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ON SOME DETERMINANT AND MATRIX INEQUALITIES WITH A GEOMETRICAL FLAVOUR

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Abstract. In this paper we study some determinant inequalities and matrix inequalities which have a geometrical flavour. We first examine some inequalities which place work of Macbeath in a more general setting and also relate to recent work of Gressman. In particular, we establish optimisers for these determinant inequalities. We then use these inequalities to establish our Main Theorem, which gives a geometric inequality of matrix type which improves and extends some inequalities of Christ.

1. Introduction

1.1. **Notation and preliminaries.** Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $n \geq 1$. $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$ and the absolute value on $\mathbb{R}$. Denote $\mathfrak{M}^{n\times n}(\mathbb{R})$ by a set of all $n \times n$ real matrices. Let $B(0, r)$ be the ball centred at 0 with radius r. For $A \subset \mathbb{R}^n$ of finite Lebesgue measure, we define the symmetric rearrangement of A as

$$A^* := \{x : |x| < r\} \equiv B(0, r), \text{ with } |A^*| = |A|.$$

That is, $v_n r^n = |A|$, where v_n is the volume of unit ball in $\mathbb{R}^n$. We then define the symmetric decreasing rearrangement of a nonnegative measurable function f as

$$f^*(x) := \int_0^\infty \chi_{\{f>t\}^*}(x)dt,$$

where $\chi_{\{f>t\}}$ is the characteristic function of the level set $\{x : f(x) > t\}$, and define the Steiner symmetrisation of f with respect to the j-th coordinate as

$$\mathcal{R}_j f(x_1, \ldots, x_n) = f^{*j}(x_1, \ldots, x_n) := \int_0^\infty \chi_{\{f(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)>t\}^*}(x_j)dt.$$

Let $u \in \mathbb{R}^n$ be a unit vector, and let $u^\perp$ be its orthogonal complement. Then for any $x \in \mathbb{R}^n$, it can be uniquely written as $x = tu + y$ where $y \in u^\perp$. We define the Steiner symmetrisation of A with respect to the direction u as

$$\mathcal{S}_u(A) := \left\{ tu + y : A \cap (\mathbb{R}u + y) \neq \phi, |t| \leq \frac{|A \cap (\mathbb{R}u + y)|}{2} \right\}.$$

Obviously, $\mathcal{R}_j \chi_A$ is the Steiner symmetrisation of A with respect to the direction e_j, $1 \leq j \leq n$. For simplicity, we denote $\mathcal{S}_{e_n}\mathcal{S}_{e_{n-1}} \cdots \mathcal{S}_{e_1}(E)$ by $\mathcal{S}E$, where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis in $\mathbb{R}^n$.

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One easily sees that for any measurable set $E \subset \mathbb{R}^n$,

$$\sup_{x\in E^*} |x| \le \sup_{x\in E} |x|, \tag{1.1}$$

and from this it is not hard to see that

$$\sup_{x,y\in E^*} |x-y| \le \sup_{x,y\in E} |x-y|. \tag{1.2}$$

One way to obtain this is as follows:

$$\sup_{x,y\in E} |x-y| = \sup_{z\in E-E} |z| \ge \sup_{z\in (E-E)^*} |z|. \tag{1.3}$$

For any $A, B \in \mathbb{R}^n$ of finite Lebesgue measure, it follows from the Brunn-Minkowski inequality that

$$A^* + B^* \subset (A+B)^*. \tag{1.4}$$

Applying (1.4) in (1.3) implies

$$\sup_{x,y\in E} |x-y| \ge \sup_{z\in (E-E)^*} |z| \ge \sup_{x\in E^*, y\in E^*} |x-y|,$$

which completes (1.2).

Let E be a measurable set of finite volume in $\mathbb{R}^n$. By the definition of the symmetric rearrangement,

$$E^* = B(0,r), \ \text{ with } \ v_n r^n = |E|.$$

Clearly,

$$\sup_{x\in E^*} |x| = r, \ \sup_{x,y\in E^*} |x-y| = 2r.$$

By (1.1) and (1.2) we have the following sharp inequalities:

$$|E| \le v_n \sup_{x\in E} |x|^n, \tag{1.5}$$

$$|E| \le \frac{v_n}{2^n} \sup_{x,y\in E} |x-y|^n. \tag{1.6}$$

Moreover, optimisers of both (1.5) and (1.6) are balls in $\mathbb{R}^n$. Inequality (1.6) is an isodiametric inequality; that is, amongst all sets with given diameter the ball has maximal volume.

1.2. **Macbeath's inequalities.** We now go on to study the analogues of (1.5) and (1.6), where we replace the distance norm by a volume or determinant, so the question becomes that of studying inequalities of the form

$$|E| \le A_n \sup_{\substack{y_j\in E\\ j=1,\dots,n}} \det(0, y_1, \dots, y_n) \tag{1.7}$$

and

$$|E| \le B_n \sup_{\substack{y_j\in E\\ j=1,\dots,n+1}} \det(y_1, \dots, y_{n+1}), \tag{1.8}$$

which are supposed to hold for any measurable set E in $\mathbb{R}^n$. Here

$$\det(y_1,\dots,y_{n+1}) := n!\mathrm{vol}(\mathrm{co}\{y_1,\dots,y_{n+1}\}).$$

So $\det(y_1,\dots,y_{n+1}) \ge 0$. The precise value of $\det(y_1,\dots,y_{n+1})$ is the absolute value of the determinant of the matrix $(y_1 - y_{n+1}, \dots, y_n - y_{n+1})_{n\times n}$. In the special case when $n = 1$, they become of the type (1.5) and (1.6) automatically. Note that both

(1.7) and (1.8) are $\mathrm{GL}_n(\mathbb{R})$ invariant, and (1.8) is translation invariant while (1.7) is not. Actually, it is enough to study convex measurable sets in $\mathbb{R}^n$, since

$$\sup_{\substack{y_j\in E\\ j=1,\dots,n}} \det(0,y_1,\dots,y_n) = \sup_{\substack{y_j\in \mathrm{co}(E)\\ j=1,\dots,n}} \det(0,y_1,\dots,y_n)$$

and

$$\sup_{\substack{y_j\in E\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}) = \sup_{\substack{y_j\in \mathrm{co}(E)\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}).$$

We are interested in the best constants A_n, B_n, and their optimisers. It is not hard to deduce that the best constants A_n and B_n are related by

$$B_n \le A_n \le (n+1)B_n. \tag{1.9}$$

Indeed, the translation invariance of (1.8) allows us to assume that $0 \in E$. Then $B_n \le A_n$ follows immediately. On the other hand, by the basic determinant property we have

$$\det(y_1,\dots,y_{n+1}) \le \sum_{j=1}^{n+1} \det(0,y_1,\dots,y_{j-1},y_{j+1},\dots,y_n),$$

which implies that

$$\sup_{\substack{y_j\in E\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}) \le (n+1) \sup_{\substack{y_j\in E\\ j=1,\dots,n}} \det(0,y_1,\dots,y_n).$$

That completes $A_n \le (n+1)B_n$. So in the special case when $n = 1$, we have $A_1 = 2$, $B_1 = 1$, which follows from (1.5) and (1.6).

Geometrically, the right side of (1.8) relates to the maximal volume of n-simplex whose vertices are in E. The relationship between the maximal volume of the n-simplex whose vertices are in E and the measure of E has been studied before (see [10], [13]). It is well known that by compactness, given a compact convex set $E \subset \mathbb{R}^n$, there exists a simplex $T \subset E$ of maximal volume. Let F be a facet of T, v the opposite vertex, and H the hyperplane through v parallel to F. Then H supports E, since otherwise one would obtain a contradiction to the maximality of the volume of T. Since F is an arbitrary facet of T, T is contained in the simplex $-n(T-c)+c$, where c is the centroid of T. See [10] for details. So $T \subset E \subset -n(T-c)+c$, and thus

$$|E| \le n^n |T|, \tag{1.10}$$

which implies that

$$B_n \le n^n,\ A_n \le (n+1)n^n.$$

In 1950, Macbeath [13] already gave the sharp version of (1.10) and (1.8) as follows. Given a compact convex set $E \subset \mathbb{R}^n$, denote by $\mathfrak{B}_m$ the set of convex polytopes with at most m vertices in E, and denote by $\mathfrak{C}_m$ the set of convex polytopes with at most m vertices in E^*. Then

$$\sup_{T'\in\mathfrak{C}_m} |T'| \le \sup_{T\in\mathfrak{B}_m} |T|. \tag{1.11}$$

So when $m = n+1$, (1.11) gives

$$\sup_{\substack{y_j\in E^*\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}) \le \sup_{\substack{y_j\in E\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}).$$

Moreover the problem is clearly affine invariant; thus the extremising sets turn out to be balls and ellipsoids for (1.8). Because the maximal simplex with vertices on a ball is the regular simplex with all sides equal, we can obtain the corresponding best constant B_n. However, we do not believe that the sharp value of A_n in (1.7) has been given previously.

1.3. **Our results.** In this paper we shall give an alternative method to derive (1.7) and (1.8) with sharp constants A_n, B_n. In Section 2, we will study some rearrangement inequalities which together with some work in [4] establish this. A key ingredient will be a rearrangement inequality of [4], stating that for any $E_j \subset \mathbb{R}$ of finite Lebesgue measure and $a_j \in \mathbb{R}$, $j = 1, \ldots, l$,

$$\sup_{x_j \in E_j^*} \left| \sum_{j=1}^{l} a_j x_j \right| \leq \sup_{x_j \in E_j} \left| \sum_{j=1}^{l} a_j x_j \right|. \tag{1.12}$$

See Lemma 2.2 for the proof.

We have described geometric inequalities in Section 1.1 and 1.2, and it is very natural to consider their functional forms and generalisations. See for example Gardner [7]. More generally, returning to the inequalities (1.1), (1.2), we see there are functional versions. One can consider a bilinear functional rearrangement version of (1.2). For all nonnegative measurable functions f, g defined on $\mathbb{R}^n$,

$$\sup_{x,y} f^*(x)g^*(y)|x-y| \leq \sup_{x,y} f(x)g(y)|x-y| \tag{1.13}$$

holds. Likewise, by the same argument as in its proof we also have

$$\sup_{x} f^*(x)|x| \leq \sup_{x} f(x)|x|. \tag{1.14}$$

For the proof, see Section 4 in [4].

In Section 2, generalizing them we arrive at the following multilinear functional rearrangement inequalities:

$$\sup_{y_j} \prod_{j=1}^{n} f_j^*(y_j) \det(0, y_1, \ldots, y_n) \leq \sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \ldots, y_n) \tag{1.15}$$

and

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^*(y_j) \det(y_1, \ldots, y_{n+1}) \leq \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1}), \tag{1.16}$$

which hold for any nonnegative measurable functions vanishing at infinity f_j defined on $\mathbb{R}^n$, in the sense that all its positive level sets have finite measure,

$$|\{x : |f(x)| > t\}| < \infty, \text{ for all } t > 0.$$

As a matter of fact, we establish much more general inequalities in Theorem 2.5 below. Then we get (1.7), (1.8) with the sharp constants by specialising to $f_j = \chi_E$ in (1.15)-(1.16), which also includes Macbeath's work (1.11) when $m = n + 1$.

There is another class of inequalities concerning analogues of (1.5), (1.6) where we replace the underlying Euclidean space $\mathbb{R}^n$ by the space of $n \times n$ real matrices and the Euclidean norm by $|\det(A)|$. For example, Christ first studied this type of inequality in [5]. Here "det" becomes ordinary determinant of a matrix.

Sublemma 14.1 ([5]). *For any $n \geq 1$ there exists $C \in \mathbb{R}^+$ with the following property. Let $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$ be a compact convex set satisfying $|E| < \infty$ and $E = -E$. Then there exists $A \in E$ satisfying*

$$|\det(A)| \geq C|E|^{\frac{1}{n}}, \tag{1.17}$$

where $|\cdot|$ denotes the Lebesgue measure on Euclidean space $\mathbb{R}^{n^2}$ and the absolute value on $\mathbb{R}$.

Lemma 13.2 ([5]). *For any $n \geq 1$ there exists $c, C \in \mathbb{R}^+$ and $k \in \mathbb{N}$ with the following property. Let E be a measurable set in $\mathfrak{M}^{n\times n}(\mathbb{R})$ satisfying $|E| < \infty$. Then there exist $T_1, \dots, T_k \in E$ and coefficients $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, $\sum_{j=1}^{k} s_j = 0$, such that*

$$\left|\det\left(\sum_{j=1}^{k} s_j T_j\right)\right| \geq C|E|^{\frac{1}{n}}. \tag{1.18}$$

Remarks 1.

(1) Let $\widetilde{E} = E - A := \{T - A : T \in E\}$ with $A \in \mathfrak{M}^{n\times n}(\mathbb{R})$. Then by Lemma 13.2 there exist $T_1, \dots, T_k \in E$ and $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, $\sum_{j=1}^{k} s_j = 0$, such that

$$\left|\det\left(\sum_{j=1}^{k} s_j(T_j - A)\right)\right| = \left|\det\left(\sum_{j=1}^{k} s_j T_j\right)\right| \geq C|E|^{\frac{1}{n}} = C|\widetilde{E}|^{\frac{1}{n}}, \tag{1.19}$$

which shows that (1.18) has a translation invariance property that (1.17) lacks.

(2) Based on the translation variance property, we have an equivalent form of Lemma 13.2: there exists $c, C \in \mathbb{R}^+$ such that for any $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$ we can always select $T_1, \dots, T_k \in E$ and coefficients $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, such that $\left|\det\left(\sum_{j=1}^{k} s_j T_j\right)\right| \geq C|E|^{\frac{1}{n}}$.

The equivalence is as follows. Supposing $A \in E$, denote $\widetilde{E} = E - A$. Then if there exist $\overline{T_1} = T_1 - A, \dots, \overline{T_k} = T_k - A \in \widetilde{E}$, where $T_j \in E$, $1 \leq j \leq k$, and there exist $s_j \in \mathbb{Z}$ satisfying $|s_j| \leq c$, such that

$$\left|\det\left(\sum_{j=1}^{k} s_j \overline{T_j}\right)\right| \geq C|\widetilde{E}|^{\frac{1}{n}}.$$

That is,

$$|\det(s_1T_1 + \cdots + s_kT_k - (s_1 + \cdots + s_k)A)| \geq C|\widetilde{E}|^{\frac{1}{n}} = C|E|^{\frac{1}{n}},$$

which satisfies the conditions of Lemma 13.2.

More specifically, when proving Lemma 13.2 Christ [5] gave that under the same hypothesis of Lemma 13.2, there exist $A_j \in E$ and $s_j \in \{0, 1\}$, $j = 1, \dots, n$, such that

$$\left|\det\left(\sum_{j=1}^{n} s_j A_j\right)\right| \geq C|E|^{\frac{1}{n}},$$

which implies that for any measurable $E \subset \mathfrak{M}^{n\times n}$,

$$\sup_{\substack{A_1,\dots,A_n\in E\\ s_1,\dots,s_n\in\{0,1\}}} |\det(s_1A_1+\cdots+s_nA_n)| \gtrsim_n |E|^{\frac{1}{n}}. \tag{1.20}$$

Christ studied forms such as $\det(A_1+\cdots+A_n)$ where A_j are all restricted to belong to the same set. It is natural to remove this restriction and allow A_j to belong to E_j; we do not assume all the E_j are equal. In this paper we will improve (1.17)-(1.18) as follows, mainly relying on the rearrangement inequality (1.12).

Main Theorem. *There exists a finite constant $\mathcal{C}_n$ such that for any measurable sets $E_j \subset \mathfrak{M}^{n\times n}(\mathbb{R})$ of finite measure, $j = 1,\dots,n$,*

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n^2}} \le \mathcal{C}_n \sup_{\substack{A_j\in E_j\\ j=1,\dots,n}} |\det(A_1+\cdots+A_n)|. \tag{1.21}$$

The Main Theorem implies that (1.17) holds for all compact convex sets in $\mathfrak{M}^{n\times n}(\mathbb{R})$ and extends Lemma 13.2 as described below. In particular, we see from the Main Theorem that all the s_j's in (1.20) can be taken to be 1.

Corollary A. *There exists a finite constant $\mathcal{A}_n$ such that for any measurable set $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$ of finite measure, for any nonzero scalar $\lambda_j \in \mathbb{R}$, $j = 1,\dots,n$,*

$$\Big(\prod_{j=1}^{n} |\lambda_j|\Big)|E|^{\frac{1}{n}} \le \mathcal{A}_n \sup_{A_j\in E} |\det(\lambda_1A_1+\cdots+\lambda_nA_n)|. \tag{1.22}$$

Corollary B. *There exists a finite constant $\mathcal{B}_n$ such that for any measurable compact convex set $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$ of finite measure,*

$$|E|^{\frac{1}{n}} \le \mathcal{B}_n \sup_{A\in E} |\det(A)|. \tag{1.23}$$

See Section 3 for the proof of Corollary B.

Remarks 2.

(1) One can easily check that

$$\sup_{A\in \mathrm{co}\{0,E\}} |\det(A)| = \sup_{A\in E} |\det(A)|.$$

This is because $|\det(\lambda A)| = \lambda^n|\det(A)|$ for any $\lambda \in [0,1]$, so we can always assume that $0 \in E$. Given a measurable $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$, by scaling let $\widetilde{E} = rE$, $0 \ne r \in \mathbb{R}$. Then

$$(|\widetilde{E}|)^{\frac{1}{n}} = (r^{n^2}|E|)^{\frac{1}{n}} = r^n|E|^{\frac{1}{n}}$$

and

$$\sup_{A\in\widetilde{E}} |\det(A)| = r^n \sup_{A\in E} |\det(A)|.$$

However, (1.23) is not translation invariant.

(2) We use a counterexample to show that (1.23) fails without the convex condition. Take $n = 2$ as an example, and let

$$E = \{(a,b,c,d) : 0 \le ad \le 1, 0 \le bc \le 1, \text{ and } 1/N \le a \le N, 1/N \le b \le N\}.$$

Then we have

$$\sup_{A\in E} |\det(A)| = \sup_{A\in E} \left|\det\begin{pmatrix} a & c\\ b & d\end{pmatrix}\right| \le 2$$

and $|E| = (2\ln N)^2$. Let $N \to \infty$. Then we get the contradiction to (1.23).

Remarks 3.

(1) An open problem is what the best constants $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ are. We prove in this paper that balls or ellipsoids are not their optimisers.

(2) Note that inequalities of matrix type introduced in this part do not enjoy an obvious affine invariance. Nevertheless, there is an important action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathfrak{M}^{n\times n}(\mathbb{R})$ by premultiplication. That is, if $T \in \mathrm{GL}_n(\mathbb{R})$, $A \in \mathfrak{M}^{n\times n}(\mathbb{R})$, and $E \subset \mathfrak{M}^{n\times n}(\mathbb{R})$, then

$$\det(TA) = \det(T)\det(A)$$

and

$$|TE| = |\det(T)|^n|E|.$$

So both matrix inequalities in this paper are invariant under premultiplication by a matrix of unimodular determinant. We do not use the invariance of the entire problem under the action of left-multiplication by members of $\mathrm{SL}_n(\mathbb{R})$ but instead the facts which underly this invariance, i.e., that this action preserves determinants of individual matrices and preserves volumes of sets. It enters as a "catalyst" in order to obtain a measure theoretic consequence, and its presence vanishes without trace.

The rest of this paper is organised as follows. In Section 2, we study the extremal sets for some generalizations of (1.7)-(1.8) via the Steiner symmetrisation procedure together with the rearrangement inequality (1.12). In this section, we also prove the multilinear functional determinant rearrangement inequalities shown in Theorem 2.5. Together with the Brascamp-Lieb-Luttinger inequality, we prove the multilinear determinant integral rearrangement inequalities in Theorem 2.7. Our Main Theorem will be proved in Section 3. We remark that the main ingredient in this paper is the rearrangement inequality (1.12). Throughout the paper, the letter C stands for positive constants, not necessarily the same at each occurrence but independent of the essential variables.

2. Determinant inequalities

In this section we study the determinant inequalities discussed in the introduction. First we recall an estimate by Gressman [8] as follows.

Lemma 2.1 ([8])**.** *There exists a finite constant C_n such that for any $y \in \mathbb{R}^n$, for any measurable sets $E_1, \ldots, E_n$ in $\mathbb{R}^n$, and for any $\delta > 0$,*

$$|\{(y_1,\ldots,y_n) \in E_1 \times \cdots \times E_n : \det(y, y_1, \ldots, y_n) < \delta\}| \le C_n\delta\prod_{j=1}^{n}|E_j|^{1-\frac{1}{n}}. \tag{2.1}$$

As an immediate consequence of (2.1), we obtain the following inequality (2.2). With the same constant C_n, we have for any $y \in \mathbb{R}^n$ and for any measurable sets $E_j \subset \mathbb{R}^n$, $1 \le j \le n$,

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n}} \le C_n \sup_{y_1\in E_1,\ldots,y_n\in E_n} \det(y, y_1, \ldots, y_n). \tag{2.2}$$

One way to see this is as follows. Let $y \in \mathbb{R}^n$ and suppose that

$$\sup_{y_1\in E_1,\ldots,y_n\in E_n} \det(y, y_1, \ldots, y_n) = s < \infty.$$

It follows from Lemma 2.1 that for all measurable sets $E_j \subset \mathbb{R}^n$, $1 \le j \le n$,

$$|\{(y_1,\dots,y_n) \in E_1 \times \cdots \times E_n : \det(y, y_1, \dots, y_n) \le s\}| \le C_n s \prod_{j=1}^{n} |E_j|^{1-\frac{1}{n}}.$$

Note that $s = \sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(y, y_1, \dots, y_n)$, so

$$|\{(y_1,\dots,y_n) \in E_1 \times \cdots \times E_n : \det(y, y_1, \dots, y_n) \le s\}| = \prod_{j=1}^{n} |E_j|.$$

Therefore,

$$\prod_{j=1}^{n} |E_j| \le C_n s \prod_{j=1}^{n} |E_j|^{1-\frac{1}{n}}.$$

That is,

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n}} \le C_n s = C_n \sup_{y_1 \in E_1, \dots, y_n \in E_n} \det(y, y_1, \dots, y_n),$$

which completes (2.2).

This motivates a multilinear perspective. Later on, we will prove the sharp version of (2.1)-(2.2). More generally, functional versions of (2.2) have been studied in [4]. As shown in Theorem 3.1 of [4], for any nonnegative measurable functions $f_j \in L^{p_j}(\mathbb{R}^n)$,

$$\prod_{j=1}^{n+1} \|f_j\|_{p_j} \le C_{n,p_j} \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \dots, y_{n+1})^{\gamma} \tag{2.3}$$

holds if and only if p_j satisfies $\frac{1}{p_j} < \frac{\gamma}{n}$ for all $1 \le j \le n+1$ and $\gamma = \sum_{j=1}^{n+1} \frac{1}{p_j}$. Lemma 3.2 in [4] gives an endpoint case of the multilinear inequality (2.3). That is, for any nonnegative measurable functions $f_j \in L^{p_j}(\mathbb{R}^n)$,

$$\prod_{j=1}^{n} \|f_j\|_{L^{n,\infty}(\mathbb{R}^n)} \|f_{n+1}\|_{L^\infty} \le C_n \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \ \det(y_1, \dots, y_{n+1}). \tag{2.4}$$

It is not hard to see that (2.4) implies for any $y \in \mathbb{R}^n$,

$$\prod_{j=1}^{n} \|f_j\|_{L^{n,\infty}(\mathbb{R}^n)} \le C_n \sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \ \det(y, y_1, \dots, y_n), \tag{2.5}$$

which also concludes (2.2) by specialising to $f_j = \chi_{E_j}$. For the proof of (2.3)-(2.5) and more general multilinear cases, we refer to [4].

Before studying the sharp versions of inequalities (2.2), we recall some useful tools in [4], which were already stated in the introduction.

Lemma 2.2 ([4])**.** *Let E_j be measurable sets in $\mathbb{R}$ and $a_j \in \mathbb{R}$, $j = 1, \dots, l$. Then*

$$\sup_{x_j \in E_j^*} \left| \sum_{j=1}^{l} a_j x_j \right| \le \sup_{x_j \in E_j} \left| \sum_{j=1}^{l} a_j x_j \right|. \tag{2.6}$$

Proof. From the Brunn-Minkowski inequality

$$|E+F| \geq |E| + |F|,$$

where $E, F \subset \mathbb{R}$, it follows that

$$|E_1 + \cdots + E_l| \geq |E_1| + \cdots + |E_l|.$$

Because $E_j^* = (-|E_j|/2, |E_j|/2)$, $1 \leq j \leq l$, then

$$E_1^* + \cdots + E_l^* = \left(-\sum_{j=1}^{l} \frac{|E_j|}{2}, \sum_{j=1}^{l} \frac{|E_j|}{2}\right).$$

Thus we have

$$|(E_1 + \cdots + E_l)^*| = |E_1 + \cdots + E_l| \geq |E_1| + \cdots + |E_l| = |E_1^* + \cdots + E_l^*|,$$

which implies that

$$(E_1 + \cdots + E_l)^* \supset E_1^* + \cdots + E_l^*. \tag{2.7}$$

Clearly, for any nonzero $a \in \mathbb{R}$ and any measurable subset E in $\mathbb{R}$,

$$(aE)^* = aE^*. \tag{2.8}$$

Combining with (2.7)-(2.8) we have

$$(a_1E_1 + \cdots + a_lE_l)^* \supset a_1E_1^* + \cdots + a_lE_l^*. \tag{2.9}$$

Applying (1.1) and (2.9), we get

$$\sup_{x_j \in E_j} \left|\sum_{j=1}^{l} a_j x_j\right| = \sup_{\bar{x} \in \sum_{j=1}^{l} a_j E_j} |\bar{x}| \geq \sup_{\bar{x} \in (\sum_{j=1}^{l} a_j E_j)^*} |\bar{x}| \geq \sup_{\bar{x} \in \sum_{j=1}^{l} a_j E_j^*} |\bar{x}|.$$

Besides,

$$\sup_{\bar{x} \in \sum_{j=1}^{l} a_j E_j^*} |\bar{x}| = \sup_{x_j \in a_j E_j^*} \left|\sum_{j=1}^{l} x_j\right| = \sup_{x_j \in E_j^*} \left|\sum_{j=1}^{l} a_j x_j\right|.$$

Therefore,

$$\sup_{x_j \in E_j} \left|\sum_{j=1}^{l} a_j x_j\right| \geq \sup_{x \in E_j^*} \left|\sum_{j=1}^{l} a_j x_j\right|.$$

□

It follows from Lemma 2.2 that we have inequalities (2.10)-(2.12). Let $E_1, \ldots, E_l$ be measurable sets in $\mathbb{R}^n$. Let $l \geq n$ and let $A = \{a_{ik}\}$ be an $l \times n$ real matrix. Then for each $1 \leq t \leq n$,

$$\sup_{\substack{y_j \in \mathcal{S}_{e_t}(E_j) \\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1} y_i, \ldots, \sum_{i=1}^{l} a_{in} y_i\right) \leq \sup_{\substack{y_j \in E_j \\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1} y_i, \ldots, \sum_{i=1}^{l} a_{in} y_i\right), \tag{2.10}$$

where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$.

Let $l = n$ and let

$$a_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

so (2.10) gives

$$\sup_{\substack{y_j\in\mathcal{S}_{e_t}(E_j)\\ j=1,\dots,n}} \det(0,y_1,\dots,y_n) \le \sup_{\substack{y_j\in E_j\\ j=1,\dots,n}} \det(0,y_1,\dots,y_n). \tag{2.11}$$

If we set $l = n+1$ and

$$a_{ik} = \begin{cases} 1 & \text{if } i = k, \\ -1 & \text{if } i = n+1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\sup_{\substack{y_j\in\mathcal{S}_{e_t}(E_j)\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}) \le \sup_{\substack{y_j\in E_j\\ j=1,\dots,n+1}} \det(y_1,\dots,y_{n+1}). \tag{2.12}$$

Proof. For simplicity, we see that (2.10) holds for e_1. Define the projection π: $\mathbb{R}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x) = (x_2,\dots,x_n),\ \forall\ x = (x_1,\dots,x_n) \in \mathbb{R}^n.$$

For any $x \in \mathbb{R}^n$, write $x = (x_1, x')$ where $x' \in \mathbb{R}^{n-1}$. For $y_j \in E_j$,

$$\det(0,y_1,\dots,y_n) = \left|\det\begin{pmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ \vdots & \vdots & & \vdots \\ y_{1n} & y_{2n} & \cdots & y_{nn}\end{pmatrix}\right| = |y_{11}A_1 + y_{21}A_2 + \cdots + y_{n1}A_n|,$$

where A_j depends only on $\{y'_1,\dots,y'_n\}$. Hence, $\det\left(0, \sum_{i=1}^{l} a_{i1}y_i,\dots,\sum_{i=1}^{l} a_{in}y_i\right)$ is the linear combination of $y_{11},\dots,y_{l1}$. That is,

$$\det\left(0, \sum_{i=1}^{l} a_{i1}y_i,\dots,\sum_{i=1}^{l} a_{in}y_i\right) = |y_{11}B_1 + y_{21}B_2 + \cdots + y_{l1}B_l|,$$

where B_j depends only on $\{y'_1,\dots,y'_l\}$. For each j, fix $y'_j := (y_{j2},\dots,y_{jn}) \in \pi(E_j)$, $1 \le j \le l$. Let

$$E_j(y'_j) = \{y_{j1} \in \mathbb{R} : (y_{j1}, y'_j) \in E_j\}.$$

It follows from Lemma 2.2 that

$$\sup_{y_{j1}\in E_j(y'_j)^*} \left|\sum_{j=1}^{l} B_j y_{j1}\right| \le \sup_{y_{j1}\in E_j(y'_j)} \left|\sum_{j=1}^{l} B_j y_{j1}\right|. \tag{2.13}$$

Since

$$\mathcal{S}_{e_1}(E_j) = \bigcup_{y'_j\in\pi(E_j)} \{(y_{j1}, y'_j) : y_{j1} \in E_j(y'_j)^*\},$$

together with (2.13) this gives

$$\sup_{\substack{y_j\in\mathcal{S}_{e_1}(E_j)\\ j=1,\dots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i,\dots,\sum_{i=1}^{l} a_{in}y_i\right) \le \sup_{\substack{y_j\in E_j\\ j=1,\dots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i,\dots,\sum_{i=1}^{l} a_{in}y_i\right).$$

□

More generally, together with the rotation invariance we have the following rearrangement theorem.

Theorem 2.3. *Let $A = \{a_{ik}\}$ be an $l \times n$ real matrix with $l \geq n$. Let u be a unit vector in $\mathbb{R}^n$. Then for any measurable sets $E_j \subset \mathbb{R}^n$, $1 \leq j \leq l$,*

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j)\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right) \leq \sup_{\substack{y_j \in E_j\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right).$$

Proof. Suppose $u = \rho e_t$, where ρ is a rotation around the origin in $\mathbb{R}^n$.
By definition,

$$\mathcal{S}_{\rho e_t}(E) = \left\{m\rho e_t + y : E \cap [\mathbb{R}(\rho e_t) + y] \neq \phi, |m| \leq \frac{|E \cap [\mathbb{R}(\rho e_t) + y]|}{2}\right\}$$
$$= \left\{\rho(me_t + \rho^{-1}y) : \rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y) \neq \phi, |m| \leq \frac{|\rho[\rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y)]|}{2}\right\}$$
$$= \left\{\rho(me_t + \rho^{-1}y) : \rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y) \neq \phi, |m| \leq \frac{|\rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y)|}{2}\right\}.$$

Note that

$$\mathcal{S}_{e_t}(\rho^{-1}(E)) = \left\{me_t + \rho^{-1}y : \rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y) \neq \phi, |m| \leq \frac{|\rho^{-1}(E) \cap (\mathbb{R}e_t + \rho^{-1}y)|}{2}\right\}.$$

Hence we obtain

$$\mathcal{S}_{\rho e_t}(E) = \rho \circ \mathcal{S}_{e_t}(\rho^{-1}(E)). \tag{2.14}$$

By the invariance under rotation ρ,

$$\sup_{\substack{y_j \in \mathcal{S}_u(E_j)\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right)$$
$$= \sup_{\substack{y_j \in \rho\circ\mathcal{S}_{e_t}(\rho^{-1}(E_j))\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right)$$
$$= \sup_{\substack{y_j \in \mathcal{S}_{e_t}(\rho^{-1}(E_j))\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right).$$

Applying (2.10) gives

$$\sup_{\substack{y_j \in \mathcal{S}_{e_t}(\rho^{-1}(E_j))\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right)$$
$$\leq \sup_{\substack{y_j \in \rho^{-1}(E_j)\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right)$$
$$= \sup_{\substack{y_j \in E_j\\ j=1,\ldots,l}} \det\left(0, \sum_{i=1}^{l} a_{i1}y_i, \ldots, \sum_{i=1}^{l} a_{in}y_i\right).$$

Therefore, we conclude that

$$\sup_{\substack{y_j\in\mathcal{S}_u(E_j)\\ j=1,\ldots,l}} \det\left(0,\sum_{i=1}^{l}a_{i1}y_i,\ldots,\sum_{i=1}^{l}a_{in}y_i\right)=\sup_{\substack{y_j\in E_j\\ j=1,\ldots,l}} \det\left(0,\sum_{i=1}^{l}a_{i1}y_i,\ldots,\sum_{i=1}^{l}a_{in}y_i\right).$$

□

Now we can decide the sharp versions of the determinant inequalities in this section. It is known that, given a compact convex set $K\subset\mathbb{R}^n$, there exists a sequence of iterated Steiner symmetrisations of K that converges in the Hausdorff metric to a ball of the same volume. For example, given a basis of unit directions $u_1,\ldots,u_n$ for $\mathbb{R}^n$ having mutually irrational multiple of π radian differences, the sequence $\mathcal{S}_{u_n}\ldots\mathcal{S}_{u_2}\mathcal{S}_{u_1}(K)$ iterated infinitely many times to K will converge to a ball of the same volume as K. For the convergence of Steiner symmetrisation, refer to [1], [2], [6], [11], [15], etc.

One can easily verify that the suprema function on the right side of inequalities (2.10) are continuous under the Hausdorff metric, and they do not change if we replace each E_j by $\overline{\mathrm{co}}(E_j)$. Therefore, applying the convergence of Steiner symmetrisation together with Theorem 2.3 we have shown the following lemma.

Lemma 2.4. *Let $l\ge n$ and let $A=\{a_{ik}\}$ be an $l\times n$ real matrix. Then for any measurable sets $E_j\subset\mathbb{R}^n$, $1\le j\le l$,*

$$\sup_{y_1\in E_1^*,\ldots,y_l\in E_l^*}\det\left(0,\sum_{i=1}^{l}a_{i1}y_i,\ldots,\sum_{i=1}^{l}a_{in}y_i\right)\le\sup_{y_1\in E_1,\ldots,y_l\in E_l}\det\left(0,\sum_{i=1}^{l}a_{i1}y_i,\ldots,\sum_{i=1}^{l}a_{in}y_i\right).$$

Obviously, it follows from Lemma 2.4 that

$$\sup_{y_1\in E_1^*,\ldots,y_n\in E_n^*}\det(0,y_1,\ldots,y_n)\le\sup_{y_1\in E_1,\ldots,y_n\in E_n}\det(0,y_1,\ldots,y_n)\tag{2.15}$$

and

$$\sup_{y_1\in E_1^*,\ldots,y_{n+1}\in E_{n+1}^*}\det(y_1,\ldots,y_{n+1})\le\sup_{y_1\in E_1,\ldots,y_{n+1}\in E_{n+1}}\det(y_1,\ldots,y_{n+1})\tag{2.16}$$

hold for any measurable sets $E_j\subset\mathbb{R}^n$, $1\le j\le n+1$.

From Lemma 2.4 we obtain the multilinear functional rearrangement inequalities.

Theorem 2.5. *Let f_j be nonnegative measurable functions vanishing at infinity on $\mathbb{R}^n$. Let $A=\{a_{ij}\}\in\mathrm{GL}_n(\mathbb{R})$. Then*

$$\sup_{y_j}\prod_{j=1}^{n}f_j^*\left(\sum_{i=1}^{n}a_{ij}y_i\right)\det(0,y_1,\ldots,y_n)\le\sup_{y_j}\prod_{j=1}^{n}f_j\left(\sum_{i=1}^{n}a_{ij}y_i\right)\det(0,y_1,\ldots,y_n).\tag{2.17}$$

Let $A=\{a_{ij}\}\in\mathrm{GL}_{(n+1)}(\mathbb{R})$. Then

$$\sup_{y_j}\prod_{j=1}^{n+1}f_j^*\left(\sum_{i=1}^{n+1}a_{ij}y_i\right)\det(y_1,\ldots,y_{n+1})\le\sup_{y_j}\prod_{j=1}^{n+1}f_j\left(\sum_{i=1}^{n+1}a_{ij}y_i\right)\det(y_1,\ldots,y_{n+1}),\tag{2.18}$$

where the sup *is the essential supremum.*

Proof. Let $\widetilde{y_j} = \sum_{i=1}^{n} a_{ij} y_i$, $1 \le j \le n$, so

$$\det(0, y_1, \dots, y_n) = \det(0, \widetilde{y_1}, \dots, \widetilde{y_n}) |\det(A)|^{-1}.$$

Then for (2.17) it suffices to prove that

$$\sup_{\widetilde{y_j}} \prod_{j=1}^{n} f_j^*(\widetilde{y_j}) \det(0, \widetilde{y_1}, \dots, \widetilde{y_n}) \le \sup_{\widetilde{y_j}} \prod_{j=1}^{n} f_j(\widetilde{y_j}) \det(0, \widetilde{y_1}, \dots, \widetilde{y_n}). \tag{2.19}$$

Similarly, for (2.18) denote $\widetilde{y_j} = \sum_{i=1}^{n+1} a_{ij} y_i$, $1 \le j \le n+1$. Since

$$\begin{pmatrix} y_1 & \dots & y_{n+1} \end{pmatrix} = \begin{pmatrix} \widetilde{y_1} & \dots & \widetilde{y_{n+1}} \end{pmatrix} A^{-1},$$

$\det(y_1, \dots, y_{n+1})$ can be written in the form

$$\det\left(0, \sum_{i=1}^{n+1} c_{i1}\widetilde{y_i}, \sum_{i=1}^{n+1} c_{i2}\widetilde{y_i}, \dots, \sum_{i=1}^{n+1} c_{in}\widetilde{y_i}\right).$$

Specifically, suppose $A^{-1} = \{b_{ij}\}_{n+1}$. Then by calculation we have $c_{ik} = b_{ik} - b_{i(n+1)}$ with $1 \le k \le n$, $1 \le i \le n+1$. Hence (2.18) becomes

$$\begin{aligned} &\sup_{\widetilde{y_j}} \prod_{j=1}^{n+1} f_j^*(\widetilde{y_j}) \det\left(0, \sum_{i=1}^{n+1} c_{i1}\widetilde{y_i}, \sum_{i=1}^{n+1} c_{i2}\widetilde{y_i}, \dots, \sum_{i=1}^{n+1} c_{in}\widetilde{y_i}\right) \\ &\le \sup_{\widetilde{y_j}} \prod_{j=1}^{n+1} f_j(\widetilde{y_j}) \det\left(0, \sum_{i=1}^{n+1} c_{i1}\widetilde{y_i}, \sum_{i=1}^{n+1} c_{i2}\widetilde{y_i}, \dots, \sum_{i=1}^{n+1} c_{in}\widetilde{y_i}\right). \end{aligned} \tag{2.20}$$

We claim that for any $l \ge n$, for any $l \times n$ real matrix $B = \{c_{ik}\}$,

$$\begin{aligned} &\sup_{y_j} \prod_{j=1}^{l} f_j^*(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1} y_i, \dots, \sum_{i=1}^{l} c_{in} y_i\right) \\ &\le \sup_{y_j} \prod_{j=1}^{l} f_j(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1} y_i, \dots, \sum_{i=1}^{l} c_{in} y_i\right) \end{aligned}$$

holds. Suppose that

$$\sup_{y_j} \prod_{j=1}^{l} f_j(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1} y_i, \dots, \sum_{i=1}^{l} c_{in} y_i\right) = s < \infty.$$

We assume for a contradiction that

$$\sup_{y_j} \prod_{j=1}^{l} f_j^*(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1} y_i, \dots, \sum_{i=1}^{l} c_{in} y_i\right) > s.$$

Then there exist positive ε and a set $G \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ such that $|G| > 0$ and for all $(x_1, \dots, x_l) \in G$ we have

$$\prod_{j=1}^{l} f_j^*(x_j) \det\left(0, \sum_{i=1}^{l} c_{i1} x_i, \dots, \sum_{i=1}^{l} c_{in} x_i\right) > s + \varepsilon, \tag{2.21}$$

which gives

$$f_1^*(x_1) > (s+\varepsilon)\left(\prod_{j=2}^{l} f_j^*(x_j)\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}. \tag{2.22}$$

Define the set

$$E_1 := \left\{y_1 : f_1(y_1) > (s+\varepsilon)\left(\prod_{j=2}^{l} f_j^*(x_j)\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}\right\},$$

so by the property of decreasing rearrangement together with (2.22) we have

$$|E_1| > v_n|x_1|^n.$$

From the definition of E_1,

$$f_2^*(x_2) > \left(s+\frac{\varepsilon}{2}\right)\left(\inf_{y_1\in E_1} f_1(y_1)\prod_{j=3}^{l} f_j^*(x_j)\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}.$$

We then define

$$\begin{aligned} E_2 = \{ & y_2 : f_2(y_2) \\ & > \left(s+\frac{\varepsilon}{2}\right)\left(\inf_{y_1\in E_1} f_1(y_1)\prod_{j=3}^{l} f_j^*(x_j)\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}\right\},\end{aligned}$$

so

$$|E_2| > v_n|x_2|^n.$$

Overall, we can take similar arguments to define sets E_t, $1 < t < l$:

$$\begin{aligned} E_t = \Bigg\{ y_t : f_t(y_t) > \left(s+\frac{\varepsilon}{t}\right)\Bigg(&\prod_{j=1}^{t-1}\inf_{y_j\in E_j} f_j(y_j)\prod_{j=t+1}^{l} f_j^*(x_j) \\ & \times\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\Bigg)^{-1}\Bigg\}\end{aligned}$$

and

$$\begin{aligned} E_l = \{ & y_l : f_l(y_l) \\ & > \left(s+\frac{\varepsilon}{l}\right)\left(\prod_{j=1}^{l-1}\inf_{y_j\in E_j} f_j(y_j)\det\left(0,\sum_{i=1}^{l} c_{i1}x_i,\dots,\sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}\right\}.\end{aligned}$$

It is easily seen that for each $j = 1,\dots,l$,

$$|E_j| > v_n|x_j|^n, \tag{2.23}$$

and thus $x_j \in E_j^*$. It follows from Lemma 2.4 that

$$\sup_{y_1\in E_1^*,\ldots,y_l\in E_l^*} \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right)$$
$$\leq \sup_{y_1\in E_1,\ldots,y_l\in E_l} \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right).$$

That together with $x_j \in E_j^*$, $j = 1, \ldots, l$, implies that

(2.24)
$$\det\left(0, \sum_{i=1}^{l} c_{i1}x_i, \ldots, \sum_{i=1}^{l} c_{in}x_i\right) \leq \sup_{y_1\in E_1,\ldots,y_n\in E_n} \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right).$$

From the definition of E_l we have for any $y_j \in E_j$, $1 \leq j \leq l$,

$$\prod_{j=1}^{l} f_j(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right)$$
$$> \left(s + \frac{\varepsilon}{l}\right)\left(\det\left(0, \sum_{i=1}^{l} c_{i1}x_i, \ldots, \sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1} \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right).$$

Therefore, together with (2.24) we obtain

$$s \geq \sup_{y_1\in E_1,\ldots,y_l\in E_l} \prod_{j=1}^{l} f_j(y_j) \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right)$$
$$> \left(s + \frac{\varepsilon}{l}\right)\left(\det\left(0, \sum_{i=1}^{l} c_{i1}x_i, \ldots, \sum_{i=1}^{l} c_{in}x_i\right)\right)^{-1}$$
$$\times \sup_{y_1\in E_1,\ldots,y_l\in E_l} \det\left(0, \sum_{i=1}^{l} c_{i1}y_i, \ldots, \sum_{i=1}^{l} c_{in}y_i\right)$$
$$> s,$$

which gives a contradiction. That completes the proof of the claim. Therefore, (2.19)-(2.20) hold. □

Remarks 4. We use a counterexample to show that Theorem 2.5 is false if $\det(A) = 0$. Let $f_1 = \chi_A$, $f_2 = \chi_B$, where A, B are disjoint measurable sets in $\mathbb{R}^2$ with non-zero measure. Obviously,

$$\sup_{y_1,y_2\in\mathbb{R}^2} f_1(y_1+y_2)f_2(y_1+y_2)\det(0, y_1, y_2) = 0,$$

while

$$\sup_{y_1,y_2} f_1^*(y_1+y_2)f_2^*(y_1+y_2)\det(0, y_1, y_2) \neq 0.$$

Likewise, for the same sets A, B above, let $f_1 = \chi_A$, $f_2 = f_3 = \chi_B$. Then

$$\sup_{y_1,y_2,y_3\in\mathbb{R}^2} f_1(y_1+y_2+y_3)f_2(y_1+y_2+y_3)f_3(y_3)\det(y_1, y_2, y_3) = 0,$$

while

$$\sup_{y_1,y_2,y_3\in\mathbb{R}^2} f_1^*(y_1+y_2+y_3)f_2^*(y_1+y_2+y_3)f_3^*(y_3)\det(y_1, y_2, y_3) \neq 0.$$

Let $A = I$. From Theorem 2.5 it is straightforward to see that

$$\sup_{y_j} \prod_{j=1}^{n} f_j^*(y_j) \det(0, y_1, \ldots, y_n) \leq \sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \ldots, y_n), \tag{2.25}$$

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^*(y_j) \det(y_1, \ldots, y_{n+1}) \leq \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1}). \tag{2.26}$$

Let $f_j = \chi_{E_j}$, and let E_j be measurable sets in $\mathbb{R}^n$. Applying (2.25)-(2.26) we obtain the following two sharp "multilinear" determinant inequalities suggested by the multilinear perspective of (2.2):

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n}} \leq A_n \sup_{y_1 \in E_1, \ldots, y_n \in E_n} \det(0, y_1, \ldots, y_n) \tag{2.27}$$

and

$$\prod_{j=1}^{n+1} |E_j|^{\frac{1}{n+1}} \leq B_n \sup_{y_1 \in E_1, \ldots, y_{n+1} \in E_{n+1}} \det(y_1, \ldots, y_{n+1}). \tag{2.28}$$

Moreover, they are both extremised by balls centred at 0. It follows from (2.25)-(2.26) that we also obtain the optimisers for (1.7) and (1.8) which is the special case when $E_j = E$.

It should be pointed out that (2.25)-(2.26) improves multilinear rearrangement inequalities (2.29), (2.30) given in [4]. For each $1 \leq i \leq n$,

$$\sup_{y_j} \prod_{j=1}^{n} f_j^{*i}(y_j) \det(0, y_1, \ldots, y_n) \leq \sup_{y_j} \prod_{j=1}^{n} f_j(y_j) \det(0, y_1, \ldots, y_n) \tag{2.29}$$

and

$$\sup_{y_j} \prod_{j=1}^{n+1} f_j^{*i}(y_j) \det(y_1, \ldots, y_{n+1}) \leq \sup_{y_j} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1}), \tag{2.30}$$

where f_j^{*i} is the Steiner symmetrisation of f_j with respect to the i-th coordinate.

Finally we give the best constant of (2.1), mainly applying the Brascamp-Lieb-Luttinger rearrangement inequality. In 1974, Brascamp, Lieb, and Luttinger [3] proved the following inequality (2.31), which is a generalisation of Riesz's rearrangement inequality [14].

Let f_j be nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity, $j = 1, \ldots, m$. Let $k \leq m$ and let $B = \{b_{ij}\}$ be a $k \times m$ matrix with $1 \leq i \leq k$, $1 \leq j \leq m$. Define

$$I(f_1, \ldots, f_m) := \int_{(\mathbb{R}^n)^k} \prod_{j=1}^{m} f_j \left(\sum_{i=1}^{k} b_{ij} x_i \right) dx_1 \ldots dx_k.$$

Then

$$I(f_1, \ldots, f_m) \leq I(f_1^*, \ldots, f_m^*). \tag{2.31}$$

Theorem 2.7. *Let f_j be nonnegative measurable functions vanishing at infinity on $\mathbb{R}^n$. Define*

$$J(f_1,\dots,f_{n+1})=\int_{(\mathbb{R}^n)^n}\prod_{j=1}^n f_j(y_j)f_{n+1}(\det(0,y_1,\dots,y_n))dy_1\dots dy_n$$

and

$$G(f_1,\dots,f_{n+2})=\int_{(\mathbb{R}^n)^{n+1}}\prod_{j=1}^{n+1} f_j(y_j)f_{n+2}(\det(y_1,\dots,y_{n+1}))dy_1\dots dy_{n+1}.$$

Then

$$J(f_1,\dots,f_{n+1})\le J(f_1^*,\dots,f_{n+1}^*) \tag{2.32}$$

and

$$G(f_1,\dots,f_{n+2})\le G(f_1^*,\dots,f_{n+2}^*). \tag{2.33}$$

Proof. By the layer cake representation, it suffices to show that for any E_j of finite volume in $\mathbb{R}^n$, $1\le j\le n+2$,

$$J(E_1,\dots,E_{n+1})\le J(E_1^*,\dots,E_{n+1}^*),\quad G(E_1,\dots,E_{n+2})\le G(E_1^*,\dots,E_{n+2}^*).$$

For any measurable $F_j\subset\mathbb{R}$, $1\le j\le n+1$, the Brascamp-Lieb-Luttinger rearrangement inequality implies that

$$\int_{(\mathbb{R}^n)^n}\prod_{j=1}^n\chi_{F_j}(x_j)\chi_{F_{n+1}}\left(\sum_{j=1}^n a_jx_j\right)dx_1\dots dx_n$$
$$\le\int_{(\mathbb{R}^n)^n}\prod_{j=1}^n\chi_{F_j^*}(x_j)\chi_{F_{n+1}^*}\left(\sum_{j=1}^n a_jx_j\right)dx_1\dots dx_n.$$

As before, since $\det(0,y_1,\dots,y_n)$ is the linear combination of $y_{11},\dots,y_{n1}$, similar to the proof of (2.10) we have

$$J(E_1,\dots,E_{n+1})\le J(\mathcal{S}_{e_1}(E_1),\dots,\mathcal{S}_{e_1}(E_{n+1})). \tag{2.34}$$

Note that $J(E_1,\dots,E_{n+1})$ is invariant under $O(n)$. By the property of

$$\mathcal{S}_{\rho e_i}(E)=\rho\circ\mathcal{S}_{e_i}(\rho^{-1}(E)),$$

we obtain for any $u\in\mathbb{S}^{n-1}$ that is a unit vector in $\mathbb{R}^n$:

$$J(E_1,\dots,E_{n+1})\le J(\mathcal{S}_u(E_1),\dots,\mathcal{S}_u(E_{n+1})). \tag{2.35}$$

Likewise, since $\det(y_1,\dots,y_{n+1})$ can be seen as the linear combination of $y_{11},\dots,$ $y_{(n+1)1}$ and by the Brascamp-Lieb-Luttinger rearrangement inequality

$$\int_{(\mathbb{R}^n)^{n+1}}\prod_{j=1}^{n+1}\chi_{F_j}(x_j)\chi_{F_{n+2}}\left(\sum_{j=1}^{n+1} a_jx_j\right)dx_1\dots dx_{n+1}$$
$$\le\int_{(\mathbb{R}^n)^{n+1}}\prod_{j=1}^{n+1}\chi_{F_j^*}(x_j)\chi_{F_{n+2}^*}\left(\sum_{j=1}^{n+1} a_jx_j\right)dx_1\dots dx_{n+1},$$

we also have

$$G(E_1,\dots,E_{n+2})\le G(\mathcal{S}_{e_1}(E_1),\dots,\mathcal{S}_{e_1}(E_{n+2})). \tag{2.36}$$

Hence by (2.14) together with the invariance of $G(E_1, \dots, E_{n+2})$,

$$G(E_1, \dots, E_{n+2}) \leq G(\mathcal{S}_u(E_1), \dots, \mathcal{S}_u(E_{n+2})). \tag{2.37}$$

Let H be the semigroup of all finite products of $\mathcal{S}_u$'s. Brascamp, Lieb, and Luttinger [3] proved for any bounded measurable $E \subset \mathbb{R}^n$ that there exists $\{h_m\}_{m=0}^{\infty} \subset G$ such that $E_m := h_m(E)$ converges to E^* in symmetric difference. That is,

$$\lim_{m \to \infty} |E_m \triangle E^*| = 0, \tag{2.38}$$

where $\triangle$ denotes the symmetric difference of two sets. Here we sketch the sequence of sets $\{E_m\}$. Let $E_0 = h_0 E = E$. Given E_m, choose unit vector u_1 such that

$$|\mathcal{S}_{u_1}(E_m) \triangle E^*| < \inf_{u \in \mathbb{S}^{n-1}} |\mathcal{S}_u(E_m) \triangle E^*| + \frac{1}{m}.$$

Hence we select $u_2, \dots, u_n \in \mathbb{S}^{n-1}$ such that $\{u_1, \dots, u_n\}$ becomes an orthonormal basis in $\mathbb{R}^n$ and then construct

$$E_{m+1} = h_{m+1}(E) = \mathcal{S}_{u_n} \mathcal{S}_{u_{n-1}} \dots \mathcal{S}_{u_1}(E_m).$$

The sequence of sets $\{E_m\}$ constructed above converges to E^* in symmetric difference. See [3] for the detailed proof. Therefore, we apply the convergence of Steiner symmetrisation together with (2.35) and (2.37) to conclude that

$$J(E_1, \dots, E_{n+1}) \leq J(E_1^*, \dots, E_{n+1}^*)$$

and

$$G(E_1, \dots, E_{n+2}) \leq G(E_1^*, \dots, E_{n+2}^*).$$

Lastly, applying the layer cake representation for f_j together with Fubini's theorem gives

$$J(f_1, \dots, f_{n+1}) = \int_0^{\infty} \cdots \int_0^{\infty} J(\chi_{\{f_1 > t_1\}}, \dots, \chi_{\{f_{n+1} > t_{n+1}\}}) dt_1 \dots dt_{n+1}.$$

Since (2.32)-(2.33) hold for characteristic functions of sets of finite Lebesgue measure, for any t_j, $1 \leq j \leq n+1$,

$$J(\chi_{\{f_1 > t_1\}}, \dots, \chi_{\{f_{n+1} > t_{n+1}\}}) \leq J(\chi^*_{\{f_1 > t_1\}}, \dots, \chi^*_{\{f_{n+1} > t_{n+1}\}}). \tag{2.39}$$

Thus

$$\begin{aligned} J(f_1, \dots, f_{n+1}) &\leq \int_0^{\infty} \cdots \int_0^{\infty} J(\chi^*_{\{f_1 > t_1\}}, \dots, \chi^*_{\{f_{n+1} > t_{n+1}\}}) dt_1 \dots dt_{n+1} \\ &= J(f_1^*, \dots, f_{n+1}^*). \end{aligned}$$

Similarly,

$$G(f_1, \dots, f_{n+2}) \leq G(f_1^*, \dots, f_{n+2}^*).$$

This completes Theorem 2.7. □

Let $f_j = \chi_{E_j}$, $1 \leq j \leq n$, and $f_{n+1} = \chi_{(|\cdot| < \delta)}$. Theorem 2.7 gives that

$$\begin{aligned} &|\{(y_1, \dots, y_n) \in E_1 \times \dots \times E_n : \det(0, y_1, \dots, y_n) < \delta\}| \\ &\leq |\{(y_1, \dots, y_n) \in E_1^* \times \dots \times E_n^* : \det(0, y_1, \dots, y_n) < \delta\}|. \end{aligned}$$

This implies that inequality (2.1) is extremised by balls centred at y, where $y \in \mathbb{R}^n$.

Let $f_{n+2} = |\cdot|^{-1}$. Then Theorem 2.7 implies that

$$\int_{(\mathbb{R}^n)^{n+1}} \prod_{j=1}^{n+1} f_j(y_j) \det(y_1, \ldots, y_{n+1})^{-1} dy_1 \ldots dy_{n+1}$$
$$\leq \int_{(\mathbb{R}^n)^{n+1}} \prod_{j=1}^{n+1} f_j^*(y_j) \det(y_1, \ldots, y_{n+1})^{-1} dy_1 \ldots dy_{n+1}.$$

3. Matrix inequalities

Now we turn to see the analogues of (1.5) and (1.6) replacing the Euclidean space $\mathbb{R}^n$ by the space of $n \times n$ real matrices. We remark that the proof of Theorem 3.1 mainly relies on the rearrangement inequality (2.6) and an invariance under the action of $O(n)$ by premultiplication as described in the introduction.

Theorem 3.1. *There exists a finite constant $\mathcal{C}_n$ such that for any measurable set $E_j \subset \mathfrak{M}^{n\times n}$ of finite measure, $j = 1, \ldots, n$,*

$$\prod_{j=1}^{n} |E_j|^{\frac{1}{n^2}} \leq \mathcal{C}_n \sup_{\substack{A_j \in E_j \\ j=1,\ldots,n}} |\det(A_1 + \cdots + A_n)|, \tag{3.1}$$

where $|\cdot|$ denotes the Lebesgue measure on Euclidean space $\mathbb{R}^{n^2}$ and the absolute value on $\mathbb{R}$.

Proof. Suppose that

$$\sup_{\substack{A_j \in E_j \\ j=1,\ldots,n}} |\det(A_1 + \cdots + A_n)| = s < \infty.$$

First we give some definitions and notation. Let $F \subset \mathfrak{M}^{n\times m}$ and define

$$v(F) = \left\{ \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{(m-1)1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{(m-1)n} \end{pmatrix} : \exists \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} \text{ such that } \begin{pmatrix} a_{11} & \ldots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{mn} \end{pmatrix} \in F \right\},$$

so $v(F) \subset \mathfrak{M}^{n\times(m-1)}$. For any n-by-$(m-1)$ matrix

$$x = \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{(m-1)1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{(m-1)n} \end{pmatrix} \in v(F),$$

we denote

$$F^x = \left\{ \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} : \begin{pmatrix} a_{11} & \ldots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{mn} \end{pmatrix} \in F \right\} \subset \mathfrak{M}^{n\times 1}.$$

Let $E \subset \mathfrak{M}^{n\times n}$. For any rotation around the origin T in $\mathbb{R}^n$, consider

$$\Phi_T : A \mapsto TA, \ \forall\, A \in E,$$

where T is an n-by-n matrix with $\det(T) = 1$. Note that Φ_T does not change $|E|$ and $\sup\limits_{A\in E} |\det(A)|$. This is because

$$\sup_{A\in\Phi_T(E)} |\det(A)| = \sup_{A\in E} |\det(TA)| = \sup_{A\in E} |\det(A)|. \tag{3.2}$$

Besides, if we see the matrix $A = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \in E$ as a vector

$$(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}) \in \mathbb{R}^{n^2},$$

then the matrix $\Phi_T(A)$ becomes

$$\begin{pmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{pmatrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Thus

$$|\Phi_T(E)| = |T|^n |E| = |E|. \tag{3.3}$$

From $|E| = \int_{v(E)} |E^x| dx$ it follows that there always exists $\overline{x} \in v(E)$ such that

$$|v(E)||E^{\overline{x}}| \gtrsim_n |E|. \tag{3.4}$$

By the John ellipsoid, for any compact convex $G \subset \mathbb{R}^n$ there exists an ellipsoid $G' \subset G$ such that

$$|G'| \gtrsim_n |G|. \tag{3.5}$$

For the John ellipsoid G', we choose a rotation $T \in O(n)$ such that TG' is an ellipsoid with principal axes parallel to the coordinate axes. As is well known, for every ellipsoid TG' with principal axes parallel to the coordinate axes, there exists an axis-parallel rectangle $H \subset TG'$ such that

$$|H| \gtrsim_n |TG'|. \tag{3.6}$$

Hence if $E^{\overline{x}}$ is convex, from (3.5)-(3.6) we may assume that there exists $T \in O(n)$ such that $E^{\overline{x}}$ is an axis-parallel rectangle in $\mathbb{R}^n$.

Take $n = 2$. By (3.4) there exists $x_{10} \in v(E_1) \subset \mathfrak{M}^{2\times 1}$, $x_{20} \in v(E_2) \subset \mathfrak{M}^{2\times 1}$ such that

$$|v(E_1)||E_1^{x_{10}}| \gtrsim |E_1|, \quad |v(E_2)||E_2^{x_{20}}| \gtrsim |E_2|. \tag{3.7}$$

Then

$$\max\{|v(E_2)||E_1^{x_{10}}|, |v(E_1)||E_2^{x_{20}}|\} \gtrsim (|E_1||E_2|)^{1/2}.$$

For simplicity, suppose that

$$|v(E_2)||E_1^{x_{10}}| \gtrsim (|E_1||E_2|)^{1/2}. \tag{3.8}$$

To study the suprema, we consider the 2-by-2 matrix

$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & (x_{10})_2 \end{pmatrix} \in E_1$$

with

$$(x_{10})_1 = x_{10} \in \mathfrak{M}^{n\times 1} \text{ and } (x_{10})_2 \in E_1^{x_{10}}.$$

For any $\overline{A}_2 := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \in E_2$, for any constructed $\overline{A}_1$ above,

$$\begin{aligned} s &\geq |\det(\overline{A}_1 + \overline{A}_2)| \\ &= |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \end{aligned}$$

So fixing the first column, we have for any $x_1 \in v(E_2)$, $x_2 \in E_2^{x_1}$,

$$s \geq \sup_{(x_{10})_2 \in E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \tag{3.9}$$

Because we fix all the columns except one, the $|\det|$ function is the convex function of the remaining column. Thus

$$s \geq \sup_{(x_{10})_2 \in \mathrm{co}E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}|. \tag{3.10}$$

By (3.5) we may assume $\mathrm{co}E_1^{x_{10}}$ is an ellipsoid in $\mathbb{R}^2$. Choose a rotation $T_0 \in O(2)$ such that $T_0\mathrm{co}E_1^{x_{10}}$ is an ellipsoid with principal axes parallel to the coordinate axes. From (3.6) we may assume $T_0\mathrm{co}E_1^{x_{10}}$ is an axis-parallel rectangle. Note that (3.10) is invariant under $O(2)$ as discussed in (3.2), so

$$\begin{aligned} s &\geq \sup_{(x_{10})_2 \in \mathrm{co}E_1^{x_{10}}} |\det \begin{pmatrix} x_1 + (x_{10})_1 & x_2 + (x_{10})_2 \end{pmatrix}| \\ &= \sup_{(x_{10})_2 \in \mathrm{co}E_1^{x_{10}}} |\det \begin{pmatrix} T_0x_1 + T_0(x_{10})_1 & T_0x_2 + T_0(x_{10})_2 \end{pmatrix}|. \end{aligned} \tag{3.11}$$

Since $T_0\mathrm{co}E_1^{x_{10}}$ is an axis-parallel rectangle in $\mathbb{R}^2$, it can be written as $A_1 \times A_2$, where A_1, A_2 are intervals in $\mathbb{R}$, and then

$$\mathcal{S}(T_0\mathrm{co}E_1^{x_{10}}) = \mathcal{S}(T_0\mathrm{co}E_1^{x_{10}} + T_0x_2) = A_1^* \times A_2^*, \ \forall\ x_2 \in E_2^{x_1}.$$

Similar to the proof of (2.10), applying (2.6) gives for any $x_1 \in v(E_2)$

$$s \geq \sup_{(x_{10})_2 \in \mathcal{S}(T_0\mathrm{co}E_1^{x_{10}})} |\det \begin{pmatrix} T_0x_1 + T_0(x_{10})_1 & (x_{10})_2 \end{pmatrix}|. \tag{3.12}$$

Therefore, by (2.2) we deduce that

$$s \geq C|T_0v(E_2) + T_0(x_{10})_1|^{1/2}|\mathcal{S}(T_0\mathrm{co}E_1^{x_{10}})|^{1/2} = C|v(E_2)|^{1/2}|\mathrm{co}E_1^{x_{10}}|^{1/2}.$$

This together with (3.8) implies that

$$s \geq C|v(E_2)|^{1/2}|\mathrm{co}E_1^{x_{10}}|^{1/2} \geq C|v(E_2)|^{1/2}|E_1^{x_{10}}|^{1/2} \geq C(|E_1||E_2|)^{1/4},$$

which completes (3.1) for $n = 2$.

Take $n = 3$. By (3.4) for each E_j there exists $x_{j0} \in v(E_j) \subset \mathfrak{M}^{3\times 2}$ such that

$$|v(E_j)||E_j^{x_{j0}}| \gtrsim |E_j|, 1 \leq j \leq 3. \tag{3.13}$$

Denote $F_j = v(E_j) \subset \mathfrak{M}^{3\times 2}$. There exists fixed $x_{j1} \in v(F_j) \subset \mathfrak{M}^{3\times 1}$ such that

$$|v(F_j)||F_j^{x_{j1}}| \gtrsim |F_j| = v(E_j). \tag{3.14}$$

From (3.13)-(3.14), we have $x_{j0} \in v(E_j), x_{j1} \in v(F_j)$, and

$$|v(F_j)||F_j^{x_{j1}}||E_j^{x_{j0}}| \gtrsim |E_j|, 1 \leq j \leq 3. \tag{3.15}$$

It is not hard to see there exists $\{i_1, i_2, i_3\}$ with $i_1 \neq i_2 \neq i_3$ such that

$$(v(F_{i_3})||F_{i_2}^{x_{i_21}}||E_{i_1}^{x_{i_10}}|)^3 \geq \prod_{j=1}^{3}(|v(F_j)||F_j^{x_{j1}}||E_j^{x_{j0}}|) \gtrsim \prod_{j=1}^{3}|E_j|. \tag{3.16}$$

For simplicity, suppose that

$$|v(F_3)||F_2^{x_{21}}||E_1^{x_{10}}| \gtrsim (|E_1||E_2||E_3|)^{1/3}. \tag{3.17}$$

Now we consider 3-by-3 matrices

$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & (x_{10})_2 & (x_{10})_3 \end{pmatrix} \in E_1$$

with $\begin{pmatrix} (x_{10})_1 & (x_{10})_2 \end{pmatrix} = x_{10} \in \mathfrak{M}^{3\times 2}$ and $(x_{10})_3 \in E_1^{x_{10}}$;

$$\overline{A}_2 := \begin{pmatrix} (x_{21})_1 & (x_{21})_2 & (x_{21})_3 \end{pmatrix} \in E_2$$

with the condition

$$(x_{21})_1 = x_{21} \in \mathfrak{M}^{3\times 1} \text{ and } (x_{21})_2 \in F_2^{x_{21}}.$$

For any $\overline{A}_3 := \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in E_3$, for any constructed $\overline{A}_1, \overline{A}_2$ above,

$$\begin{aligned} s &\geq |\det(\overline{A}_1 + \overline{A}_2 + \overline{A}_3)| \\ &= |\det\begin{pmatrix} x_1 + (x_{10})_1 + (x_{21})_1 & x_2 + (x_{10})_2 + (x_{21})_2 & x_3 + (x_{10})_3 + (x_{21})_3 \end{pmatrix}|. \end{aligned}$$

So fixing all columns except the third column, we have

$$s \geq \sup_{(x_{10})_3 \in E_1^{x_{10}}} |\det\big(x_1 + (x_{10})_1 + (x_{21})_1 \quad x_2 + (x_{10})_2 + (x_{21})_2 \quad x_3 + (x_{10})_3 + (x_{21})_3\big)|.$$

Obviously,

$$s \geq \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x_{10}}} |\det\big(x_1 + (x_{10})_1 + (x_{21})_1 \quad x_2 + (x_{10})_2 + (x_{21})_2 \quad x_3 + (x_{10})_3 + (x_{21})_3\big)|.$$

As before, by (3.5) we assume there exists $T_0\mathrm{co}E_1^{x_{10}}$, an ellipsoid with principal axes parallel to the coordinate axes in $\mathbb{R}^3$. From (3.6) we may assume $T_0\mathrm{co}E_1^{x_{10}}$ is an axis-parallel rectangle. Because of the invariance under $O(3)$,

$$\begin{aligned} s &\geq \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x_{10}}} |\det\big(x_1 + (x_{10})_1 + (x_{21})_1 \quad x_2 + (x_{10})_2 + (x_{21})_2 \quad x_3 + (x_{10})_3 + (x_{21})_3\big)| \\ &= \sup_{(x_{10})_3 \in \mathrm{co}E_1^{x_{10}}} |\det\big(T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_0(x_2 + (x_{10})_2 + (x_{21})_2) \quad T_0(x_3 + (x_{10})_3 + (x_{21})_3)\big)|. \end{aligned}$$

Since $T_0\mathrm{co}E_1^{x_{10}}$ is an axis-parallel rectangle in $\mathbb{R}^3$, it can be written as $A_1 \times A_2 \times A_3$, where A_1, A_2, A_3 are intervals in $\mathbb{R}$. Similar to the proof of (2.10) together with

$$\mathcal{S}(T_0\mathrm{co}E_1^{x_{10}}) = \mathcal{S}(T_0\mathrm{co}E_1^{x_{10}} + h) = A_1^* \times A_2^* \times A_3^*, \ \forall\, h \in \mathbb{R}^3,$$

applying (2.6) gives for any $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \in v(E_3)$,

$$s \geq \sup_{\substack{(x_{10})_3 \\ \in \mathcal{S}(T_0\mathrm{co}E_1^{x_{10}})}} |\det\big(T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_0(x_2 + (x_{10})_2 + (x_{21})_2) \quad (x_{10})_3\big)|.$$

Then fixing all columns except the second column, we have that

$$s \geq \sup_{\substack{(x_{21})_2 \\ \in F_2^{x_{21}}}} |\det\big(T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_0(x_2 + (x_{10})_2 + (x_{21})_2) \quad (x_{10})_3\big)|$$

holds for any $(x_{10})_3 \in \mathcal{S}(T_0 \mathrm{co} E_1^{x_{10}})$. Similarly, by the convex property of $|\det|$ function when fixing other columns

$$s \geq \sup_{\substack{(x_{21})_2 \\ \in \mathrm{co} F_2^{x_{21}}}} |\det (T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_0(x_2 + (x_{10})_2 + (x_{21})_2) \quad (x_{10})_3)|.$$

By (3.5) we may assume that $T_0 \mathrm{co} F_2^{x_{21}}$ is an ellipsoid in $\mathbb{R}^3$. Choose a rotation $T_1 \in O(3)$ such that $T_1 T_0 \mathrm{co} F_2^{x_{21}}$ is an ellipsoid with principal axes parallel to the coordinate axes. From (3.6) we may assume that $T_1 T_0 \mathrm{co} F_2^{x_{21}}$ is an axis-parallel rectangle. By the invariance of $O(3)$,

$$\begin{aligned} s \geq & \sup_{\substack{(x_{21})_2 \\ \in \mathrm{co} F_2^{x_{21}}}} |\det (T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_0(x_2 + (x_{10})_2 + (x_{21})_2) \quad (x_{10})_3)| \\ = & \sup_{\substack{(x_{21})_2 \\ \in \mathrm{co} F_2^{x_{21}}}} |\det (T_1 T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad T_1 T_0(x_2 + (x_{10})_2 + (x_{21})_2) \\ & \qquad T_1(x_{10})_3)|. \end{aligned}$$

Since $T_1 T_0 \mathrm{co} F_2^{x_{21}}$ is an axis-parallel rectangle, together with

$$\mathcal{S}(T_1 T_0 \mathrm{co} F_2^{x_{21}}) = \mathcal{S}(T_1 T_0 \mathrm{co} F_2^{x_{21}} + h), \quad \forall\, h \in \mathbb{R}^3,$$

apply inequality (2.6) again to obtain that

$$s \geq \sup_{\substack{(x_{10})_3 \in \mathcal{S}(T_0 \mathrm{co} E_1^{x_{10}}) \\ (x_{21})_2 \in \mathcal{S}(T_1 T_0 \mathrm{co} F_2^{x_{21}})}} |\det \big(\; T_1 T_0(x_1 + (x_{10})_1 + (x_{21})_1) \quad (x_{21})_2 \quad T_1(x_{10})_3 \; \big)|$$

holds for any $x_1 \in v(F_3) \subset \mathfrak{M}^{3\times 1}$.

Lastly, applying (2.2) we conclude that

$$\begin{aligned} s &\geq C|T_1 T_0 v(F_3) + T_1 T_0 (x_{10})_1 \\ &\quad + T_1 T_0 (x_{21})_1|^{1/3} |\mathcal{S}(T_1 T_0 \mathrm{co} F_2^{x_{21}})|^{1/3} |T_1 \mathcal{S}(T_0 \mathrm{co} E_1^{x_{10}})|^{1/3} \\ &= C|v(F_3)|^{1/3} |\mathrm{co} F_2^{x_{21}})|^{1/3} |\mathrm{co} E_1^{x_{10}})|^{1/3}. \end{aligned}$$

This together with (3.17) implies that

$$\begin{aligned} s &\geq C|v(F_3)|^{1/3} |\mathrm{co} F_2^{x_{21}}|^{1/3} |\mathrm{co} E_1^{x_{10}}|^{1/3} \\ &\geq C|v(F_3)|^{1/3} |F_2^{x_{21}}|^{1/3} |E_1^{x_{10}}|^{1/3} \\ &\geq C(|E_1||E_2||E_3|)^{1/9}. \end{aligned}$$

This completes (3.1) for $n = 3$.

For the general n, for each E_j, denote $F_{j0} = E_j$, $1 \leq j \leq n$. Given $1 \leq k \leq n-2$, let

$$F_{jk} = v(F_{j(k-1)}) \subset \mathfrak{M}^{n\times(n-k)}.$$

Then by (3.4) there exists fixed $x_{jk} \in v(F_{jk}) \subset \mathfrak{M}^{n\times(n-k-1)}$, $0 \leq k \leq n-2$, such that

$$|v(F_{jk})||F_{jk}^{x_{jk}}| \gtrsim |F_{jk}| = |v(F_{j(k-1)})|. \tag{3.18}$$

That is, for each E_j there exist $\{x_{j0}, \ldots, x_{j(n-2)}\}$ such that for each $k = 0, \ldots, n-2$,

$$x_{jk} \in v(F_{jk}) \subset \mathfrak{M}^{n\times(n-k-1)}$$

and

$$|v(F_{j(n-2)})||F_{j(n-2)}^{x_{j(n-2)}}||F_{j(n-3)}^{x_{j(n-3)}}|\dots|F_{j1}^{x_{j1}}||F_{j0}^{x_{j0}}| \gtrsim_n |E_j|. \tag{3.19}$$

It is not hard to see that there exist $\{i_j\}_{j=1}^n$ with $1 \le i_j \le n$ and $i_j \ne i_k$ for $j \ne k$ such that

$$\begin{aligned}&(|v(F_{i_n(n-2)})||F_{i_{n-1}(n-2)}^{x_{i_{n-1}(n-2)}}||F_{i_{n-2}(n-3)}^{x_{i_{n-2}(n-3)}}|\dots|F_{i_2 1}^{x_{i_2 1}}||F_{i_1 0}^{x_{i_1 0}}|)^n\\ \ge &\prod_{j=1}^n(|v(F_{j(n-2)})||F_{j(n-2)}^{x_{j(n-2)}}||F_{j(n-3)}^{x_{j(n-3)}}|\dots|F_{j1}^{x_{j1}}||F_{j0}^{x_{j0}}|) \gtrsim_n \prod_{j=1}^n |E_j|.\end{aligned}$$

For simplicity, suppose $i_j = j, 1 \le j \le n$. That is,

$$|v(F_{n(n-2)})||F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|\dots|F_{21}^{x_{21}}||F_{10}^{x_{10}}| \gtrsim_n \prod_{j=1}^n |E_j|^{1/n}. \tag{3.20}$$

To study the suprema, we consider the n-by-n matrices

$$\overline{A}_1 := \begin{pmatrix} (x_{10})_1 & \dots & (x_{10})_n \end{pmatrix} \in E_1$$

with $\begin{pmatrix} (x_{10})_1 & \dots & (x_{10})_{(n-1)} \end{pmatrix} = x_{10} \in \mathfrak{M}^{n\times(n-1)}$ and $(x_{10})_n \in F_{10}^{x_{10}}$ and

$$\overline{A}_2 := \begin{pmatrix} (x_{21})_1 & \dots & (x_{21})_n \end{pmatrix} \in E_2$$

with $\begin{pmatrix} (x_{21})_1 & \dots & (x_{21})_{(n-2)} \end{pmatrix} = x_{21} \in \mathfrak{M}^{n\times(n-2)}$ and $(x_{21})_{n-1} \in F_{21}^{x_{21}}$. That is, construct $\{\overline{A}_1, \dots, \overline{A}_{n-1}\}$ such that for each $1 \le k \le n-1$,

$$\overline{A}_k := \begin{pmatrix} (x_{k(k-1)})_1 & \dots & (x_{k(k-1)})_n \end{pmatrix} \in E_k,$$

with the condition that

$$\big((x_{k(k-1)})_1 \dots (x_{k(k-1)})_{n-k}\big) = x_{k(k-1)} \in \mathfrak{M}^{n\times(n-k)}, \quad (x_{k(k-1)})_{n-k+1} \in F_{k(k-1)}^{x_{k(k-1)}}.$$

For any $\overline{A}_n := \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \in E_n$, for any constructed $\overline{A}_1, \dots, \overline{A}_{n-1}$ above,

$$\begin{aligned} s &\ge |\det(\overline{A}_1 + \dots + \overline{A}_{n-1} + \overline{A}_n)| \\ &= |\det\begin{pmatrix} x_1 + \sum_{k=1}^{n-1}(x_{k(k-1)})_1 & \dots & x_n + \sum_{k=1}^{n-1}(x_{k(k-1)})_n \end{pmatrix}|. \end{aligned}$$

Taking the same arguments as in the case $n = 3$, there exist $T_0, T_1 \in O(n)$,

$$s \ge \sup_{\substack{(x_{10})_n \in \mathcal{S}(T_0 \mathrm{co} F_{10}^{x_{10}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(T_1 T_0 \mathrm{co} F_{21}^{x_{21}})}} |\det\begin{pmatrix} B & B' \end{pmatrix}|, \tag{3.21}$$

where

$$B = T_1 T_0 \begin{pmatrix} x_1 + \sum_{k=1}^{n-1}(x_{k(k-1)})_1 & \dots & x_{n-2} + \sum_{k=1}^{n-1}(x_{k(k-1)})_{n-2} \end{pmatrix} \in \mathfrak{M}^{n\times(n-2)},$$

$$B' = \begin{pmatrix} (x_{21})_{(n-1)} & T_1(x_{10})_n \end{pmatrix} \in \mathfrak{M}^{n\times 2}.$$

Applying the same arguments again to (3.21), there exist $T_2 \in O(n)$,

$$s \ge \sup_{\substack{(x_{10})_n \in \mathcal{S}(T_0 \mathrm{co} F_{10}^{x_{10}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(T_1 T_0 \mathrm{co} F_{21}^{x_{21}}) \\ (x_{32})_{n-2} \in \mathcal{S}(T_2 T_1 T_0 \mathrm{co} F_{32}^{x_{32}})}} |\det\begin{pmatrix} C & C' \end{pmatrix}|, \tag{3.22}$$

where

$$C = T_2T_1T_0 \begin{pmatrix} x_1 + \sum_{k=1}^{n-1}(x_{k(k-1)})_1 & \dots & x_{n-3} + \sum_{k=1}^{n-1}(x_{k(k-1)})_{n-3} \end{pmatrix} \in \mathfrak{M}^{n\times(n-3)},$$

$$C' = \begin{pmatrix} (x_{32})_{(n-2)} & T_2(x_{21})_{(n-1)} & T_2T_1(x_{10})_n \end{pmatrix} \in \mathfrak{M}^{n\times 3}.$$

Keep repeating the same arguments above, and finally we have that there exists $T_0, \dots, T_{n-2} \in O(n)$, such that for any $x_1 \in v(F_{n(n-2)}) \subset \mathfrak{M}^1$,

$$s \geq \sup_{\substack{(x_{10})_n \in \mathcal{S}(T_0 \mathrm{co} F_{10}^{x_{10}}) \\ (x_{21})_{(n-1)} \in \mathcal{S}(T_1T_0 \mathrm{co} F_{21}^{x_{21}}) \\ \dots\dots \\ (x_{(n-1)(n-2)})_2 \in \mathcal{S}(T_{n-2}T_{n-3}\dots T_0 \mathrm{co} F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}})}} |\det \begin{pmatrix} D & D' \end{pmatrix}|, \tag{3.23}$$

where $D \in \mathfrak{M}^{n\times 1}$, $D' \in \mathfrak{M}^{n\times(n-1)}$:

$$D = (T_{n-2}\dots T_0)(x_1 + \sum_{k=1}^{n-1}(x_{k(k-1)})_1),$$

$$D' = \big((x_{(n-1)(n-2)})_2 \quad T_{n-2}(x_{(n-2)(n-3)})_3 \quad (T_{n-2}T_{n-3})(x_{(n-3)(n-4)})_4 \cdots \\ (T_{n-2}\dots T_1)(x_{10})_n\big).$$

It follows from (2.2) together with the invariance under $O(n)$ that

$$s \geq C|v(F_{n(n-2)})|^{1/n}|\mathrm{co}F_{(n-1)(n-2)}^{x_{n-1)(n-2)}}|^{1/n}|\mathrm{co}F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|^{1/n}\dots|\mathrm{co}F_{21}^{x_{21}}|^{1/n}|\mathrm{co}F_{10}^{x_{10}}|^{1/n}.$$

Obviously,

$$|\mathrm{co}F_{k(k-1)}^{x_{k(k-1)}}| \geq |F_{k(k-1)}^{x_{k(k-1)}}|, \ 1 \leq k \leq n-1.$$

This together with (3.20) implies that

$$\begin{aligned} s &\geq C(|v(F_{n(n-2)})||\mathrm{co}F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||\mathrm{co}F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|\dots|\mathrm{co}F_{21}^{x_{21}}||F_{10}^{x_{10}}|)^{1/n} \\ &\geq C(|v(F_{n(n-2)})||F_{(n-1)(n-2)}^{x_{(n-1)(n-2)}}||F_{(n-2)(n-3)}^{x_{(n-2)(n-3)}}|\dots|F_{21}^{x_{21}}||F_{10}^{x_{10}}|)^{1/n} \geq C\prod_{j=1}^{n}|E_j|^{\frac{1}{n^2}}. \end{aligned}$$

This completes Theorem 3.1. □

Corollary 3.2. *There exists a finite constant $\mathcal{A}_n, \mathcal{B}_n$ such that for any measurable set $E \subset \mathfrak{M}^{n\times n}$ of finite measure, for any nonzero scalar $\lambda_j \in \mathbb{R}$, $j = 1, \dots, n$,*

$$(\prod_{j=1}^{n}|\lambda_j|)|E|^{\frac{1}{n}} \leq \mathcal{A}_n \sup_{\substack{A_j \in E \\ j=1,\dots,n}} |\det(\lambda_1A_1 + \dots + \lambda_nA_n)|. \tag{3.24}$$

If E is a compact convex set in $\mathfrak{M}^{n\times n}$, then

$$|E|^{\frac{1}{n}} \leq \mathcal{B}_n \sup_{A\in E} |\det(A)|. \tag{3.25}$$

Proof. To see (3.24), let $E_j = \lambda_jE$. Applying Theorem 3.1 gives

$$\prod_{j=1}^{n}|\lambda_jE|^{\frac{1}{n^2}} \leq \mathcal{C}_n \sup_{\substack{A_j \in E \\ j=1,\dots,n}} |\det(\lambda_1A_1 + \dots + \lambda_nA_n)|,$$

which implies (3.24). In particular, if $E \subset \mathfrak{M}^{n\times n}$ is a compact convex set, setting $\lambda_j = \frac{1}{n}$, $j = 1, \dots, n$, it follows from (3.24) that

$$\frac{1}{n^n}|E|^{1/n} \le \mathcal{A}_n \sup_{\substack{A_j\in E\\ j=1,\dots,n}} \left|\det\left(\frac{1}{n}A_1 + \cdots + \frac{1}{n}A_n\right)\right|.$$

On the other hand, since E is convex,

$$\sup_{A\in E}|\det(A)| \ge \sup_{\substack{A_j\in E\\ j=1,\dots,n}} \left|\det\left(\frac{1}{n}A_1 + \cdots + \frac{1}{n}A_n\right)\right|.$$

Thus we get (3.25). □

Here we give a direct way to see Lemma 13.2 of [5], which follows from (3.25). Let $E \subset \mathfrak{M}^{n\times n}$ be a measurable set. The inequality (1.18) in Lemma 13.2 has translation invariance property, so we assume that $0 \in E$. Given any matrices $A_1, \dots, A_{n^2}$ in E, from (3.25) it follows that

$$|\mathrm{co}\{0, A_1, \dots, A_{n^2}\}|^{\frac{1}{n}} \lesssim_n \sup_{A\in \mathrm{co}\{0,A_1,\dots,A_{n^2}\}} |\det(A)|. \tag{3.26}$$

By (2.2), there exist $A_1, \dots, A_{n^2}$ such that

$$|E| \lesssim_n |\mathrm{co}\{0, A_1, \dots, A_{n^2}\}|;$$

together with (3.26) we obtain that

$$|E|^{\frac{1}{n}} \lesssim_n \sup_{A\in \mathrm{co}\{0,A_1,\dots,A_{n^2}\}} |\det(A)|. \tag{3.27}$$

For any convex set $F \subset \mathfrak{M}^{n\times n}$,

$$\sup_{A\in \mathrm{co}\{0,F\}} |\det(A)| = \sup_{A\in F} |\det(A)|,$$

since $|\det(\lambda A)| = \lambda^n |\det(A)| \le |\det(A)|$ for any $\lambda \in [0, 1]$. So

$$\sup_{A\in \mathrm{co}\{0,A_1,\dots,A_{n^2}\}} |\det(A)| = \sup_{A\in \mathrm{co}\{A_1,\dots,A_{n^2}\}} |\det(A)|. \tag{3.28}$$

Denote $A^{(k)}$ by the k-th column vector of the matrix A, $1 \le k \le n$. Then there exist $\widetilde{A}_1, \dots, \widetilde{A}_n \in \{A_1, \dots, A_{n^2}\}$ ($\widetilde{A}_i$, $\widetilde{A}_j$ might be the same matrix), such that for any $\{\lambda_1, \dots, \lambda_{n^2}\}$ satisfying $\sum_{j=1}^{n^2} \lambda_j = 1$ and $0 \le \lambda_j \le 1$,

$$|\det(\lambda_1 A_1 + \cdots + \lambda_{n^2} A_{n^2})| \le \left|\sum_{\substack{i_j\in\{1,\dots,n\}\\ i_j\ne i_k,\forall j\ne k}} \det\left(\widetilde{A}_{i_1}^{(1)}, \dots, \widetilde{A}_{i_n}^{(n)}\right)\right| \tag{3.29}$$

holds. This is because

$$\begin{aligned}\sum_{1\le l_1,\dots,l_n\le n^2} \lambda_{l_1}\dots\lambda_{l_n} &\le \sum_{1\le l_1,\dots,l_{n-1}\le n^2} \lambda_{l_1}\dots\lambda_{l_{n-1}}\\ &\le\dots\le \sum_{1\le l_1,l_2\le n^2} \lambda_{l_1}\lambda_{l_2}\\ &\le \sum_{1\le l_1\le n^2} \lambda_{l_1}=1.\end{aligned}$$

Hence from (3.27)-(3.29)

$$|E|^{\frac{1}{n}} \lesssim_n \left| \sum_{\substack{i_j\in\{1,\dots,n\}\\ i_j\neq i_k,\forall j\neq k}} \det\left(\widetilde{A}_{i_1}^{(1)},\dots,\widetilde{A}_{i_n}^{(n)}\right)\right|. \tag{3.30}$$

As mentioned in the proof of Lemma 13.2 of [5], $\sum\limits_{\substack{i_j\in\{1,\dots,n\}\\ i_j\neq i_k,\forall j\neq k}} \det\left(\widetilde{A}_{i_1}^{(1)},\dots,\widetilde{A}_{i_n}^{(n)}\right)$ is a $\mathbb{Z}$-linear combination of $\left\{\det\left(\sum\limits_{j=1}^n s_j\widetilde{A}_j\right) : s_j\in\{0,1\}\right\}$. This gives (1.20):

$$|E|^{\frac{1}{n}} \lesssim_n \sup_{\substack{A_1,\dots,A_n\in E\\ s_1,\dots,s_n\in\{0,1\}}} |\det(s_1A_1+\dots+s_nA_n)|.$$

Obviously, (3.25) is not affine invariant. The following example shows that balls or ellipsoids are not the optimisers.

Example 3.3.

(i) Let $n=2$, $E=B(0,r)$, $A=\begin{pmatrix} a & c\\ b & d\end{pmatrix}\in E$.

Then $\sup\limits_{A\in E}|\det(A)|=\frac{r^2}{2}$ by calculation. Consider the ellipsoid F in $\mathbb{R}^4$ with $|F|=|B(0,r)|$,

$$F=\left\{\begin{pmatrix} a & c\\ b & d\end{pmatrix} : \frac{a^2}{l_1^2}+\frac{b^2}{l_2^2}+\frac{c^2}{l_3^2}+\frac{d^2}{l_4^2}\le 1\right\}.$$

It is easy to obtain $\sup\limits_{A\in F}|\det(A)|\ge\frac{l_1l_4+l_2l_3}{4}\ge\frac{r^2}{2}$ by GM-AM inequality.

(ii) Let $r=1$. Since $A\mapsto|\det(A)|$ is a continuous function on $E=B(0,1)$ under the natural topology on Euclidean space $\mathbb{R}^4$, there exists $0<\delta<\frac{1}{25}$ such that $|\det(A)|\le\frac{1}{4}$ for all $A\in E$ satisfying

$$\left|A-\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right| = (a-1)^2+b^2+c^2+d^2\le 2\delta.$$

Then for all $A\in E$ satisfying $\sqrt{1-\delta}\le a\le 1$, we have

$$b^2+c^2+d^2\le 1-a^2\le 1-(1-\delta)=\delta.$$

Thus

$$\left|A-\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right| = (a-1)^2+b^2+c^2+d^2\le(1-\sqrt{1-\delta})^2+\delta\le 2\delta,$$

which implies that $|\det(A)|\le\frac{1}{4}$ for any $A\in E$ satisfying $\sqrt{1-\delta}\le a\le 1$.

Let $P = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ with $p = \frac{1}{\sqrt{1-\delta}}$ and then consider $\sup\limits_{A \in \mathrm{co}\{P \cup E\}} |\det(A)|$:

$$\begin{aligned}
\sup_{A \in \mathrm{co}\{P \cup E\}} |\det(A)| &= \sup_{A \in E, \lambda \in [0,1]} |\det(\lambda A + (1-\lambda)P)| \\
&= \sup_{A \in E, \lambda \in [0,1]} \left| \det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d + (1-\lambda)p \end{pmatrix} \right| \\
&= \sup_{A \in E, \lambda \in [0,1]} \left| \det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix} \right|.
\end{aligned}$$

When $a \notin [\sqrt{1-\delta}, 1]$,

$$\begin{aligned}
&\sup_{A \in E, \lambda \in [0,1]} \left| \det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix} \right| \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda) a p \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda)\sqrt{1-\delta}\frac{1}{\sqrt{1-\delta}} \\
&= \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{2} + \lambda(1-\lambda) \leq \frac{1}{2}.
\end{aligned}$$

When $a \in [\sqrt{1-\delta}, 1]$,

$$\begin{aligned}
&\sup_{A \in E, \lambda \in [0,1]} \left| \det \begin{pmatrix} \lambda a & \lambda c \\ \lambda b & \lambda d \end{pmatrix} + \det \begin{pmatrix} \lambda a & 0 \\ \lambda b & (1-\lambda)p \end{pmatrix} \right| \\
&\leq \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda) p \\
&= \sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda)\frac{1}{\sqrt{1-\delta}}.
\end{aligned}$$

It is easy to see that for $0 < \delta < \frac{1}{25}$ given above

$$\sup_{\lambda \in [0,1]} \lambda^2 \frac{1}{4} + \lambda(1-\lambda)\frac{1}{\sqrt{1-\delta}} \leq \frac{1}{2}.$$

Therefore,

$$\sup_{A \in \mathrm{co}\{P \cup E\}} |\det(A)| = \sup_{A \in E} |\det(A)|,$$

which implies that balls cannot be the optimisers.

Remarks 5. Let $E \subset \mathfrak{M}^{n \times n}$ be a compact convex set. If we compare the maximal volume of simplicies $\sup\limits_{A_0, \ldots, A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \ldots, A_{n^2}\})$ contained in E with $\sup\limits_{A \in E} |\det(A)|$, it follows from (3.25) that

$$\sup_{A_0, \ldots, A_{n^2} \in E} \mathrm{vol}(\mathrm{co}\{A_0, \ldots, A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n. \tag{3.31}$$

Indeed by John ellipsoids, it is enough to consider the case when E is a ellipsoid in $\mathfrak{M}^{n\times n}$. For any ellpsoid

$$E \equiv \left\{ x \in \mathbb{R}^{n^2} : \sum_{i}^{n^2} \frac{|\langle x - x_0, \omega_i \rangle|^2}{l_i^2} \leq 1 \right\},$$

where $x_0 \in \mathbb{R}^{n^2}$, $\{\omega_i\}$ is an orthonormal basis in $\mathbb{R}^{n^2}$. By the affine invariance of $\sup_{A_0,\dots,A_{n^2} \in E} \operatorname{vol}(\operatorname{co}\{A_0,\dots,A_{n^2}\})$, it is enough to see balls centred at 0. Apply the Hadamard inequality for any $A_j \in B(0,r) \subset \mathbb{R}^{n^2}$, $j = 0,\dots,n^2$,

$$\operatorname{vol}(\operatorname{co}\{A_0,\dots,A_{n^2}\}) \leq |A_0 - A_1||A_0 - A_2| \dots |A_0 - A_{n^2}| \lesssim_n r^{n^2} \sim |B(0,r)|.$$

Hence for any ellipsoid $E \subset \mathbb{R}^{n^2}$,

$$\sup_{A_0,\dots,A_{n^2} \in E} \operatorname{vol}(\operatorname{co}\{A_0,\dots,A_{n^2}\}) \lesssim_n |E|.$$

On the other hand, by (3.25)

$$|E| \lesssim_n \sup_{A \in E} |\det(A)|^n.$$

Therefore, we have the following relation:

$$\sup_{A_0,\dots,A_{n^2} \in E} \operatorname{vol}(\operatorname{co}\{A_0,\dots,A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n.$$

Similarly, we have

$$\sup_{A_1,\dots,A_{n^2} \in E} \operatorname{vol}(\operatorname{co}\{0, A_1,\dots,A_{n^2}\}) \lesssim_n \sup_{A \in E} |\det(A)|^n. \tag{3.32}$$

If $0 \in E$, this is true, which is mainly due to the Hadamard inequality and the $\mathrm{GL}_n(\mathbb{R})$ invariance of $\sup_{A_1,\dots,A_{n^2} \in E} \operatorname{vol}(\operatorname{co}\{0, A_1,\dots,A_{n^2}\})$. If $0 \notin E$, the relation above still holds because of the fact that

$$\sup_{A \in E} |\det(A)|^n = \sup_{A \in \operatorname{co}\{0,E\}} |\det(A)|^n.$$

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