A NOTE ON HIGHER EXTREMAL METRICS

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ABSTRACT. In this paper we introduce "higher extremal Kähler" metrics. We provide an example of the same on a minimal ruled surface. We also prove a perturbation result that implies that there are non-trivial examples of "higher constant scalar curvature" metrics, which are basically metrics where the top Chern form is harmonic. We also give a relatively short proof of Liu's formula for the Bando-Futaki invariants (which are obstructions for the existence of harmonic Chern forms) of hypersurfaces of projective space.

1. INTRODUCTION

The problem of finding Kähler-Einstein metrics and, more generally, extremal Kähler metrics is of active interest (for instance see [16] and the references therein). Extremal metrics may be characterised as Kähler metrics for which the gradient of the scalar curvature (expressed as $S = \frac{nc_1 \wedge \omega^{n-1}}{\omega^n}$) is a holomorphic vector field. Special cases of these are the constant scalar curvature Kähler (cscK) metrics, which we interpret as those metrics for which the first Chern form is harmonic [2].

The Chern classes are important objects in algebraic geometry. In addition to the classes, the first Chern-Weil form itself is quite natural to study because it is the Ricci form for a Kähler manifold. Indeed, the first Chern form was used by Yau to prove the Bogomolov-Miyaoka-Yau inequality as a consequence of the Calabi conjecture [18]. As Yau stated in [19], the higher Chern-Weil forms are quite mysterious. That being said, we note that at the level of classes the top Chern class is the Euler class. Therefore, studying the top Chern form might potentially lead to interesting consequences. We are thus led to study the equation

$$(1.1) \qquad c_n(\omega) = \lambda \omega^n,$$

where the gradient of λ is a holomorphic vector field. We call these metrics *higher extremal Kähler*, and if λ is a constant, i.e., the top Chern form is harmonic, then we dub them as *higher constant scalar curvature (hcscK)*.

The hcscK metrics and their avatars were considered earlier by Bando [2], who came up with an obstruction for their existence. Another version of the higher extremal metrics was studied by Futaki [8,9], where he considered the perturbed scalar curvature $S(J,t) = \frac{c_1 + tc_2 + t^2c_3 + \cdots}{\omega^n}$, where t is a small real number. Our question is the case for large t in a sense. So Futaki's results do not apply in any direct manner that the author can see.

In this paper we study examples of higher extremal and hcscK metrics. Our first example comes from a minimal ruled surface. For the usual extremal Kähler metrics this example was first studied in [17], and more general results were proven in [1].

Theorem 1.1. *Let X be $\mathbb{P}(L \oplus \mathcal{O})$ where L is a degree -1 line bundle over a genus 2 surface Σ. Let C be the Poincaré dual of any fibre and let S_∞ be the copy of Σ corresponding to the line $L \oplus \{0\}$. There exists a Kähler metric ω in the class $2\pi(C + S_\infty)$ such that*

$$(1.2) \qquad\qquad c_2(\omega) = \frac{\lambda}{2(2\pi)^2}\omega^2,$$

where $\nabla^{(1,0)}\lambda$ is a holomorphic non-zero vector field on X; i.e., it is higher extremal Kähler but not hcscK.

Remark 1.1. The aforementioned theorem does not assert that for $2\pi(C + mS_\infty)$, where $m > 1$, there are no extremal metrics. The author suspects that there might be a maximum m (just as in the usual extremal Kähler case) beyond which there may not exist a solution. The proof is by reducing the equation to an ODE[1] that unfortunately is not integrable and is non-autonomous. The analysis of the ODE is somewhat delicate. In contrast to the usual case [17] where the corresponding ODE always has a solution satisfying the desired boundary conditions (but it is not clear that the solution actually gives rise to a Kähler metric), in our case the difficulty lies with the existence of a solution to the ODE satisfying the boundary conditions. Also, the proof of Theorem 1.1 shows that the assertion of the higher extremal metric not being hcscK is true regardless of m.

In our quest to find more examples, we note that the hermitian symmetric spaces are hcscK. This is because their metric, curvature, and hence characteristic forms are constant linear combinations of invariant differential forms. Actually, in the case of a surface X with ample canonical bundle, Yau [18] showed that if $c_1^2 = 3c_2$ numerically, then indeed it admits hcscK metrics and that they are all Kähler-Einstein as well (by virtue of them being ball quotients).

It is natural to wonder if there are non-trivial (i.e., not $X_1 \times X_2$ with product Kähler-Einstein metrics) examples of hcscK metrics. Also, near the symmetric Kähler-Einstein metrics are there any other hcscK metrics; i.e., does local uniqueness hold? The following perturbation result addresses these questions in some cases.

Theorem 1.2. *Suppose that either $(X, \omega) = (\mathbb{D}^1/\Gamma_1 \times \mathbb{D}^1/\Gamma_2, \pi_1^*\omega_1 + \pi_2^*\omega_2)$, where ω_1, ω_2 are constant curvature metrics, or $X = \mathbb{D}^2/\Gamma$ equipped with a metric ω of constant holomorphic sectional curvature. Suppose $\tilde{\omega}$ is any closed real $(1,1)$-form. There exists an $\epsilon_1, \epsilon_2 > 0$ such that for $|t| < \epsilon_1$ there exists a unique smooth function ϕ of zero average (with respect to ω) depending smoothly on t satisfying $\|\phi\|_{C^{4,\alpha}} < \epsilon_2$ such that $\omega + t\tilde{\omega} + \sqrt{-1}\partial\bar\partial\phi$ is hcscK.*

Remark 1.2. Consider surfaces of general type satisfying $c_1^2 = 3c_2$. Noether's formula, Hodge theory, and the fact that c_2 is the Euler characteristic allow us to prove that $h^{1,1} = h^{2,0} + h^{1,0} + 1 > 1$ for any such surface (of which there are infinitely many [3]) other than the 100 fake projective planes [5, 15]. For instance, the Cartwright-Steger surface [4] is a concrete example. For such surfaces one can

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[1]It is a version of Chini's equation.

come up with non-trivial examples of $\tilde{\omega}$ and hence by Theorem 1.2 find non-trivial hcscK metrics that are not Kähler-Einstein.

As pointed out earlier, whenever holomorphic vector fields exist the Bando-Futaki invariant provides an obstruction for the existence of hcscK metrics. It has been computed explicitly in very few cases, most notably by Liu [12] for hypersurfaces in $\mathbb{CP}^n$. Her formula can be used to come up with examples of non-existence of hcscK metrics. The theorem we are alluding to is Theorem 1.1 of [12].

Theorem 1.3 (Liu)**.** *Let M be a hypersurface in $\mathbb{CP}^n$ defined by a homogeneous polynomial F of degree $d \leq n$. Let Y be a holomorphic vector field on $\mathbb{CP}^n$ such that $YF = \kappa F$ for a constant κ. Then the q-th Bando-Futaki invariant is*

$$\mathcal{F}_q(Y, \omega_{FS}) = -(n+1-d)^{n-q}\frac{(d-1)(n+1)}{n}\sum_{j=0}^{q-1}(-d)^j(j+1)\binom{n}{q-j-1}\kappa.$$

In this paper we give a simplified proof of Liu's formula (whilst adhering to her basic strategy). The technique of computation (relying on generating series) might potentially be useful in calculating Bando-Futaki invariants in other cases. The crucial simplification comes from a linear algebra lemma (Lemma 4.1) that was used to similar effect in [14].

It is interesting to see if the Lebrun-Simanca kind of deformation results can be proven for these objects. We hope to explore this and other questions in later works.

2. A HIGHER EXTREMAL METRIC ON A RULED SURFACE

First we give a high-level overview of this section. The aim is to produce a higher extremal metric on a manifold with a lot of symmetry. Akin to [17] an ansatz reduces the problem to finding a parameter C and solving an ODE depending on C for a function ϕ on $[1, m+1]$, where m is a given integer (that specifies the Kähler class under consideration) satisfying $\phi(m+1) = 0$. The ODE being non-integrable poses difficulties with regard to existence. It turns out that for a connected set of C the ODE does have a smooth solution depending smoothly on C, but it is not clear whether $\phi(m+1) = 0$. So we produce a value of C so that $\phi(m+1) > 0$ and another value so that $\phi(m+1) < 0$. Thus there is some admissible C for which $\phi(m+1) = 0$. In our proof we can do everything with the exception of producing a C so that $\phi(m+1) < 0$. We can do this rigorously only for $m = 1$. However, numerically solving the ODE using the Runge-Kutta method on Wolfram Alpha seems to suggest that this is true for higher values of m too. It is just that one does not know explicit error bounds on the numerical solution and hence cannot "trust" it for a proof. With this bird's eye view in mind we proceed further.

Let (Σ, ω_Σ) be a genus 2 Riemann surface equipped with a metric of constant scalar curvature -2. Let L be a degree -1 holomorphic line bundle on Σ equipped with a metric h such that $-\omega_\Sigma$ is the curvature of h. Let X be the ruled surface $\mathbb{P}(L \oplus \mathcal{O})$. Just as in [1, 10, 17] we will construct extremal Kähler metrics on X. In whatever follows we follow the exposition of Székelyhidi [16].

The strategy is to first consider an ansatz on the total space of L minus the zero section and then extend the resulting metric to all of X. One way to potentially produce a metric is to pull back L to its total space and add the curvature of the resulting bundle to the pullback of ω_Σ. Motivated by this observation one writes

the following ansatz (let $p : X \to \Sigma$ be the projection map, let z be a coordinate on Σ, and let w be a coordinate on the fibres L):

$$(2.1) \qquad \omega = p^*\omega_\Sigma + \sqrt{-1}\partial\bar{\partial}f(s),$$

where $s = \ln|(z,w)|_h^2 = \ln|w|^2 + \ln h(z)$ and f is a strictly convex function that makes ω a metric. We choose coordinates (z_0, w_0) around a point Q such that $dh(z_0) = 0$. Therefore at Q we have the following equalities:

$$\partial s(Q) = \frac{dw}{w} \ , \ \bar{\partial}s(Q) = \frac{d\bar{w}}{\bar{w}},$$

$$\sqrt{-1}\partial\bar{\partial}s(Q) = p^*\omega_\Sigma,$$

$$(2.2) \qquad \omega(Q) = (1 + f'(s))p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2}.$$

The last equation is easily seen to hold at points other than Q as well.

Proceeding to study the Kähler class of ω we see that by the Leray-Hirsch theorem $H^2(X, \mathbb{R}) = \mathbb{R}C \oplus \mathbb{R}S_\infty$, where C is the Poincaré dual of a fibre (i.e., C is a sphere) and S_∞ is a copy of Σ sitting in X as the "infinity section", i.e., the line $L \oplus \{0\}$. It is clear that $C.C = 0, C.S_\infty = 1 = S_\infty.S_\infty$. We wish our ansatz to be in the cohomology class $[\omega] = 2\pi(C + mS_\infty)$, where m is a positive integer. Therefore $[\omega].C = 2\pi m$ and $[\omega].S_\infty = 2\pi(1+m)$. Indeed,

$$\int_C \omega = \int_{\mathbb{C}-\{0\}} f''(s)\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2} = 2\pi(\lim_{s\to\infty} f'(s) - \lim_{s\to-\infty} f'(s)) = 2\pi m,$$

$$(2.3) \qquad \int_{S_\infty} \omega = \int_\Sigma \lim_{s\to\infty}(1 + f'(s))\omega_\Sigma = (1+m)\int_\Sigma \omega_\Sigma = 2\pi(1+m).$$

Thus $0 \leq f'(s) \leq m$.

Returning back to the metric ω we see that

$$(2.4) \qquad \omega^2 = 2(1 + f'(s))f''(s)p^*\omega_\Sigma\sqrt{-1}\frac{dw \wedge d\bar{w}}{|w|^2}.$$

Calculating the curvature matrix of forms $\Theta = \bar{\partial}(h^{-1}\partial h)$ we obtain the following.

$$(2.5) \qquad \Theta = \left[\begin{array}{cc} -\partial\bar{\partial}\ln(1 + f'(s)) - 2p^*\omega_\Sigma & 0 \\ 0 & -\partial\bar{\partial}\ln(f''(s)) \end{array} \right].$$

At this point we appeal to the unreasonable effectiveness of the Legendre transform and define

$$(2.6) \qquad \tau = f'(s) \ , \ f(s) + F(\tau) = s\tau.$$

Therefore, $s = F'(\tau), \frac{ds}{d\tau} = F''(\tau)$. Since $f''(s)$ seems to crop up often, define (as Hwang-Singer did in [10]) the so-called momentum profile $\phi(\tau) = f''(s) = \frac{1}{F''(\tau)}$. Hence $\frac{d\tau}{ds} = \frac{1}{F''(\tau)} = \phi(\tau)$. Moreover, $f'''(s) = \frac{df''(s)}{d\tau}\phi(\tau) = \phi'\phi$.

In terms of $\gamma = \tau + 1 \in [1, m+1]$ the curvature form reads as

$$\sqrt{-1}\Theta = \begin{bmatrix} -\sqrt{-1}\partial\bar{\partial}\ln(\gamma) - 2p^*\omega_\Sigma & 0 \\ 0 & -\sqrt{-1}\partial\bar{\partial}\ln(\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{-1}\frac{\partial\gamma\bar{\partial}\gamma}{\gamma^2} - \frac{1}{\gamma}\sqrt{-1}\partial\bar{\partial}\gamma - 2p^*\omega_\Sigma & 0 \\ 0 & -\left(\frac{\phi'}{\phi}\right)'\sqrt{-1}\partial\gamma\bar{\partial}\gamma - \frac{\phi'}{\phi}\sqrt{-1}\partial\bar{\partial}\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\phi}{\gamma}\left[\frac{\phi}{\gamma} - \phi'\right]\frac{dwd\bar{w}}{|w|^2} - (\frac{\phi}{\gamma} + 2)p^*\omega_\Sigma & 0 \\ 0 & -\phi''\phi\sqrt{-1}\frac{dwd\bar{w}}{|w|^2} - \phi'p^*\omega_\Sigma \end{bmatrix}.$$

The top Chern form is $c_2 = \frac{1}{(2\pi)^2}\det(\sqrt{-1}\Theta)$, which is

$$(2.7) \qquad c_2 = \frac{1}{(2\pi)^2}p^*\omega_\Sigma\frac{\sqrt{-1}dwd\bar{w}}{|w|^2}\frac{\phi}{\gamma^2}\left(\gamma(\phi + 2\gamma)\phi'' + \phi'(\phi'\gamma - \phi)\right).$$

We want

$$(2.8) \qquad c_2 = \frac{1}{(2\pi)^2}\frac{\lambda}{2}\omega^2$$

to hold for some λ whose gradient is a holomorphic vector field, i.e.,

$$\nabla^{(1,0)}\lambda = \lambda'\nabla^{(1,0)}\tau = \lambda'w\frac{\partial}{\partial w},$$

which is a holomorphic vector field if and only if λ' is a constant; i.e., $\lambda = A\gamma + B$ for some A and B.

So our equation (2.8) boils down to an ODE for $\phi(\gamma)$:

$$\gamma(\phi + 2\gamma)\phi'' + \phi'(\phi'\gamma - \phi) = (A\gamma + B)\gamma^3$$

$$\Rightarrow 2\gamma^2\phi'' + (\frac{\phi\phi'}{\gamma})'\gamma^2 = (A\gamma + B)\gamma^3$$

$$\Rightarrow 2\phi' + \frac{\phi\phi'}{\gamma} = A\frac{\gamma^3}{3} + B\frac{\gamma^2}{2} + C$$

$$(2.9) \qquad \Rightarrow (2\gamma + \phi)\phi' = A\frac{\gamma^4}{3} + B\frac{\gamma^3}{2} + C\gamma,$$

where A, B, C are constants. It can be easily seen that [16] for ω to extend across the zero and infinity sections the following boundary conditions have to be met by $\phi(\gamma)$:

$$\phi(1) = \phi(m+1) = 0,$$
$$(2.10) \qquad \phi'(1) = -\phi'(m+1) = 1.$$

So we need to solve (2.9) for ϕ as well as for A, B, C so that the boundary conditions (2.10) are met and $\phi > 0 \ \forall \ \gamma \in [1, m+1]$. Unfortunately the form $(2x+y)dy - p(x)dx$ is not closed, and hence equation 2.9 cannot be integrated. Nevertheless, one can still prove Theorem 1.1 for $m = 1$. In order to do so we prove the following preliminary result about the ODE (2.9) with boundary conditions (2.10).

Theorem 2.1. *Given a positive integer m, consider the ODE*

$$(2.11) \qquad (2\gamma + \phi)\phi' = A\frac{\gamma^4}{3} + B\frac{\gamma^3}{2} + C\gamma$$

with the boundary conditions

$$\phi(1) = \phi(m+1) = 0,$$
$$(2.12) \qquad \phi'(1) = -\phi'(m+1) = 1.$$

If $C < M$ (where $M > 2$) then there exist linear functions $A(C), B(C)$ depending on a parameter C and a smooth solution ϕ to (2.11) on $[1, m+1]$ depending smoothly on C satisfying all the conditions of (2.12) except $\phi(m+1) = 0$. There exists a $C < M$ such that $\phi(m+1, C) > 0$. Moreover, if there exists a smooth solution satisfying all the boundary conditions, then $\phi > 0$ on $[1, m+1]$.

Proof. We impose the boundary conditions (2.10) on equation (2.9) to get the following relations between A, B, C:

$$2 = \frac{A}{3} + \frac{B}{2} + C,$$

$$-2 = \frac{A(m+1)^3}{3} + \frac{B(m+1)^2}{2} + C$$

$$\Rightarrow A(C) = \frac{3C}{m}\left[1 - \frac{1}{(m+1)^2}\right] - \frac{6}{m}\left[\frac{1}{(m+1)^2} + 1\right],$$

$$(2.13) \qquad B(C) = -2C\left[1 + \frac{1}{m} - \frac{1}{m(m+1)^2}\right] + 4 + \frac{4}{m}\left[1 + \frac{1}{(m+1)^2}\right].$$

Thus $A(C)$ and $B(C)$ are linear functions of C. Moreover, given C, if we manage to solve (2.9) on $[1, m+1]$ with the initial condition $\phi(1) = 0$, then (2.13) imply that $\phi' = 1$ and if we further ensure that $\phi(m+1) = 0$, then $\phi'(m+1) = -1$ automatically. The bottom line is that we have to prove that given C, there exists a smooth positive solution depending smoothly on C to the initial value problem

$$\phi' = \frac{A(C)\frac{\gamma^4}{3} + B(C)\frac{\gamma^3}{2} + C\gamma}{2\gamma + \phi} \text{ on } [1, m+1],$$

$$(2.14) \qquad \phi(1) = 0$$

and that there exists a $C = C_m$ such that $\phi(m+1) = 0$.

Near $\gamma = 1$ since the right-hand side of (2.14) is locally Lipschitz we have a unique smooth solution locally. At this point it is convenient to change variables. Let $v = \frac{(2\gamma+\phi)^2}{2}$. Equation (2.14) turns into the following:

$$v' = 2\sqrt{2}\sqrt{v} + p(\gamma)\gamma,$$

$$(2.15) \qquad v(1) = 2.$$

We want to find a smooth solution to (2.15) on $[1, m+1]$ so that $v(m+1) = 2(m+1)^2$ and $v(\gamma) > 2\gamma^2$ on $(1, m+1)$.

As before we have a unique smooth solution depending smoothly on parameters near $\gamma = 1$. If there is a solution on $[1, \gamma_*)$ such that $M \geq v \geq \epsilon > 0$, then since the right-hand side is C^1, by standard ODE theory the solution can be continued past γ_*. An easy comparison argument using $\sqrt{v} \leq kv$ and Gronwall's inequality shows that v is always bounded above. In order to prove lower bounds on v we need to study $p(\gamma)$.

Lemma 2.1. *Let $m \geq 1$ be a given positive integer and C be a real number. The polynomial $p(\gamma) = A(C)\frac{\gamma^3}{3} + B(C)\frac{\gamma^2}{2} + C$ (and hence $p(\gamma)\gamma$) satisfying $p(m+1) = -2$ and $p(1) = 2$ has exactly one root in $[1, m+1]$. Moreover $p(\gamma)$ has at most one critical point $\gamma = -\frac{B}{A}$ in $[1, m+1]$. As a consequence on $[1, m+1]$ we have the following:*

(2.16)
$$\int_1^\gamma p(t)t\,dt \geq \min(0, \int_1^{m+1} p(t)t\,dt) \quad and$$

$$\int_1^{m+1} p(t)t\,dt = LC + N, \quad where$$

$$L = \frac{m^2 + 2m}{2} - \frac{(m+1)^4 - 1}{4}\left[1 + \frac{1}{m} - \frac{1}{m(m+1)^2}\right]$$
$$+ \frac{(m+1)^5 - 1}{5m}\left[1 - \frac{1}{(m+1)^2}\right],$$
$$N = -\frac{(m+1)^5 - 1}{5}\frac{2}{m}\left[1 + \frac{1}{(m+1)^2}\right] + \frac{1}{2}((m+1)^4 - 1)\left[1 + \frac{1}{m} + \frac{1}{m(m+1)^2}\right].$$

If $C \leq 2$, then $LC + N > 0$, which implies that $\int_1^\gamma p(t)t\,dt > 0$.

Proof. Since $p(m+1) = -2$ and $p(1) = 2$, p has an odd number of roots (counted with multiplicity) in $[1, m+1]$. Now $p' = \gamma(A(C)\gamma + B(C))$, which has at most one root in $[1, m+1]$. This implies that p has exactly one root γ_0 in $[1, m+1]$. This also means that if there exists a smooth solution of (2.14) on $[1, m+1]$ satisfying $\phi(m+1) = 0$, then $\phi > 0$ on $(1, m+1)$.

Notice that $\gamma \to \int_1^\gamma p(t)t\,dt$ assumes its minimum over $[1, m+1]$ on the boundary because its only critical point is a local maximum. An easy calculation shows that indeed $\int_1^{m+1} p(t)t\,dt = LC + N$ where L and N are as above. The following proves that indeed $L < 0$ and $N > 0$ for $m \geq 1$:

(2.17)
$$L = \frac{(m+1)^2 - 1}{2} - \frac{(m+1)^4 - 1}{4m}\left[m + 1 - \frac{1}{(m+1)^2}\right] + \frac{(m+1)^4}{5}\left[1 - \frac{1}{(m+1)^2}\right]$$
$$+ \frac{(m+1)^4 - 1}{5m}\left[1 - \frac{1}{(m+1)^2}\right]$$
$$= \frac{3}{10}(m+1)^2 - \frac{1}{20}(m+1)^4 - \frac{1}{4} - \frac{(m+1)^4 - 1}{20m}\left[1 - \frac{1}{(m+1)^2}\right] < 0 \ \forall \ m \geq 1,$$
$$N = -\frac{m(m+1)^4 - 1 + (m+1)^4}{5}\frac{2}{m}\left[1 + \frac{1}{(m+1)^2}\right]$$
$$+ \frac{(m+1)^4 - 1)}{2m}\left[m + 1 + \frac{1}{(m+1)^2}\right]$$
$$= \frac{1}{10}(m+1)^4 - \frac{1}{2} - \frac{2}{5}(m+1)^2 + \frac{(m+1)^4 - 1}{10m}\left[1 + \frac{1}{(m+1)^2}\right] > 0 \qquad \forall \ m \geq 1.$$

Let $C = 2 - \delta$ where $\delta \geq 0$. Then

$$LC + N = 2L + N - \delta L > 2L + N$$

$$= (m+1)^2 - 1 + ((m+1)^4 - 1)\frac{1}{m(m+1)^2} - \frac{4}{5}\frac{(m+1)^5 - 1}{m(m+1)^2}$$

$$= (m+1)^2 - 1 + ((m+1)^4 - 1)\frac{1}{m(m+1)^2} - \frac{4}{5}\frac{(m+1)^4 - 1}{m(m+1)^2} - \frac{4}{5}(m+1)^2$$

$$= \frac{(m+1)^2}{5} - 1 + \frac{1}{5}((m+1)^4 - 1)\frac{1}{m(m+1)^2}$$

$$(2.18) \quad = \frac{(m+1)^2}{5} - \frac{4}{5} + \frac{m+1}{5} + \frac{1}{5(m+1)} + \frac{1}{5(m+1)^2} > \frac{2}{5}.$$

$\square$

We now conclude the proof of Theorem 2.1. Given m, if C is chosen so that

$$\int_1^{m+1} p(\gamma)\gamma d\gamma \geq -2 + \epsilon,$$

(2.19) \qquad i.e., \quad $LC + N \geq -2 + \epsilon,$

then

$$v(\gamma) - v(1) = \int_1^\gamma 2\sqrt{2}\sqrt{v} + \int_1^\gamma p(t)t dt$$

(2.20) \qquad $\Rightarrow v(\gamma) > 2 - 2 + \epsilon = \epsilon.$

This implies that for C satisfying (2.19) (in particular, by Lemma 2.1 $C \leq 2$ satisfies (2.19) for all $m \geq 1$) we have a smooth solution to (2.15), hence to (2.14), on $[1, m+1]$. Now we have to somehow choose a C so that $\phi(m+1) = 0$, i.e., $v(m+1) = 2(m+1)^2$. One possible strategy is to show that there is a C satisfying (2.19) such that $v(m+1, C) < 2(m+1)^2$ and likewise another C for which $v(m+1, C) > 2(m+1)^2$. Thus there will exist a C so that $v(m+1, C) = 2(m+1)^2$.

If C is very negative, then $LC + N$ can be made as large as we want. Thus $v(m+1, C) > 2 + LC + N > 2(m+1)^2$. This completes the proof of Theorem 2.1. $\square$

We proceed further to prove Theorem 1.1. As mentioned earlier, this reduces to choosing C so that $LC + N \geq -2 + \epsilon$ for some $\epsilon > 0$ so that $v(m+1, C) < 2(m+1)^2$. This is a tricky business. Here is where we use the assumption that $m = 1$. For this we need to choose $\delta > 0$ to be very small so that among other things $C = 2 + \delta$ satisfies $LC + N = -\frac{33}{20}$, $A(C) > 0$, and $B(C) < 0$. Upon calculation we have the following:

$$\frac{A}{3} = \frac{\delta}{m}\left[1 - \frac{1}{(m+1)^2}\right] - \frac{4}{m(m+1)^2},$$

$$\frac{B}{2} = -\delta\left[1 + \frac{1}{m} - \frac{1}{m(m+1)^2}\right] + \frac{4}{m(m+1)^2}$$

$$\Rightarrow \text{if } \delta > \frac{4}{(m+1)^2 - 1}, \quad \text{then } A > 0, \ B < 0,$$

$$LC + N = \delta L + \frac{(m+1)^2}{5} - \frac{4}{5} + \frac{m+1}{5} + \frac{1}{5(m+1)} + \frac{1}{5(m+1)^2}$$

$$= \delta \left(\frac{3}{10}((m+1)^2 - 1) - \frac{1}{20}((m+1)^4 - 1) \right.$$

$$\left. - \frac{(m+1)^4 - 1}{20m} \left[1 - \frac{1}{(m+1)^2} \right] \right)$$

$$+ \frac{(m+1)^2}{5} - \frac{4}{5} + \frac{m+1}{5} + \frac{1}{5(m+1)} + \frac{1}{5(m+1)^2}$$

$$= \delta' L, \quad \text{where } \delta = \delta' + \frac{4}{(m+1)^2 - 1}$$

$$\Rightarrow \frac{A}{3} = \frac{\delta'}{m} \left[1 - \frac{1}{(m+1)^2} \right],$$

$$(2.21) \qquad \frac{B}{2} = -\delta' \frac{(m+1)^3 - 1}{m(m+1)^2} - \frac{4}{(m+1)^2 - 1}.$$

Therefore $\delta' = \frac{-33}{20L}$.

We now prove that for $m = 1$ and the chosen value of $C = 2 + \frac{4}{(m+1)^2 - 1} - \frac{1.5}{L} = \frac{22}{3}$, the solution v satisfies $v' > 0$ on $[1, 2]$. Before this we note that $A = 9$ and $B = \frac{50}{3}$. If γ_0 is the root of $p(\gamma)$ on $[1, 2]$, then on $[1, \gamma_0]$ we see that

$$v' \geq 2\sqrt{2}\sqrt{v}$$

$$\Rightarrow (\sqrt{v})' \geq \sqrt{2} \Rightarrow \sqrt{v}(\gamma_0) \geq \sqrt{2} + \sqrt{2}(\gamma_0 - 1) = \sqrt{2}\gamma_0.$$

Therefore, $v' > 0$ on $[1, \gamma_0]$. On the other hand, the root γ_0 in $[1, 2]$ of the polynomial $p(\gamma)\gamma = 3\gamma^4 - \frac{25}{3}\gamma^3 + \frac{22}{3}\gamma$ is clearly larger than 1.2. Therefore $\sqrt{v(\gamma_0)} > 1.2\sqrt{2}$. Moreover, one can also see (by graphing for instance) that $p(\gamma)\gamma > -4.5$ on $[1, 2]$. But $v'(\gamma_0) = 2\sqrt{2}\sqrt{v(\gamma_0)} = 4.8$, and hence when $\gamma > \gamma_0$ we see that $v'(\gamma) > -4.5 + 4.8 = 0.3$. This proves that $v' > 0$ on $[1, 2]$.

As a consequence, for $a, a + h \in [1, 2]$ we see that

$$2\sqrt{2v(a)}h + \int_a^{a+h} p(\gamma)\gamma d\gamma < v(a + h) - v(a) < 2\sqrt{2v(a+h)}h + \int_a^{a+h} p(\gamma)\gamma d\gamma$$

$$\Rightarrow 2\sqrt{2v(a)} + \int_a^{a+h} p(\gamma)\gamma d\gamma \leq v(a + h)$$

$$(2.22) \quad \leq 4h^2 + v(a) + \int_a^{a+h} p(\gamma)\gamma d\gamma + 2h \sqrt{4h^2 + 2 \left(v(a) + \int_a^{a+h} p(\gamma)\gamma d\gamma \right)}.$$

Using inequality (2.22) twice with $h = \frac{1}{2}$ and $a = 1$ we see that $v(2) \leq 7.5 < 2(1+1)^2 = 8$. This proves that for $m = 1$ indeed there exists a C so that $\phi(m+1) = 0$ thus almost proving Theorem 1.1. The only thing left is to prove that there cannot exist any hcscK metrics.

Indeed, if such a metric exists, then there is a solution to (2.14) satisfying $\phi > 0$ (and hence $v > 2\gamma^2$) and $A = 0$. In this case $B = -\frac{12}{(m+1)^2 - 1}$ and $C = 4 + \frac{8}{(m+1)^2 - 1}$.

This implies that $\int_1^{m+1} p(\gamma)\gamma d\gamma = 2$. Therefore,

$$v(m+1) = 4 + \int_1^{m+1} 2\sqrt{2v} d\gamma > 4 + 4 \int_1^{m+1} \gamma d\gamma$$

$$= 4 + 2((m+1)^2 - 1) > 2(m+1)^2.$$

This is a contradiction.

3. Perturbation results

In this section we prove Theorem 1.2. Let (X, ω) be a compact Kähler surface, let $\tilde{\omega}$ be any closed real $(1,1)$-form, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be spaces of $C^{4,\alpha}$ functions on X with zero average and $C^{0,\alpha}$ $(2,2)$-forms on X with zero average respectively. Denote by U an open subset of $\mathbb{R} \times \mathcal{B}_1$ consisting of $(t, \phi) \in \mathbb{R} \times \mathcal{B}_1$ such that $\omega + t\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\phi > 0$. Consider the following map $L : U \to \mathcal{B}_2$:

$$(3.1) \qquad L(t, \phi) = c_2(\omega + t\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\phi) - \frac{\displaystyle\int_X c_2}{\displaystyle\int_X (\omega + t\tilde{\omega})^2} (\omega + t\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\phi)^2.$$

Clearly $L^{-1}(0)$ consists of hcscK metrics in the Kähler class $[\omega + t\tilde{\omega}]$. Assume now that ω is an hcscK metric satisfying $c_2(\omega) = \frac{\lambda}{2(2\pi)^2}\omega^2$. In order to apply the implicit function theorem on Banach manifolds, we will linearise L with respect to ϕ at $\phi = 0, t = 0$. Indeed,

$$(3.2) \qquad DL_{t=0,\phi=0}(\psi) = \frac{d}{ds}|_{s=0} c_2(\omega + s\sqrt{-1}\partial\bar{\partial}\psi) - \frac{\lambda}{(2\pi)^2}\omega\sqrt{-1}\partial\bar{\partial}\psi.$$

We have a small lemma in the making.

Lemma 3.1. *The linearisation DL given by equation (3.2) is uniformly elliptic in ψ if the holomorphic sectional curvature has a definite sign throughout X.*

Proof. Let $P(A, B)$ be the polarisation of the determinant of 2×2 matrices A and B; i.e., if A and B are thought of as 2-forms, then $P(A, B) = \frac{A \wedge B}{2}$. Proposition 6 of [6] states (in this special case) that there exists a smoothly varying family of Bott-Chern forms $bc_2(h, k)$ such that the following holds:

$$c_2(\omega + s\sqrt{-1}\partial\bar{\partial}\psi) - c_2(\omega) = -\frac{\sqrt{-1}\partial\bar{\partial}}{2\pi} bc_2(\omega + s\partial\bar{\partial}\psi, \omega), \text{ and}$$

$$\frac{d}{ds} bc_2(\omega + s\sqrt{-1}\partial\bar{\partial}\psi, \omega) = -2\sqrt{-1}P\left(h^{-1}\frac{dh}{ds}, \frac{\sqrt{-1}}{2\pi}\Theta_h\right),$$

where $h = \omega + s\sqrt{-1}\partial\bar{\partial}\psi$ and Θ_h is the curvature of h. Using this result, we may compute the linearisation of L to be the following:

$$(3.3) \quad DL_{\phi=0,t=0}(\psi) = -2\frac{1}{(2\pi)^2}\bar{\partial}\partial P\left(\omega^{i\bar{k}}\sqrt{-1}\frac{\partial^2\psi}{dz^j d\bar{z}^k}, \Theta\right) - \frac{\lambda}{(2\pi)^2}\omega\sqrt{-1}\partial\bar{\partial}\psi,$$

where Θ is the curvature of ω. In order to find the principal symbol let us choose coordinates such that $\omega = \sqrt{-1}\sum dz^i \wedge d\bar{z}^i$. Replacing ∂ by a covector ξ we see that the prinicipal symbol is $\frac{2}{(2\pi)^2}\Theta(\xi \wedge \bar{\xi}, \xi \wedge \bar{\xi})dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2$, which is just the holomorphic sectional curvature. Hence, it having a definite sign (along with compactness of X) implies uniform ellipticity. $\square$

From now on we will specialise to (X, ω) being one of the symmetric surfaces in the statement of Theorem 1.2. In the cases considered in Theorem 1.2 the holomorphic sectional curvature has a sign, and hence by Lemma 3.1 equation (3.3) is uniformly elliptic of the fourth order. By the Fredholm alternative, it is surjective if and only if the kernel of its formal adjoint operator is trivial. It is easy to see that $DL_{\phi=0,t=0}$ is symmetric on the space of smooth functions. The following lemma implies that DL is an isomorphism.

Lemma 3.2. *If (X, ω) is a Kähler surface in Theorem* 1.2*, then the kernel of $DL_{\phi=0}$ is trivial.*

Proof. Suppose $DL(\psi) = 0$. Multiplying and integrating by parts we see that (implicitly writing in terms of normal coordinates)

$$(3.4) \qquad 2 \int_X \partial\bar{\partial}\psi \wedge P(\psi_{i\bar{j}}, \Theta) + \lambda \int_X \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega = 0.$$

For the surfaces in question it is clear that $\lambda > 0$. Suppose we choose normal coordinates such that $\psi_{i\bar{j}} = diag(\mu_1, \mu_2)$. Then

$$\partial\bar{\partial}\psi \wedge P(\psi_{i\bar{j}}, \Theta) = \sum \mu_i dz^i \wedge d\bar{z}^i \wedge \frac{\mu_1 \Theta_{2\bar{2}} + \mu_2 \Theta_{1\bar{1}}}{2}$$

$$= -(\mu_1^2 \Theta_{2\bar{2}2\bar{2}} + 2\mu_1\mu_2 \Theta_{1\bar{1}2\bar{2}} + \mu_2^2 \Theta_{1\bar{1}1\bar{1}})\sqrt{-1}^2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2$$

$$(3.5) \qquad = -\Theta\left(\sum \mu_i \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial \bar{z}_i}, \sum \mu_i \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial \bar{z}_i}\right)\sqrt{-1}^2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2.$$

For $X = \mathbb{D}^2/\Gamma$ and $X = \mathbb{D}^1/\Gamma_1 \times \mathbb{D}^1/\Gamma_2$ equipped with their "canonical" metrics, the curvature operator is non-positive. Hence $\nabla\psi = 0$, and thus $\psi = 0$. $\qquad\square$

By the Fredholm alternative and the Schauder estimates DL is indeed an isomorphism. Therefore by the implicit function theorem on Banach spaces, for small t there exists a unique hcscK metric in a $C^{2,\alpha}$ neighbourhood of ω in the class $[\omega + t\tilde{\omega}]$ depending smoothly on t. In particular, for some ball quotients we can choose $\tilde{\omega}$ to be in a cohomology class that is not a multiple of the first Chern class and therefore get a non-Kähler-Einstein example of an hcscK metric.

4. Bando-Futaki invariants of projective hypersurfaces

Let M be a compact Kähler manifold. The Bando-Futaki invariants associated to a given Kähler class ω and a given holomorphic vector field Y (henceforth denoted as $\mathcal{F}_k(Y, \omega)$) are obstructions to the harmonicity of the Chern forms c_k of the holomorphic tangent bundle. By Hodge theory there exists a smooth function g_k such that

$$c_k - H(c_k) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}g_k,$$

where $H(c_k)$ is the harmonic projection of c_k. The Bando-Futaki invariants are defined as

$$\mathcal{F}_k(Y, \omega) = \int_M L_Y g_k \wedge \omega^{n-k+1},$$

where L_Y is the Lie derivative with respect to Y.

The fact that these functions are actually invariants of the Kähler class was proven by Bando [2]. In Liu's paper [12] these invariants were computed for a smooth, degree d hypersurface M of $\mathbb{CP}^n$ for the Fubini-Study Kähler class. Liu

speculated that an "abstraction" of the procedure used is desirable (in order to compute the same for complete intersections). We simplify some aspects of Liu's proof (whilst following the same basic strategy), thus providing a possible abstraction of that method.

An important tool in our calculations is the following linear algebra lemma, which has proven to be quite useful in the calculation of characteristic forms [14].

Lemma 4.1. *Let A be a matrix over $\mathbb{C}$ or over a commutative algebra $\mathcal{A}$ over $\mathbb{C}$, where in the latter case all its matrix elements are nilpotent. Suppose that $A^2 = aA$ for some $a \in \mathcal{A}$ and that $1 - \lambda a$ is invertible for all λ in some domain $D \subset \mathbb{C}$ containing 0. Then for such λ we have*

$$(I - \lambda A)^{-1} = I + \frac{\lambda}{1 - \lambda a} A$$

and

$$\det(I - \lambda A) = \exp\left\{\frac{\operatorname{Tr} A}{a} \log(1 - \lambda a)\right\}.$$

In particular, if α_i, β_i, $i = 1, \ldots, k$, are odd elements in some graded-commutative algebra over $\mathbb{C}$ (e.g., the algebra of complex differential forms on X) and $A_{ij} = \alpha_i \beta_j$, then $A^2 = aA$ where $a = -\operatorname{Tr} A = -\sum_{i=1}^{k} \alpha_i \beta_i$, and

$$\det(I - \lambda A) = \frac{1}{1 - \lambda a}.$$

Proof. For $\lambda \in D$ we have

$$(I - \lambda A)^{-1} = I + \frac{\lambda}{1 - \lambda a} A.$$

To prove the formula for the determinant, we use the identity

$$\frac{d}{d\lambda} \log \det(I - \lambda A) = -\operatorname{Tr}\left\{A(I - \lambda A)^{-1}\right\}, \quad \lambda \in D.$$

It is well-known for matrices over $\mathbb{C}$ (and easily proved using the Jordan canonical form), and for matrices with nilpotent entries it easily follows from the definition of the determinant. Using the formula for the inverse, we obtain

$$\frac{d}{d\lambda} \log \det(I - \lambda A) = -\frac{\operatorname{Tr} A}{1 - \lambda a} = \frac{d}{d\lambda} \frac{\operatorname{Tr} A}{a} \log(1 - \lambda a),$$

and integrating from 0 to λ using $\det I = 1$ gives the result. $\square$

Our starting point of Liu's formula is the expression for the curvature of the induced metric on the hypersurface M defined by $F(Z_0, Z_1, \ldots, Z_n) = 0$ where F is a homogeneous polynomial with non-zero gradient. On the set where $Z_0 \neq 0$, define the complex coordinates $z_i = \frac{Z_i}{Z_0}$ for $i \geq 1$. Defining $f = F[1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}]$, if $\frac{\partial f}{\partial z_1} \neq 0$, then by the implicit function theorem z_1 is a holomorphic function of the other coordinates. Let $a_i = \frac{\partial z_1}{\partial z_i}$, $\widetilde{g}$ be the metric on M induced by the Fubini-study metric $\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} \left(\frac{\delta_{ij}}{1+|z|^2} - \frac{z_i \bar{z}_j}{(1+|z|^2)^2}\right) dz_i \wedge d\bar{z}_j$, $F_k = \frac{\partial F}{\partial Z_k}$, and

$\rho = \frac{\sum_{k=0}^{k=n}|F_k|^2}{(1+|z|^2)|F_1|^2}$. It is easy to see that[2]

$$\widetilde{g_{\mu\nu}} = \frac{\delta_{\mu\nu} + a_\mu \bar{a}_\nu}{1 + |z|^2} - \frac{(\bar{z}_\mu + \bar{z}_1 a_\mu)(z_\nu + z_1 \bar{a}_\nu)}{(1 + |z|^2)^2},$$

$$\Theta_{\mu\nu} = \widetilde{g_{ij}} dz_i \wedge d\bar{z}_j \delta_{\mu\nu} - \widetilde{g_{\mu j}} d\bar{z}_j \wedge dz_\nu - \frac{1}{\rho}(\frac{\partial a_\mu}{\partial z_i} dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j} \widetilde{g^{\nu s}} d\bar{z}_j).$$

Now, we shall state and prove Lemma 2.3 of [12].

Lemma 4.2. *The qth Chern form of the degree d hypersurface M is*

$$c_q(\Theta) = \sum_{k=0}^{q} \alpha_{qk} (\frac{\sqrt{-1}}{2\pi}\omega)^k \wedge (\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\xi)^{q-k},$$

where $\xi = \log(\frac{\sum_{k=0}^{n}|F_k|^2}{(\sum_{k=0}^{k=n}|Z_k|^2)^{d-1}})$, and

$$\alpha_{00} = 1,$$
$$\alpha_{qq} = \binom{n+1}{q} - d\alpha_{(q-1)(q-1)},$$
$$\alpha_{q(q-k)} = -[d\alpha_{(q-1)(q-k-1)} + \alpha_{(q-1)(q-k)}] \quad \text{for } k = 1, \ldots, q-1,$$
$$\alpha_{q0} = (-1)^q,$$

where q ranges from 1 to $n-1$.

Proof. We use Lemma 4.1 quite often in what follows. For the sake of brevity we denote $a \wedge b$ by ab from now on:

$$\Theta_{ij} = \omega\delta_{ij} + v_i w_j + \alpha_i \beta_j,$$
$$\det(I + t\Theta) = \det(\delta_{ij}(1 + t\omega) + t(v_i w_j + \alpha_i \beta_j))$$
$$= (1 + t\omega)^{n-1} \det(\delta_{ij} + \frac{t}{1 + t\omega}(v_i w_j + \alpha_i \beta_j))$$

$$= (1 + t\omega)^{n-1} \det(\delta_{ij} + \frac{t}{1 + t\omega}v_i w_j) \times \det(\delta_{ij} + (\delta_{ab} + \frac{t}{1 + t\omega}v_a w_b)^{-1}\frac{t}{1 + t\omega}\alpha_i \beta_j)$$
$$= (1 + t\omega)^{n-1} \det(\delta_{ij} + \lambda v_i w_j) \det(I + \lambda A),$$

where $\omega = \widetilde{g}_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu$, $v_\mu = -\widetilde{g}_{\mu j} d\bar{z}_j$, $w_\nu = dz_\nu$, $\alpha_\mu = -\frac{1}{\rho}\frac{\partial a_\mu}{\partial z_i} dz_i$, $\beta_\nu = \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\widetilde{g}^{\nu s} d\bar{z}_j$, $\lambda = \frac{t}{1+t\omega}$, $u_i = \frac{t}{(1+t\omega)+tw_j \wedge v_j}v_i$, and $A_{ij} = (\delta_{ab} + \frac{t}{1+t\omega}v_a w_b)^{-1}\alpha_i \beta_j$.

Now notice that $A^2 = (\beta_i \alpha_i - \beta_j u_j w_k \alpha_k)A = -tr(A)A$. Using Lemma 4.1, we see that

$$\det(1 + t\Theta) = (1 + t\omega)^{n+1}\frac{1}{1 + t\omega + \frac{t}{\rho}\frac{\partial a_\mu}{\partial z_i} dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\widetilde{g}^{\mu s} d\bar{z}_j}.$$

From [12] we see that $\frac{1}{\rho}\frac{\partial a_\mu}{\partial z_i} dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\widetilde{g}^{\mu s} d\bar{z}_j = (d-1)\omega + \partial\bar{\partial}\xi$. Hence, we see that the coefficient of t^k in the above expression is

$$c_a(\Theta) = \sum_b \sum_l \binom{n+1}{b} d^l \omega^{b+l} (-1)^{a-b} \binom{a-b}{l} (\partial\bar{\partial}\xi)^{a-b-l}$$

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[2]Either using [12] or by noting that one may compute the inverse of the metric and hence the curvature by using Lemma 4.1.

$$= \sum_{k=0}^{a}\sum_{b=0}^{k}\binom{n+1}{b}d^{k-b}\omega^{k}(-1)^{a-b}\binom{a-b}{k-b}(\partial\bar{\partial}\xi)^{a-k}.$$

From this, the lemma follows. $\square$

At this juncture we may compute the Bando-Futaki invariants using a generating series version of Liu's approach. Our basic strategy of proof is the same as Liu's, in that we shall not compute the invariant directly. Instead, we observe that $i_Y(c(\Theta)) - i_Y(H(c(\Theta))) - \bar{\partial}(i_Y(\partial f)) = 0$ (where c and f are the Chern and the Futaki polynomials respectively). This shall be rewritten as $\bar{\partial}\eta = 0$, and after that we shall find the harmonic part of η to finally compute the integral. In the course of the proof we use Lemma 4.1 repeatedly.

Proof of Theorem 1.3. First, we recall that $\det(I + t\Theta) = \frac{(1+t\omega)^{n+1}}{1+t(\omega d + \partial\bar{\partial}\xi)}$. The harmonic part of the same may be obtained by putting $\xi = 0$. Hence,

$$\begin{aligned} c - Hc &= (1+t\omega)^{n+1}\left(\frac{1}{1+t(\omega d + \partial\bar{\partial}\xi)} - \frac{1}{1+t\omega d}\right) \\ &= -t\partial\bar{\partial}\left(\frac{\xi(1+t\omega)^{n+1}}{(1+t(\omega d + \partial\bar{\partial}\xi))(1+t\omega d)}\right) \\ &= \partial\bar{\partial}f. \end{aligned}$$

In what follows, θ is the "Hamiltonian" function [12] such that $i_Y\omega = -\bar{\partial}\theta$. We shall use the fact that i_Y is a derivation (and hence the quotient and the product rules for derivatives may be used when interpreted suitably):

$$\begin{aligned} i_Y(Hc) &= \frac{(n+1)(1+t\omega)^n t i_Y(\omega)(1+t\omega d) - t i_Y(\omega)d(1+t\omega)^{n+1}}{(1+t\omega d)^2} \\ &= \bar{\partial}\left(\frac{t(1+t\omega)^n\theta(d-(n+1)-nt\omega d)}{(1+t\omega d)^2}\right) \\ &= \bar{\partial}\alpha_2, \end{aligned}$$

$$(I+t\Theta)^{-1} = \frac{1}{1+t\omega}\left(\delta_{ij} + \frac{t}{1+t\omega}(v_i w_j + \alpha_i\beta_j)\right)^{-1}$$

$$= \frac{1}{1+t\omega}\left(\delta_{ij} + \frac{t}{1+t\omega}v_i w_j\right)^{-1}\left(\delta_{ij} + \frac{t}{1+t\omega}\left(\left(\delta_{ab} + \frac{t}{1+t\omega}v_a w_b\right)^{-1}\right)_{ik}\alpha_k\beta_j\right)^{-1}.$$

Using Lemma 4.1 and noticing that $w_k v_k = -\omega$ and $\beta_k\alpha_k = (d-1)\omega + \partial\bar{\partial}\xi$ we see that

$$((I+t\Theta)^{-1})_{ab} = \frac{1}{1+t\omega}(\delta_{ac} - tv_a w_c)\left(\delta_{cb} - \frac{t(\alpha_c\beta_b - tv_c w_k\alpha_k\beta_b)}{1+t\omega d + t\partial\bar{\partial}\xi - t^2\beta_k v_k w_l\alpha_l}\right),$$

$$\begin{aligned} i_Y(c) &= \det(I+t\Theta)\mathrm{tr}(t i_Y(\Theta)(I+t\Theta)^{-1}) \\ (4.1) &\qquad = -t\bar{\partial}(\det(I+t\Theta)\mathrm{tr}(\nabla Y(I+t\Theta)^{-1})). \end{aligned}$$

We use the following equations from [12]:

$$\begin{aligned} (\nabla Y)_k^l &= -\tilde{g}^{lj}\partial_k\bar{\partial}_j\theta, \\ \Phi &= -\frac{1}{\rho}Y_{;k}^l\frac{\partial a_l}{\partial z^p}\frac{\partial\bar{a}_s}{\partial\bar{z}^q}\tilde{g}^{k\bar{s}}dz^p \wedge d\bar{z}^q \\ &= \mathrm{div}(Y)((d-1)\omega + \partial\bar{\partial}\xi) - \partial\bar{\partial}\theta + \partial\bar{\partial}\Delta\theta \\ &\quad - (n+1)\theta((d-1)\omega + \partial\bar{\partial}\xi). \end{aligned}$$

Upon simplification of equation (4.1) (recall that since $a_i = \frac{\partial z_1}{\partial z_i}$, $w_k \alpha_k = 0$) we get

$$i_Y(c) = -t\bar{\partial}\left(\frac{(1+t\omega)^n}{1+t\omega d + t\partial\bar{\partial}\xi}(\operatorname{div}(Y) + t\bar{\partial}\partial\theta - \frac{t\Phi}{1+t\omega d + t\partial\bar{\partial}\xi})\right)$$
$$= \bar{\partial}\alpha_1,$$

$$i_Y(\partial f) = t\frac{-Y(\xi)(1+t\omega)^{n+1}}{(1+t(\omega d + \partial\bar{\partial}\xi))(1+t\omega d)}$$
$$- \frac{t^2(1+t\omega)^{n+1}\partial\xi}{(1+t(\omega d + \partial\bar{\partial}\xi))^2(1+t\omega d)^2}\bar{\partial}[\theta((n+1-d+nt\omega d)(1+t(\omega d + \partial\bar{\partial}\xi))$$
$$- (1+t\omega)(1+t\omega d)d) + Y(\xi)((1+t\omega)(1+t\omega d))].$$

It is easy to see that for an appropriate form γ, we have

$$\alpha_1 - \alpha_2 - i_Y(\partial f) = \bar{\partial}\gamma + \frac{t(1+t\omega)^n}{(1+t\omega d)^2}\theta(nt\omega d + n + 1 - d) - i_Y(\partial f)$$

$$-t\frac{(1+t\omega)^n}{1+t(\omega d + \partial\bar{\partial}\xi)^2}(\operatorname{div}(Y)(1+t\omega) + (n+1)t\theta((d-1)\omega + \partial\bar{\partial}\xi)).$$

We shall use this identity [13]:

$$\operatorname{div}(Y) - Y(\xi) - (n - d + 1)\theta = -\kappa.$$

Replacing $\operatorname{div}(Y)$ by the above identity and simplifying we have

$$\alpha_1 - \alpha_2 - i_Y(\partial f) = \bar{\partial}\gamma + t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d + \partial\bar{\partial}\xi))^2}$$

$$-\bar{\partial}\left(\frac{t^2(1+t\omega)^{n+1}\partial\xi}{(1+t(\omega d + \partial\bar{\partial}\xi))^2(1+t\omega d)}\right.$$

$$\times\left.\left[\frac{\theta((n+1-d+nt\omega d)(1+t(\omega d + \partial\bar{\partial}\xi)) - (1+t\omega)(1+t\omega d)d)}{1+t\omega d} + (1+t\omega)Y(\xi)\right]\right).$$

Thus, the harmonic part is $t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d + \partial\bar{\partial}\xi))^2}$. Notice that (the integral of a non-top form is defined to be zero)

$$\int_M L_Y f \wedge \frac{1}{1-\omega} = \int_M (di_Y + i_Y\partial)f \wedge \frac{1}{1-\omega}$$

$$= \int_M (\alpha_1 - \alpha_2 - t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d + \partial\bar{\partial}\xi))^2}) \wedge \frac{1}{1-\omega}$$

$$= \int_M (\alpha_1 - \frac{t(1+t\omega)^n\theta(d-(n+1)-nt\omega d)}{(1+t\omega d)^2} - t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t\omega d)^2}) \wedge \frac{1}{1-\omega},$$

where Stokes' theorem was used to deduce that $\int_M di_Y f \wedge \frac{1}{1-\omega} = 0$, to replace $1 + t(\omega d + \partial\bar{\partial}\xi)$ by $1 + t\omega d$, and to ignore the integral of the anharmonic part of $\alpha_1 - \alpha_2 - i_Y(\partial f)$.

From Lemma 2.6 of [12], it follows that $\int_M \frac{\alpha_1}{1-\omega} = 0$. After replacing t by $\frac{\sqrt{-1}}{2\pi}$, one may easily compute the integral using the facts that $\int_M \theta\omega^{n-1} = \frac{\kappa}{n}$ and $\int \omega^{n-1} = d$. This completes the proof. $\square$

Acknowledgments

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