ such that $\lim_{z\to\infty} |\Phi_M(z)| = \infty$ and $\lim_{z\to\infty} \Phi_M(z)/z = 1$.

Let us define the *main cardioid* $\heartsuit$ of the Mandelbrot set as

$$\heartsuit := \{c \in \mathbb{C} \ : \ f_c(z) \text{ has an indifferent fixed point}\}.$$

A simple computation shows that $\heartsuit$ can be parametrized as $c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$ where $\theta \in [0,1]$; in this parametrization, for each $c \in \heartsuit$ the map f_c has multiplier $e^{2\pi i\theta}$ at the α-fixed point. Let Ω denote the set of angles of external rays landing on the main cardioid of the Mandelbrot set:

$$\Omega := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ R_M(\theta) \text{ lands on } \heartsuit\}.$$

The following proposition makes precise the connection between Farey words and the set of rays landing on the cardioid (see Figure 10).

Proposition 8.2. *Let $r = \frac{p}{q} \in \mathbb{Q} \cap (0,1)$, and let $w = W_r$ be the corresponding Farey word. Let us now define the pair of angles (θ^-, θ^+) as*

$$\begin{array}{rcl} \theta^- & = & 0.\overline{\tau({}^t w)}, \\ \theta^+ & = & 0.\overline{\tau w}, \end{array}$$

and let $c \in \heartsuit$ denote the parameter on the main cardioid for which the α-fixed point of f_c has multiplier $e^{2\pi i r}$. Then we have the following properties:

1. *in the Julia set of f_c, the set of external rays landing at the α-fixed point is the set*
$$C(w) = \{0.\overline{\tau^k w} \ : \ 0 \le k \le q-1\}$$
whose binary expansions are all cyclic permutations of w;
2. *in parameter space, the pair of angles of external rays (θ^-, θ^+) lands on the main cardioid at the parameter c;*
3. *we have the identity*
$$\Omega = 2\mathcal{E}_B.$$

As an example, if $r = \frac{2}{5}$, then $w = 00101$ and $\theta^- = 0.\overline{01001} = \frac{9}{31}$, while $\theta^+ = 0.\overline{01010} = \frac{10}{31}$. In the dynamical plane, the set of rays landing at the α-fixed point is $C(w) = (\frac{5}{31}, \frac{9}{31}, \frac{10}{31}, \frac{18}{31}, \frac{20}{31})$.

Proof. (1) Let $c \in \heartsuit$ be the parameter for which the map f_c has an indifferent fixed point of multiplier $e^{2\pi i r}$. It is known that its Julia set $J(f_c)$ is locally connected. Hence all external rays land in the dynamical plane of f_c, and the *landing map* $L(\theta) : \mathbb{R}/\mathbb{Z} \to J(f_c)$ defined as $L(\theta) := \lim_{\rho\to 1^+} \Phi(\rho e^{2\pi i\theta})$ is a continuous semiconjugacy between the doubling map and f_c; that is, we have the commutative diagram (see also [33])

$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[loop above, "D"] \arrow[r, "L"] & J(f_c) \arrow[loop above, "f_c"] \end{tikzcd}$$

Let $S := L^{-1}(\alpha)$ be the set of angles of external rays landing at the α-fixed point. The map f_c permutes the rays landing at α and preserves their cyclic orientation in the plane. Moreover, since the multiplier of f_c at α is $e^{2\pi i r}$, the set S has rotation number r under the doubling map. Hence by Lemma 2.7 the set S equals $C(w)$.

To pass to the statement in parameter space, note that it is known that the pair of rays landing on c in parameter space corresponds to the elements of S which

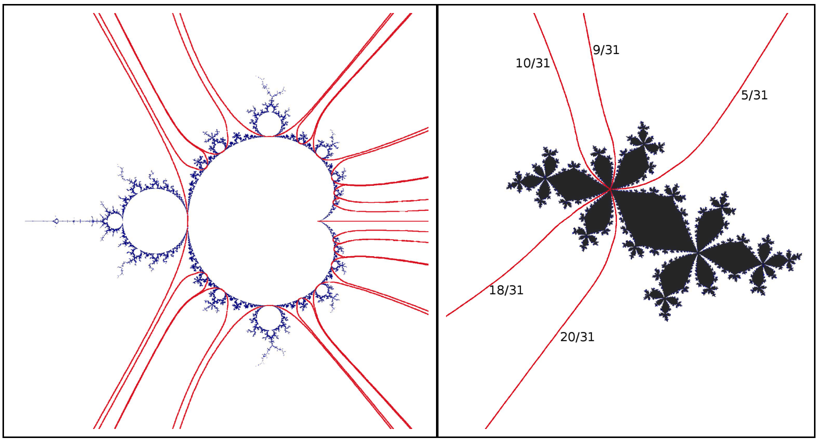

FIGURE 10. Left: the set Ω of external rays landing on the main cardioid of the Mandelbrot set. Right: the set $C(w)$ of external rays landing on the α-fixed point of a Julia set for the center of a hyperbolic component tangent to the main cardioid (here, the rotation number is $r = 2/5$ and the corresponding Farey word is $w = 00101$).

delimit a sector which (in the dynamical plane) contains the critical value. Now, the set $S^1 \setminus S$ is the union of q (connected) arcs, and the doubling map permutes their endpoints. Hence, D maps each arc of length smaller than $1/2$ homeomorphically to its image, and there is a unique arc ℓ_0 of length at least $1/2$. Since the map f_c is a local homeomorphism away from its critical point, the component in the dynamical plane corresponding to ℓ_0 must contain the critical point. Now note that by Lemma 2.6(1) and (3), the arc $\ell = (0.\overline{{}^t w}, 0.\overline{w})$ is a connected component of $S^1 \setminus S$, and by Proposition 8.1 and the definition of $\mathcal{E}_B$, the length of ℓ is more than $1/2$, so it must be $\ell = \ell_0$, the one which contains the critical point. As a consequence, the arc which contains the critical value is delimited by taking the forward image of the endpoints of ℓ_0; thus, it is the arc $\ell_1 = (0.\overline{\tau({}^t w)}, 0.\overline{\tau w}) = (\theta^-, \theta^+)$, and claim (2) is proven.

As for the last statement, the previous construction implies the correspondence $\Omega \cap \mathbb{Q} = 2(\mathcal{E}_B \cap \mathbb{Q})$. Claim (3) follows by taking closures, as it is known that the set of angles of rays landing on the main cardioid is the closure of the set of rational angles of rays landing on the main cardioid (see [19], Corollary 4.4). □

We now have the tools to prove Theorem 1.3 in the introduction.

Proof of Theorem 1.3. By comparing the definitions of $\mathcal{E}_{KU}$ and $\mathcal{E}_B$ one has $\mathcal{E}_{KU} = \phi(\mathcal{E}_B)$; then by using equation (21) and Proposition 8.2(3) we get

$$Q(\mathcal{E}_{KU}) = Q(\phi(\mathcal{E}_B)) = 2\mathcal{E}_B = \Omega.$$

□

8.3. **The magic formula.** Let us conclude this section by comparing the bifurcation set $\mathcal{E}_{KU}$ with the bifurcation set (or *exceptional set*) $\mathcal{E}_N$ for Nakada's α-continued fraction transformations (see [10]). By comparing equation (25) with the definition[3] of $\mathcal{E}_N$ from [10], one can easily check the inclusion

$$\mathcal{E}_{KU} \cap [0,1/2] \subset \mathcal{E}_N. \tag{59}$$

Note that the inclusion is strict (and actually, the Hausdorff dimension of $\mathcal{E}_N$ is 1, while the dimension of $\mathcal{E}_{KU}$ is 0).

Since both sets in (59) are related to sets of rays landing in the Mandelbrot set, it is interesting to see what our dictionary tells us when we transport the previous inclusion to the world of complex dynamics. First, using Theorem 1.3, the Minkowski question mark Q maps $\mathcal{E}_{KU}$ homeomorphically to the set Ω of external angles of rays landing on the main cardioid. Meanwhile, by the main theorem of [6], the set $\mathcal{E}_N$ is related to the set of rays landing on the real slice of the Mandelbrot set. Indeed, if we let $\mathcal{R}$ be the set of external angles of rays whose impression intersects the real slice of the Mandelbrot set, then we have the homeomorphism ([6], Theorem 1.1)

$$\psi(\mathcal{E}_N) = \mathcal{R} \cap [1/2, 1),$$

where $\psi(x) := \frac{1}{2} + \frac{Q(x)}{4}$. Thus we have the following commutative diagram, where i is the inclusion map:

$$\begin{tikzcd}
\mathcal{E}_{KU} \cap [0,1/2] \arrow[r, "i"] \arrow[d, "Q"] & \mathcal{E}_N \arrow[d, "\psi"] \\
\Omega \cap [0,1/2] \arrow[r, dotted, "T"] & \mathcal{R}
\end{tikzcd}$$

As a consequence, the map $T(\theta) := \psi(Q^{-1}(\theta))$, which can be expressed as just

$$T(\theta) = \frac{1}{2} + \frac{\theta}{4},$$

maps the set of rays landing on the upper half of the main cardioid into the set of real rays, i.e.,

$$T(\theta) \subseteq \mathcal{R}$$

for each $\theta \in \Omega \cap [0,1/2]$. This fact is known in the folklore as "Douady's magic formula" (see [4], Theorem 1.1).

Appendix

We shall now give the proofs of a few technical lemmas we postponed in the main body of the article.

Proof of Lemma 6.2. Since the proof follows closely the same strategy as in [35] we give here just a sketch of the main steps. The idea is based on comparing the return times of K_α on the intervals $[\alpha'-1, \alpha-1]$ and $[\alpha', \alpha]$: by the ergodic theorem, these give information on how the invariant measure changes.

Let J_w be the quamterval labelled by the Farey word w and let $\alpha, \alpha' \in J_w$ be such that either

$$s \leq \alpha' \leq \alpha < \alpha^+, \quad S \cdot \alpha' > \alpha$$

or

$$\alpha^- < \alpha' < \alpha < s, \quad {}^tS' \cdot \sigma(\alpha) < \alpha'.$$

[3]In [10], the bifurcation set is simply denoted by $\mathcal{E}$, and its complement is denoted by $\mathcal{M}$. See also [12, section 3].

Then, for every $x \in [\alpha', \alpha]$ there exist two increasing sequences (visiting times) $(n_0(k))_{k\in\mathbb{N}}$, $(n_1(k))_{k\in\mathbb{N}}$ such that

(1) $n_0(k)$ and $n_1(k)$ are k^{th}-return times on $(\alpha'-1, \alpha-1)$ and (α', α), respectively:

$$\begin{aligned}&K^n_{\alpha'}(x-1) \in (\alpha'-1, \alpha-1) \iff \quad n = n_0(k) \text{ for some } k \in \mathbb{N},\\ &K^n_{\alpha}(x) \in (\alpha', \alpha) \iff \quad n = n_1(k) \text{ for some } k \in \mathbb{N};\end{aligned} \tag{60}$$

(2) although the return times may depend on x, their difference just depends on w: $n_0(k) - n_1(k) = k(|w|_0 - |w|_1)$;

(3) the matching property induces a synchronization of k^{th}-returns:

$$\begin{aligned}&K^{n_0(k)}_{\alpha'}(x-1) = K^{n_1(k)}_{\alpha}(x) - 1,\\ &\mathbf{T}\mathbf{M}_{\alpha',x-1,n_0(k)} = \mathbf{M}_{\alpha,x,n_1(k)}\mathbf{T};\end{aligned} \tag{61}$$

(4) just before the k^{th} return the two orbits are together:

$$K^{n_0(k)-1}_{\alpha'}(x-1) = K^{n_1(k)-1}_{\alpha}(x) - 1, \mathbf{T}\mathbf{M}_{\alpha',x-1,n_0(k)-1} = \mathbf{M}_{\alpha,x,n_1(k)-1}.$$

It is now possible to choose $x \in (\alpha', \alpha)$ such that both of the following conditions hold:

(a) x is a typical point for K_α, namely,

$$\begin{aligned}&\lim_{x\to+\infty} \tfrac{1}{n}\#\{i < n \ : \ K^i_\alpha(x) \in (\alpha', \alpha)\} = \mu_\alpha([\alpha', \alpha]),\\ &2\lim_{x\to+\infty} \tfrac{1}{n}\log q_{n,\alpha}(x) = h(\alpha).\end{aligned}$$

(b) $x-1$ is typical for $K_{\alpha'}$, that is:

$$\begin{aligned}&\lim_{x\to+\infty} \tfrac{1}{n}\#\{j < n \ : \ K^j_{\alpha'}(x-1) \in (\alpha'-1, \alpha-1)\} = \mu_{\alpha'}([\alpha'-1, \alpha-1]),\\ &2\lim_{x\to+\infty} \tfrac{1}{n}\log q_{n,\alpha'}(x-1) = h(\alpha').\end{aligned}$$

Therefore, on one hand we have

$$\lim_{k\to+\infty} \frac{k}{n_0(k)} = \mu_{\alpha'}([\alpha'-1, \alpha-1]), \quad \lim_{k\to+\infty} \frac{k}{n_1(k)} = \mu_\alpha([\alpha', \alpha]),$$

which implies by taking the quotient and using (2) that

$$\lim_{k\to+\infty} \frac{n_0(k)}{n_1(k)} = \lim_{k\to+\infty} 1 + \frac{k}{n_1(k)}(|w|_0 - |w|_1) = 1 + (|w|_0 - |w|_1)\mu_\alpha([\alpha', \alpha]).$$

Then, putting everything together we get

$$\begin{aligned}h(\alpha) &= \lim_{k\to+\infty} \tfrac{2}{n_1(k)-1}\log q_{n_1(k)-1,\alpha}(x)\\ &= \lim_{k\to+\infty} \tfrac{n_0(k)-1}{n_1(k)-1}\tfrac{2}{n_0(k)-1}\log q_{n_0(k)-1,\alpha'}(x-1)\\ &= (1 + (|w|_0 - |w|_1)\mu_\alpha([\alpha', \alpha]))h(\alpha'),\end{aligned}$$

which proves the claim. □

Proof of Lemma 6.5. We may assume, without loss of generality, that $I = [0, 1]$ and that the intervals are indexed in decreasing size: $|I_1| \geq |I_2| \geq \dots$. By definition the upper box-dimension of $\mathcal{G}$ is given by

$$\overline{\text{B.dim}}\,\mathcal{G} := \limsup_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}, \tag{62}$$

where $N(\epsilon)$ is the minimum cardinality of a cover of $\mathcal{G}$ with intervals of diameter less than or equal to ϵ. Let us now define

$$M(\epsilon) := \sup\{i \ : \ |I_i| \geq \epsilon\}.$$

Notice now that any cover of $[0,1] \setminus \bigcup_{i=1}^{M(\epsilon)} I_i$ with intervals of diameter less than ϵ necessarily must have cardinality at least $M(\epsilon)+1$, because any such interval intersects at most one connected component. Hence

$$M(\epsilon) \leq N(\epsilon).$$

If we now fix $\delta \in (\delta_0, \eta)$, by (62) there is some $C > 0$ such that

$$M(\epsilon) \leq N(\epsilon) \leq C\epsilon^{-\delta} \qquad \forall \epsilon \leq 1.$$

Now

$$\sum_{M(\epsilon/2^k)+1}^{M(\epsilon/2^{k+1})} |I_i|^\eta \leq \left(\frac{\epsilon}{2^k}\right)^\eta M(\epsilon/2^{k+1}) \leq C 2^\delta \epsilon^{\eta-\delta} 2^{k(\delta-\eta)};$$

hence summing over $k \geq 0$ we get

$$\sum_{M(\epsilon)+1}^{\infty} |I_i|^\eta \leq \epsilon^{\eta-\delta} \frac{C2^\delta}{1-2^{\delta-\eta}}.$$

Now, given any subsequence $J_1, J_2, \ldots$, if we set $\epsilon := \sum_{i=1}^\infty |J_i|$, then all elements in the subsequence have length smaller than ϵ, hence

$$\sum_{i=1}^\infty |J_i|^\eta \leq \sum_{M(\epsilon)+1}^{\infty} |I_i|^\eta \leq \left(\sum_{i=1}^\infty |J_i|\right)^{\eta-\delta} \frac{C2^\delta}{1-2^{\delta-\eta}}.$$

□

Proof of Proposition 4.3. Let us pick $s' \in (0,1) \cap \mathbb{Q}$, since $\mathbb{Q} \cap \mathcal{E}_{KU} = \{0,1\}$; then $s' \notin \mathcal{E}_{KU}$. Therefore there is $w \in FW^*$ such that $s' \in J_w$, and Proposition 4.3 will be proved once we prove that

$$s' \in J_w \quad \Rightarrow \widetilde{J}_{s'} \subset J_w. \tag{63}$$

Now, let $S := RL(w)$ and let $s := [0; S]$ be the pseudocenter of J_w (so that, by virtue of equation (22), $\widetilde{J}_s = J_w$). Let us first assume that $s' > s$: this means that $s' = [0; ST]$ with $|T|$ even. Since $S' \ll S$, it is clear that the left endpoint $\sigma\beta(\sigma s')$ of $\widetilde{J}_{s'}$ belongs to $\widetilde{J}_s$. To prove that the right endpoint $\beta(s')$ satisfies $\beta(s') \leq \beta(s)$, let us set $m := \max\{j \ : \ |S|^j < |T|\}$. Then, either (a) $T \ll S^{m+1}$ or (b) $T = S^m P$ with $S = PZ$ (i.e., T is a prefix of S^{m+1}). We claim that in both cases $\beta(s') = [0; \overline{ST}] \leq [0; \overline{S}] = \beta(s)$. In case (a) this claim is trivial, and the same is true in case (b) if $P = S$. On the other hand, in the case when P is a proper prefix of S, by virtue of Lemma 3.1(iii) one gets that $PZ < ZP$ so that using the lemma of (16) we get

$$PPZ < PZP \iff S^{m+1}PS < SS^{m+1}P \iff [0; \overline{ST}] = [0; \overline{S^{m+1}P}] < [0; \overline{S}],$$

completing the proof of the inclusion $\widetilde{J}_{s'} \subset \widetilde{J}_s = J_w$.

On the other hand, if $s' < s$, then $\sigma(s') > \sigma(s)$ and $\sigma(s') \in \widetilde{J}_{\sigma s} = J_{{}^t\check{w}}$, so the previous case implies that

$$\widetilde{J}_{s'} = \sigma(\widetilde{J}_{\sigma s'}) \subset \sigma(\widetilde{J}_{\sigma s}) = \widetilde{J}_s,$$

and (63) is thus proven. □

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