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A RIGIDITY THEOREM ON THE SECOND FUNDAMENTAL FORM FOR SELF-SHRINKERS

QI DING

Abstract. In Theorem 3.1 of *The rigidity theorems of self-shrinkers* (2014), the author and Y. L. Xin proved a rigidity result for self-shrinkers under the integral condition on the norm of the second fundamental form. In this paper, we relax such a bound to any finite constant (see Theorem 4.4 for details).

1. Introduction

Self-similar solutions for mean curvature flow play a key role in understanding the possible singularities that the flow goes through. Self-shrinkers are type I singularity models of the flow. Huisken made pioneer work on self-shrinking solutions of the flow [22, 23]. Colding and Minicozzi [8] gave a comprehensive study for self-shrinking hypersurfaces and solved a long-standing conjecture raised by Huisken.

Colding–Ilmanen–Minicozzi [9] showed that cylindrical self-shrinkers are rigid in a very strong sense. Namely, any other shrinker that is sufficiently close to one of them on a large but compact set, must itself be a round cylinder. See [25] by Guang–Zhu for further results. Lu Wang in [37, 38] proved strong uniqueness theorems for self-shrinkers asymptotic to regular cones or generalized cylinders of infinite order.

For Bernstein type theorems, Ecker–Huisken [17] and Wang [36] showed the nonexistence of nontrivial graphic self-shrinking hypersurfaces in Euclidean space. For $2 \leq n \leq 6$, Guang–Zhu showed that any smooth complete self-shrinker in $\mathbb{R}^{n+1}$ which is graphical inside a large, but compact, set must be a hyperplane. Ding–Xin–Yang [14] studied the sharp rigidity theorems with the condition on Gauss map of self-shrinkers. In high codimensions, see [2, 3, 10, 13, 26] for more Bernstein type theorems.

Le-Sesum [30] showed that any complete embedded self-shrinking hypersurface with polynomial volume growth must be a hyperplane provided the squared norm of the second fundamental form $|B|^2 < \frac{1}{2}$. Cao–Li [1] showed that any complete self-shrinker (with high codimension) with polynomial volume growth must be a generalized cylinder provided $|B|^2 \leq \frac{1}{2}$. Later, Cheng–Peng [5] removed the condition of polynomial volume growth in the case of $|B|^2 < \frac{1}{2}$ (see [4, 6, 12, 42] for more results on the gap theorems of the norm of the second fundamental form). In [12], Ding–Xin proved a rigidity result for self-shrinkers if the integration of $|B|^n$ is small. In this paper, we improve the small constant to any finite constant.

For a complete properly immersed self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$, Ilmanen showed that there exists a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with the cross section being a compact set in

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$\mathbb{S}^n$ such that $\lambda\Sigma^n \to \mathcal{C}$ as $\lambda \to 0_+$ locally in the Hausdorff metric on closed sets (see [28, Lecture 2, B, remark on p. 8]). In [35], Song gave a simple proof by a "maximum principle for self-shrinkers". For high codimensions, with backward heat kernel (see [8]) we show the uniqueness of tangent cones at infinity for self-shrinkers with Euclidean volume growth in the current sense with the condition on mean curvature (see Theorem 3.3).

ϵ-regularity theorems for the mean curvature flow have been studied by Ecker [15,16], Han-Sun [19], Ilmanen [27], Le-Sesum [29]. Now we use the one showed by Ecker [16] starting from self-similar solutions, and obtain the curvature estimates for self-shrinkers, see Theorem 4.2. Combining Theorem 3.3, Theorem 4.2 and backward uniqueness for parabolic operators [18], we can show that self-shrinkers with finite integration on $|B|^n$ must be planes, which improves a previous rigidity theorem in [12]. A litter more, we obtain the following Theorem.

Theorem 1.1. *Let M be an n-dimensional properly noncompact self-shrinker with compact boundary in $\mathbb{R}^{n+m}$, and let B denote the second fundamental form of M. If*

$$\lim_{r\to\infty}\int_{M\cap B_{2r}\setminus B_r}|B|^n d\mu = 0, \tag{1.1}$$

M is contained in an n-plane through the origin.

2. Preliminary

Let M be an n-dimensional C^2-submanifold in $\mathbb{R}^{n+m}$ with the induced metric. Let ∇ and $\overline{\nabla}$ be the Levi–Civita connections on M and $\mathbb{R}^{n+m}$, respectively. We define the second fundamental form B of M by

$$B(V,W) = (\overline{\nabla}_V W)^N = \overline{\nabla}_V W - \nabla_V W$$

for any $V, W \in \Gamma(TM)$, where the mean curvature vector H of M is given by $H = \mathrm{trace}(B) = \sum_{i=1}^n B(e_i, e_i)$, where $\{e_i\}$ is a local orthonormal frame field of M.

In this paper, M^n is said to be a *self-shrinker* in $\mathbb{R}^{n+m}$ if its mean curvature vector satisfies

$$H = -\frac{X^N}{2}, \tag{2.1}$$

where $X = (x_1, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ is the position vector of M in $\mathbb{R}^{n+m}$, and $(\cdots)^N$ stands for the orthogonal projection into the normal bundle NM. Let $(\cdots)^T$ denote the orthogonal projection into the tangent bundle TM.

We define a second-order differential operator $\mathcal{L}$ as in [8] by

$$\mathcal{L}f = e^{\frac{|X|^2}{4}}\mathrm{div}\left(e^{-\frac{|X|^2}{4}}\nabla f\right) = \Delta f - \frac{1}{2}\langle X, \nabla f\rangle$$

for any $f \in C^2(M)$. Let Δ be the Laplacian of M, then for self-shrinkers,

$$\Delta|X|^2 = 2\langle X, \Delta X\rangle + 2|\nabla X|^2 = 2\langle X, H\rangle + 2n = -|X^N|^2 + 2n. \tag{2.2}$$

In [8], Colding and Minicozzi defined a function F_{X_0,t_0} for self-shrinking hypersurfaces in Euclidean space. Obviously, hypersurfaces can be generalized to submanifolds naturally in this definition. Set $\Phi_t \in C^\infty(\mathbb{R}^{n+m})$ for any $t > 0$ by

$$\Phi_t(X) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|X|^2}{4t}}.$$

For an n-complete submanifold M in $\mathbb{R}^{n+m}$, we define a functional F_t on M by

$$F_t(M) = \int_M \Phi_t d\mu = \frac{1}{(4\pi t)^{n/2}} \int_M e^{-\frac{|X|^2}{4t}} d\mu \quad \text{for} \quad t > 0,$$

where $d\mu$ is the volume element of M. Sometimes, we write F_t for simplicity if there is nothing ambiguous in the text. If a self-shrinker is proper, then it is equivalent to the fact that it has Euclidean volume growth at most by [7] and [11]. We shall only consider proper self-shrinkers in the following text.

Now we use the backward heat kernel to give a monotonicity formula for self-shrinkers with arbitrary codimensions, which is essentially the same as the self-shrinking hypersurfaces established by Colding–Minicozzi in [8].

Lemma 2.1. *For any $0 < t_1 \le t_2 \le \infty$, each complete immersed self-shrinker M^n with boundary ∂M (may be empty) in $\mathbb{R}^{n+m}$ satisfies*

$$\begin{aligned} F_{t_2}(M) - F_{t_1}(M) = &- \int_{t_1}^{t_2} \left(\int_{\partial M} \langle X^T, \nu_{\partial M}\rangle \frac{\Phi_s(X)}{2s} \right) ds \\ &+ \int_{t_1}^{t_2} \frac{1}{4s}\left(1 - \frac{1}{s}\right) \left(\int_M |X^N|^2 \Phi_s(X) d\mu \right) ds. \end{aligned} \tag{2.3}$$

Proof. We differentiate $F_t(M)$ with respect to t,

$$F_t' = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \left(-\frac{n}{2} + \frac{|X|^2}{4t} \right) e^{-\frac{|X|^2}{4t}} d\mu. \tag{2.4}$$

A straightforward calculation shows (see also [11])

$$\begin{aligned} -e^{\frac{|X|^2}{4t}} \operatorname{div}\left(e^{-\frac{|X|^2}{4t}} \nabla |X|^2 \right) =& - \Delta|X|^2 + \frac{1}{4t} \nabla|X|^2 \cdot \nabla|X|^2 \\ =& - 2\langle H, X\rangle - 2n + \frac{1}{t}|X^T|^2 \\ =& |X^N|^2 + \frac{|X^T|^2}{t} - 2n \\ =& \left(1 - \frac{1}{t}\right)|X^N|^2 + \frac{|X|^2}{t} - 2n, \end{aligned} \tag{2.5}$$

where the third equality above uses the self-shrinkers' equation (2.1). Then

(2.6)
$$\begin{aligned}F_t' =&(4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+1)}\int_M\left(-\frac{1}{4}\mathrm{div}\left(e^{-\frac{|X|^2}{4t}}\nabla|X|^2\right)-\frac{1}{4}\left(1-\frac{1}{t}\right)|X^N|^2e^{-\frac{|X|^2}{4t}}\right)d\mu\\ =&\frac{1}{4}(4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+1)}\left(-2\int_{\partial M}\langle X^T,\nu_{\partial M}\rangle e^{-\frac{|X|^2}{4t}}-\left(1-\frac{1}{t}\right)\int_M|X^N|^2e^{-\frac{|X|^2}{4t}}d\mu\right)\\ =&-\frac{1}{2t}\int_{\partial M}\langle X^T,\nu_{\partial M}\rangle\Phi_t(X)-\frac{1}{4t}\left(1-\frac{1}{t}\right)\int_M|X^N|^2\Phi_t(X)d\mu,\end{aligned}$$

where $\nu_{\partial M}$ is the normal vector of ∂M in $\Gamma(TM)$. Then we complete the proof by integration from t_1 to t_2. □

Denote

$$\begin{aligned}G_t(M)\triangleq& F_t'(M)+\frac{1}{2t}\int_{\partial M}\langle X^T,\nu_{\partial M}\rangle\Phi_t(X)\\ =&-\frac{1}{4t}\left(1-\frac{1}{t}\right)\int_M|X^N|^2\Phi_t(X)d\mu.\end{aligned}\tag{2.7}$$

The above lemma implies $G_t(M)\le 0$ for each self-shrinker and $t\ge 1$. If ∂M is bounded and has finite $(n-1)$-dimensional Hausdorff measure, then the limit

$$\lim_{t\to\infty}\left(\int_1^t G_s(M)ds\right)$$

always exists and is a finite negative number. Hence, it's clear that $\lim_{t\to\infty}F_t(M)$ exists.

3. Uniqueness of tangent cones at infinity for self-shrinkers

For any n-rectifiable varifold $V\subset\mathbb{R}^{n+m}$, we define a functional Ξ_t by

$$\Xi_t(V,f)=\frac{1}{(4\pi t)^{n/2}}\int_{\mathrm{spt}V}fe^{-\frac{|X|^2}{4t}}d\mu_V$$

for any $t>0$, where μ_V is a measure on $\mathbb{R}^{n+m}$ associated with the Radon measure of V in $\mathbb{R}^{n+m}\times G(n,n+m)$.

We suppose that M is a self-shrinker in $\mathbb{R}^{n+m}\setminus B_R$ with boundary $\partial M\subset\partial B_R$ for some $R\ge 1$ and $\mathcal{H}^{n-1}(\partial M)<\infty$. Let $\phi\in C^1(\mathbb{R}^{n+m}\setminus\{0\})$ be a homogeneous function of degree zero. Namely, for any $0\ne X\in\mathbb{R}^{n+m}$,

$$\phi(X)=\phi(|X|\xi)=\phi(\xi)$$

with $\xi=\frac{X}{|X|}$. Then

$$\partial_{x_i}\phi=\sum_j\left(\frac{\delta_{ij}}{|X|}-\frac{x_ix_j}{|X|^3}\right)\partial_{\xi_j}\phi\tag{3.1}$$

and

$$|\overline{\nabla}\phi|^2=\sum_{j,k}\left(\frac{\delta_{jk}}{|X|^2}-\frac{x_jx_k}{|X|^4}\right)\partial_{\xi_j}\phi\partial_{\xi_k}\phi\le|X|^{-2}\sum_j\left(\partial_{\xi_j}\phi\right)^2\triangleq|X|^{-2}|\phi|_1^2.\tag{3.2}$$

Taking the derivative of $\Xi_t(M,\phi)$ on t obtains

$$
\begin{aligned}
&\partial_t \Xi_t(M,\phi) = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \left(-\frac{n}{2} + \frac{|X|^2}{4t}\right) \phi e^{-\frac{|X|^2}{4t}} d\mu \\
=&(4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \left(-\frac{\phi}{4}\mathrm{div}\left(e^{-\frac{|X|^2}{4t}}\nabla|X|^2\right) - \frac{\phi}{4}\left(1-\frac{1}{t}\right)|X^N|^2 e^{-\frac{|X|^2}{4t}}\right) d\mu.
\end{aligned} \tag{3.3}
$$

Combining $X \cdot \overline{\nabla}\phi = 0$, we have

$$
\begin{aligned}
&\int_M -\frac{\phi}{4}\mathrm{div}\left(e^{-\frac{|X|^2}{4t}}\nabla|X|^2\right) d\mu \\
=&\int_M -\frac{1}{4}\mathrm{div}\left(\phi e^{-\frac{|X|^2}{4t}}\nabla|X|^2\right) d\mu + \int_M \frac{1}{4}\nabla\phi\cdot\nabla|X|^2 e^{-\frac{|X|^2}{4t}} d\mu \\
=&-\frac{1}{2}\int_{\partial M} \phi\langle X^T, \nu_{\partial M}\rangle e^{-\frac{|X|^2}{4t}} + \int_M \frac{1}{2} X\cdot\nabla\phi e^{-\frac{|X|^2}{4t}} d\mu \\
=&-\frac{1}{2}\int_{\partial M} \phi\langle X^T, \nu_{\partial M}\rangle e^{-\frac{R^2}{4t}} - \frac{1}{2}\int_M X^N\cdot\overline{\nabla}\phi e^{-\frac{|X|^2}{4t}} d\mu.
\end{aligned} \tag{3.4}
$$

Set $c_R = 2^{-1}(4\pi)^{-\frac{n}{2}} R\cdot\mathcal{H}^{n-1}(\partial M)$. Substituting (3.2) and (3.4) into (3.3) obtains

$$
\begin{aligned}
&|\partial_t \Xi_t(M,\phi)| \le 2^{-1}(4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \Bigg(\int_M |X^N|\cdot|\overline{\nabla}\phi| e^{-\frac{|X|^2}{4t}} d\mu \\
&\qquad\qquad + |\phi|_0 R e^{-\frac{R^2}{4t}} \mathcal{H}^{n-1}(\partial M)\Bigg) + |\phi|_0 |G_t(M)| \\
\le& 2^{-1}(4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \frac{|X^N|}{|X|}|\phi|_1 e^{-\frac{|X|^2}{4t}} d\mu + |\phi|_0\left(|G_t(M)| + c_R t^{-(\frac{n}{2}+1)}\right) \\
\le& |\phi|_0\left(|G_t(M)| + c_R t^{-(\frac{n}{2}+1)}\right) \\
&+ 2^{-1}(4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} |\phi|_1 \left(\int_M |X^N|^2 e^{-\frac{|X|^2}{4t}} d\mu\right)^{1/2} \left(\int_M |X|^{-2} e^{-\frac{|X|^2}{4t}} d\mu\right)^{1/2} \\
\le& |\phi|_0\left(|G_t(M)| + c_R t^{-(\frac{n}{2}+1)}\right) \\
&+ |\phi|_1 |G_t(M)|^{1/2} \sqrt{\frac{t}{t-1}} \left((4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+2)} \int_M |X|^{-2} e^{-\frac{|X|^2}{4t}} d\mu\right)^{1/2}.
\end{aligned} \tag{3.5}
$$

Put $D_r = M \cap B_r$ for every $r > 0$. There is a constant $c_0 > 0$ depending only on M such that for all $r > 0$

$$\int_{D_r} 1 d\mu < c_0 r^n.$$

Note that $M \subset \mathbb{R}^{n+m} \setminus B_R$. Then for $n \geq 2$, $t \geq R^2$, one has

(3.6)

$$\begin{aligned}
t^{-\frac{n}{2}} \int_M \frac{t}{|X|^2} e^{-\frac{|X|^2}{4t}} d\mu \leq & t^{-\frac{n}{2}} \sum_{k=-1-[\frac{\log(tR^{-2})}{2\log 2}]}^{\infty} \int_{D_{2^{k+1}\sqrt{t}} \setminus D_{2^k\sqrt{t}}} \frac{t}{|X|^2} e^{-\frac{|X|^2}{4t}} d\mu \\
\leq & t^{-\frac{n}{2}} \sum_{k=-1-[\frac{\log(tR^{-2})}{2\log 2}]}^{\infty} \frac{1}{4^k} e^{-4^{k-1}} \int_{D_{2^{k+1}\sqrt{t}} \setminus D_{2^k\sqrt{t}}} 1 d\mu \\
\leq & c_0 \sum_{k=0}^{\infty} 4^{-k} e^{-4^k} 2^{(k+1)n} + c_0 \sum_{k=-1-[\frac{\log(tR^{-2})}{2\log 2}]}^{-1} 4^{-k} 2^{(k+1)n} \\
\leq & c_0 \sum_{k=0}^{\infty} 2^{k(n-2)+n} e^{-4^{k-1}} + c_0 \sum_{k=1}^{1+[\frac{\log(tR^{-2})}{2\log 2}]} 2^{-k(n-2)+n} \\
\leq & (4\pi)^{\frac{n}{2}} c_1 \left(1 + \log t - 2\log R\right),
\end{aligned}$$

where c_1 is a constant depending only on n, c_0. Therefore

(3.7)

$$\begin{aligned}
|\partial_t \Xi_t(M,\phi)| \leq & \sqrt{c_1} \frac{\sqrt{1+\log t}}{t} |\phi|_1 \left| \frac{t}{t-1} G_t(M) \right|^{1/2} + |\phi|_0 \left(|G_t(M)| + c_R t^{-(\frac{n}{2}+1)} \right) \\
\leq & c_1 \frac{1+\log t}{4t(t-1)} |\phi|_1 + c_R t^{-(\frac{n}{2}+1)} |\phi|_0 + \left(|\phi|_0 + |\phi|_1 \right) |G_t(M)|.
\end{aligned}$$

Theorem 3.1. *Let M be an n-dimensional self-shrinker in $\mathbb{R}^{n+m}$ with Euclidean volume growth and boundary $\partial M \subset \partial B_R$. If*

$$\limsup_{r\to\infty} \left(r^{1-n} \int_{M\cap B_r} |H| \right) < \infty, \tag{3.8}$$

then there is a sequence $t_i \to \infty$ such that

$$M_{t_i} \triangleq t_i^{-1} M = \{X \in \mathbb{R}^{n+m} |\ t_i X \in M\}$$

converges to a cone C in $\mathbb{R}^{n+m}$.

Proof. By co-area formula, we can choose $R' > 0$ so that $\mathcal{H}^{n-1}(\partial M) < \infty$ with $\partial M \subset \partial B_{R'}$. Denote R' by R for convenience. Let $M_t \triangleq t^{-1} M = \{X \in \mathbb{R}^{n+m} |\ tX \in M\}$ for any $t > 0$. Since M has Euclidean volume growth and (3.8) holds, then by compactness of varifolds, there exists an n-rectifiable varifold T in $\mathbb{R}^{n+m}$ with integer multiplicity and a sequence of t_i such that $M_{t_i} = t_i^{-1} M \rightharpoonup T$ in the sense of Radon measure (see 42.7 Theorem of [34] for example).

Denote ϕ and $\Xi_t(M,\phi)$ as above. Set μ_t to be the volume element of M_t. Since

(3.9)

$$\Xi_{t^2}(M,\phi) = \frac{1}{(4\pi t^2)^{n/2}} \int_M \phi e^{-\frac{|X|^2}{4t^2}} d\mu = \frac{1}{(4\pi)^{n/2}} \int_{M_t} \phi e^{-\frac{|X|^2}{4}} d\mu_t = \Xi_1(M_t,\phi),$$

then for all $R > 0$,

(3.10)

$$\lim_{i\to\infty} \Xi_1(M_{t_i R},\phi) = \lim_{i\to\infty} \Xi_{R^2}(M_{t_i},\phi) = \frac{1}{(4\pi R^2)^{n/2}} \int_T \phi\ e^{-\frac{|X|^2}{4R^2}} d\mu_T = \Xi_{R^2}(T,\phi).$$

Note that $G_t(M)$ does not change sign for $t > 1$. Fixing $0 < r < R < \infty$, from (3.7) we have

(3.11)

$$\begin{aligned}&\left|\Xi_{t_i^2 r^2}(M,\phi) - \Xi_{t_i^2 R^2}(M,\phi)\right| \le \int_{t_i^2 r^2}^{t_i^2 R^2} |\partial_s \Xi_s(M,\phi)| ds \\ \le& \int_{t_i^2 r^2}^{t_i^2 R^2} \left(c_1 \frac{1+\log s}{4s(s-1)}|\phi|_1 + c_R|\phi|_0 s^{-(\frac{n}{2}+1)} + (|\phi|_0 + |\phi|_1)\,|G_s(M)| \right) ds \\ \le& \frac{c_1}{4}|\phi|_1 \int_{t_i^2 r^2}^{\infty} \frac{1+\log s}{s(s-1)} ds + \frac{2}{n}\,(t_i r)^{-n-2}\, c_R|\phi|_0 + (|\phi|_0 + |\phi|_1)\left| \int_{t_i^2 r^2}^{t_i^2 R^2} G_s(M) ds \right|\end{aligned}$$

for all t_i with $rt_i \ge 2$. Since

(3.12)

$$\begin{aligned}&\left| \int_{t_i^2 r^2}^{t_i^2 R^2} G_s(M) ds \right| \le \left| \int_{t_i^2 r^2}^{t_i^2 R^2} F_t'(M) ds + \int_{t_i^2 r^2}^{t_i^2 R^2} \left(\frac{1}{2s} \int_{\partial M} \langle X^T, \nu_{\partial M}\rangle \Phi_s(X) \right) ds \right| \\ \le& \left| F_{t_i^2 r^2}(M) - F_{t_i^2 R^2}(M) \right| + \int_{t_i^2 r^2}^{t_i^2 R^2} \left(\frac{R}{2s} \mathcal{H}^{n-1}(\partial M)(4\pi s)^{-n/2} \right) ds \\ =& \left| F_{t_i^2 r^2}(M) - F_{t_i^2 R^2}(M) \right| + \frac{R}{n}(4\pi)^{-n/2}\mathcal{H}^{n-1}(\partial M)(t_i r)^{-n}\end{aligned}$$

and $\lim_{t\to\infty} F_t$ exists, we obtain

$$\lim_{i\to\infty} \Xi_1(M_{t_i r},\phi) = \lim_{i\to\infty} \Xi_1(M_{t_i R},\phi) = \Xi_{R^2}(T,\phi). \tag{3.13}$$

Hence

$$\Xi_t(T,\phi) = \frac{1}{(4\pi t)^{n/2}} \int_T \phi e^{-\frac{|X|^2}{4t}} d\mu_T \tag{3.14}$$

is independent of $t \in (0,\infty)$.

Clearly,

$$0 < \mathcal{H}^n(T \cap B_r) \le c_2 r^n$$

for some constant $c_2 > 0$ and all $r > 0$. By the following lemma for $V(r) = \int_{T\cap B_r} \phi\, d\mu_T$, we conclude that

$$r^{-n} \int_{T\cap B_r} \phi\, d\mu_T \tag{3.15}$$

is a constant independent of r. An analog argument as the proof of 19.3 in [34] implies that T is a cone. □

Lemma 3.2. *Let $V(r)$ be a monotone nondecreasing continuous function on $[0,\infty)$ with $V(0) = 0$ and $V(r) \le c_3 r^n$ for some constant $c_3 > 0$. If the quantity*

$$\frac{1}{(4\pi t)^{n/2}} \int_0^\infty e^{-\frac{r^2}{4t}} dV(r) \tag{3.16}$$

is a constant for any $t > 0$, then $r^{-n}V(r)$ is a constant.

Proof. There are constants $\kappa_0, \kappa_1 > 0$ such that for all $t > 0$,

$$\int_0^\infty e^{-\frac{r^2}{t}} dV(r) = \kappa_0 t^{n/2} = \kappa_1 \int_0^\infty e^{-\frac{r^2}{t}} dr^n, \tag{3.17}$$

namely,

$$\int_0^\infty e^{-\frac{r^2}{t}} d\left(V(r)-\kappa_1 r^n\right)=0. \tag{3.18}$$

Integrating by parts implies

$$\int_0^\infty \left(V(r)-\kappa_1 r^n\right) r e^{-\frac{r^2}{t}} dr=0. \tag{3.19}$$

Suppose that there is a constant $r_0>0$ such that $V(r_0)-\kappa_1 r_0^n>0$ (or else we complete the proof by (3.19)). Then there is a $0<\delta<\frac{r_0}{2}$ and $\epsilon>0$ such that $V(r)-\kappa_1 r^n\ge\epsilon$ for all $r\in(r_0-\delta,r_0+\delta)$. Set $t_p=\frac{2}{p}r_0^2$; then in $(0,\infty)$ the function

$$r^p e^{-\frac{r^2}{t_p}}$$

attains its maximal value at $r=r_0$.

Now we claim

$$\lim_{p\to\infty}\frac{p^{\frac12}e^{\frac{p}{2}}}{r_0^{p+1}}\int_{r_0-\delta}^{r_0+\delta} r^p e^{-\frac{r^2}{t_p}} dr=\int_{-\infty}^\infty e^{-t^2}dt=\sqrt{\pi}. \tag{3.20}$$

In fact,

$$\begin{aligned} I(p)\triangleq&\frac{p^{\frac12}e^{\frac{p}{2}}}{r_0^{p+1}}\int_{r_0-\delta}^{r_0+\delta} r^p e^{-\frac{r^2}{t_p}} dr=p^{\frac12}e^{\frac{p}{2}}\int_{-\frac{\delta}{r_0}}^{\frac{\delta}{r_0}}(1+s)^p e^{-\frac{p}{2}(1+s)^2}ds\\ =&\int_{-\frac{\delta}{r_0}\sqrt{p}}^{\frac{\delta}{r_0}\sqrt{p}}\left(1+\frac{t}{\sqrt{p}}\right)^p e^{-\frac{p}{2}\left(\frac{2t}{\sqrt{p}}+\frac{t^2}{p}\right)}dt\\ =&\int_{-\frac{\delta}{r_0}\sqrt{p}}^{\frac{\delta}{r_0}\sqrt{p}} e^{p\log\left(1+\frac{t}{\sqrt{p}}\right)}e^{-\sqrt{p}t-\frac{t^2}{2}}dt. \end{aligned} \tag{3.21}$$

When $-\frac12\le s<\infty$, a simple calculation implies

$$\min\left\{0,\frac{8}{3}s^3\right\}\le\log(1+s)-s+\frac{s^2}{2}\le\frac{s^3}{3}.$$

Combining the above inequality, we get

$$\begin{aligned} \limsup_{p\to\infty} I(p)\le&\limsup_{p\to\infty}\int_{-\frac{\delta}{r_0}\sqrt{p}}^{\frac{\delta}{r_0}\sqrt{p}} e^{-t^2+\frac{t^3}{3\sqrt{p}}}dt\\ =&\lim_{p\to\infty}\int_{-\frac{\delta}{r_0}\sqrt{p}}^{\frac{\delta}{r_0}\sqrt{p}} e^{-t^2(1-\frac{t}{3\sqrt{p}})}dt=\int_{-\infty}^\infty e^{-t^2}dt \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \liminf_{p\to\infty} I(p)\ge&\lim_{p\to\infty}\int_0^{\frac{\delta}{r_0}\sqrt{p}} e^{-t^2}dt+\liminf_{p\to\infty}\int_{-\frac{\delta}{r_0}\sqrt{p}}^0 e^{-t^2+\frac{8t^3}{3\sqrt{p}}}dt\\ =&\int_0^\infty e^{-t^2}dt+\lim_{p\to\infty}\int_{-\frac{\delta}{r_0}\sqrt{p}}^0 e^{-t^2\left(1-\frac{8t}{3\sqrt{p}}\right)}dt=\int_{-\infty}^\infty e^{-t^2}dt. \end{aligned} \tag{3.23}$$

Hence we have shown (3.20).

For $p > 1$,

$$
\begin{aligned}
&\frac{p^{\frac{1}{2}} e^{\frac{p}{2}}}{r_0^{p+1}} \int_{r_0+\delta}^{\infty} r^{n+p} e^{-\frac{r^2}{t_p}} dr = r_0^n \int_{\frac{\delta}{r_0}\sqrt{p}}^{\infty} e^{(n+p)\log\left(1+\frac{t}{\sqrt{p}}\right)} e^{-\sqrt{p}t-\frac{t^2}{2}} dt \\
\le & r_0^n \int_{\frac{\delta}{r_0}\sqrt{p}}^{\infty} e^{(n+p)\frac{t}{\sqrt{p}}} e^{-\sqrt{p}t-\frac{t^2}{2}} dt \le r_0^n \int_{\frac{\delta}{r_0}\sqrt{p}}^{\infty} e^{\frac{n}{\sqrt{p}}t-\frac{t^2}{2}} dt.
\end{aligned} \tag{3.24}
$$

Then

$$
\begin{aligned}
&\liminf_{p\to\infty} \frac{p^{\frac{1}{2}} e^{\frac{p}{2}}}{r_0^{p+1}} \int_0^{\infty} (V(r) - \kappa_1 r^n) r^p e^{-\frac{r^2}{t_p}} dr \\
\ge & \liminf_{p\to\infty} \frac{p^{\frac{1}{2}} e^{\frac{p}{2}}}{r_0^{p+1}} \left(\epsilon \int_{r_0-\delta}^{r_0+\delta} r^p e^{-\frac{r^2}{t_p}} dr - \kappa_1 \int_0^{r_0-\delta} r^{n+p} e^{-\frac{r^2}{t_p}} dr - \kappa_1 \int_{r_0+\delta}^{\infty} r^{n+p} e^{-\frac{r^2}{t_p}} dr \right) \\
\ge & \epsilon\sqrt{\pi} - \kappa_1 r_0^n \limsup_{p\to\infty} \left(\frac{p^{\frac{1}{2}} e^{\frac{p}{2}}}{r_0^{p+1}} \int_0^{r_0-\delta} r^p e^{-\frac{pr^2}{2r_0^2}} dr + \int_{\frac{\delta}{r_0}\sqrt{p}}^{\infty} e^{\frac{n}{\sqrt{p}}t-\frac{t^2}{2}} dr \right) \\
= & \epsilon\sqrt{\pi} - \kappa_1 r_0^n \limsup_{p\to\infty} \left(\int_{-\sqrt{p}}^{-\frac{\delta}{r_0}\sqrt{p}} e^{p\log\left(1+\frac{t}{\sqrt{p}}\right)} e^{-\sqrt{p}t-\frac{t^2}{2}} dt + \int_{\frac{\delta}{r_0}\sqrt{p}}^{\infty} e^{-t^2\left(\frac{1}{2}-\frac{n}{\sqrt{p}t}\right)} dr \right) \\
\ge & \epsilon\sqrt{\pi} - \kappa_1 r_0^n \limsup_{p\to\infty} \left(\int_{-\sqrt{p}}^{-\frac{\delta}{r_0}\sqrt{p}} e^{\sqrt{p}t} e^{-\sqrt{p}t-\frac{t^2}{2}} dt \right) = \epsilon\sqrt{\pi}.
\end{aligned} \tag{3.25}
$$

Taking the derivative of t in (3.19) yields

$$
\int_0^{\infty} (V(r) - \kappa_1 r^n) r^{2k+1} e^{-\frac{r^2}{t}} dr = 0 \tag{3.26}
$$

for any $t > 0$ and $k = 0, 1, 2, \ldots$. If we choose $p = 2k+1$, $r_0^2 > e$, and $t_p = \frac{2}{p} r_0^2$ in (3.25), then we get the contradiction provided k is sufficiently large. Hence $V(r) - \kappa_1 r^n \equiv 0$. □

Theorem 3.3. *Let M be an n-dimensional smooth self-shrinker with Euclidean volume growth and boundary $\partial M \subset \partial B_R$ in $\mathbb{R}^{n+m}$. If (3.8) holds, then the limit $\lim_{r\to\infty} r^{-1}M$ exists and is a cone, namely, the tangent cone at infinity of M is a unique cone.*

Proof. We claim

$$
\lim_{r\to\infty} \left(r^{-n} \int_{M\cap B_r} \phi d\mu \right) \tag{3.27}
$$

exists for every homogeneous function $\phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\})$ with degree zero. Suppose

$$
\limsup_{r\to\infty} r^{-n} \int_{M\cap B_r} \phi d\mu > \liminf_{r\to\infty} r^{-n} \int_{M\cap B_r} \phi d\mu \tag{3.28}
$$

for some homogeneous function $\phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\})$ with degree zero. Then there exist two sequences $p_i \to \infty$ and $q_i \to \infty$ such that

$$
\lim_{i\to\infty} p_i^{-n} \int_{M\cap B_{p_i}} \phi d\mu > \lim_{i\to\infty} q_i^{-n} \int_{M\cap B_{q_i}} \phi d\mu. \tag{3.29}
$$

By compactness of varifolds and Theorem 3.1, there exist two cones C_1, C_2 in $\mathbb{R}^{n+m}$ with integer multiplicities and subsequences p_{k_i} of p_i and q_{k_i} of q_i such that $M_{p_{k_i}} \rightharpoonup C_1$ and $M_{q_{k_i}} \rightharpoonup C_2$ in the sense of Radon measure. So we have

$$\begin{aligned}\int_{C_1\cap B_1}\phi d\mu_{C_1} &= \lim_{i\to\infty}\int_{M_{p_{k_i}}\cap B_1}\phi d\mu_{p_{k_i}} = \lim_{i\to\infty} p_{k_i}^{-n}\int_{M\cap B_{p_{k_i}}}\phi d\mu \\ &> \lim_{i\to\infty} q_{k_i}^{-n}\int_{M\cap B_{q_{k_i}}}\phi d\mu = \lim_{i\to\infty}\int_{M_{q_{k_i}}\cap B_1}\phi d\mu_{q_{k_i}} \\ &= \int_{C_2\cap B_1}\phi d\mu_{C_2},\end{aligned} \tag{3.30}$$

which implies

$$\int_{C_1}\phi e^{-\frac{|X|^2}{4}}d\mu_{C_1} > \int_{C_2}\phi e^{-\frac{|X|^2}{4}}d\mu_{C_2} \tag{3.31}$$

by co-area formula.

From the previous argument, the limit

$$\lim_{t\to\infty}\Xi_t(M,\phi) = \lim_{t\to\infty}\frac{1}{(4\pi t)^{n/2}}\int_M \phi e^{-\frac{|X|^2}{4t}}d\mu \tag{3.32}$$

exists. It infers that

$$\begin{aligned}\int_{C_1}\phi e^{-\frac{|X|^2}{4}}d\mu_{C_1} &= \lim_{i\to\infty}\int_{M_{p_{k_i}}}\phi e^{-\frac{|X|^2}{4}} = \lim_{t\to\infty}\frac{1}{t^{n/2}}\int_M \phi e^{-\frac{|X|^2}{4t}}d\mu \\ &= \lim_{i\to\infty}\int_{M_{q_{k_i}}}\phi e^{-\frac{|X|^2}{4}} = \int_{C_2}\phi e^{-\frac{|X|^2}{4}}d\mu_{C_2}.\end{aligned} \tag{3.33}$$

However, (3.33) contradicts (3.31). Hence, the claim (3.27) holds.

If $\lim_{i\to\infty} r_i^{-1}M \rightharpoonup C^+$, $\lim_{i\to\infty} s_i^{-1}M \rightharpoonup C^-$, and $C^+ \neq C^-$ are cones, then from (3.33) one has

$$\int_{C^+}\phi e^{-\frac{|X|^2}{4}}d\mu_{C^+} = \int_{C^-}\phi e^{-\frac{|X|^2}{4}}d\mu_{C^-} \tag{3.34}$$

for every homogeneous function $\phi \in C^1(\mathbb{R}^{n+m}\setminus\{0\})$ with degree zero. It's clear that

$$\int_{C^+\cap\partial B_1}\phi = \int_{C^-\cap\partial B_1}\phi. \tag{3.35}$$

Arbitrariness of ϕ implies $C^+ = C^-$. Therefore, the tangent cone at infinity of M is a unique cone. □

4. A rigidity theorem for self-shrinkers

Let us recall an ϵ-regularity theorem for mean curvature flow showed by Ecker (a little different from Theorem 1.8 in [16]).

Theorem 4.1. *For $p \in [n, n+2]$, there exists a constant $\epsilon_0 > 0$ such that for any smooth properly immersed solution $\mathcal{M} = (\mathcal{M}_t)_{t\in(-4,0)}$ of mean curvature flow in $\mathbb{R}^{n+m}$ and every X_0 which the solution reaches at time $t_0 \in [-1, 0)$, the assumption*

$$I_{X_0,t_0} \triangleq \sup_{\sqrt{-t_0}\le\rho<\rho'\le 2}\frac{1}{(\rho'^2-\rho^2)^{\frac{n+2-p}{2}}}\int_{-\rho'^2}^{-\rho^2}\int_{\mathcal{M}_t\cap B_2(X_0)}|B|^p \le \epsilon_0 \tag{4.1}$$

implies

$$\sup_{\sigma\in[0,1]}\left(\sigma^2 \sup_{t\in(t_0-(1-\sigma)^2,t_0)} \sup_{\mathcal{M}_t\cap B_{1-\sigma}(X_0)} |B|^2\right) \le \left(\epsilon_0^{-1} I_{X_0,t_0}\right)^{\frac{2}{p}}. \tag{4.2}$$

For completeness, we give a proof in the Appendix which is based on Ecker's proof. Let us consider the mean curvature flow in Theorem 4.1 which starts from a self-shrinker. Let M be a self shrinker; then the one-parameter family $\mathcal{M}_t = \sqrt{-t}M$ is a mean curvature flow for $-4 \le t < 0$. In this case,

(4.3)

$$\begin{aligned} I_{X_0,t_0} &= \sup_{\sqrt{-t_0}\le\rho<\rho'\le 2} \left(\rho'^2-\rho^2\right)^{-\frac{n+2-p}{2}} \int_{-\rho'^2}^{-\rho^2}\left(\int_{\sqrt{-t}M\cap B_2(X_0)} |B|^p\right) dt \\ &= \sup_{\sqrt{-t_0}\le\rho<\rho'\le 2} \left(\rho'^2-\rho^2\right)^{-\frac{n+2-p}{2}} \int_{\frac{1}{\rho'}}^{\frac{1}{\rho}}\left(\int_{\frac{1}{r}M\cap B_2(X_0)} |B|^p\right) \frac{2}{r^3} dr \\ &= \sup_{\sqrt{-t_0}\le\rho<\rho'\le 2} 2\left(\rho'^2-\rho^2\right)^{-\frac{n+2-p}{2}} \int_{\frac{1}{\rho'}}^{\frac{1}{\rho}}\left(r^{p-n-3}\int_{M\cap B_{2r}(rX_0)} |B|^p d\mu\right) dr. \end{aligned}$$

For any $-\frac{1}{4} < t_0 < 0$ and $X_0 \in \sqrt{-t_0}M$, $I_{X_0,t_0} \le \epsilon_0$ implies

$$\frac{1}{4}\sup_{t\in(t_0-\frac{1}{4},t_0)} \sup_{\sqrt{-t}M\cap B_{\frac{1}{2}}(X_0)} |B|^2 \le \left(\epsilon_0^{-1} I_{X_0,t_0}\right)^{\frac{2}{p}}. \tag{4.4}$$

Hence

$$\sup_{t\in(2,(-t_0)^{-1/2})}\left(\sup_{\frac{1}{t}M\cap B_{\frac{1}{2}}(X_0)} |B|^2\right) \le 4\left(\epsilon_0^{-1} I_{X_0,t_0}\right)^{\frac{2}{p}}. \tag{4.5}$$

Now we have the following curvature estimates for self-shrinkers.

Theorem 4.2. *Let M be an n-dimensional proper self-shrinker in $\mathbb{R}^{n+m}$. If for some $p \in [n, n+2)$ there is*

$$\lim_{R\to\infty}\int_{M\cap B_{2R}\setminus B_R} |B|^p d\mu = 0, \tag{4.6}$$

then there exist constants $c, r_0 > 0$ such that for all $r \ge r_0$ and $t > 4$ we have

$$\sup_{M\cap\partial B_{(r+1)t}} |B| \le \frac{c}{t}\left(\sup_{s\ge r}\int_{M\cap B_{2s}\setminus B_s} |B|^p d\mu\right)^{\frac{1}{p}}. \tag{4.7}$$

Proof. For any $\epsilon > 0$, there exists a constant $r_0 \ge 2$ such that for any $r_1 \ge r_0$ we have

$$\sup_{r\ge r_1}\int_{M\cap B_{2r}\setminus B_r} |B|^p d\mu < \epsilon.$$

For any vector $X_0 \in \mathbb{R}^{n+m}$ with $|X_0| \ge 2r_1 + 2$, it is clear that

$$B_{2r}(rX_0) \subset \left(B_{(|X_0|+2)r} \setminus B_{(|X_0|-2)r}\right) \subset \left(B_{2(|X_0|-2)r} \setminus B_{(|X_0|-2)r}\right).$$

Let $X \in \sqrt{-t}M$ with $|X| \ge 2r_1 + 2$ and $t < 0$; then

(4.8)

$$\int_{M\cap B_{2r}(rX)} |B|^p d\mu \le \int_{M\cap\left(B_{2(|X|-2)r}\setminus B_{(|X|-2)r}\right)} |B|^p d\mu \le \sup_{s\ge r_1}\int_{M\cap B_{2s}\setminus B_s} |B|^p d\mu < \epsilon.$$

In view of (4.3), one has

(4.9)

$$I_{X,t} \leq \sup_{0\leq\rho<\rho'\leq 2}\left(\rho'^2-\rho^2\right)^{-\frac{n+2-p}{2}}\int_{\frac{1}{\rho'}}^{\frac{1}{\rho}} 2r^{p-n-3}dr\cdot\sup_{r\geq r_1}\int_{M\cap B_{2r}\setminus B_r}|B|^p d\mu$$
$$\leq \frac{2}{2+n-p}\sup_{0\leq\rho<\rho'\leq 2}\left(\rho'^2-\rho^2\right)^{-\frac{n+2-p}{2}}\left(\rho'^{2+n-p}-\rho^{2+n-p}\right)\sup_{r\geq r_1}\int_{M\cap B_{2r}\setminus B_r}|B|^p d\mu.$$

Since for each fixed $\alpha\in(0,1]$ and each $s\geq 1$,

$$\frac{\partial}{\partial s}\left(\frac{s^{2\alpha}-1}{(s^2-1)^{\alpha}}\right)=2\alpha\frac{s-s^{2\alpha-1}}{(s^2-1)^{\alpha}}\geq 0, \tag{4.10}$$

then

$$\sup_{s\geq 1}\frac{s^{2\alpha}-1}{(s^2-1)^{\alpha}}=\lim_{s\to\infty}\frac{s^{2\alpha}-1}{(s^2-1)^{\alpha}}=1.$$

So we obtain

$$I_{X,t}\leq\frac{2}{2+n-p}\sup_{r\geq r_1}\int_{M\cap B_{2r}\setminus B_r}|B|^p d\mu<\frac{2\epsilon}{2+n-p}. \tag{4.11}$$

Let $\epsilon=\frac{2+n-p}{2}\epsilon_0$, let $|X|\geq 2r_1+2$, and let $-\frac{1}{4}<t<0$. Then combining (4.5) we have

$$\sup_{s\in(2,(-t)^{-1/2})}\left(\sup_{\frac{1}{s}M\cap B_{\frac{1}{2}}(X)}|B|\right)\leq 2\left(\epsilon^{-1}\sup_{r\geq r_1}\int_{M\cap B_{2r}\setminus B_r}|B|^p d\mu\right)^{\frac{1}{p}}, \tag{4.12}$$

which implies

$$\begin{aligned}2\left(\epsilon^{-1}\sup_{r\geq r_1}\int_{M\cap B_{2r}\setminus B_r}|B|^p d\mu\right)^{\frac{1}{p}} &\geq \sup_{X\in\frac{1}{t}M\cap\partial B_{2r_1+2}}\left(\sup_{\frac{1}{t}M\cap B_{\frac{1}{2}}(X)}|B|\right)\\ &=\sup_{|X|=2r_1+2,tX\in M}\left(t\sup_{M\cap B_{\frac{t}{2}}(tX)}|B|\right)\\ &\geq t\sup_{M\cap\partial B_{2t(r_1+1)}}|B|\end{aligned} \tag{4.13}$$

for any $r\geq r_1$ and $t>2$. This suffices to complete the proof. □

Lemma 4.3. *Let M be an n-dimensional proper noncompact self-shrinker in $\mathbb{R}^{n+m}$ with*

$$\limsup_{r\to\infty}\int_{M\cap B_{2r}\setminus B_r}|H|^p d\mu<\infty \tag{4.14}$$

for some $p\geq 2$. Then every end of M has at least Euclidean volume growth.

Proof. For any end E of M, there is a constant $r_0>0$ such that $\partial E\subset B_{r_0}$. Replacing E by $E\setminus B_{r_0}$ if necessary, we have $\partial E\subset\partial B_{r_0}$. Set $E_r=E\cap B_r$. For

$0 \le s < 1$ and $r \ge r_0$, we have

$$
\begin{aligned}
&\frac{\partial}{\partial r}\left(r^{-n+s}\int_{E_r} 1 d\mu\right) = -(n-s)r^{-n+s-1}\int_{E_r} 1 d\mu + r^{-n+s}\int_{E\cap\partial B_r}\frac{|X|}{|X^T|}\\
\ge& -(n-s)r^{-n+s-1}\int_{E_r} 1d\mu + r^{-n+s-1}\int_{E\cap\partial B_r}|X^T|\\
=& -(n-s)r^{-n+s-1}\int_{E_r} 1d\mu + \frac{1}{2}r^{-n+s-1}\int_{E_r}\Delta|X|^2 + r^{-n+s-1}\int_{\partial E}|X^T|\\
\ge& sr^{-n+s-1}\int_{E_r} 1d\mu - 2r^{-n+s-1}\int_{E_r}|H|^2 d\mu\\
\ge& sr^{-n+s-1}\int_{E_r} 1d\mu - 2r^{-n+s-1}\left(\int_{E_r}|H|^p d\mu\right)^{\frac{2}{p}}\left(\int_{E_r} 1d\mu\right)^{1-\frac{2}{p}}.
\end{aligned}
\tag{4.15}
$$

Set

$$\widetilde{V}_s(r) = r^{-n+s}\int_{E_r} 1d\mu;$$

then

$$
\begin{aligned}
\partial_r\widetilde{V}_s \ge& \frac{s}{r}\widetilde{V}_s - 2r^{-\frac{2}{p}(n-s)-1}\widetilde{V}_s^{1-\frac{2}{p}}\left(\int_{E_r}|H|^p d\mu\right)^{\frac{2}{p}}\\
=&\frac{\widetilde{V}_s}{r}\left(s - 2\left(\int_{E_r}|H|^p d\mu\right)^{\frac{2}{p}}\left(\int_{E_r} 1d\mu\right)^{-\frac{2}{p}}\right).
\end{aligned}
\tag{4.16}
$$

For any $r > 0$, let $q \in \mathbb{N}$ with $2^q \le r < 2^{q+1}$. By (4.14), there is a constant $c > 0$ such that

$$\int_{E_r}|H|^p d\mu \le \sum_{k=0}^{q}\int_{E_{2^{k+1}}\setminus E_{2^k}}|H|^p d\mu + \int_{E_1}|H|^p d\mu \le c(q+2) \le c\left(\frac{\log r}{\log 2}+2\right). \tag{4.17}$$

From [31, 33], every end of any self-shrinker has at least linear growth. For any $\delta > 0$, there exists a constant $r_\delta > 0$ such that for all $r \ge r_\delta$,

$$\left(\int_{E_r}|H|^p d\mu\right)^{\frac{2}{p}}\left(\int_{E_r} 1d\mu\right)^{-\frac{2}{p}} \le \frac{\delta}{4};$$

then (4.16) implies

$$\partial_r\widetilde{V}_\delta \ge \frac{\delta\widetilde{V}_\delta}{2r}. \tag{4.18}$$

By the Newton–Leibniz formula,

$$\log\widetilde{V}_\delta(r) \ge \log\widetilde{V}_\delta(r_\delta) + \int_{r_\delta}^{r}\frac{\partial_s\widetilde{V}_\delta(s)}{\widetilde{V}_\delta(s)}ds \ge \log\widetilde{V}_\delta(r_\delta) + \frac{\delta}{2}\log\frac{r}{r_\delta}. \tag{4.19}$$

Denote $\widetilde{V}(r) = \widetilde{V}_0(r)$. By (4.16),

$$\partial_r\widetilde{V}^{\frac{2}{p}} \ge -\frac{4}{p}\left(\int_{E_r}|H|^p d\mu\right)^{\frac{2}{p}} r^{-\frac{2n}{p}-1}. \tag{4.20}$$

There is a constant $s_0 > e$ such that for all $s \geq s_0$ the inequality $\log s < s^{\frac{n}{p}}$ holds. Hence combining (4.14) and (4.20), for any $r_2 \geq r_1 \geq \max\{s_0, r_0\}$ we have

(4.21)
$$\widetilde{V}^{\frac{2}{p}}(r_2) - \widetilde{V}^{\frac{2}{p}}(r_1) \geq -\frac{nc'}{p}\int_{r_1}^{r_2} r^{-\frac{2n}{p}-1}\log r dr \geq -\frac{nc'}{p}\int_{r_1}^{r_2} r^{-\frac{n}{p}-1}dr \geq -c' r_1^{-\frac{n}{p}}$$

for some constant $c' > 0$. (4.19) infers

$$\lim_{r\to\infty} r^{\delta}\widetilde{V}(r) = \infty$$

for any $\delta > 0$. Combining (4.21), we obtain

$$\widetilde{V}^{\frac{2}{p}}(r_2) \geq \frac{1}{2}\widetilde{V}^{\frac{2}{p}}(r_1) > 0 \tag{4.22}$$

for some fixed sufficiently large $r_1 \geq \max\{s_0, r_0\}$. This suffices to complete the proof. □

Now let us prove the following rigidity theorem.

Theorem 4.4. *Let M be an n-dimensional properly noncompact self-shrinker with compact boundary in $\mathbb{R}^{n+m}$. If*

$$\lim_{r\to\infty}\int_{M\cap B_{2r}\setminus B_r} |B|^n d\mu = 0, \tag{4.23}$$

M is contained in an n-plane through the origin.

Proof. From Theorem 4.2, we obtain

$$\lim_{r\to\infty}\left(r\sup_{B_{5r}}|B|\right) = 0. \tag{4.24}$$

Let $M_r = r^{-1}M$ for any $r > 0$; then $M_t \cap \left(B_K \setminus B_{\frac{1}{K}}\right)$ for any $K > 0$ has bounded sectional curvature. On the one hand, $M_r \cap \left(B_K \setminus B_{\frac{1}{K}}\right)$ converges to a smooth manifold with a $C^{1,\alpha}$ metric in the Gromov–Hausdorff sense. On the other hand, Theorem 3.3 implies that M_r converges to a unique cone C in $\mathbb{R}^{n+1}$ in the current sense. Hence for any $x \in C \setminus \{0\}$, there is a neighborhood Ω_x of x such that $\Omega_x \cap C$ can be represented as a graph with a $C^{1,\alpha}$ graphic function. Hence by Fatou's lemma, $\Omega_x \cap C$ is flat by (4.24). So we conclude that M_r converges to a union of finite n-planes through the origin as $r \to \infty$. Note that every end of M converges to a union of finite n-planes through the origin by Lemma 4.3. Therefore, up to rotation there are a constant $R > 0$ and a smooth graph $\mathrm{graph}_u \subset M$ over $\mathbb{R}^n \setminus B_R$ with the graphic function $u = (u^1, \ldots, u^m)$. Moreover, there is a constant c_M such that

$$|D^j u^\alpha(x)| \leq c_M |x|^{-j+1} \tag{4.25}$$

on $\mathbb{R}^n \setminus B_R$ for any $j = 0, 1, 2$ and $1 \leq \alpha \leq m$. Here, c_M is a general constant, which may change from line to line.

Let $g_{ij} = \delta_{ij} + \sum_{1\leq\alpha\leq m} u_i^\alpha u_j^\alpha$, and let (g^{ij}) be the inverse matrix of (g_{ij}). From the equation of self-shrinkers (see [10] for instance)

$$\sum_{1\leq i,j\leq n} g^{ij}u_{ij}^\alpha = \frac{-u^\alpha + x\cdot Du^\alpha}{2}, \tag{4.26}$$

we have

$$
\begin{aligned}
\Delta_M u^\alpha =& \frac{1}{\sqrt{\det g_{ij}}}\partial_{x_i}\left(g^{kl}\sqrt{\det g_{ij}}u_j^\alpha\right) \\
=& \frac{1}{\sqrt{\det g_{ij}}}\partial_{x_i}\left(g^{ij}\sqrt{\det g_{kl}}\right)u_j^\alpha + \frac{1}{2}x\cdot Du^\alpha - \frac{u^\alpha}{2}.
\end{aligned} \tag{4.27}
$$

Denote $g_t^{ij}(x) = g^{ij}(x,t) = g^{ij}\left(\frac{x}{\sqrt{t}}\right)$; then

$$
\left|\delta_{ij} - g_t^{ij}\right| \le c_1 \sum_\beta |\nabla_{\mathbb{R}^n} u^\beta|, \tag{4.28}
$$

where c_1 is a constant. Let $Q(x,t,Du^\beta,D^2u^\gamma) = \frac{1}{\sqrt{t}}\left(\delta_{ij} - g_t^{ij}\right)u_{ij}^\alpha|_{\frac{x}{\sqrt{t}}}$; then on $(\mathbb{R}^n \setminus B_R)\times\mathbb{R}^+$, from (4.25) one has

$$
|Q(x,t,Du^\beta,D^2u^\gamma)| \le \frac{c_2}{|x|}\sum_\beta |\nabla_{\mathbb{R}^n} u^\beta|, \tag{4.29}
$$

where c_2 is a constant.

Denote $a^{ij}(x,t) = a_0^{ij}\left(\frac{x}{\sqrt{t}}\right)$ and $U^\alpha(x,t) = \sqrt{t}u^\alpha\left(\frac{x}{\sqrt{t}}\right)$. Then

$$
\begin{aligned}
\frac{\partial}{\partial t}U^\alpha + \Delta_{\mathbb{R}^n}U^\alpha =& \frac{1}{2\sqrt{t}}u^\alpha\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2}Du^\alpha\left(\frac{x}{\sqrt{t}}\right)\cdot\frac{x}{t} + \frac{1}{\sqrt{t}}\Delta_{\mathbb{R}^n}u^\alpha\Big|_{\frac{x}{\sqrt{t}}} \\
=& -\frac{1}{\sqrt{t}}g_t^{ij}u_{ij}^\alpha + \frac{1}{\sqrt{t}}\Delta_{\mathbb{R}^n}u^\alpha\Big|_{\frac{x}{\sqrt{t}}} = Q(x,t,Du^\beta,D^2u^\gamma).
\end{aligned} \tag{4.30}
$$

Hence for any $(x,t) \in (\mathbb{R}^n\setminus B_R)\times\mathbb{R}^+$, by combining (4.29) we have

$$
\left|\frac{\partial}{\partial t}U^\alpha + \Delta_{\mathbb{R}^n}U^\alpha\right| \le \frac{c_2}{|x|}\sum_\beta |\nabla_{\mathbb{R}^n}U^\beta|. \tag{4.31}
$$

Due to Theorem 1 (with the version of vector-valued functions) by Escauriaza–Seregin–Šverák in [18] (see the following content in Theorem 1 of [18]), we obtain

$$
U^\alpha \equiv 0 \qquad \text{on } \ \mathbb{R}^n \setminus B_R,
$$

and then graph_u is contained in an n-plane through the origin. Hence M is also contained in an n-plane through the origin by the rigidity of elliptic equations, and then we complete the proof. □

5. Appendix

Let us prove Theorem 4.1. There exist $\sigma_1 \in (0,1)$, $t_1 \in [t_0 - (1-\sigma_1)^2, t_0]$, and $X_1 \in \mathcal{M}_{t_1} \cap \overline{B}_{1-\sigma_1}(X_0)$ such that

$$
\sigma_1^2|B|^2\Big|_{(X_1,t_1)} = \sup_{\sigma\in[0,1]}\left(\sigma^2 \sup_{t\in(t_0-(1-\sigma)^2,t_0)}\ \sup_{\mathcal{M}_t\cap B_{1-\sigma}(X_0)}|B|^2\right).
$$

Denote $\lambda_1 = |B|^{-1}\Big|_{(X_1,t_1)}$. Then

$$
\sup_{t\in(t_0-(1-\frac{\sigma_1}{2})^2,t_0)}\ \sup_{\mathcal{M}_t\cap B_{1-\frac{\sigma_1}{2}}(X_0)}|B|^2 \le \frac{4}{\lambda_1^2}.
$$

Since

$$B_{\frac{\sigma_1}{2}}(X_1)\times\left(t_1-\frac{\sigma_1^2}{4},t_1\right)\subset B_{1-\frac{\sigma_1}{2}}(X_0)\times\left(t_0-\left(1-\frac{\sigma_1}{2}\right)^2,t_0\right),$$

then

$$\sup_{t\in(t_1-\frac{\sigma_1^2}{4},t_1)}\ \sup_{\mathcal{M}_t\cap B_{\frac{\sigma_1}{2}}(X_1)}|B|^2\le\frac{4}{\lambda_1^2}.$$

Let I_{X_0,t_0} be as in (4.1). It is sufficient to prove

$$\sigma_1\lambda_1^{-1}\le\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{1}{p}}$$

for a certain uniform constant $\epsilon_0>0$ depending only on n provided $I_{X_0,t_0}\le\epsilon_0$. By contradiction, we assume

$$\sigma_1\lambda_1^{-1}>\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{1}{p}}.$$

Denote $\lambda\triangleq\lambda_1\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{1}{p}}<\sigma_1$.

Define

$$\widetilde{M}_s=\lambda^{-1}\left(M_{\lambda^2s+t_1}-X_1\right)$$

for $s\in\left(-\frac{4+t_1}{\lambda^2},\frac{t_0-t_1}{\lambda^2}\right)$, where we have changed variables by setting $X=\lambda Y+X_1$ and $t=\lambda^2s+t_1$. Then $\widetilde{M}_s$ is a smooth solution of mean curvature flow satisfying

$$0\in\widetilde{M}_0,\qquad |B|\Big|_{(0,0)}=\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{1}{p}}\le 1$$

and

$$\sup_{s\in(-\frac{\sigma_1^2}{4\lambda^2},0)}\ \sup_{\widetilde{M}_s\cap B_{\frac{\sigma_1}{2\lambda}}}|B|^2\le 4\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{2}{p}}.$$

Since $\sigma_1>\lambda$, then

$$\sup_{s\in(-\frac{1}{4},0)}\ \sup_{\widetilde{M}_s\cap B_{\frac{1}{2}}}|B|^2\le 4\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{2}{p}}.$$

By scaling, it follows that

$$I_{X_0,t_0}=\sup_{\sqrt{-t_0}\le\rho<\rho'\le 2}\left(\frac{\lambda^2}{\rho'^2-\rho^2}\right)^{\frac{n+2-p}{2}}\int_{-\frac{\rho'^2+t_1}{\lambda^2}}^{-\frac{\rho^2+t_1}{\lambda^2}}\int_{\widetilde{M}_s\cap B_{\frac{2}{\lambda}}\left(\frac{X_0-X_1}{\lambda}\right)}|B|^p. \tag{5.1}$$

Since $-1<t_0<0$ and $t_0-(1-\sigma_1)^2\le t_1\le t_0$, we choose $\rho^2=-t_1$, $\rho'^2-\rho^2=\rho'^2+t_1=2\lambda^2>0$. Noting that $X_1\in\mathcal{M}_{t_1}\cap\overline{B}_{1-\sigma_1}(X_0)$, we have

$$I_{X_0,t_0}\ge 2^{-\frac{n+2-p}{2}}\int_{-2}^{0}\int_{\widetilde{M}_s\cap B_{\frac{1}{\lambda}}(0)}|B|^p\ge 2^{-\frac{n+2-p}{2}}\int_{-\frac{1}{4}}^{0}\int_{\widetilde{M}_s\cap B_{\frac{1}{2}}}|B|^p. \tag{5.2}$$

Now let's recall the evolution equation for the norm of the second fundamental form in [41]:

$$\left(\frac{d}{ds}-\Delta_{\widetilde{M}_s}\right)|B|^2=-2|\nabla B|^2+2|R^N|+2\sum_{\alpha,\beta}S_{\alpha\beta}^2\le 3|B|^4. \tag{5.3}$$

Since

$$\sup_{s\in(-\frac{1}{4},0)}\ \sup_{\widetilde{M}_s\cap B_{\frac{1}{2}}}|B|^2\le 4\left(\epsilon_0^{-1}I_{X_0,t_0}\right)^{\frac{2}{p}}\le 4,$$

then

$$\left(\frac{d}{ds} - \Delta_{\widetilde{M}_s}\right)|B|^p \leq \frac{3p}{2}|B|^{p+2} \leq 6p|B|^p. \tag{5.4}$$

By the mean value inequality for mean curvature flow in [15], [16] (where the case of submanifolds is similar to the case of hypersurfaces), there exists a constant $c(n)$ such that

$$|B|^p\Big|_{(0,0)} \leq c(n)\int_{-\frac{1}{4}}^{0}\int_{\widetilde{M}_s\cap B_{\frac{1}{2}}}|B|^p, \tag{5.5}$$

which implies

$$\epsilon_0^{-1} I_{X_0,t_0} \leq c(n)2^{\frac{n+2-p}{2}} I_{X_0,t_0}. \tag{5.6}$$

This is impossible for the sufficiently small ϵ_0. Hence we complete the proof of Theorem 4.1.

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Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China

Email address: dingqi@fudan.edu.cn, dingqi09@fudan.edu.cn