THE STRUCTURE THEORY OF NILSPACES II: REPRESENTATION AS NILMANIFOLDS

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ABSTRACT. This paper forms the second part of a series of three papers by the authors concerning the structure of nilspaces of Antolín Camarena and Szegedy. A nilspace is a compact space $X$ together with closed collections of cubes $C_n(X) \subseteq X^{2^n}$, $n = 1, 2, \ldots$ satisfying some natural axioms. From these axioms it follows that a nilspace can be built as a finite tower of extensions where each of the successive fibers is a compact abelian group.

Our main result is a new proof of a result due to Antolín Camarena and Szegedy [Nilspaces, nilmanifolds and their morphisms, arXiv:1009.3825v3 (2012)] stating that if each of these groups is a torus, then $X$ is isomorphic (in a strong sense) to a nilmanifold $G/\Gamma$. We also extend the theorem to a setting where the nilspace arises from a dynamical system $(X,T)$. These theorems are a key stepping stone towards the general structure theorem in [The structure theory of nilspaces III: Inverse limit representations and topological dynamics, arXiv:1605.08950v1 [math.DS] (2016)] (which again closely resembles the main theorem of Antolín Camarena and Szegedy).

The main technical tool, enabling us to deduce algebraic information from topological data, consists of existence and uniqueness results for solutions of certain natural functional equations, again modelled on the theory in Antolín Camarena and Szegedy’s paper.

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1. Introduction

This paper forms part of a series by the authors [GMV16a, GMV16b] concerning the structure theory of nilspaces. Much of this is concerned with the approach of Szegedy [Sze12] and Antolín Camarena and Szegedy [ACS12] to the inverse theorem for the Gowers norms, as well as with relations to dynamics and in particular works of Host and Kra [HK05] and Host, Kra, and Maass [HKM10].

An extensive introduction to this project from the viewpoint of higher order Fourier analysis and the inverse theorem for the Gowers norms is contained in [GMV16a]. Similarly, [GMV16b] introduces the project from a dynamical perspective. We will not repeat the bulk of these introductions here, nor offer much motivation for the definition or study of nilspaces and related constructs, but instead refer the reader to these companion works.

Nilspaces originate in a work of Host and Kra [HK08], where these objects appeared under the name of “parallelepipeds structures”. The study of these objects was furthered by Antolín Camarena and Szegedy [ACS12], who in the same work formulated a strong structure theorem for nilspaces, subject to certain further hypotheses.

The papers of Candela [Can17b, Can17a] expand on [ACS12], providing more detailed proofs. He also includes several additional results implicit in [ACS12], particularly about continuous systems of measures.

The goals of this work are as follows.

- We prove a structure theorem for nilspaces with certain additional topological assumptions, which allow us to deduce that they are isomorphic (in a suitable sense) to nilmanifolds $G/\Gamma$. This is a key stepping stone towards the general structure theorem.
- Along the way, we prove some rather technical results concerning “cocycles” on cubespaces (closely related to [ACS12 section 3.6]). The resulting primitive “cohomology” theory is a powerful tool for deducing algebraic information from topological data, and it will be invaluable both here and elsewhere in the project [GMV16b].

The structural result we will prove here roughly has the following flavor: if a (compact, ergodic) nilspace $X = (X, C^n(X))$, satisfying some connectivity hypotheses, has any chance of being a nilmanifold topologically—for instance, it had better be a topological manifold, i.e., locally homeomorphic to a subset of $\mathbb{R}^n$—then it is a nilmanifold.

One possible formulation of the statement is the following theorem. (For the definition of nilspaces we refer the reader to [GMV16a section 3.1] or [GMV16b section 1.3], and for that of Host–Kra cubes, see [GMV16a section 2 and Appendix A] or [GMV16b section 1.4]).

**Theorem 1.1.** Let $X = (X, C^n(X))$ be a (compact, ergodic)\footnote{The assumption that a nilspace be compact and ergodic is in force universally, so it usually appears in parentheses to downplay its particular significance in any given context. However, these conditions are certainly not optional.} nilspace of degree $s$. Suppose $X$ is locally connected and of finite Lebesgue covering dimension, and further that all the spaces $C^n(X)$ are connected.
Then $X$ is isomorphic to a nilmanifold $G/\Gamma$. That is, there exists a filtered connected Lie group $G_\bullet$, a discrete co-compact subgroup $\Gamma$ of $G$, and a homeomorphism $\phi: X \leftrightarrow G/\Gamma$ that identifies the cubes $C^k(X)$ with the Host–Kra cubes $HK^k(G_\bullet)/\Gamma$.

We note that the conditions on local connectedness and finite dimensionality hold particularly if $X$ is a topological manifold.

The topological conditions in the above theorem can be replaced by other sets of assumptions involving the so-called structure groups of the nilspace $X$. Such variants will be formulated below, including the one proved by Antolín Camarena and Szegedy [ACS12, Theorem 7]. In fact, we prove only one of these variants in this paper and the proof of Theorem 1.1 is completed in [GMV16b, Theorem A.1] using the general structure theory of nilspaces.

Both the overall structure of these arguments and a good part of the fine detail are modelled closely on the work of Antolín Camarena and Szegedy, and we will attempt to make this dependence explicit wherever possible. However, in some aspects we have deviated from their approach. Our reasons for doing so are some mixture of the following.

- In places, we obtain slightly stronger results, which will be useful especially in applications to dynamics. Obtaining these can require a modified approach.
- Our arguments stay entirely within the topological category, avoiding reference to objects that are merely measurable rather than continuous. This eliminates some subtleties encountered in [ACS12] concerning the replacement of measurable objects by continuous ones—albeit arguably at the expense of introducing other subtleties in different places. In particular, this strategy requires a different treatment of the part of the argument covered in section 4. Similarly, for some results in section 5 we will have to prove strengthened versions, which are not required here but are needed to keep the arguments in [GMV16b] in the topological category.
- In some cases we find that our alternatives are simpler or easier to understand, or that they yield a fuller understanding of the methods and structures involved.

1.1. Structure of the paper. As we have said, we will not repeat in detail the relevant definitions of cubespaces, nilspaces, ergodicity, and so on, that appear in [GMV16a, section 3.1], instead referring the reader to that paper. Similarly, we will assume some familiarity with the crucial subject of the structure groups $A_k(X)$ of a nilspace, and with the “weak structure theorem” of [GMV16a, section 5.1].

Alternatively, a reader mainly interested in dynamics may read [GMV16b, sections 1.3–1.6]. The same notions and their properties motivated from a dynamical viewpoint are introduced in [GMV16b]. However, the proofs are available only in [GMV16a].

We will recall only some of the more specialized definitions relating to Theorem 1.1 and its generalizations in section 2. The same section will introduce formally the group $\text{Aut}_k(X)$ of $k$-translations of a nilspace, which will play a key role in the arguments. Section 2 concludes with a discussion of the various variations on the statement of Theorem 1.1.
Section 3 explains the high-level steps of the proof of the structural results. This reduces the structural result to a statement (see Proposition 3.3) that a nilmanifold $X$ has “enough” automorphisms in a certain technical sense.

We establish this technical statement in section 4 conditional on a “cohomological” theorem (see Theorem 4.11) that will arise fairly naturally in the course of the proof.

We introduce the cocycle theory, and give a proof of this remaining theorem, in section 5.

2. Definitions and statements

The flavor of all of our structural statements is to find conditions one can impose on a cubespace $X = (X, C^n(X))$ that are sufficient to ensure that it is actually a nilmanifold $G/\Gamma$.

The various “algebraic” constraints we impose are discussed at length in [GMV16a]. In short, it is fairly hopeless to ask for a rigid structure theorem unless we insist that $X$ is an ergodic nilspace.

However, we also require some topological input (beyond the standard assumption that $X$ is compact). For instance, a nilmanifold is certainly a smooth real manifold, so we need to rule out examples such as the solenoid $\mathbb{R} / \mathbb{Z}/ \mathbb{Z}$, which is a compact abelian group and hence a nilspace of degree 1, but not a manifold.

In the previous section we formulated Theorem 1.1 under topological conditions on the space $X$ and the space of cubes $C^n(X)$. We will now state a similar result in which these conditions are replaced by conditions on the structure groups of $X$. We recall that these are a sequence of compact abelian groups $A_k(X)$, defined canonically in terms of the cubestructure on $X$. The key topological hypothesis we can impose is that these are Lie groups.

These formulations turn out to fit more naturally in the theory, and we prove only these statements in this paper.

Unfortunately the condition that $A_k(X)$ are Lie still does not suffice. Our method also requires some fairly strong assumptions on connectivity, to which we now turn.

2.1. Connectivity hypotheses. The simplest connectivity hypothesis is just that $X$ itself is connected as a topological space. It turns out that this condition becomes inadequate fairly quickly because it fails to establish the connectedness “on all levels” that is required to explore the structure of $X$ inductively.

Sticking with the idea of imposing conditions on the structure groups $A_k(X)$, one can insist that these groups are connected for all $k$'s. Since we have already assumed that they are Lie, this is equivalent to asking that they be tori $(\mathbb{R}/\mathbb{Z})^d$ for some integer $d$.

Definition 2.1. We say a (compact, ergodic) nilspace $X$ is toral if for each $k \geq 1$ the structure group $A_k(X)$ is isomorphic to a torus $(\mathbb{R}/\mathbb{Z})^d$ for some integer $d$ (which depends on $X$ and $k$).

This is the approach taken in [ACS12].
Theorem 2.2 ([ACS12, Theorem 7]). Let \( X = (X, C^n(X)) \) be a (compact, ergodic) toral nilspace of degree \( s \). Then \( X \) is isomorphic to a nilmanifold \( G/\Gamma \) in the sense of Theorem 1.1.

For the purposes of our core structural result, Theorem 2.18, we will work with a slightly different topological condition, this time relating to the spaces of cubes \( C^k(X) \). This is needed for compatibility with our statement of the final structure theorem for general nilspaces ([GMV16a, Theorem 4.1]).

Definition 2.3. We say a nilspace \( X \) is strongly connected if the topological space \( C^k(X) \) is connected for all \( k \)'s \( \geq 0 \). (In particular, \( X = C^0(X) \) itself is connected.)

It turns out that these two connectivity conditions are equivalent. While it is easy to see that a toral nilspace is strongly connected (see Proposition 2.4), the converse appears to be much more difficult. In fact, the only proof we are aware of is based on the full force of our structure theorem. Indeed, Theorem 2.18 implies that a strongly connected nilspace with Lie structure groups is isomorphic to a Host–Kra nilspace of a connected nilpotent Lie group endowed with a filtration of connected subgroups, and hence it is a toral nilspace a fortiori.

Proposition 2.4. A toral (compact, ergodic) nilspace is strongly connected.

Proof. By the weak structure theorem ([GMV16a, Theorem 5.4]) the space \( C^n(X) \) is expressible as a tower of extensions (more correctly, principal bundles)

\[
C^n(X) = C^n(X_s) \rightarrow C^n(X_{s-1}) \rightarrow \cdots \rightarrow C^n(X_1) \rightarrow \{\ast\},
\]

where each fiber is the compact abelian group \( C^n(D_k(A_k(X))) \), which in turn is isomorphic to \( A_k(X)^d \) for some \( d \geq 0 \), and in particular is connected. Since a tower of extensions by connected fibers is connected, we deduce that \( C^n(X) \) is connected.

In fact, our techniques allow us to say something even in the absence of any such connectivity hypothesis, and we will make statements in this setting as well (see Theorem 2.19 and Corollary 2.20 in what follows). These statements are especially interesting from the viewpoint of topological dynamics.

2.2. Some notation. We recall some miscellaneous notation from [GMV16a]. Given two configurations \( c, c' : \{0, 1\}^n \rightarrow X \), we denote by \([c, c']\) the “concatenated” configuration

\[
[c, c'] : \{0, 1\}^{n+1} \rightarrow X, \quad \omega \mapsto \begin{cases} c(\omega_1, \ldots, \omega_n) : \omega_{n+1} = 0, \\ c'(\omega_1, \ldots, \omega_n) : \omega_{n+1} = 1. \end{cases}
\]

We will not be too concerned about which coordinate in \( \{1, \ldots, n + 1\} \) is the preferred one along which the concatenation occurs; i.e., some other coordinate may play the role of \( (n + 1) \) in this expression, and we will still denote the final configuration by \([c, c']\).
We also use $\square^n(x)$ to denote the constant configuration $\{0,1\}^n \to X$, sending every coordinate to $x$. Similarly, the notation $\sqcap^n(x; y)$ denotes the configuration $\sqcap^n(x; y) : \{0,1\}^n \to X$

$$\omega \mapsto \begin{cases} y & : \omega = \bar{1}, \\ x & : \omega \neq \bar{1}. \end{cases}$$

More generally, we use the notation $\square^n(c)$, where $c \in X^{(0,1)^k}$, to denote the $(n+k)$-configuration $(\omega_1, \ldots, \omega_{n+k}) \mapsto c(\omega_1, \ldots, \omega_k)$. We use the notation $\sqcap^n(c_1; c_2)$ in a similar manner for $c_1, c_2 \in X^{(0,1)^k}$.

We reserve the right to mix and abuse notation freely, as in $\sqcap^2(\square^3(x); \square^3(y))$. Hopefully the meaning will always be clear.

Finally, we use the standard notation $X \lesssim_{a,b,...} Y$ to denote that $X \leq C(a,b,...)Y$ for some constant $C$ depending only on the variables $(a,b,...)$.

2.3. Automorphisms and translations. The heart of the problem of identifying a suitable nilspace $X$ with a nilmanifold $G/\Gamma$ is in recovering the group $G$ and its group law.

The approach to doing this taken in [ACSI12], which we follow, is to consider a suitable group of automorphisms of $X$. Since automorphism groups are clearly groups, we will be in good shape if we can take $G$ to be this group and then identify $X$ with a suitable quotient of $G$.

However, we want to describe not just $X$ as a topological space but also its cube structure. The cubes on a nilmanifold $G/\Gamma$ are given by the Host–Kra construction, which requires the additional data of a filtration on the group $G$. (For a detailed account of this theory, again see [GMV16a section 2 and Appendix A].) So, we need to find not just a suitable group of automorphisms of $X$ but also a filtration on that group.

Fortunately, there are natural definitions of all of these objects, which we now describe. The automorphism group itself is straightforward.

**Definition 2.5.** Let $X$ be a compact cubespaces. We write $\text{Aut}(X)$ for the group of all automorphisms of $X$ in the category of cubespaces; that is, the collection of homeomorphisms $\phi : X \to X$ such that, for any configuration $c : \{0,1\}^k \to X$, $c \in C^k(X)$ if and only if $\phi(c) \in C^k(X)$.

We endow $\text{Aut}(X)$ with the usual supremum metric

$$d(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x)) = \|\psi \circ \phi^{-1}\|,$$

where we use the notation

$$\|\phi\| = \sup_{x \in X} d(x, \phi(x)),$$

which generates the usual compact-open topology.

**Remark 2.6.** It is straightforward to verify that $\text{Aut}(X)$ is a closed subgroup of the group of homeomorphisms of $X$ and hence is completely metrizable. It follows from

\[\text{Note that we do not claim that the supremum metric itself on Aut}(X) \text{ is complete, merely that Aut}(X) \text{ is complete with respect to some metric which generates the same topology, e.g., } d'(\phi, \psi) = d(\phi, \psi) + d(\phi^{-1}, \psi^{-1}) \text{ [BK56 Corollary 1.2.2]. The choice of the metric is not important, but we stick with the supremum metric for convenience of notation.}\]
a standard theorem (see [HK06, Example 1.3(v)]) that Aut(X) is also separable, or equivalently second countable.

We now consider the filtration.

**Definition 2.7.** Again, let X be a compact cubespace, and fix k ≥ 0. Given \( \phi \in \text{Aut}(X) \) and a face F of \( \{0,1\}^n \), let \( [\phi]_F \) denote the element of Aut(X)\(^{\{0,1\}^n} \) given by

\[
\omega \mapsto \begin{cases} 
\phi : \omega \in F, \\
\text{id} : \omega \notin F.
\end{cases}
\]

We write \( \text{Aut}_k(X) \) for the collection of \( \phi \in \text{Aut}(X) \) with the following additional property. For any integer \( n \geq k \), any face F of \( \{0,1\}^n \) of codimension k, and any \( c \in C^n(X) \), the configuration \( [\phi]_F \cdot c \)

\[
\{0,1\}^n \to X,
\omega \mapsto \begin{cases} 
\phi(c(\omega)) : \omega \in F, \\
c(\omega) : \omega \notin F
\end{cases}
\]

is in \( C^m(X) \).

We refer to the elements of \( \text{Aut}_k(X) \) as \( k \)-translations. Clearly \( \text{Aut}_0(X) = \text{Aut}(X) \).

The notion of translations originates from the work of Host and Kra [HK08, Definition 6], and they play a pivotal role in the program of Antolín Camarena and Szegedy [ACS12], which we discuss now.

The motivation for this definition is perhaps not entirely clear\(^6\). We will illustrate it somewhat with some facts and examples.

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\(^6\)Below is one way to arrive at this definition. We stress that the following discussion is purely for motivation and is not logically necessary for the argument (although some ideas discussed now will be relevant later; see, for instance, Proposition 2.17).

We define a *cubegroup* as an object \( G \) that is both a topological group and a cubespace, with the added requirement that \( C^k(G) \) is a (closed) subgroup of \( G^{\{0,1\}^k} \) under pointwise operations for all k’s.

It turns out that Aut(X) is a natural example of a cubegroup; that is, there is a canonical categorial notion of when \( 2^n \) automorphisms of X form a k-cube, and this notion is closed under pointwise composition. The cubespace structure can be described informally as the largest possible one such that \( \omega \mapsto \phi_\omega(c(\omega)) \) is a cube of \( X \) for every \( (\omega \mapsto \phi_\omega) \in C^k(\text{Aut}(X)) \) and every \( c \in C^k(X) \), i.e., such that the pointwise action of a cube on \( X \) sends cubes to cubes. More precisely, one should define an element \( \psi : \{0,1\}^k \to \text{Aut}(X) \) to be in \( C^k(\text{Aut}(X)) \) if and only if \( \Box^\ell(\psi)(c) \in C^{k+\ell}(X) \) for every \( \ell \geq 0 \) and \( c \in C^{k+\ell}(X) \), and then one can check that this defines a cubegroup structure on Aut(X).

It is a fact (which we will not prove because we do not need it anywhere in the paper) that—in complete generality—all cubegroups arise from the Host–Kra construction applied to a filtered group. That is, given a cubegroup \( G \), there is a unique filtration \( G_\bullet \) on \( G \) such that \( C^k(G) = \text{HK}^k(G_\bullet) \) for all k’s, so specifying a cubegroup structure on \( G \) is equivalent to specifying a filtration. Under this correspondence the filtration Aut\(_k\)(X) from Definition 2.7 is precisely the one giving rise to the cubegroup structure on Aut(X) discussed above.

In the interests of concreteness and simplicity we will suppress explicit discussion of the cube structure on Aut(X) in what follows (referring equivalently instead to HK\(_k\)(Aut(X)\(_\bullet\)) and phrase everything in terms of filtrations, taking Definition 2.7 as the logical starting point.
Proposition 2.8. Let $X$ be a compact cubespace. The groups $(\text{Aut}_k(X))_{k \geq 0}$ form a filtration. That is, $\text{Aut}_k(X)$ represents a decreasing sequence of closed subgroups of $\text{Aut}(X)$, and for any $i, j$ we have the commutator inclusion $[\text{Aut}_i(X), \text{Aut}_j(X)] \subseteq \text{Aut}_{i+j}(X)$.

Proof. That $\text{Aut}_k(X)$ forms a subgroup is clear, and it is similarly straightforward to argue that it is closed in $\text{Aut}(X)$ (since $C^n(X)$ is closed in $X^{(0,1)^n}$ for all $n$’s). Similarly, by modifying the same cube twice on an adjacent pair of faces, it is easy to see that a $(k+1)$-translation is also a $k$-translation for all $k$’s.

For the commutator result, take any $\phi \in \text{Aut}_i(X)$, $\psi \in \text{Aut}_j(X)$, and $c \in \{0,1\}^n$ for some $n \geq i + j$, and let $F$ be a face of $\{0,1\}^n$ of codimension $(i+j)$. Pick faces $F_1, F_2 \subseteq \{0,1\}^n$ of codimensions $i, j$, respectively, such that $F_1 \cap F_2 = F$ and note that

$$[\phi|_{F_1}, \psi|_{F_2}][\phi|_{F_1}^{-1}[\psi|_{F_2}^{-1}[\phi|_{F_1}^{-1}][\psi|_{F_2}^{-1}]]_F$$

(with operations applied pointwise). But clearly the configuration

$$[\phi|_{F_1}, \psi|_{F_2}][\phi|_{F_1}^{-1}[\psi|_{F_2}^{-1}]], c$$

is in $C^n(X)$ by hypothesis on $\phi$ and $\psi$, as required. \hfill \Box

Proposition 2.9. Suppose $X$ is a compact cubespace with $k$-uniqueness. Then $\text{Aut}_k(X) = \{\text{id}\}$.

Proof. For any $x \in X$ and $\phi \in \text{Aut}_k(X)$, let $F = \{1\} \subseteq \{0,1\}^k$, and apply $[\phi|_F$ to the constant cube $c = \Box^k(x)$ in $C^k(X)$. Then $[\phi|_F.c$ is a cube, agreeing with $c$ on all but the topmost vertex of $\{0,1\}^k$, and so is equal to $c$ by $k$-uniqueness. Hence, $\phi(x) = x$. \hfill \Box

Note that in general we may have $\text{Aut}_0(X) \neq \text{Aut}_1(X)$, and hence even if $X$ is a nilspace of degree $s$, this filtration is not proper; hence, it does not guarantee that $\text{Aut}_0(X)$ is a nilpotent group; indeed, we will see below that it usually is not. However, it does guarantee that $\text{Aut}_1(X)$ is nilpotent (of a nilpotency class of at most $s$) under these conditions. Hence, the group of 1-translations, equipped with the filtration

$$(1) \quad \text{Aut}_1(X) = \text{Aut}_2(X) \supseteq \text{Aut}_3(X) \supseteq \cdots \supseteq \text{Aut}_k(X) \supseteq \cdots,$$

is a much better candidate for our group $G$ used to construct the nilmanifold $G/\Gamma$. We will never have much need to consider the full automorphism group $\text{Aut}(X)$ again, restricting our attention to $\text{Aut}_1(X)$. In what follows we denote by $\text{Aut}_C(X)$ the group $\text{Aut}_1(X)$ endowed with the filtration $\Box$.

Example 2.10. Suppose that $X = (\mathbb{R}/\mathbb{Z})^2$, with cubespace structure coming from the usual degree 1 filtration. Then $\text{Aut}(X) \cong GL_2(\mathbb{Z}) \ltimes (\mathbb{R}/\mathbb{Z})^2$, the group of affine automorphisms of $(\mathbb{R}/\mathbb{Z})^2$, acting by

$$(M, a).x = Mx + a.$$

Indeed, since the $k$-cubes of $X$ are the $k$-dimensional parallelepipeds, i.e., configurations of the form

$$\{0,1\}^k \to (\mathbb{R}/\mathbb{Z})^2,$$

$$\omega \mapsto x + \sum_{i=1}^k \omega_i h_i$$
for some coefficients $x, h_i \in (\mathbb{R}/\mathbb{Z})^2$, it is easy to see that such maps send cubes to cubes. Conversely, since the 2-cubes of $X$ are the configurations $[[x, y], [z, w]]$ such that $x + w = y + z$, we have, for any $\phi \in \text{Aut}(X)$ and $x, y, z, w \in X$ such that $x + w = y + z$,

$$\phi(x) + \phi(w) = \phi(y) + \phi(z)$$

and so in particular taking $z = 0$, we have

$$(\phi(x + w) - \phi(0)) = (\phi(x) - \phi(0)) + (\phi(w) - \phi(0)),$$

so we deduce that $(\phi - \phi(0))$ is necessarily linear and hence $\phi$ is affine linear.

The subgroup of 1-translations is precisely $\text{Aut}_1(X) = \{\text{id}\} \times (\mathbb{R}/\mathbb{Z})^2$. To see this, we observe that if $\phi \in \text{Aut}^1(X)$ and $[[x, y], [z, w]] \in C^2(X)$, then so is $[[x, y], [\phi(z), \phi(w)]]$; equivalently, if $x, y, z, w \in (\mathbb{R}/\mathbb{Z})^2$ satisfies $x - y = z + w$, then $x - y = \phi(z) - \phi(w)$. So, $\phi(z) - \phi(w) = z - w$ for all $z$'s, $w$'s in $(\mathbb{R}/\mathbb{Z})^2$, and hence $\phi$ is just a translation $x \mapsto x + t$.

It is easy to check that such maps obey the 1-translation property for higher-dimensional cubes.

For $k \geq 2$ $\text{Aut}_k(X)$ is the trivial group.

**Example 2.11.** Let $X = G/\Gamma$, where $G$ is the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

with its central series filtration, and $\Gamma$ is

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

the usual discrete co-compact subgroup.

Then, for any $g \in G$, the map

$$x\Gamma \mapsto gx\Gamma$$

is a 1-translation on $G/\Gamma$. If $g$ lies in $G_2$ (the center of $G$), then this map is moreover a 2-translation. It turns out in this case that these are the only elements of $\text{Aut}_1(X)$ and $\text{Aut}_2(X)$.

However, if we replace $\Gamma$ with

$$\Gamma' = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in 2\mathbb{Z}, z \in \mathbb{Z} \right\},$$

then the map

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\Gamma' \mapsto \begin{pmatrix} 1 & x & z+x \\ 0 & 1 & y+1 \\ 0 & 0 & 1 \end{pmatrix}\Gamma' = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Gamma'$$

is a 1-translation not of the form $x\Gamma' \mapsto gx\Gamma'$. However, the connected component of the identity in $\text{Aut}_1(G/\Gamma')$ still consists entirely of maps $x\Gamma' \mapsto gx\Gamma'$.

The full automorphism group $\text{Aut}(X)$ is rather complicated (again, one can argue that it contains a copy of $\text{SL}_2(\mathbb{Z})$) and will not concern us.

**Remark 2.12.** Generalizing the previous example, for any filtered nilmanifold $G/\Gamma$ it turns out that maps $x\Gamma \mapsto gx\Gamma$ are always in $\text{Aut}_k(X)$ provided that $g \in G_k$. As we saw above, these need not be all the elements of $\text{Aut}_k(G/\Gamma)$, but they will yield all of the connected component of the identity in $\text{Aut}_k(G/\Gamma)$.

This is of course encouraging for the approach of using $\text{Aut}_1(X)$ as a proxy for $G$. 
The definition of $k$-translation we have given is in some sense the most natural, but it can be cumbersome in practice because of the need to consider rather general configurations, and in particular cubes of arbitrarily large dimension.

The following proposition establishes a convenient alternative definition that is equivalent under certain circumstances.

**Proposition 2.13.** Suppose $X$ is a (compact, ergodic) nilspace of degree $s$, and fix $k$, $0 \leq k \leq s + 1$. Then $\phi \in \text{Homeo}(X)$ is a $k$-translation if and only if for any $c \in C^{s+1-k}(X)$ the configuration $\cup^k(c; \phi(c))$ is an $(s + 1)$-cube.

Note this condition is a special case of that from Definition 2.7 specializing to $n = s + 1$ and $c = \Box^k(c')$ for some $c' \in C^{s+1-k}(X)$. Hence, the “only if” direction is clear; the content is that it suffices to check the definition on configurations of this form.

Before we give the proof, we recall some facts and terminology from [GMV16a]. We say that a cubespace obeys the **glueing axiom** if the following holds: for any triple $c_1, c_2, c_3$ of cubes, the fact that $[c_1, c_2]$ and $[c_2, c_3]$ are cubes implies that $[c_1, c_3]$ is a cube also. This can be visualized as the cubes $[c_1, c_2]$ and $[c_2, c_3]$ being glued along the common face $c_2$. We recall from [GMV16a, Proposition 6.2] that nilspaces obey the glueing axiom. Readers unfamiliar with these ideas are advised to consult [GMV16a, section 6.1].

**Proof.** We first recall [GMV16a, Proposition 3.11] that in a nilspace of degree $s$, a configuration $\{0,1\}^n \to X$ for $n \geq s + 1$ is a cube if and only if every face of dimension $(s + 1)$ is a cube.

Our first objective is to prove that it is enough to prove the condition from Definition 2.7 for cubes $c$ of dimension $s + 1$.

Let $F$ be a face of $\{0,1\}^n$ of codimension $k$ and $c \in C^n(X)$ for some $n \geq s + 1$. Every face of $[\phi]_F(c)$ of dimension $(s + 1)$ has the form $[\phi]_F(c')$, where $c'$ is a face of $c$ and $F'$ is a face of $\{0,1\}^{s+1}$ with a codimension of at most $k$. So, the condition from Definition 2.7 for $n = s + 1$ implies all cases in which $n \geq s + 1$.

If $n < s + 1$ and $c \in C^n(X)$, we note that the duplicated configuration $\Box^{s+1-n}([\phi]_F(c))$ has the form $[\phi]_{F'}([\phi]_{F'}([\phi]_{F'}(c)))$ required by Definition 2.7 where $F' = \Box^{s+1-n}(F)$ (abusing notation somewhat) is a corresponding face of $\{0,1\}^{s+1}$ of codimension $k$, so this configuration lies in $C^{s+1-k}(X)$. Restricting ourselves to an appropriate face, we recover the fact that $[\phi]_{F'}(c) \in C^n(X)$. So, the condition for $n = s + 1$ implies the condition for all $n$'s.

We now have to show that it suffices to consider cubes of the form $\Box^k(c')$ for $c' \in C^{s+1-k}(X)$. This is by a “glueing argument”, which is very much related to the “universal replacement property” of the canonical equivalence relation in [GMV16a, Proposition 6.3]. There we prove that in a fibrant cubespace $Y$ the fact that $\cup^k(y; y')$ is a cube implies the following: if $c \in C^k(Y)$ is such that $c(\tilde{1}) = y$, then the configuration $c'$ defined by $c'(\omega) = c(\omega)$ for $\omega \neq \tilde{1}$ and $c'(\tilde{1}) = y'$ is also a cube. A full proof of this is given in [GMV16a, Proposition 7.12, second claim]. One can turn that into a proof of our claim by replacing vertices by cubes of dimension $s + 1 - k$ in a straightforward manner.

Here we give some diagrams for the case $s = 1$ and $k = 2$ only for the reader’s convenience. The general case is very similar but notationally awkward.
Let $c \in C^{s+1}(X) = C^2(X)$. We draw this as

\[
\begin{array}{ccc}
  & c(01) & c(11) \\
 c(00) & \hline & c(10) \\
  & c(01) & c(11) \\
\end{array}
\]

and we wish to show that

\[
\begin{array}{ccc}
  & \phi(c(11)) & \\
 c(00) & \hline & c(10) \\
  & c(00) & \phi(a) \\
\end{array}
\]

is a cube. Our hypothesis states that for every $a \in C^{s+1-k}(X) = C^0(X) = X$ the following configuration is an $(s+1)$-cube, i.e., a 2-cube:

\[
\begin{array}{ccc}
  & a & \\
 a & \hline & a \\
\end{array}
\]

We consider

\[
\begin{array}{ccc}
  & c(01) & c(11) & \phi(c(11)) \\
 c(01) & \hline & c(11) & c(11) \\
 c(00) & c(10) & c(10) & c(10) \\
\end{array}
\]

and note that the upper-right square is a cube by the hypothesis and the remaining small squares are cubes by the cubespace axioms; hence, the outer square is a cube by gluing (twice) as required. 

Recalling Remark 2.12, the connected component of the identity in $\text{Aut}_k(X)$ may be a better object to work with than $\text{Aut}_k(X)$ itself. We consider this object briefly now.

**Definition 2.14.** For any topological group $G$ we let $G^\circ \leq G$ denote the connected component of the identity in $G$. In particular, $\text{Aut}_k^\circ(X)$ denotes the connected component of the identity in the group of $k$-translations.
Remark 2.15. Note that the connected component is taken in \( \text{Aut}_k(X) \), not in the larger group \( \text{Aut}(X) \) or \( \text{Aut}_1(X) \). Indeed, these need not agree in general; i.e., we may have \( \text{Aut}_k^\circ(X) \subseteq \text{Aut}_k(X) \cap \text{Aut}_1^\circ(X) \).

We should really check that this revised sequence of groups \( \text{Aut}_1^\circ(X) \supseteq \text{Aut}_2^\circ(X) \supseteq \ldots \) is still well-behaved.

Proposition 2.16. Let \( X \) be a compact cubespace. Then the sequence \((\text{Aut}_k^\circ(X))_{k \geq 0}\) is still a filtration of closed groups in the sense of Proposition 2.8.

Proof. Since \( \text{Aut}_{k+1}^\circ(X) \) is a connected subset of \( \text{Aut}_k(X) \) containing the identity, it is contained in \( \text{Aut}_k^\circ(X) \). Certainly all of these groups are closed. Since the commutator map is continuous, the set

\[
\{(g, h) : g \in \text{Aut}_k^\circ(X), g \in \text{Aut}_j^\circ(X)\}
\]

is a connected subset of \( \text{Aut}_{i+j}^\circ(X) \) and so is contained in \( \text{Aut}_{i+j}^\circ(X) \); hence, so is the closed subgroup it generates. \( \square \)

2.4. A discrete subgroup and the evaluation map. Having defined the group \( \text{Aut}_1^\circ(X) \) that we hope will take the role of \( G \) in the construction of the nilmanifold \( G/\Gamma \), we now consider the question of how \( \Gamma \), and the isomorphism \( X \leftrightarrow G/\Gamma \), will arise.

Suppose we fix an element \( x_0 \in X \). There is a natural map

\[
ev_{x_0} : \text{Aut}_1(X) \to X
\]

\[
\phi \mapsto \phi(x_0),
\]

and this gives rise to an identification of the orbit of \( x \) under \( \text{Aut}_1(X) \) with \( \text{Aut}_1(X)/\text{Stab}(x_0) \), where \( \text{Stab}(x) = \{ \phi \in \text{Aut}_1(X) : \phi(x) = x \} \) denotes the stabilizer. Clearly the same story makes sense when restricting to \( \text{Aut}_1^\circ(X) \). For notational convenience we denote by \( \text{Stab}(x) \) also \( \{ \phi \in \text{Aut}_1^\circ(X) : \phi(x) = x \} \). The meaning of \( \text{Stab}(x) \) is always clear from the context.

Suppose we knew that \( \text{Stab}(x_0) \) was a discrete and co-compact subgroup of \( \text{Aut}_1^\circ(X) \), and also that the action of \( \text{Aut}_1^\circ(X) \) on \( X \) was transitive. Then we would have a homeomorphism \( X \cong \text{Aut}_1^\circ(X)/\text{Stab}(x_0) \), which is already enough to identify \( X \) (as a topological space) with a nilmanifold.

However, this transitivity assumption is a very significant one. So far, we have made no progress towards even showing that \( \text{Aut}_1(X) \) is nontrivial. Showing that 1-translations (and, more generally, \( k \)-translations) are fairly abundant is both the core and the hardest aspect of the whole argument.

If the identification of \( X \) with \( \text{Aut}_1^\circ(X)/\text{Stab}(x_0) \) should hold true in the category of cubespace, we would need to know that \( C^n(X) \) were identified under this bijection with the Host–Kra cubes \( \text{HK}^n(\text{Aut}_1^\circ(X))/\text{Stab}(x_0) \), where here \( \text{Stab}(x_0) = \{ \phi \in \text{HK}^n(\text{Aut}_1^\circ(X)) : \phi(\square^n(x_0)) = \square^n(x_0) \} \). Unwrapping the definitions, this says that a configuration \( c : \{0, 1\}^n \to X \) is a cube if and only if it has the form \( \omega \mapsto \phi_\omega \cdot x_0 \) for some \( (\phi_\omega) \in \text{HK}_k(\text{Aut}_1^\circ(X)) \).

It is certainly not hard to show the “if” direction (see also footnote 6).
Proposition 2.17. Suppose that \((\phi, \omega)_{\omega \in \{0,1\}^n} \in \text{HK}^n(\text{Aut}_c^s(X))\) is an element of the Host–Kra cube group, and suppose that \(c \in C^n(X)\) is a cube. Then the configuration
\[
\omega \mapsto (\phi_\omega(c(\omega)))
\]
is a cube in \(C^n(X)\).

In particular, the map \(\text{Aut}_1^c(X)/\text{Stab}(x_0) \to X\) is a cubespace morphism.

Proof. The Host–Kra cube group is generated (by definition) by elements \([\phi]_F\), where \(F \subseteq \{0,1\}^n\) has codimension \(k\) and \(\phi \in \text{Aut}_k^c(X)\). By definition of \(\text{Aut}_k^c\), applying any such configuration pointwise to a cube \(c \in C^n(X)\) yields another cube in \(C^n(X)\). The result follows from repeated application of this fact. \(\square\)

However, the “only if” direction is not clear. Indeed, the case \(n = 0\) corresponds to the statement that the action of \(\text{Aut}_1^c(X)\) on \(X\) is transitive; for higher \(n\)'s, this statement is some kind of assertion of “higher transitivity” or “transitivity on all levels”. Proving this is strongly analogous to proving transitivity, although one needs an analogous strengthening of the hypotheses.

2.5. Structural results. We are now in a position to state more precise versions of the main structural result (Theorem 1.1), whose proof will occupy us for the rest of this article.

The first is a very slight strengthening of Theorem 2.2 using the different connectivity hypothesis explained above. Part of our proof was reported in [Gut13].

Theorem 2.18. Let \(X = (X, C^n(X))\) be a (compact, ergodic) nilspace of degree \(s\). Suppose that all of the structure groups \(A_k(X)\) are Lie groups. Finally, suppose \(X\) is strongly connected.

Fix an element \(x_0 \in X\). Then the following hold.

(i) The group \(\text{Aut}_1^c(X)\) is Lie, and the filtration \((\text{Aut}_k^c(X))_{k \geq 1}\) is Lie and has a degree of at most \(s\).

(ii) The subgroup \(\text{Stab}(x_0) \leq \text{Aut}_1^c(X)\) is discrete and co-compact in \(\text{Aut}_1^c(X)\).

Moreover, \(\text{Stab}(x_0) \cap \text{Aut}_k^c(X)\) is co-compact in \(\text{Aut}_k^c(X)\) for all \(k\)'s \(\geq 1\).

(iii) The natural map
\[
\text{Aut}_1^c(X) \to X
\]
\[
\phi \mapsto \phi(x_0)
\]
gives rise to an isomorphism of cubespaces \(\text{Aut}_1^c(X)/\text{Stab}(x_0) \to X\). That is, this map is a homeomorphism, and \(C^n(X)\) is identified with the Host–Kra cubes \(\text{HK}^n(\text{Aut}_1^c(X))/\text{Stab}(x_0)\) on the nilmanifold \(\text{Aut}_1^c(X)/\text{Stab}(x_0)\).

One can summarize these conclusions more briefly by saying that \(X\) is a nilmanifold with a degree of at most \(s\).

By Proposition 2.4, this implies Theorem 2.2.

As mentioned above, it is instructive to record some of what our method provides when the connectivity hypothesis is not assumed.

Theorem 2.19. Let \(X = (X, C^n(X))\) be a compact ergodic nilspace of degree \(s\) whose structure groups are Lie groups. Then \(X\) has finitely many connected components, each of which is open in \(X\), and each of which is homeomorphic to a nilmanifold.
Furthermore, for any \( x_0 \in X \), the natural evaluation map
\[
ev_{x_0} : \text{Aut}_1^0(X) \to X
\]
\[
\phi \mapsto \phi(x_0)
\]
induces an homeomorphism \( S \cong \text{Aut}_1^0(X)/\text{Stab}(x_0) \), where \( \text{Aut}_1^0(X) \) is a nilpotent Lie group of degree at most \( s \), \( \text{Stab}(x_0) \) is a discrete and co-compact subgroup and \( S \) denotes the connected component of \( x_0 \) in \( X \).

Finally, if \( \text{Aut}_1(X) \) is known a priori to act transitively on the set of connected components of \( X \), then there is a stronger identification (of topological spaces) \( X \cong \text{Aut}_1(X)/\text{Stab}(x_0) \) for any \( x_0 \in X \), where again \( \text{Aut}_1(X) \) is a nilpotent Lie group with a degree of at most \( s \) and \( \text{Stab}(x_0) \) is a discrete co-compact subgroup.

Note that this result provides information only about the topological structure of the nilspace \( X \). We are able to say something about the cubespace structure as well, but we will postpone this discussion. See Lemma 3.12 below.

The primary interest in the last statement of the above theorem comes from the case of dynamical systems, where this condition arises naturally. In particular, we record the following corollary.

**Corollary 2.20.** Let \( (G, X) \) be a minimal topological dynamical system, where \( X \) is a (compact, ergodic) nilspace of degree \( s \), whose structure groups \( A_k(X) \) are Lie groups, and suppose \( G \) acts on \( X \) through a continuous group homomorphism
\[
\sigma : G \to \text{Aut}_1(X).
\]

Then \( X \) is homeomorphic to a nilmanifold as in the previous theorem; that is, \( X \cong \text{Aut}_1(X)/\text{Stab}(x_0) \) for any \( x_0 \in X \). Under this identification, \( G \) acts by left translations \( x\Gamma \mapsto \sigma(g)x\Gamma \).

**Proof.** Since \( (G, X) \) is minimal, one may map any element of \( X \) onto any open set by an element of \( G \). But the connected components of \( X \) are open by the theorem, so we conclude that the group of translations \( \text{Aut}_1(X) \) acts transitively on the set of connected components of \( X \). \[ \square \]

To motivate the above result further, we mention that for any minimal group action \( (G, X) \), one can associate a natural cubespace structure \( \{C^0_G\}_{n \in \mathbb{Z}_{\geq 0}} \) such that the acting group \( G \) immerses into \( \text{Aut}_1(X, C_G) \). This construction is due to Host, Kra, and Maass [HKM10]. Given a minimal group action \( (G, X) \), Gutman, Glasner, and Ye [GGY18] defined the regional proximal relation \( \text{RP}^n_G(X) \), which is a closed equivalence relation on \( X \) for each \( n \in \mathbb{Z}_{\geq 0} \) and extends the definition of Host, Kra, and Maass [HKM10] for actions of \( \mathbb{Z} \) to general group actions. It is proved in [GGY18] that the cubespace structure \( (X, C_G) \) is a nilspace with a degree of at most \( s \) for some \( s \in \mathbb{Z}_{\geq 0} \) if and only if \( \text{RP}^s_G(X) \) is trivial. For a detailed exposition and further results we refer the reader to the companion paper [GMV16b] and to [GGY18].

Based on these results, the following is a special case of Corollary 2.20.

**Corollary 2.21.** Let \( (G, X) \) be a minimal topological dynamical system. Suppose that the regional proximal relation \( \text{RP}^s_G(X) \) is trivial for some \( s \in \mathbb{Z}_{\geq 0} \), and suppose that the structure groups of \( (X, C_G) \) are Lie groups.

Then \( (G, X) \) is a nilsystem; that is, \( X \) can be identified with a nilmanifold \( N/\Gamma \), where \( N \) is a nilpotent Lie group with a step of at most \( s \), and the action of \( G \) can be realized through a continuous homomorphism \( G \to N \).
We note that the hypothesis on the structure groups can be replaced by some assumptions on the topology of $X$ that are similar to Theorem 1.1, and therefore the corollary can be stated without any reference to the cubespace structure. However, the proof of that result requires the structure theory for general nilspaces developed in [GMV16b] Appendix, and we refer the interested reader to that paper.

3. THE MAIN STEPS IN THE STRUCTURAL RESULT

We continue our outline of the proof of the main structure theorems of this paper following [ACS12]. From the discussion above it is clear that our main task is to show that $X$ has “enough” $k$-translations in some sense. At this stage, it is not particularly obvious that even a single nontrivial $k$-translation exists for any $k$, so we first need to find a source of these automorphisms.

The source is ultimately based on the weak structure theory expounded on in [GMV16a]. We recall that if $X$ is a (compact, ergodic) nilspace of degree $s$, then there is a canonical factor

$$
\pi_{s-1}: X \rightarrow \pi_{s-1}(X)
$$

where $\pi_{s-1}(X)$ is a (compact, ergodic) nilspace with a degree of at most $(s - 1)$, and the fibers of $\pi_{s-1}$ are identified with the $s$th structure group $A_s(X)$ of $X$, which is some compact abelian group. More precisely, there is an action of $A_s(X)$ on the whole space $X$, whose orbits are precisely these fibers. For full statements see [GMV16a] Theorem 5.4.

The key observation is the following.

**Proposition 3.1.** Let $X$ be a compact ergodic nilspace of degree $s$, and let $a \in A_s(X)$. Write $t_a: X \rightarrow X$ for the action $x \mapsto a.x$ of $A_s$ on $X$ described above. Then $t_a \in \text{Aut}_s(X)$.

**Proof.** After unwrapping the definitions, this is immediate from the weak structure theory. Indeed, what we need—that applying a fixed element $a \in A_s$ to a face of $c \in C^n(X)$ of codimension $s$ yields another cube—is a special case of [GMV16a] Theorem 5.4 (ii)].

So, although we still cannot prove that $\text{Aut}_1(X)$ acts transitively, we at least know that it acts transitively on each fiber of $\pi_{s-1}$ (and in fact this is true even of $\text{Aut}_s(X)$).

The second idea is then to induct on the degree. Suppose we had already established the relevant transitivity statement for nilspaces of degree $(s - 1)$; so in particular we knew that $\text{Aut}_1(\pi_{s-1}(X))$ acts transitively on $\pi_{s-1}(X)$ (given suitable assumptions). Then we might hope to achieve transitivity on $X$ by first moving to the correct fiber of $\pi_{s-1}$ (by inductive assumption) and then moving to the correct point in that fiber using an element of $A_s$.

Of course, the challenge for this plan is that there is no obvious way to relate $\text{Aut}_1(\pi_{s-1}(X))$ to $\text{Aut}_1(X)$. Specifically, to make this approach work, we will need to be able to lift a 1-translation of $\pi_{s-1}(X)$ to one of $X$, under suitable hypotheses,
i.e., given $\phi \in \text{Aut}_1(\pi_{s-1}(X))$, to find $\tilde{\phi} \in \text{Aut}_1(X)$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{\phi}} & X \\
\downarrow^{\pi_{s-1}} & & \downarrow^{\pi_{s-1}} \\
\pi_{s-1}(X) & \xrightarrow{\phi} & \pi_{s-1}(X)
\end{array}
$$

commutes.

Going in the other direction (i.e., that given $\tilde{\phi} \in \text{Aut}_k(X)$, we can push it down to $\phi \in \text{Aut}_k(\pi_{s-1}(X))$) is straightforward, and we verify this now.

**Proposition 3.2.** Let $X$ be a compact, ergodic nilspace of degree $s$, and fix $k \geq 0$. Then there is a canonical (continuous) group homomorphism $\pi_s : \text{Aut}_k(X) \to \text{Aut}_k(\pi_{s-1}(X))$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & X \\
\downarrow^{\pi_{s-1}} & & \downarrow^{\pi_{s-1}} \\
\pi_{s-1}(X) & \xrightarrow{\pi_s(\phi)} & \pi_{s-1}(X)
\end{array}
$$

commutes for every $\phi \in \text{Aut}_k(X)$.

The proof is completely mechanical.

**Proof.** Note that if $k > s$, the result is trivial (Proposition 2.3), so we assume $k \leq s$. We do the only thing possible: given $y \in \pi_{s-1}(X)$, we pick $x \in \pi_{s-1}^{-1}(y)$ arbitrarily, and we define $\pi_s(\phi))(y) := \pi_{s-1}(\phi(x))$. It suffices to check that this is well-defined and a $k$-translation; continuity is then straightforward to check.

If $x' \in \pi_{s-1}^{-1}(y)$ is another lift of $y$, then there is some $a \in A_s$ such that $x' = a.x$ (see [GMV16a, Theorem 5.4(i)]). But $t_a$ is an element of $\text{Aut}_s(X)$ (Proposition 3.1), and this commutes with $\phi$ (Proposition 2.3 again, and Proposition 2.8). So, $\phi(x') = a.\phi(x)$ and $\pi_{s-1}(\phi(x')) = \pi_{s-1}(a.\phi(x)) = \pi_{s-1}(\phi(x))$ since $t_a$ preserves fibers of $\pi_{s-1}$. This shows well-definedness.

Now suppose that $c \in C^n(\pi_{s-1}(X))$, and suppose that $F \subseteq \{0,1\}^n$, a face of codimension $k$, is given. By definition there is a cube $\hat{c} \in C^n(X)$ such that $\pi_{s-1}(\hat{c}) = c$. So,

$$\pi_s(\phi)(F).c = \pi_{s-1}((\phi)(F).\hat{c})$$

and the right-hand side is a cube as required, since $\phi$ is a $k$-translation and $\pi_{s-1}$ is a cubespaces morphism. \qed

The statement for identity components follows trivially from this, i.e., $\pi_s(\text{Aut}_k^0(X)) \subseteq \text{Aut}_k^0(\pi_{s-1}(X))$.

The “translation lifting” statement we require is precisely that this group homomorphism is surjective. This is not quite true in general. However, it is the case that any sufficiently small element of $\text{Aut}_k(\pi_{s-1}(X))$ can be lifted to a $k$-translation on $X$, i.e., lies in the image of $\pi_s$. In fact, we will show the following.

**Proposition 3.3.** Let $X$ be a (compact, ergodic) nilspace of degree $s$ whose structure groups $A_s(X)$ are all Lie. For any $k \geq 1$, the homomorphism $\pi_s : \text{Aut}_k(X) \to \text{Aut}_k(\pi_{s-1}(X))$ is an open map.
In particular, the image of \( \pi_s \) is an open subgroup of \( \text{Aut}_k(\pi_{s-1}(X)) \), meaning exactly that all sufficiently small translations on \( \pi_{s-1}(X) \) lift to \( X \). More precisely, there exists a \( \delta > 0 \) (depending only on \( X \) and \( k \)) such that if \( \phi \in \text{Aut}_k(\pi_{s-1}(X)) \) is a “perturbation of the identity” in the sense that \( \| \phi \| \leq \delta \), then there is a lift \( \tilde{\phi} \in \text{Aut}_k(X) \) such that \( \pi_s(\tilde{\phi}) = \phi \), i.e., \( \pi_{s-1} \circ \tilde{\phi} = \phi \circ \pi_{s-1} \).

Proposition 3.3 is at the heart of the argument and will occupy us for the majority of the rest of the paper. We will return to its proof in section 4.

For the remainder of this section, we will fill in the outstanding gaps in the deduction of Theorem 2.18, and the other results stated in section 2.5, from Proposition 3.3.

Some of this work is concerned with making the previous discussion rigorous, and also with extending all mention of transitivity of the action of \( \text{Aut}_k(\pi_{s-1}(X)) \) to the higher transitivity required to get the corresponding statement about cubes.

A large part of the rest is of a strongly topological nature: we will have to check that certain groups are Lie groups, certain maps are open maps, and so forth. To some extent, this material is technical and could be skipped on first reading. However, we caution that it cannot easily be separated from the rest of the argument: without this topological input, we could not prove even a much weakened version of Theorem 2.18.

Here and later in the paper we will have to draw on a couple of fairly powerful results about topological groups: Gleason’s lemma (Theorem 4.1) concerning principal bundles, and the Gleason–Kuranishi extension theorem (Theorem 3.6) that asserts that an extension of a Lie group by a Lie group is Lie.

However, we remark that we will not use the full power of either of these. Specifically, Gleason’s lemma is required only in the compact abelian Lie setting, where the proof is essentially an application of Fourier analysis. For the Gleason–Kuranishi extension theorem, we require only the case of central extensions. Both of these facts are given a self-contained treatment in [Tao14] which is substantially easier than the original references. See Remark 3.8 for more details.

3.1. The stabilizer is discrete. To construct our nilmanifold \( G/\Gamma \), recall that we plan to use \( G = \text{Aut}_k(\pi_{s-1}(X)) \) and \( \Gamma = \text{Stab}(x_0) \) for some \( x_0 \in X \). So, we will need to know that this \( \Gamma \) is discrete.

**Lemma 3.4.** Let \( X \) be a (compact, ergodic) nilspace of degree \( s \) whose structure groups \( A_r(X) \) are Lie, and fix any \( x_0 \in X \). Then \( \text{Stab}(x_0) \subseteq \text{Aut}_1(X) \) is discrete.

Equivalently (since \( \text{Aut}_1(X) \) carries the supremum metric), we wish to show that if \( \phi \) is a 1-translation other than the identity and \( \phi(x_0) = x_0 \), then \( \| \phi \| = \sup_{x \in X} d(x, \phi(x)) \geq \delta \) for some constant \( \delta \) independent of \( \phi \) (and in fact \( x_0 \)).

**Proof of Lemma 3.4.** We proceed by induction on \( s \). The case \( s = 0 \) is a triviality (since a 0-step compact ergodic nilspace is just the one-point space \( \{ \ast \} \)).

Again, we consider \( \pi_s : \text{Aut}_1(X) \to \text{Aut}_1(\pi_{s-1}(X)) \). It is clear from the definition that \( \pi_s \) maps \( \text{Stab}(x_0) \) onto \( \text{Stab}(\pi_{s-1}(x_0)) \). By inductive hypothesis \( \text{Stab}(\pi_{s-1}(x_0)) \) is discrete, so we conclude that if \( \phi \in \text{Aut}_1(X) \) and \( \| \phi \| \) is sufficiently small, then \( \phi \in \ker(\pi_s) \). Equivalently, \( y \) and \( \phi(y) \) lie in the same fiber for all \( y \in X \).

By the weak structure theory, for each \( y \in X \) there is a unique \( a \in A_s \) such that \( \phi(y) = a.y \). Hence, there is a function \( \tau : X \to A_s \) such that \( \phi(y) = \tau(y).y \). It is
clear that $\tau$ is continuous. Since $\phi$ commutes with the action of $A_s \subseteq \text{Aut}_s(X)$ (by Propositions 3.1 and 2.8), in fact $\tau$ factors through $\pi_{s-1}$, i.e., as a map $X \to \pi_{s-1}(X) \to A_s$. By abuse of notation we write $\tau$ for this map $\pi_{s-1}(X) \to A_s$.

We now want to unwrap the fact that $\phi$ is a 1-translation to a condition on $\tau$. By Proposition 2.13, this is equivalent to saying that $[c, \tau(c), c] \in C^{s+1}(X)$ for every $c = C^s(X)$ (where $\tau(c)$ denotes a pointwise application). By the weak structure theorem [GMV16a, Theorem 5.4] this is equivalent to saying that

$$[\pi^s(0), \tau(c)] \in C^{s+1}(D_s(A_s(X)))$$

for every $c \in C^m(X)$.

We recall that $D_s(A_s)$ is the cubespace whose base space is the group $A_s$ and whose cubes are $HK^\pi(A_s)$, where $A_s$ is given the degree $s$ filtration $A_s = \cdots = A_1 \supseteq \{0\}$. We also recall ([GMV16a, Proposition 5.1]) that a configuration $a: \{0, 1\}^{s+1} \to A_s$ lies in $C^{s+1}(D_s(A_s))$ if and only if

$$\sum_{\omega \in \{0, 1\}^{s+1}} (-1)^{|\omega|} a(\omega) = 0 .$$

It follows that $\phi$ is a 1-translation if and only if the function

$$\rho: C^s(\pi_{s-1}(X)) \to A_s, \quad c \mapsto \sum_{\omega \in \{0, 1\}^s} (-1)^{|\omega|} \tau(c)(\omega)$$

is identically 0.

We will see in section 5 that this is a natural statement in terms of the rudimentary cohomology theory expounded there. Moreover, if $\|\phi\|$ is sufficiently small (i.e., less than some absolute constant), then $\tau$ also takes values close to 0 in $A_s$ (i.e., $d(0, \tau(y)) \leq \varepsilon$ for any fixed small $\varepsilon$ and for all $y$'s). Under these conditions a rigidity result, Theorem 5.2, states that $\tau$ must in fact be constant.

Since $\phi(x_0) = x_0$ and hence $\tau(x_0) = 0$ by assumption, we must therefore have $\tau$ be identically 0. But then $\phi = \text{id}$, as required. \hfill $\Box$

3.2. The group $\text{Aut}_1(X)$ is Lie. Recall that we wish to use $\text{Aut}_1^G(X)$ as the group $G$ in the definition of a nilmanifold $G/T$. Hence, it will be important to verify that this is a Lie group. In fact, we will prove that $\text{Aut}_1(X)$ is a Lie group, which implies that $\text{Aut}_1^G(X)$ is also a Lie group.

Once again, we stress that our definitions do not require a Lie group to be connected.

**Lemma 3.5.** Let $X = (X, C^s(X))$ be a (compact, ergodic) nilspace of degree $s$ whose structure groups $A_s(X)$ are Lie. Then $\text{Aut}_1(X)$ is a Lie group.

Our approach will be to apply induction on $s$, with the case $s = 0$ being a trivial base case. We again consider the map

$$\pi_s: \text{Aut}_1(X) \to \text{Aut}_1(\pi_{s-1}(X)).$$

Since $\pi_{s-1}(X)$ is a compact, ergodic nilspace with a degree of at most $(s-1)$, by inductive hypothesis $\text{Aut}_1(\pi_{s-1}(X))$ is Lie. By Proposition 3.3 the image of $\pi_s$ is an open subgroup and so is also Lie.

We will use the following result.
Theorem 3.6 (Gleason–Kuranishi extension theorem). Let $G$ be a topological group, and suppose that there exists a closed normal subgroup $N$ of $G$ such that $N$ and $G/N$ are Lie groups. Then $G$ is also Lie.

Given this, it suffices to verify that $\ker(\pi_s) \leq \text{Aut}_1(X)$ is a Lie group. We know that the copy of $A_s(X)$ in $\text{Aut}_1(X)$ (acting by $t_a : x \mapsto a.x$) is contained in $\ker(\pi_s)$ since it acts on each fiber of $\pi_{s-1}$. Hence, it will suffice to show the following.

Lemma 3.7. The subgroup $A_s(X) \leq \ker(\pi_s)$ is open.

Proof. Suppose that $\phi \in \ker(\pi_s) \setminus A_s$. Fix any point $y \in X$. Since $y$ and $\phi(y)$ lie in the same fiber of $\pi_{s-1}$, there is an unique $a \in A_s(X)$ such that $y = a.\phi(y)$. As before, write $t_a$ for the translation $x \mapsto a.x$ on $X$, and define $\phi'(y) = t_a \circ \phi(y)$; hence, $\phi'(x) = x$, so $\phi' \in \text{Stab}(x) \subseteq \text{Aut}_1(X)$.

By Lemma 3.4 $\|\phi'\| \geq \delta$ for some constant $\delta$ independent of $\phi$. If $\|t_a\| \leq \delta/2$, then $\|\phi\| \geq \delta - \delta/2 = \delta/2$ and we are done. If not, i.e., $\|t_a\| > \delta/2$, then since $A_s(X)$ is compact and acts freely on $X$, we have $d(x, t_a(x))$ bounded below by a constant depending only on $\delta/2$, and hence $\|\phi\| \geq d(x, t_a(x))$ is bounded below as required.

We noted above (Remark 2.6) that $\text{Aut}(X)$ is second countable, and hence clearly so is the closed subgroup $\ker(\pi_s)$. Moreover, $A_s(X)$ is an open subgroup of $\ker(\pi_s)$ that is Lie, and so by standard results in Lie theory we may extend the differentiable structure on $A_s(X)$ to one on all of $\ker(\pi_s)$, making the latter into a Lie group. This completes the proof of Lemma 3.7.

Remark 3.8. Noting that the copy of $A_s(X)$ in $\text{Aut}_1(X)$ is central, we have only really used the fact that a central extension of a Lie group by a Lie group is Lie, and that a discrete extension of a Lie group is Lie. The latter fact is straightforward given elementary Lie theory (e.g., assuming something of equivalent strength to Cartan’s closed subgroup theorem). Hence, we may rely on the proof in [Tao14, Theorem 2.6.1] rather than the one in [Gle51] in this special case, but we have not pursued this.

3.3. Completing the proof of Theorem 2.18. We now have all the ingredients in place to deduce Theorem 2.18 as well as the other results stated in subsection 2.5.

Recall that we wished to know that $\text{Aut}_1(X)$ acts transitively on $X$ under suitable hypotheses. Again, it turns out that the most natural statement of this is in terms of openness of some map.

Lemma 3.9. Let $X$ be a (compact, ergodic) nilspace of degree $s$ whose structure groups $A_r(X)$ are Lie, and let $x_0 \in X$ be fixed. Then the evaluation map

\[
ev_{x_0} : \text{Aut}_1(X) \to X,
\]

\[
\phi \mapsto \phi(x_0)
\]

is continuous and open.
Equivalently, this states that if two points \( x, y \in X \) are very close together, then there is a small 1-translation \( \phi \) such that \( \phi(x) = y \).

For inductive reasons we will verify the following slight strengthening.

**Lemma 3.10.** Let \( X \) be as in Lemma 3.9. Let \( 0 \leq k < s \) be fixed. For all \( \varepsilon \)'s > 0 there exists \( \delta > 0 \) such that the following holds: if \( x, y \in X \), \( d(x, y) \leq \varepsilon \) and \( \pi_k(x) = \pi_k(y) \), then there exists a \( \phi \in \text{Aut}_{k+1}(X) \), \( \|\phi\| \leq \varepsilon \) such that \( \phi(x) = y \).

**Proof.** Essentially, this just combines Propositions 3.1 and 3.3. We proceed by induction on \( s \), and we note that the case \( s = 0 \) is trivial.

If \( k = s - 1 \), then the result is immediate from Proposition 3.1, we know \( \pi_{s-1}(x) = \pi_{s-1}(y) \), so we can set \( \phi = t_a \) for the unique \( a \in A_s \) such that \( a.x = y \), and so \( \phi \in \text{Aut}_s(X) \), as required. The bound on \( \|t_a\| \) follows from Proposition 3.1.

If \( k < s - 1 \), we apply induction on \( s \) to obtain \( \phi \in \text{Aut}_{k+1}(\pi_{s-1}(X)) \) such that \( \phi(\pi_{s-1}(x)) = \pi_{s-1}(y) \), and \( \|\phi\| \) can be made arbitrarily small if \( d(x, y) \) is small enough.

By Proposition 3.3, we may obtain a lift \( \phi' \in \text{Aut}_{k+1}(X) \) of \( \phi \) (i.e., such that \( \pi_s(\phi'(x)) = \pi_s(y) \)) where again \( \|\phi'\| \) can made as small as we like.

We now know that \( \pi_{s-1}(\phi'(x)) = \pi_{s-1}(y) \). Finally, we note that \( d(\phi'(x), y) \leq d(x, y) + \|\phi'\| \), which is still as small as we like, so we again apply the case \( k = s - 1 \) to obtain \( \psi \in \text{Aut}_s(X) \) such that \( \psi(\phi'(x)) = y \). So, \( \phi := \psi \circ \phi' \) is in \( \text{Aut}_{k+1}(X) \) and is still arbitrarily small.

**Proof of Lemma 3.10.** Continuity is immediate by the choice of metric on \( \text{Aut}_1(X) \). Since \( \pi_0(X) = \{s\} \), openness follows immediately from the case \( k = 0 \) of Lemma 3.10.

Our final conclusions are phrased in terms of the identity component \( \text{Aut}_1^0(X) \). It is fairly painless to deduce useful facts about this from what we have already shown about \( \text{Aut}_1(X) \).

**Corollary 3.11.** Let \( X \) be as in the statement of Lemma 3.9. Then the identity component \( \text{Aut}_1^0(X) \) acts transitively on each connected component of \( X \).

Moreover, there are only finitely many connected components of \( X \), and each one is open and closed in \( X \).

**Proof.** Since \( \text{Aut}_1(X) \) is Lie (Lemma 3.5), its identity component \( \text{Aut}_1^0(X) \) is open. Therefore by Lemma 3.9 the orbits of \( \text{Aut}_1^0(X) \) in \( X \) are open, and hence closed (since the orbits partition the space). It follows that any connected subset of \( X \) is contained in a single orbit.

Since each orbit of \( \text{Aut}_1^0(X) \) is the image of a connected set under a continuous map and hence is connected, we further deduce that every point of \( X \) has a connected open neighborhood, so all connected components are open. Since they partition the space, they are therefore closed; since \( X \) is compact, it follows that there are only finitely many.

Combined with Lemmas 3.5 and 3.4 this is enough to complete the proof of Theorem 2.19.

We now turn to the outstanding statements from Theorem 2.18. Essentially, we still have said nothing about the cubes \( C^n(X) \) in terms of the filtration \( \text{Aut}_k(X) \).
To do so, we need to generalize several of the arguments above that concerned 1-translations acting on $X$ to statements about the Host–Kra cube group $HK^n(Aut_s(X))$ acting on $C^n(X)$.

For instance, the following generalizes Lemma 3.9.

**Lemma 3.12.** Let $X$ be as in Lemma 3.9 and fix $c \in C^n(X)$. Then the evaluation map

$$ev_c: HK^n(Aut_s(X)) \to C^n(X),$$

$$(\phi_\omega)_{\omega \in \{0,1\}^n} \mapsto (\omega \mapsto \phi_\omega(c(\omega)))$$

is open.

**Remark 3.13.** Recall that this definition makes sense by virtue of Proposition 2.17.

Note that this lemma (as well as the previous ones) requires no connectivity assumptions. Hence, this lemma provides a lot of information about cubes in not necessarily strongly connected nilspaces. In particular, it implies that all sufficiently small cubes can be obtained from a constant cube using the action of the Host–Kra cubegroups of $Aut_s(X)$. On the other hand, we have no information about large cubes, which is why we need the connectivity hypotheses in the main results.

The missing ingredients that prevent us from simply repeating the proof of Lemma 3.9 are all concerned with topological facts about Host–Kra cubegroups. Everything we will need follows cleanly in turn from the algebraic theory of $HK^n(G_\bullet)$ expounded on in [GMV16a, Appendix]. We recall these results now.

**Lemma 3.14 ([GMV16a, Lemma A.12]).** Let $G_\bullet, H_\bullet$ be two filtered topological groups. Let $\tau: G \to H$ be a homomorphism such that $\tau(G_i) \subseteq \tau(H_i)$. Then $\tau$ induces a homomorphism $\tau: HK^n(G_\bullet) \to HK^n(H_\bullet)$ for each $n$ by pointwise application on the vertices.

If $\tau: G_i \to H_i$ is open for each $i$, then so is the induced homomorphism $\tau: HK^n(G_\bullet) \to HK^n(H_\bullet)$ for each $n$.

**Lemma 3.15 ([GMV16a, Lemma A.13]).** Let $G_\bullet, H_\bullet$ be two filtered topological groups. Suppose $G_i \subseteq H_i$ for each $i$. If $G_i$ are open in $H_i$ (resp., connected) for each $i$, then $HK^n(G_\bullet)$ is also open in $HK^n(H_\bullet)$ (resp., connected) for each $n$.

We note the following consequence.

**Lemma 3.16.** Let $X$ be a (compact, ergodic) nilspace of degree $s$ whose structure groups $A_r(X)$ are Lie groups. Let $\pi_{s-1}: X \to \pi_{s-1}(X)$ denote the canonical projection, and let $\pi_s: Aut_k(X) \to Aut_k(\pi_{s-1}(X))$ denote the canonical homomorphism described previously.

Then

$$\pi_s: HK^n(Aut_\bullet(X)) \to HK^n(Aut_\bullet(\pi_{s-1}(X)))$$

is an open map.

**Proof.** This follows from Proposition 3.3 and Lemma 3.14.

**Proof of Lemma 3.12.** We follow the proof of Lemma 3.9 very closely; in fact, that result corresponds precisely to the case $n = 0$ of this. As ever, we induct on $s$, the degree of $X$, with the case $s = 0$ being trivial.

It suffices to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $c_1, c_2 \in C^n(X)$ and $d(c_1, c_2) \leq \delta$, then there exists a $\phi \in HK^n(Aut_\bullet(X))$, $\|\phi\| \leq \varepsilon$ such that $\phi(c_1) = c_2$. 

By inductive hypothesis the map $\text{ev}_{\pi_{s-1}(c)}: HK^n(\text{Aut}_s(\pi_{s-1}(X))) \to C^n(\pi_{s-1}(X))$ is open, and by Lemma 3.10 so is $\pi_s: HK^n(\text{Aut}_s(X)) \to HK^n(\text{Aut}_s(\pi_{s-1}(X)))$. As before, we consider the composite and deduce that, for a suitably chosen $\delta$, there exists a $\phi' \in HK^n(\text{Aut}_s(X))$, $\|\phi'\| \leq \eta$ such that $\pi_{s-1}(\phi'(c_1)) = \pi_{s-1}(c_2)$, where $0 < \eta \leq \varepsilon/2$ is some parameter to be determined.

Now we have $\delta(\phi'(c_1), c_2) \leq \delta + \eta$ and moreover these lie in the same fiber of $\pi_{s-1}$. But we recall that the fibers of $C^n(X) \to C^n(\pi_{s-1}(X))$ are completely described by the weak structure theory ([GMV16a, Theorem 5.4(ii)]): specifically, $C^n(\text{D}_s(A_n(X)))$ acts simply transitively on each fiber. We again use $t_a$ to denote this action and crucially observe that it corresponds to an element of $HK^n(\text{Aut}_s(X))$ (since $C^n(\text{D}_s(A_n(X)))$ is defined as the Host–Kra group on $A_n(X)$ with the degree $s$ filtration, which is a subfiltration of $\text{Aut}_s(X)$.

So, there is an unique $a \in C^n(\text{D}_s(A_n(X)))$ such that $t_a(\phi'(c_1)) = c_2$. By Proposition 3.12 for a suitably chosen $\eta$ and $\delta$, we may ensure that $\|t_a\| \leq \varepsilon/2$. Hence, taking $\phi = t_a \circ \phi'$, we have $\phi(x) = y$ and $\|\phi\| \leq \varepsilon$, as required.

Finally, we can conclude the proof of Theorem 2.18.

Proof of what is left of Theorem 2.18. What is left to prove is that $HK^n(\text{Aut}_s^0(X))$ acts transitively on $C^n(X)$, using in particular that $X$ is strongly connected.

Since $\text{Aut}_k(X)$ is a closed subgroup of $\text{Aut}_1(X)$, it is Lie (by Cartan’s closed subgroup theorem), and hence $\text{Aut}_k^0(X)$ is open in $\text{Aut}_k(X)$. By Lemma 3.12 it follows that $HK^n(\text{Aut}_k^0(X))$ is an open subgroup of $HK^n(\text{Aut}_k(X))$. Hence, by Lemma 3.12 the orbits of the action of this group on $C^n(X)$ are open, and hence closed (as they partition the space). But since $C^n(X)$ was assumed to be connected, this means the action is transitive, as required.

Given this, it follows that the map $\text{ev}_{\pi\{n(x_0)}: HK^n(\text{Aut}_s^0(X)) \to C^n(X)$ given by $(\phi_\omega)_{\omega \in \{0, 1\}^n} \mapsto (\omega \mapsto \phi_\omega(x_0))$ is continuous, open, and surjective and so induces a homeomorphism $HK^n(\text{Aut}_s^0(X))/\text{Stab}(x_0) \cong C^n(X)$ for each $n \geq 0$. In particular, when $n = 0$ we have $\text{Aut}_s^0(X)/\text{Stab}(x_0) \cong X$.

Finally, we must check to see that $\text{Stab}(x_0) \cap \text{Aut}_s^0(X)$ is co-compact in $\text{Aut}_s^0(X)$ for all $k's \geq 1$. By Lemma 3.10 the orbits of the action of $\text{Aut}_s^0(X)$ on the closed equivalence class $\pi_{s-1}^{-1}(\pi_{s-1}(x_0))$ are open (noting that $\text{Aut}_s^0(X)$ is open in $\text{Aut}_k(X)$ as above). Hence, these orbits are also closed (as they partition the space) and therefore compact. If $S$ is the orbit containing $x_0$, then the restricted evaluation map $\text{ev}_{x_0}: \text{Aut}_k(X) \to S$ is continuous, open, and surjective and so induces an homeomorphism $\text{Aut}_s^0(X)/(\text{Stab}(x_0) \cap \text{Aut}_s^0(X)) \cong S$, so the left-hand side is compact, as required.

4. Enough $k$-translations

We now return to the proof of Proposition 3.3 the translation lifting statement.

We recall the setup. We have a (compact, ergodic) nilspace $X = (X, C^n(X))$ of degree $s$ and are considering the canonical factor map $\pi_{s-1}: X \to \pi_{s-1}(X)$. For notational brevity we write $\pi$ for $\pi_{s-1}$ and $\pi(X)$ for $\pi_{s-1}(X)$ for the remainder of this section. We are assuming in particular that the top structure group $A_s(X)$ is a Lie group.

We wish to show the following: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\phi \in \text{Aut}_k(\pi(X))$ with $\|\phi\| \leq \delta$, there exists $\hat{\phi} \in \text{Aut}_k(X)$ with $\|\hat{\phi}\| \leq \varepsilon$ such that $\pi \circ \hat{\phi} = \phi \circ \pi$.  

The strategy is roughly as follows.

1. We first seek some function $\psi: X \to X$ such that $\pi \circ \psi = \phi \circ \pi$ which behaves nicely with respect to the action of $A_s(X)$. We will also be able to choose $\psi$ to be small. Crucially, though, it may not be a $k$-translation on $X$.

2. We then argue that we can “repair” $\psi$ to be a genuine $k$-translation $\tilde{\phi}$. Since necessarily $\pi \circ \psi = \pi \circ \tilde{\phi}$, this amounts to finding a function $f: X \to A_s$ from $X$ to the structure group $A_s$ and setting $\tilde{\phi}(x) = f(x)\psi(x)$ (where as usual this denotes the action of $A_s(X)$ on $X$).

Here we deviate from the approach taken in [ACS12]. Both proofs rely on the cocycle theory developed in [ACS12] and expounded upon in section 5. (This is step (2) in the above program.) However, the lifts of small translations are constructed in different ways. The main difference is that in [ACS12] a measurable, but not necessarily continuous, lift is constructed first, while our argument stays in the continuous category.

We recall that $A_s(X)$ acts freely on $X$, with orbits being precisely the fibers of $\pi$. The standard name for this setup is that $X \to \pi(X)$ is a principal bundle or $A_s$-principal bundle. However, this will not typically mean that $X$ is homeomorphic as a topological space to $A_s \times \pi(X)$, i.e., this bundle need not be the trivial bundle. In our setting it turns out that this bundle is nonetheless locally trivial in the sense that it locally resembles such a direct product. This result is due to Gleason.

**Theorem 4.1 ([Gle50], Theorem 3.3]).** Suppose that $A$ is a compact Lie group acting freely on a completely regular topological space $Y$. Let $\tau: Y \to Y/A$ denote the quotient map. Then for every point $x \in Y/A$ there is a neighborhood $U$ of $x$ and a local section $\sigma: U \to \tau^{-1}(U)$, i.e., a continuous map such that $\tau \circ \sigma = \text{id}_U$.

Although this is a significant result, we remark that the proof simplifies somewhat when the acting group $A$ is abelian, which is the only case we use.

The next result is the key ingredient for part (1) of our argument. It can be extracted from the proof of the first covering homotopy theorem ([Ste51], Theorem 11.3]), but we include a proof for completeness.

**Lemma 4.2.** Let $X$ be a (compact, ergodic) nilspace of degree $s$ whose structure group $A_s(X)$ is Lie. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds. Given any homeomorphism $f: \pi(X) \to \pi(X)$ with $\|f\| \leq \delta$, there exists a homeomorphism $f': X \to X$ such that

1. $\|f'\| \leq \varepsilon$,
2. $f'$ is a lift of $f$, i.e., $\pi \circ f' = f \circ \pi$, and
3. $f'$ is a bundle map, meaning that $f'(a.x) = a.f'(x)$ for all $a \in A_s(X)$.

**Proof.** Let $\varepsilon > 0$ be fixed. By Theorem 4.1 we may choose an open cover $\{U_i\}_{i=1}^m$ of $\pi(X)$ and a family of local sections $\sigma_i: U_i \to \pi^{-1}(U_i)$. We may assume that $\sigma_i$'s are uniformly continuous, so for any fixed $\eta > 0$ (to be specified later) we may refine the cover if necessary so that $d(\sigma_i(x), \sigma_i(x')) \leq \eta$ for all $i$'s and $x, x' \in U_i$.

We now choose $\delta$ to be a Lebesgue number for the cover $U_i$; i.e., for all $x \in \pi(X)$ there is an $i$ such that $\{y: d(x, y) \leq \delta\} \subseteq U_i$. Write $U'_i$ for the set of all $x$'s such that $\{y: d(x, y) \leq \delta\} \subseteq U_i$; this is another open cover of $\pi(X)$.

We would like to define $f'$ as follows. Given that $x \in \pi^{-1}(U'_i)$, there is an unique element $a_i(x) \in A_s(X)$ such that $x = a_i(x).\sigma_i(\pi(x))$. (If we use the local section $\sigma_i$
to identify $\pi^{-1}(U_i')$ with $U_i' \times A_s(X)$, then $a_i$ is simply the projection to the $A_s(X)$ component.) Then we set for $x \in U_i'$

$$f'_i(x) = a_i(x) \cdot \sigma_i(f(\pi(x))).$$

Note that $d(\pi(x), f(\pi(x))) \leq \delta$; hence, $f(\pi(x)) \in U_i'$ for all $x$'s in $U_i'$.

Note that $d(\sigma_i(\pi(x)), \sigma_i(f(\pi(x)))) \leq \eta$ by assumption, and by uniform continuity of the action of $A_s(X)$, we may choose $\eta$ so that

$$d(a \cdot \sigma_i(\pi(x)), a \cdot \sigma_i(f(\pi(x)))) \leq \varepsilon'/2$$

for all $a$'s in $A_s(X)$, where $\varepsilon'$ is some constant to be determined; so in particular when $a = a_i(x)$, we have

$$d(x, f'_i(x)) \leq \varepsilon'/2.$$ 

Properties (ii) and (iii) for $f'_i$ are clear by construction.

Unfortunately we are not done: $f'_i$ is defined only on $U_i'$, and the functions for different $i$'s are not compatible. So, we have not constructed a function on all of $X$ with the desired properties.

To fix this, we will take an average of the values $f'_i(x)$ at each point $x$, in a sense to be made precise. We first choose a continuous partition of unity $\nu_i : \pi(X) \to \mathbb{R}_{\geq 0}$ adapted to $U_i'$; that is,

$$\sum_{i=1}^{m} \nu_i(y) = 1$$

for all $y$'s in $\pi(X)$, and $\nu_i$ is supported on $U_i'$ for each $i$.

We now define

$$f'(x) = \sum_{i=1}^{m} \nu_i(\pi(x)) f'_i(x).$$

Strictly speaking, this definition makes no sense whatsoever: $f'_i(x)$ is an element of $X$ on which neither addition nor multiplication by real numbers is legitimate. We now justify what is actually meant by it as follows.

* For a fixed $x$ all of the points $\{f'_i(x) : 1 \leq i \leq m, \pi(x) \in U_i\}$ lie in the same fiber of $\pi$ and hence are related to each other by the action of elements of $A_s(X)$. To fix the notation, let $i_0$ be such that $\pi(x) \in U_{i_0}'$, and write $\alpha_i(x) \in A_s(X)$ for the unique element such that $\alpha_i(x) \cdot f'_{i_0}(x) = f'_i(x)$ for each $i$ such that $\pi(x) \in U_i'$.

* Since $d(f'_i(x), f'_j(x)) \leq \varepsilon'$ for all $i$'s, $j$'s, we find that the elements $\alpha_i(x) \in A_s(X)$ may be assumed to be small with respect to our preferred metric on $A_s(X)$ (by Proposition A.1).

* Write $A_s(X) \cong (\mathbb{R}/\mathbb{Z})^d \times K$ for some $d \geq 0$ and some finite group $K$. The $\alpha_i$'s all lie in some small neighborhood of the identity, which in turn is locally isomorphic to a small ball in $\mathbb{R}^d \times \{0\}$ for a small enough $\varepsilon'$. Write $\tilde{\alpha}_i(x) \in \mathbb{R}^d$ for the lift of $\alpha_i(x)$.

* It now makes sense to define

$$\tilde{\alpha}(x) = \sum_{i=1}^{m} \nu_i(x) \tilde{\alpha}_i(x) \in \mathbb{R}^d,$$
which also lies in the same small neighborhood of the identity, and then to let \( \alpha(x) \in (\mathbb{R}/\mathbb{Z})^d \times \{0\} \) be the projection in \( A_s(X) \). We let

\[
   f'(x) = \alpha(x).f'_s(x)
\]

for all \( x \)'s.

- Finally, we remark that this process was independent of the choice of \( i_0 \), and hence \( f' \) defines a function on all of \( X \).

In short, we are doing a kind of integration of functions with values in a compact abelian Lie group \( A_s(X) \), or more precisely on the space \( \pi^{-1}(f(x)) \) which is identified with \( A_s(X) \), and claiming this makes sense whenever all of the points being averaged are sufficiently close together. We will need to repeat this kind of process in what follows, and a further discussion of closely related issues appears in subsection 5.2.

We conclude by arguing that \( f' \) as defined has the required properties. Since \( f'(x) \) lies in the same fiber as each \( f'_s(x) \) by construction, it is clear that (ii) holds. Since the elements \( \alpha_s(x) \), and hence \( \alpha(x) \), are all arbitrarily close to \( 0 \in A_s(X) \) provided \( \epsilon' \) is sufficiently small, we can ensure that \( d(f'_s(x), \alpha(x).f'_s(x)) \leq \epsilon/2 \) and hence (i) holds, given the previous discussion. It is also not hard to argue that \( \alpha_s(x) \) and \( \alpha(x) \) are continuous functions of \( x \) for each choice of \( i_0 \), and hence \( f' \) is locally continuous and therefore continuous.

Property (iii) holds because this integration process commutes with translation by \( A_s(X) \); indeed, it is clear from the definitions that \( \alpha_s(x) = \alpha_i(a.x) \) for any \( a \in A_s(X) \). It is a consequence of (ii) and (iii) that \( f' \) is a bijection and hence a homeomorphism.

We have now completed part (1) of our strategy: we write \( \psi: X \to X \) for the function \( f' \) returned by applying Lemma 4.2 to \( \phi \). To recap, we know that

- \( \pi \circ \psi = \phi \circ \pi \);
- \( \|\psi\| \to 0 \) as \( \|\phi\| \to 0 \); and
- \( \psi \) commutes with \( A_s(X) \), i.e., \( \psi \circ t_a = t_a \circ \psi \) for all \( a \)'s \( \in A_s(X) \).

We now wish to attack part (2); that is, “fixing” \( \psi \) to make it into a \( k \)-translation. Just as in the proof of Lemma 3.3, it is helpful to unwrap what it would mean for a function \( \psi \) with these properties to be a \( k \)-translation on \( X \). More accurately, we want a condition that measures the failure of \( \psi \) to be a \( k \)-translation. The hope is then to construct a correction to this failure, and thereby to construct \( \phi \).

In the interests of generality, let \( \chi: X \to X \) denote an arbitrary homeomorphism such that \( \pi \circ \chi(x) = \phi \circ \pi(x) \) for all \( x \)'s \( \in X \).

By Proposition 2.13 we have \( \chi \) as a \( k \)-translation if and only if for every \( c \in C^{s+1-k}(X) \) we have \( \iota^k(c; \chi(c)) \in C^{s+1}(X) \).

By assumption the projection \( \pi(\iota^k(c; \chi(c))) \) is a cube of \( \pi(X) \). Hence—by the weak structure theory—the obstructions to this configuration that is a cube in \( X \) can be expressed in terms of the structure group \( A_s \). We recall the following precise statement of this.

**Proposition 4.3.** Let \( c \in C^{s+1}(X) \) be some cube, and let \( c': \{0,1\}^{s+1} \to X \) be a configuration such that \( \pi(c) = \pi(c') \). Let \( \alpha: \{0,1\}^{s+1} \to A_s(X) \) denote the unique configuration such that \( \alpha.c = c' \).
Then \(c' \in C^{s+1}(X)\) if and only if \(\alpha \in C^{s+1}(D_\alpha (A_\alpha (X)))\). This holds if and only if the alternating sum

\[
\sum_{\omega \in \{0,1\}^{s+1}} (-1)^{|\omega|}\alpha(\omega)
\]

is 0.

Proof. See [GMV16a, Theorem 5.4 and Proposition 5.1]. \(\square\)

We term this single element of \(A_\alpha (X)\) the discrepancy of the configuration, which we now define formally.

**Definition 4.4.** Let \(c' : \{0,1\}^{s+1} \to X\) be a configuration such that \(\pi(c') \in C^{s+1}(\pi(X))\). Let \(c\) be some element of \(C^{s+1}(X)\) such that \(\pi(c) = \pi(c')\) (which always exists by the definition of quotient cubes—see [GMV16a, Definition 5.2]).

As above, define \(\alpha : \{0,1\}^{s+1} \to A_\alpha (X)\) as the unique configuration such that \(\alpha \cdot c = c'\). Then we define the discrepancy \(\Delta(c')\) of \(c'\) by

\[
\Delta(c') = \sum_{\omega \in \{0,1\}^{s+1}} (-1)^{|\omega|}\alpha(\omega).
\]

**Proposition 4.5.** The discrepancy is well-defined, and \(\Delta(c') = 0\) if and only if \(c' \in C^{s+1}(X)\).

*Proof.* The second statement is taken directly from Proposition [1.3] To verify well-definedness, note that if \(c_1, c_2 \in C^{s+1}(X)\) are cubes with \(\pi(c_1) = \pi(c_2) = \pi(c')\) and \(\alpha_1, \alpha_2 : \{0,1\}^{s+1} \to A_\alpha (X)\) satisfies \(\alpha_1 \cdot c_1 = \alpha_2 \cdot c_2 = c'\), then \((\alpha_1 - \alpha_2) \cdot c_1 = c_2\) and, invoking the proposition again, we find that

\[
\sum_{\omega \in \{0,1\}^{s+1}} (-1)^{|\omega|} (\alpha_1(\omega) - \alpha_2(\omega)) = 0,
\]

as required. \(\square\)

We note one more elementary fact about discrepancies.

**Proposition 4.6.** The discrepancy is additive in the following sense: if \(c' = [c_0, c_1]\) and \(c'' = [c_1, c_2]\) are configurations such that \(\pi(c'), \pi(c'') \in C^{s+1}(\pi(X))\) are cubes, then \(\pi([c_0, c_2]) \in C^{s+1}(\pi(X))\) and \(\Delta([c_0, c_2]) = \Delta([c_0, c_1]) + \Delta((c_1, c_2))\).

*Proof.* The first fact is immediate from glueing (see [GMV16a, Proposition 6.2]). The discrepancy identity is clear from considering the alternating sums. \(\square\)

We now return to considering our function \(\chi\). With the above discussion in mind, we define a function

\[
\rho_\chi : C^{s+1-k}(X) \to A_\alpha (X),
\]

\(c \mapsto \Delta(\rho^{k}(c; \chi(c)))\),

i.e., the discrepancy of the configuration \(\rho^{k}(c; \chi(c))\) that arose while characterizing \(k\)-translations.

Our observations can be summarized by the following result.

**Lemma 4.7.** Given \(\chi\) as above, we have \(\chi\) being a \(k\)-translation if and only if \(\rho_\chi\) is identically 0.

*Proof.* This is immediate from Propositions [2.13, 4.5, 4.13] \(\square\)
So, the failure of \( \chi \) to be a \( k \)-translation is entirely captured by the function \( \rho_\psi \). Now suppose that we take our original function \( \psi: X \to X \) and attempt to repair it by setting
\[
\tilde{\phi}(x) = f(x) \cdot \psi(x)
\]
for some function \( f: X \to A_s(X) \), as suggested originally. Then by the lemma we will have succeeded if and only if \( \rho_\psi \equiv 0 \), which in turn holds if and only if for every \( c \in C^{s+1-k}(X) \),
\[
\rho_\psi(c) + \sum_{\omega \in \{0,1\}^{s+1-k}} (-1)^{|\omega|+k} f(c(\omega)) = 0.
\]

The crucial point is now the following. The function \( \rho_\psi \) is some kind of “cocycle”: a function on cubes with certain properties that we will describe. The above equation states that \( \rho_\psi \) is some kind of “coboundary”, i.e., equal to the “derivative”
\[
\pm \sum_{\omega \in \{0,1\}^{s+1-k}} (-1)^{|\omega|} f(c(\omega)).
\]

So, what we want is to show some kind of “cohomological triviality”: that every cocycle \( \rho_\psi \) will be a coboundary, and therefore that \( \psi \) can be corrected to a \( k \)-translation.

This will not always be true. However, it is true provided that \( \rho_\psi \) is sufficiently small in the sense of its image being contained in a small ball in \( A_s(X) \)\(^8\).

It remains only to define what we mean by a cocycle and a coboundary, and to state the cohomological triviality result that we will prove in the next section.

**Definition 4.8.** Let \( \ell \geq 0 \) be an integer, \( X \) a cubespace, and \( A \) an abelian group. By an \( \ell \)-cocycle on \( X \) with values in \( A \), we mean a continuous function
\[
\rho: C^\ell(X) \to A
\]
satisfying the following conditions:

1. (additivity) if \( c' = [c_0, c_1], c'' = [c_1, c_2], \) and \( c = [c_0, c_2] \) are all \( \ell \)-cubes, then 
   \[
   \rho(c) = \rho(c') + \rho(c'') ;
   \]
2. (reflections) if \( c = [c_0, c_1], \) then \( \rho(c) = -\rho([c_1, c_0]); \)
3. (degenerate cubes) if \( c = [c_0, c_0], \) then \( \rho(c) = 0. \)

We say \( \rho \) is a coboundary if there exists a continuous function \( f: X \to A \) such that
\[
\rho(c) = \partial^\ell f(c) := \sum_{\omega \in \{0,1\}^\ell} (-1)^{|\omega|} f(c(\omega)).
\]

**Remark 4.9.**
- A 0-cocycle is the same thing as a continuous function \( X \to A \).
- We stress that our rather vague notation allows the concatenation operation \([-,-]\) to occur on any coordinate \( \{1, \ldots, \ell\} \), not just the last one. For instance, for a 2-cococycle \( \rho \) it is the case that
  \[
  \rho([[a, b], [c, d]]) = -\rho([[c, d], [a, b]]) = -\rho([[b, a], [d, c]]) = \rho([[d, c], [b, a]]).
  \]

\(^8\)An equivalent statement is that the “cohomology in \( \mathbb{R}^d \)” is trivial, or that the “cohomology group is discrete” in an appropriate sense. We will not make these statements rigorous, though one could certainly do so.
Note that properties (i), (ii), (iii) are not logically independent; i.e., our definition is not minimal. Indeed, item (iii) is an immediate consequence of item (i) with the substitution \( c_0 = c_1 = c_2 = c_0 \). Then item (ii) can also be deduced by taking \( c_2 = c_0 \) in (i) and using (iii).

As well as taking the derivative \( \partial^k \) of a function on \( X \), one may analogously take the derivative of a function of cubes; e.g., given \( \rho : C^n(X) \to A \), we may write

\[
\partial \rho([c, c']) = \rho(c) - \rho(c').
\]

This definition does now depend implicitly on which coordinate \( \{1, \ldots, n+1\} \) is used for the concatenation; we will introduce more precise conventions below when this is necessary to avoid confusion.

It is trivial to verify that any coboundary is indeed a cocycle. More generally, if \( \rho \) is an \( \ell \)-cocycle, then \( \partial^k \rho \) is an \( (\ell + k) \)-cocycle.

**Lemma 4.10.** For any \( \chi \) having the properties discussed above, the function \( \rho_\chi \) is indeed a cocycle.

**Proof.** Additivity follows directly from the additivity of discrepancies, proven in Proposition [4.6]. The other two properties follow from this. \( \square \)

In summary, at long last, it will suffice to prove the following fact, which is a special case of Theorem [5.1].

**Theorem 4.11** ([ACS12 Lemma 3.19]). Suppose that \( X \) is a compact ergodic nilspace of degree \( s \), suppose that \( A \) is a compact abelian Lie group (equipped with a metric \( d_A \)), suppose that \( \ell \geq 0 \) is an integer, and suppose that \( \rho : C^\ell(X) \to A \) is an \( \ell \)-cocycle.

Then there exists an \( \varepsilon = \varepsilon(s, \ell, A) > 0 \), depending only on \( s \), \( \ell \), and \( A \) (but not on \( X \)), such that the following holds. Suppose that \( \delta \leq \varepsilon \) and \( d_A(\rho(c), \rho(c')) \leq \delta \) for all cases in which \( c, c' \in C^\ell(X) \). Then there exists an \( f : X \to A \) such that \( \rho = \partial^\ell f \) and also \( d_A(f(x), f(x')) \lesssim_s \delta \) for all cases in which \( x, x' \in X \).

We now bring everything together to conclude the proof of Proposition [5.2]. If the original map \( \phi \) has \( \|\phi\| \) being small enough, then \( \|\psi\| \) is also arbitrarily small. Hence, the cocycle \( \rho_\psi \) can be taken to be as small as we like with respect to the metric

\[
\sup_{c, c' \in C^\ell(X)} d_{A^c(X)}(\rho_\psi(c), \rho_\psi(c')).
\]

Indeed, if \( \psi \) is small, then \( \psi^k(c; \psi(c)) \) is close to a genuine cube \( \Box^k(c) \), and since the discrepancy map \( \Delta \) is continuous (where here we have used Proposition [A.1]), it follows that \( \rho_\psi(c) \) is close to 0 for every \( c \).

Hence, the correction \( f \) from Theorem [4.11] can be chosen to be arbitrarily small, meaning the final \( k \)-translation \( \hat{\phi} \in \text{Aut}_k(X) \) that we obtain is also arbitrarily small, as required.

5. **Cocycle theory**

The purpose of this section is to prove Theorem [4.11] concerning the triviality of small cocycles.

It will be necessary elsewhere in this project to have a version of this result in greater generality than is stated therein. The proof of this more general version is very similar to a direct proof of the specialized version; the only difference is that we
need to draw on the relative weak structure theory, as expounded upon in [GMV16a section 7], which generalizes (again in a fairly routine way) the absolute version from [ACS12] (or [GMV16a section 6]).

The technical generalization is as follows. The reader is advised to recall the notion of a fibration, and other related terminology, from [GMV16a section 7].

**Theorem 5.1.** Let $A$ be a compact abelian Lie group (equipped with a metric $d_A$), and let $s \geq 0$, $\ell \geq 1$ be given. Then there exists an $\varepsilon = \varepsilon(s, \ell, A) > 0$ such that the following holds.

Let $\beta : X \to Y$ be any fibration of degree $s$ between compact, ergodic cubespaces $X$ and $Y$ that obey the gluing axiom.

Now let $\rho$ be a continuous $\ell$-cocycle on $X$ with values in $A$, let $0 < \delta \leq \varepsilon$ be given, and suppose that $d(\rho(c), \rho(c')) \leq \delta$ whenever $\beta(c) = \beta(c')$.

Then $\rho = \partial^\ell f + \tilde{\rho} \circ \beta$, where $f : X \to A$ is continuous and satisfies $d(f(x), f(y)) \leq s, \ell \delta$ (that is, there exists a constant $c = c(s, \ell) > 0$ such that $d(f(x), f(y)) \leq c\delta$) whenever $\beta(x) = \beta(y)$, and $\tilde{\rho}$ is a continuous $\ell$-cocycle on $Y$.

(Recall that $\partial^\ell$ was defined in Definition 4.8.)

We briefly pause to explain why this does in fact generalize Theorem 4.11. Letting $X$, $A$, and $\rho$ be as in the statement of Theorem 4.11 we may invoke Theorem 5.1 where

- $X$ is still the same (compact, ergodic, degree $s$) nilspace $X$;
- $Y = \{\ast\}$, the one-point space;
- $\beta : X \to \{\ast\}$ is the trivial map; and
- the cocycle $\rho$ is unchanged.

The statement that $\beta$ is a fibration of finite degree is precisely saying that $X$ is a nilspace of finite degree. Hence, all of the hypotheses of Theorem 5.1 in this special case follow from those of Theorem 4.11. Given the conclusion of Theorem 5.1 note that the only $\ell$-cocycle $C^\ell(\{\ast\}) \to A$ is the zero function, so $\tilde{\rho} = 0$ and we recover the conclusion of Theorem 4.11.

Even though Theorem 5.1 does indeed generalize Theorem 4.11 since its proof will involve some additional notational and technical difficulties, we will continue to provide sketches of the argument in the special case of Theorem 4.11. The hope is that the intuition will be easier to grasp in this model setting.

Note that, in section 3, we needed a kind of dual to Theorem 4.11, stating that the function $f : X \to A$ that is used to construct the coboundary $\partial^\ell f$ is itself essentially uniquely determined. The precise statement is the following.

**Theorem 5.2.** Let $A$ be a compact abelian Lie group, equipped with the metric $d_A$, and let $s \geq 0$, $\ell \geq 1$ be given. Then there exists an $\varepsilon = \varepsilon(s, \ell, A) > 0$ such that the following holds.

Suppose $X = (X, C^n(X))$ is a compact, ergodic nilspace of degree $s$, and $f : X \to A$ is a continuous function such that $d_A(f(x), f(x')) \leq \varepsilon$ for all cases in which $x, x' \in X$, and $\partial^\ell f : C^\ell(X) \to A$ is the zero function.

Then $f$ is constant.

Again, we would like a version that holds in the setting of Theorem 5.1 concluding that the function $f : X \to A$ obtained there is essentially unique. It turns out that, in this case, this generalization follows easily from the original version.
Corollary 5.3. Suppose $A, \beta, X, Y$ are as in the statement of Theorem 5.1. Then there exists an $\varepsilon = \varepsilon(A) > 0$ such that the following holds.

Suppose $\ell \geq 0$ is an integer, and suppose that $f : X \to A$ is a continuous function such that $d_A(f(x), f(x')) \leq \varepsilon$ for all cases in which $x, x' \in X$ and furthermore in which $\partial^\ell f$ is constant on fibers of $\beta$ (meaning that if $c, c' \in C^\ell(X)$ and $\beta(c) = \beta(c')$, then $\partial^\ell f(c) = \partial^\ell f(c')$).

Then $f$ is also constant on fibers of $\beta$, i.e., $f = \tilde{f} \circ \beta$ for some continuous $\tilde{f} : Y \to A$.

Proof of Corollary 5.3 from Theorem 5.2. Given a $y \in Y$, its preimage $Z := f^{-1}(y) \subseteq X$ is a nilspace of degree $s$. Moreover, by hypothesis $\partial^\ell f$ is constant on $C^\ell(Z)$, and therefore 0 on $C^\ell(Z)$, since clearly $\partial^\ell f(\Box^\ell(z)) = 0$ for any $z \in Z$.

Applying Theorem 5.2, we deduce that $f$ is constant on $Z$. But since $y$ was arbitrary, this suffices. □

We will spend a while building up the proof of Theorem 5.1 in stages. The uniqueness result (Theorem 5.2) is somewhat easier, and we will return to this at the end.

5.1. The case $\ell = 1$. We briefly contemplate the first even slightly nontrivial case of Theorem 4.11, i.e., when $\ell = 1$. Since $X$ is ergodic, $C^1(X) = X \times X$, which means that we have a function

$$
\rho : X \times X \to A
$$

whose image is contained in a small ball of $A$, such that

$$
\rho([x, y]) + \rho([y, z]) = \rho([x, z])
$$

for all cases in which $x, y, z \in X$ (this is just from the definition of a 1-cocycle). Also, $\rho([x, y]) = -\rho([y, x])$ and $\rho([x, x]) = 0$ for all $x$'s, $y$'s.

Finally, recall that we wish to construct $f : X \to A$ such that $\rho([x, y]) = f(x) - f(y)$ for all cases in which $x, y \in X$. It turns out this is not a terribly deep statement; one can simply fix $x_0 \in X$ and define

$$
f(x) := \rho([x, x_0])
$$

noting that

$$
f(x) - f(y) = \rho([x, x_0]) - \rho([y, x_0]) = \rho([x, x_0]) + \rho(x_0, y) = \rho([x, y]).
$$

Observe that we have not even really used our “small image” assumption on $\rho$, or that $A$ is Lie, in this argument.

When we come to prove more general cases of the result, it will be important for inductive reasons that we can find a canonical function $f$ to define our coboundary; here, $f$ depends on the arbitrary choice of $x_0$. To set the scene for what follows, we note that we can achieve this if, rather than fixing one particular $x_0$, we instead average over all choices. That is, we define

$$
(2) \quad f(x) := \int_X \rho([x, y])dy,
$$
which is indeed canonical. To show that this is still a valid choice, we work through the calculation

\[
\begin{align*}
f(x) - f(y) &= \int_X \rho([x, z]) \, dz - \int_X \rho([y, z]) \, dz \\
&= \int_X \rho([x, z]) + \rho([z, y]) \, dz \\
&= \int_X \rho([x, y]) \, dz \\
&= \rho([x, y]).
\end{align*}
\]

However, we have overlooked two important subtleties here.

- We are integrating expressions with values in \( A \), a compact abelian Lie group.
- We have not defined a probability measure on the space \( X \), so the integration is currently meaningless.

The first item will be discussed in subsection 5.2. In brief, we can lift the integrand to the universal cover of (the respective connected component of) \( A \), then project back the result. It turns out that this can be defined canonically provided that the integrand takes values in a sufficiently small ball in \( A \).

The second item could be addressed using the weak structure theorem, which implies that \( X \) can be obtained by successive principal bundle extensions

\[
X \to \pi_{s-1}(X) \to \cdots \to \pi_0(X) = \{ \ast \}.
\]

This allows the construction of a probability measure on \( X \) from the Haar measures on the structure groups. This is the approach taken in [ACS12].

We found it easier to carry out the averaging in stages and prove Theorem 5.1 by a double induction on \( s \) and \( \ell \). In the special case \( \ell = 1 \) currently discussed, this takes the following shape. We put

\[
f' = \int_{A_s} \rho([x, a.x]) d\mu_{A_s}(a);
\]

i.e., we integrate only on the fiber of \( \pi_s \) that contains \( x \) using the Haar measure on the structure group \( A_s \).

It is no longer reasonable to expect that \( \partial f' \) agrees with \( \rho \), since the definition of \( f' \) depends only on the values of \( \rho \) on edges whose endpoints lie in the same fiber of \( \pi_{s-1} \). However, it turns out that \( \rho - \partial f' \) is constant on the fibers of \( \pi_{s-1} \), and hence it can be pushed down to a cocycle on \( \pi_{s-1}(X) \), and the argument can be completed by induction. We omit the details, but they are given in section 5.4 in a greater generality.

We finish this section with a comment on the proof in the general case \( \ell \geq 1 \). Antolín Camarena and Szegedy define the function

\[
f(x) = \int_{c \in C^\ell(X), c(\vec{0}) = x} \rho(c) dc
\]

using a system of suitable probability measures that are defined on the space of cubes whose vertex at \( \vec{0} \) is the point \( x \). (Compare this formula with (2).)

Our proof carries out the above averaging in several steps using a double induction on \( s \) and \( \ell \). If one combines the averaging in the inductive steps, the resulting formula will be the same as in the approach of Antolín Camarena and Szegedy.
However, we believe that the technical details of the argument become simpler in our approach. In particular, we can avoid any discussion of continuous systems of measures, and moreover the combinatorial properties we need to verify are also easier.

5.2. A remark on integration. We discuss now the issue raised above about integration of functions taking values in Lie groups.

Suppose that \((Y, \mu)\) is some probability space, and suppose that \(f : Y \to \mathbb{R}/\mathbb{Z}\) is some measurable function. It is clear that the integral \(\int f(x)d\mu(x)\) does not make sense in general, since averaging on \(\mathbb{R}/\mathbb{Z}\) is not well-defined.

However, suppose that we know that \(f\) takes values in some specified interval \((a, b) \subseteq \mathbb{R}/\mathbb{Z}\) of a width of at most \(1/10\) (say). Then we can make sense of \(\int f(x)d\mu(x)\) by identifying \((a, b) \subseteq \mathbb{R}/\mathbb{Z}\) bijectively with a suitable interval \((\tilde{a}, \tilde{b}) \subseteq \mathbb{R}\), lifting \(f\) to a function \(Y \to (\tilde{a}, \tilde{b})\), performing normal real integration, and then projecting the result (which lies in \((\tilde{a}, \tilde{b})\) by convexity) back down to \((a, b) \subseteq \mathbb{R}/\mathbb{Z}\). It is straightforward to check that this operation does not depend on the choice of \(\tilde{a}, \tilde{b}\).

We will abuse notation to write \(\int f(x)d\mu(x)\) for this element of \(\mathbb{R}/\mathbb{Z}\) whenever it makes sense to do so. Specifically, we will write this only when \(\sup\{|f(x) - f(y)| : x, y \in Y\} \leq \delta\) for some suitably small absolute constant \(\delta\) (in this case taken to be \(1/10\)).

More generally, the same remarks hold for any compact abelian Lie group \(A = (\mathbb{R}/\mathbb{Z})^d \times K\), with \(K\) being a finite group. For any reasonable metric on this space, integration is well-defined for functions whose image lies in a sufficiently small \(\delta\)-ball, with \(\delta\) depending only on \(A\), and the integral will also lie in the same \(\delta\)-ball.

For this to hold, we do need to make some choices about the metric we impose on \(A\). Specifically, it will be convenient to use a metric \(d_A\) on \(A \cong (\mathbb{R}/\mathbb{Z})^d \times K\) that is induced from the standard Euclidean metric on \(\mathbb{R}^d\) (and the standard discrete metric on \(K\)). This choice is not canonical because it requires us to fix an isomorphism \(A \cong (\mathbb{R}/\mathbb{Z})^d \times K\), though of course all metrics on a compact space are equivalent, so we can simply fix a choice and the ambiguity need not concern us.

**Definition 5.4.** For any compact abelian Lie group \(A\) that appears, we assume that an identification \(A \cong (\mathbb{R}/\mathbb{Z})^d \times K\) has been fixed and define

\[
d_A((t, k), (t', k')) = (1 - |k - k'|) + \inf \{ \|x - x\|_2 : x, x' \in \mathbb{R}^d, x, x' \equiv t, t' \pmod{1} \}.
\]

Similarly, we write \(\|\cdot\|_A\) for the “norm” \(d_A(0, -)\).

(Here the expression \(|k - k'|\) takes the value 1 if \(k = k'\) and 0 otherwise.)

Finally, we observe that the integral discussed above is finitely additive wherever this makes sense; i.e., if \(f, f', f + f'\) all have images within (possibly different) \(\delta\)-balls in \(A\), then \(\int (f + f') = \int f + \int f'\).

5.3. A slightly less easy case. As one further stepping stone towards Theorems 4.11 and 5.1 we consider the case where \(X = D_s(H)\) for some compact abelian group \(H\). Recall that this means that \(H\) carries the degree \(s\) filtration

\[
H = H_0 = H_1 = \cdots = H_s \supseteq \{0\},
\]

and \(C^k(X)\) represents the Host–Kra cubes \(HK^k(H_\bullet)\) with respect to this filtration. (We use additive notation for \(H\) and also for the Host–Kra groups we derive from it.)
We prove Theorem 4.11 in the special case considered in this section by induction on \( \ell \). In the inductive step, we are looking for an \((\ell - 1)\)-cocycle \( \rho' : C^{\ell - 1} \to A \) such that \( \partial \rho' = \rho \). Motivated by the previously discussed case of \( \ell = 1 \), we intend to define \( \rho'(c) \) by taking an average of \( \rho([c, c']) \) over the space
\[
\{ c' : [c, c'] \in C^\ell(X) \}.
\]

In order to compute the average, we need to specify a probability measure. Luckily, the space can be identified with a compact abelian Lie group, and we can make use of the Haar measure on this group. We denote
\[
T_1^\ell := \{ c : [0, 1]^{\ell - 1} \to H : [0, c] \in C^\ell(X) \}
\]
and observe that \([\bar{0}, T_1^\ell]\) is a closed subgroup of \( HK^\ell(H_*) \). It is clear that \([c_1, c_2] \in C^\ell(D_8(H))\) if and only if \( c_2 = c_1 + t \) for some \( t \in T_1^\ell \); indeed, the Host–Kra cube group is a group, so since \([c_1, c_2]\) and \([c_1, c_1]\) are both cubes, so is \([c_1, c_2] - [c_1, c_1] = [\bar{0}, c_2 - c_1] \) for \( t = c_2 - c_1 \) is in \( T_1^\ell \).

In fact, it turns out that \( T_1^\ell \) is just the Host–Kra cubespace of dimension \((\ell - 1)\) filtration, but we will not actually need to know this.

The special case of Theorem 4.11 considered in this section follows from the following proposition by induction on \( \ell \).

**Proposition 5.5.** Suppose that \( X = D_8(H) \) as above, suppose that \( A \) is a compact abelian Lie group, suppose that \( \ell \geq 1 \), and suppose that \( \rho : C^\ell(X) \to A \) is a cocycle such that \( d_A(\rho(c_1), \rho(c_2)) \leq \delta \) for all \( c_1 \)'s and \( c_2 \)'s, and for some suitably small \( \delta \)'s (i.e., \( \delta \leq \varepsilon(\ell, A) \)).

We define
\[
\rho' : C^{\ell - 1}(X) \to A,
\]
\[
c' \mapsto \int_{T_1^\ell} \rho([c', c' + t]) d\mu_{T_1^\ell}(t).
\]

Then
(i) the function \( \rho' \) is continuous and \( d_A(\rho'(c_1'), \rho'(c_2')) \leq \delta \) for all cases in which \( c_1', c_2' \in C^{\ell - 1}(X) \);
(ii) moreover, \( \rho' \) is an \((\ell - 1)\)-cocycle; and
(iii) we have \( \rho([c_1, c_2]) = \rho'(c_1) - \rho'(c_2) \) for all cubes \( c = [c_1, c_2] \in C^\ell(X) \).

**Proof.** Note that we are drawing on subsection 5.2 to make sense of the integral in the definition of \( \rho' \). To see (i), we note that
\[
\rho'(c_1') - \rho'(c_2') = \int_{T_1^\ell} \rho([c_1', c_1' + t]) - \rho([c_2', c_2' + t]) d\mu_{T_1^\ell}(t),
\]
and since \( \|\rho(c_1) - \rho(c_2)\|_A \leq \delta \) for all cases in which \( c_1, c_2 \in C^\ell(X) \) by assumption, the integrand is bounded in norm pointwise by \( \delta \); hence, so is the integral. Moreover, since the maps
\[
c' \mapsto [c', c' + t]
\]
are equicontinuous for \( t \in T_1^\ell \), and since \( \rho \) is uniformly continuous, we deduce that the integrand becomes arbitrarily small when \( d(c_1', c_2') \) is arbitrarily small. Again, an average of small values in \( A \) is small, so \( \rho' \) is (uniformly) continuous.
We now consider (ii). We need some further definitions. Let
\[ T_2^\ell := \{ c: \{0,1\}^{\ell-2} \to H : [\bar{0}, \bar{0}], [\bar{0}, c] \in C^\ell(H) \} \]
and note as before that \([t_0, t_1] \in T_2^1\) if and only if \(t_1 = t_0 + u\) for some \(u \in T_2^1\) (again, just using the fact that the Host–Kra cube group is a group). In other words, every \(t \in T_2^1\) has a unique decomposition as
\[ t = [v, v] + [\bar{0}, u] \]
for \(v \in T_1^{\ell-1}\) and \(u \in T_2^1\), and this establishes a group isomorphism \(T_2^1 \leftrightarrow T_1^{\ell-1} \times T_2^1\).

(Indeed, it is easy to see that \([0,0]\) has a unique decomposition as \([1,0] + [0,1]\) just using the fact that the Host–Kra cube group is a group). In other words, every \(t \in T_2^1\) has a unique decomposition as
\[ t = [v, v] + [\bar{0}, u] \]
for \(v \in T_1^{\ell-1}\) and \(u \in T_2^1\), and this establishes a group isomorphism \(T_2^1 \leftrightarrow T_1^{\ell-1} \times T_2^1\).

To prove the additivity of \(\rho\), suppose that \([c_0, c_1], [c_1, c_2] \in C^{\ell-1}(X)\), and write
\[
\rho'(\{c_0, c_2\}) = \int_{T_1^1} \rho([c_0, c_2], [c_0, c_2] + t) d\mu_{T_2^1}(t)
\]
\[
= \int_{T_1^{\ell-1} \times T_2^1} \rho([c_0, c_2], [c_0 + v, c_2 + u + v]) d\mu_{T_1^{\ell-1}}(v) d\mu_{T_2^1}(u)
\]
\[
= \int_{T_1^{\ell-1} \times T_2^1 \times T_2^1} \left( \rho([c_0, c_1], [c_0 + v, c_1 + u' + v]) + \rho([c_1, c_2], [c_1 + u' + v, c_2 + u + v]) \right) d\mu_{T_1^{\ell-1}}(v) d\mu_{T_2^1}(u) d\mu_{T_2^1}(u').
\]

In the last line we used the additivity of \(\rho\) and that the cubes in the last line can be glued along the common face \([c_1, c_1 + u' + v]\) to obtain the cube in the penultimate line.

We integrate out the first term and continue the calculation as follows.
\[
\rho'(\{c_0, c_2\}) = \rho'(\{c_0, c_1\}) + \int_{T_2^1} \left( \int_{T_1^{\ell-1} \times T_2^1} \rho([c_1, c_2], [c_1 + (v + u'), c_2 + (u - u') + (v + u')]) d\mu_{T_1^{\ell-1}}(v) d\mu_{T_2^1}(u) \right) d\mu_{T_2^1}(u')
\]
\[
= \rho'(\{c_0, c_1\}) + \int_{T_2^1} \left( \int_{T_1^{\ell-1} \times T_2^1} \rho([c_1, c_2], [c_1 + v'', c_2 + u'' + v'']) d\mu_{T_1^{\ell-1}}(v'') d\mu_{T_2^1}(u'') \right) d\mu_{T_2^1}(u')
\]
\[
= \rho'(\{c_0, c_1\}) + \rho'(\{c_1, c_2\}),
\]
as required. For the penultimate equation we used the substitution \(v'' = v + u'\) and \(u'' = u - u'\) and the translation invariance of the Haar measure.

We finish the proof with (iii). Given \([c_1, c_2] \in C^\ell(X)\), we may write \(c_2 = c_1 + t\) for some \(t \in T_1^1\) and then observe
\[
\rho([c_1, c_2]) = \int_{T_1^1} \left( \rho([c_1, c_1 + w]) + \rho([c_1 + w, c_1 + t]) \right) d\mu_{T_1^1}(w)
\]
\[
= \int_{T_1^1} \rho([c_1, c_1 + w]) d\mu_{T_1^1}(w) - \int_{T_1^1} \rho([c_1 + t, c_1 + w]) d\mu_{T_1^1}(w),
\]
5.4. The general case. The key ideas for the general case, Theorem 5.1, have already appeared in the special cases we have just discussed. The very vague strategy they suggest for proving the general case is as follows.

- As in subsection 5.3, we will perform an averaging over a space of cubes
  \[ \{c^\ell: [c, c'] \in C^\ell(X)\} \]
  to obtain an \((\ell - 1)\)-cocycle \(\rho'\) from an \(\ell\)-cocycle \(\rho\), such that \(\rho([c_1, c_2]) = \rho'(c_1) - \rho'(c_2)\). Iterating this argument \(\ell\) times will give the result.
- Unfortunately we do not currently have measures defined on these spaces, so we will do as is suggested in subsection 5.1 and appeal to the weak structure theory. This will enable us to express the averaging as a sequence of integrals over each of the structure groups \(A_s(X)\) in turn, or more accurately over Host–Kra configurations built out of these groups.

With some thought this approach is seen to be essentially equivalent to performing a double induction on both \(\ell\) (the order of the cocycle) and \(s\) (the degree of the nilspace \(X\) in Theorem 4.11, or of the fibration \(\beta\) in the case of Theorem 5.1).

In each stage of the induction, we perform some integration over groups related to \(A_s(X)\) in the spirit of the previous subsections, and we are left at the end with some simpler objects on which to iterate.

We need to introduce some notation for technical reasons that will become clear later. Let \(c_1, c_2 : \{0, 1\}^\ell \to X\) be two configurations. Until now, we have defined the concatenation by the same symbol irrespective of the coordinate on which the concatenation takes place. Now we wish to designate this in our notation, and we write

\[ [c_1, c_2]_k: \{0, 1\}^{\ell + 1} \to X, \]

\[ \omega \mapsto \begin{cases} c_1(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{\ell + 1}) & \text{if } \omega_k = 0, \\ c_2(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_{\ell + 1}) & \text{if } \omega_k = 1. \end{cases} \]

We also introduce some notation for the derivative along a specific coordinate. If \(\rho: C^\ell \to A\) is a function on cubes, we write

\[ \partial_k \rho([c_1, c_2]_k) = \rho(c_1) - \rho(c_2). \]

Observe that the identity

\[ \partial^{\ell + 1} f = \partial_k (\partial^\ell f) \]

holds irrespective of the value of \(k\). Hence, if we are differentiating a coboundary \(\partial^\ell f\), the derivative is independent of the direction \(k\); however, this will not be the case in general.

Armed with this notation, we can now formulate the inductive step in the proof of Theorem 5.1.

Lemma 5.6. Let \(A\) be a compact abelian Lie group with the metric \(d_A\) as above, and let \(s \geq 1, \ell \geq 1\) be given. Then there exists an \(\varepsilon = \varepsilon(s, \ell, A)\) such that the following holds.

Suppose that \(X\) is a (compact, ergodic) nilspace of degree \(s\), and suppose that \(\rho: C^\ell(X) \to A\) is an \(\ell\)-cocycle such that \(d_A(\rho(c), \rho(c')) \leq \delta\) for all cases in which \(c, c' \in C^\ell(X)\) and for some \(\delta \leq \varepsilon\).
Write $\pi_{s-1} : X \to \pi_{s-1}(X)$ for the canonical factor. Then we may decompose

$$\rho(c) = \sum_{k=1}^{\ell} \partial_k \rho'_k(c) + \tilde{\rho}(\pi_{s-1}(c)),$$

where

- $\rho'_k : C^{\ell-1}(X) \to A$ is an $(\ell - 1)$-cocycle for each $k$, and
- $\tilde{\rho} : C^{\ell}(\pi_{s-1}(X)) \to A$ is an $\ell$-cocycle on $\pi_{s-1}(X)$

such that both $\rho'_k$ and $\tilde{\rho}$ take images in a small ball in $A$, i.e., $d_A(\rho'_k(c_1), \rho'_k(c_2)) \lesssim_{s,\ell} \delta$ for all cases in which $c_1, c_2 \in C^{\ell-1}(X)$ and $k$, and $d_A(\tilde{\rho}(c_1), \tilde{\rho}(c_2)) \lesssim_{s,\ell} \delta$ for all cases in which $c_1, c_2 \in C^{\ell}(\pi_{s-1}(X))$.

We verify that this is enough to prove Theorem 4.11.

**Proof of Theorem 4.11 assuming Lemma 5.6.** We remark again that an ergodic nilspace of degree 0 is just the one-point space $\{\ast\}$, so a cocycle of any positive order on this space is identically 0. Also recall that a 0-cocycle is the same thing as a continuous function.

So, we may proceed by induction on $s$ and $\ell$, where the cases $\ell = 0$ and $s = 0$ are both clear. Given $s, \ell > 0$ and a cocycle $\rho : C^{\ell}(X) \to A$ as in Theorem 4.11 we decompose

$$\rho(c) = \sum_{k=1}^{\ell} \partial_k \rho'_k(c) + \tilde{\rho}(\pi_{s-1}(c))$$

as in Lemma 5.6.

Temporarily we will say that a function $f : Y \to A$ has a small image if $d_A(f(y), f(y')) \lesssim_{s,\ell} \delta$ for all cases in which $y, y' \in X$. By inductive hypothesis, $\rho'_k = \partial^{\ell-k} g_k$ for some continuous $g_k : X \to A$ with a small image, and similarly $\tilde{\rho} = \partial^{\ell} h$ for some continuous $h : \pi_{s-1}(X) \to A$ with a small image.

Setting $f = \sum_{k=1}^{\ell} g_k + h \circ \pi_{s-1}$, we see that this is again a continuous function $X \to A$ with a small image, and moreover

$$\partial^{\ell} f = \sum_{k=1}^{\ell} \partial^{\ell} g_k + \partial^{\ell} (h \circ \pi_{s-1}) = \sum_{k=1}^{\ell} \partial_k \rho'_k + \tilde{\rho} = \rho,$$

which completes the proof. \hfill $\square$}

Lemma 5.6 is proved by iterating the following result. Recall from subsection 5.3 that we write $T^t_1$ for the set of configurations $t : \{0,1\}^{\ell-1} \to A_s(X)$ that satisfy

$$[0,t] \in C^{\ell}(D_s(A_s)).$$

(Here we did not indicate in which coordinate the concatenation takes place, as the resulting group is independent of this choice.)

**Lemma 5.7.** Let $A$ be a compact abelian Lie group with the metric $d_A$ as above, and let $s \geq 1$, $\ell \geq 1$ be given. Then there exists an $\epsilon = \epsilon(s,\ell,A)$ such that the following holds.

Suppose $X$ is a (compact, ergodic) nilspace of degree $s$, and suppose that $\rho : C^{\ell}(X) \to A$ is an $\ell$-cocycle such that $d_A(\rho(c), \rho(c')) \leq \delta$ for all cases in which $c, c' \in C^{\ell}(X)$ and for some $\delta \leq \epsilon$.  

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Write $\pi_{s-1} : X \to \pi_{s-1}(X)$ for the canonical factor. Fix a number $1 \leq k \leq t$. Then we may decompose
\[
\rho(c) = \partial_k \rho'(c) + \tilde{\rho}(c),
\]
where
- $\rho' : C^{\ell-1}(X) \to A$ is an $(\ell - 1)$-cocycle, and
- $\tilde{\rho} : C^\ell(X) \to A$ is an $\ell$-cocycle which is invariant under the action of the group $[0, T_1^1]_k$, such that both $\rho'$ and $\tilde{\rho}$ take images in a small ball in $A$, i.e., $d_A(\rho'(c_1), \rho'(c_2)) \lesssim_{s, \ell} \delta$ for all cases in which $c_1, c_2 \in C^{\ell-1}(X)$ and $d_A(\tilde{\rho}(c_1), \tilde{\rho}(c_2)) \lesssim_{s, \ell} \delta$ for all cases in which $c_1, c_2 \in C^\ell(X)$.

The proof of Lemma \ref{5.7} follows subsection \ref{5.3} very closely, which is possible by virtue of the weak structure theory. There will be some additional complications caused by the relative nature of Lemma \ref{5.7} compared to Proposition \ref{5.5}.

**Proof of Lemma \ref{5.7}**. The definition of $\rho'$ will be similar to that in Proposition \ref{5.5}. Instead of integrating over the set
\[
\{ c' : [c, c']_k \in C^\ell(X) \},
\]
we will integrate over all such $c'$'s such that additionally $c, c'$ lie in the same fiber of $\pi_{s-1}$. By the weak structure theory this is equivalent to saying
\[
\rho'(c) = \int_{T_1^1} \rho([c, t.c]_k) d\mu_{T_1^1}(t),
\]
where as always $t.c$ denotes the action of $A_s(X)$ on $X$, applied pointwise to the configurations $t : \{0,1\}^{\ell-1} \to A_s(X)$, $c : \{0,1\}^{\ell-1} \to X$.

It suffices to check that
(i) the function $\rho' : C^{\ell-1}(X) \to A$ is continuous and has a small image (in the above sense);
(ii) $\rho'$ is an $(\ell - 1)$-cocycle on $X$; and
(iii) the function
\[
\tilde{\rho}([c_1, c_2]_k) := \rho([c_1, c_2]_k) - (\rho'(c_1) - \rho'(c_2))
\]
is a continuous $\ell$-cocycle on $X$ with a small image, and furthermore is invariant under the action of $[0, T_1^1]_k$.

Part (i) follows by the same arguments as in Proposition \ref{5.5}. Furthermore, the proof of (ii) is identical to the corresponding part of Proposition \ref{5.5} where we simply replace all expressions of the form $(c + t)$ with $(t.c)$, where $c$ is a cube of $X$ and $t$ a configuration with values in $A_s(X)$.

There is a bit more to say for (iii). First, note that since $\rho$ and $\partial_k \rho'$ are continuous $\ell$-cocycles with small image, it is immediate that $\tilde{\rho}$ inherits these properties (see Remark \ref{4.9}); the challenge is to show the invariance property.

We have
\[
\tilde{\rho}([c, t.c']_k) = \tilde{\rho}([c, c']_k) + \tilde{\rho}([c', t.c']_k),
\]
so it suffices to show that
\[
\rho([c', t.c']_k) = \rho'(c') - \rho'(t.c')
\]
for all $c'$'s and for $t \in T_1^\beta$. (Indeed, this implies that $\tilde{\rho}([c', t.c']_k) = 0$.) Again, this is very similar to Proposition 5.5 we compute
\[
\rho([c', t.c']_k) = \int_{T_1^\beta} (\rho([c', w.c']_k) + \rho([w.c', t.c']_k)) \, d\mu_{T_1}(w)
\]
\[
= \int_{T_1^\beta} \rho([c', w.c']_k) \, d\mu_{T_1}(w) - \int_{T_1^\beta} \rho([t.c', w.c']_k) \, d\mu_{T_1}(w)
\]
\[
= \rho'(c') - \rho'(t.c'),
\]
as required, once again after reparameterization of the last integral. \hfill \square

Proof of Lemma 5.7. We iterate Lemma 5.7. We first write \( \rho = \partial_1 \rho_1' + \tilde{\rho}_1 \), then in the $k$th step apply Lemma 5.7 to the cocycle $\tilde{\rho}_k - 1$ and write $\tilde{\rho}_k - 1 = \partial_k \rho_k' + \tilde{\rho}_k$. Combining these equations, we obtain the desired decomposition
\[
\rho = \sum_{k=1}^\ell \partial_k \rho_k' + \tilde{\rho}_\ell.
\]

It remains to show that $\tilde{\rho}_\ell$ descends to a cocycle on $\pi_{s-1}(X)$. We first observe that $\tilde{\rho}_k$ is invariant under $[0, T_1^\beta]_k$ and that this property is inherited by $\tilde{\rho}_j$ for all $j$'s $> k$. Indeed, the cocycle $\tilde{\rho}_j$ is constructed in the proof of Lemma 5.7 as the difference of a $[0, T_1^\beta]_k$-invariant cocycle and an average of $[0, T_1^\beta]_k$-invariant cocycles.

Thus $\tilde{\rho}_\ell$ is invariant under $[0, T_1^\beta]_k$ for all $k$'s. Recalling that reflection simply negates the cocycle, i.e., $\tilde{\rho}_\ell([c_0, c_1]_k) = -\tilde{\rho}_\ell([c_1, c_0]_k)$, it follows that $\tilde{\rho}_\ell$ is also invariant under $[T_1^\beta, 0]_k$. It is clear that these groups generate $\mathcal{D}_s(A_s(X))$, and hence $\tilde{\rho}_\ell$ is invariant under the action of this group.

Now it follows that $\tilde{\rho}_\ell$ descends to a function on $C^\ell(\pi_{s-1}(X))$. However, it is not completely obvious that this yields a cocycle. One needs to check that configurations $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$ on $\pi_{s-1}$ such that $[\tilde{c}_0, \tilde{c}_1]$ and $[\tilde{c}_1, \tilde{c}_2]$ are cubes of $\pi_{s-1}(X)$ lift to corresponding configurations $c_0, c_1, c_2$ on $X$ in a compatible way. Fortunately, this is always true; see [GMV16a] Lemma 7.5. \hfill \square

This completes the proof of Theorem 4.11. The generalization to a proof of Theorem 5.3 involves no new ideas at all, and only a small amount of further justification.

We will have to recall some notions from the relative structure theory expounded upon in [GMV16a, section 7]. What we shall need is that the canonical factor $X \to \pi_{s-1}(X)$ of a nilspace $X$ of degree $s$ has an analogue for fibrations.

Proposition 5.8. Let $\beta : X \to Y$ be a fibration of degree $s$ between compact, ergodic cubespaces $X$ and $Y$ that obey the glueing axiom. Then there is a map $\pi_{\beta,s-1} : X \to \pi_{\beta,s-1}(X)$ where

- the fibration $\beta$ factors as $X \xrightarrow{\pi_{\beta,s-1}} \pi_{\beta,s-1}(X) \xrightarrow{\beta} Y$;
- the map $\tilde{\beta}$ is a fibration of degree $(s-1)$;
- there is a compact abelian group $A_s(\beta)$, the structure group of the fibration, which acts continuously and freely on all of $X$ and whose orbits are precisely the fibers of $\pi_{\beta,s-1}$;
- a similar statement holds for cubes; specifically, $C^k(D_s(A_s(\beta)))$ acts pointwise on $C^k(X)$, and its orbits are precisely the fibers of the map $\pi_{\beta,s-1} : C^k(X) \to C^k(\pi_{\beta,s-1}(X))$. 

For details see [GMV16a, Proposition 7.12 and Theorem 7.19]. (Apply both results with \(s + 1\) in place of \(s\).)

We can now state the inductive step that provides a technical generalization of Lemma 5.6.

**Lemma 5.9.** Let \(A\) be a compact abelian Lie group with the metric \(d_A\) as above, and let \(s \geq 1\), \(\ell \geq 1\) be given. Then there exists an \(\varepsilon = \varepsilon(s, \ell, A)\) such that the following holds.

Suppose \(X, Y, \beta, \rho\) are as in Theorem 5.1. Then there is a decomposition

\[
\rho(c) = \sum_{k=1}^{\ell} \partial_k \rho_k'(c) + \tilde{\rho}(\pi_{\beta,s-1}(c))
\]

where

- \(\rho_k' : C^{\ell-1}(X) \to A\) is an \((\ell - 1)\)-cocycle for each \(k\), and
- \(\tilde{\rho} : C^\ell(\pi_{\beta,s-1}(X)) \to A\) is an \(\ell\)-cocycle on \(\pi_{\beta,s-1}(X)\)

such that both \(\rho'\) and \(\tilde{\rho}\) have a small image in the usual sense.

The deduction of Theorem 5.1 is again very similar.

**Proof of Theorem 5.1 assuming Lemma 5.9.** The only thing that changes is the base case; the rest of the argument is the same as for Theorem 4.11. If we have a fibration \(\beta : X \to Y\) of degree 0 between two ergodic cubespaces \(X\) and \(Y\), then \(\beta\) is an isomorphism. In particular, an \(\ell\)-cocycle on \(X\) is the same thing (under \(\beta\)) as an \(\ell\)-cocycle on \(Y\), so again Theorem 5.1 is trivial in this case.

Finally, the proof of Lemma 5.9 is unchanged from that of Lemma 5.6, replacing all appearances of \(\pi_{s-1}\) with \(\pi_{\beta,s-1}\) and \(A_s(X)\) with \(A_s(\beta)\).

All appeals to the weak structure theory are legitimized by Proposition 5.3.

5.5. The uniqueness result. We now prove Theorem 5.2. Our proof is very closely modelled on that of [ACS12, Lemma 3.2].

We work up to the result in several incremental stages, each of which is in fact a special case of the theorem. The first is classical.

**Lemma 5.10.** Suppose \(A\) is a compact abelian Lie group equipped with the metric \(d_A\). Then there exists an \(\varepsilon = \varepsilon(A) > 0\) such that the following holds.

If \(G\) is an abelian group and \(\phi : G \to A\) is a group homomorphism such that \(d_A(0,\phi(g)) \leq \varepsilon\) for all \(g\)’s \(\in G\), then \(\phi\) is the trivial homomorphism.

**Proof.** This is equivalent to saying that \(A\) has “no small subgroups”; there is some neighborhood of the identity in \(A\) containing no nontrivial subgroups.

Since we know that \(A\) is isomorphic to \((\mathbb{R}/\mathbb{Z})^d \times K\) for some finite \(K\), we may argue this directly. Let

\[
\pi : \mathbb{R}^d \times K \to (\mathbb{R}/\mathbb{Z})^d \times K
\]

denote the projection map. Choose some \(\delta > 0\) such that \(\pi\) identifies \(B_\delta(0) \subseteq (\mathbb{R}/\mathbb{Z})^d \times K\) bijectively with the corresponding open ball \(B_\delta(0) \times \{0\}\) in \(\mathbb{R}^d \times K\).

Let \(x \in B_{\delta/2}(0) \setminus \{0\} \subseteq (\mathbb{R}/\mathbb{Z})^d \times K\), and let \(\tilde{x}\) be the corresponding lift to \(\mathbb{R}^d\). Then there is some positive integer \(n\) such that \(\delta/2 \leq |n\tilde{x}| < \delta\). It follows that \(nx \in B_{\delta}(0) \subseteq B_{\delta/2}(0) \subseteq (\mathbb{R}/\mathbb{Z})^d \times K\), and therefore \(x\) is not contained in any subgroup of \(B_{\delta/2}(0)\).

\(\square\)
Note that if we interpret $G$ as a nilspace of degree 1 (i.e., considering $D_1(G)$), then any homomorphism $\phi: G \to A$ has $\partial^2 \phi = 0$, so this is a special case of Theorem 5.2.

Having shown that the space of linear maps $G \to A$ is discrete, we bootstrap this to a statement about polynomial maps.

**Lemma 5.11.** Suppose that $A$ is a compact abelian Lie group equipped with the metric $d_A$. Fix $\ell \geq 1$. Then there exists an $\varepsilon = \varepsilon(\ell, A) > 0$ such that the following holds.

Let $G$ be any abelian group, and consider the cubespace structure $D_1(G)$ on $G$. Let $\gamma: G \to A$ be a map such that $\partial^\ell \gamma \equiv 0$ and that $d_A(\gamma(g), \gamma(g')) \leq \varepsilon$ for all cases in which $g, g' \in G$.

Then $\gamma$ is constant.

**Proof.** We proceed by induction on $\ell$. The case $\ell = 1$ is trivial; since $[x, y] \in C^1(G)$ for every case in which $x, y \in G$, we have $\partial^1 \gamma([x, y]) = \gamma(x) - \gamma(y) = 0$ for all $x$'s and $y$'s, and hence $\gamma$ is constant.

Now take $\ell > 1$. Note that an $\ell$-cube in $D_1(G)$ has the form $[c, t + c]$ for some $t \in G$, where $t + c$ denotes the pointwise shift $\omega \mapsto t + c(\omega)$ of $c$. Defining the derivative

$$\gamma_t(x) = \gamma(x) - \gamma(x + t),$$

we note that

$$\partial^\ell \gamma([c, t + c]) = \partial^{\ell-1} \gamma_t(c).$$

By hypothesis, for all $t$'s we have $\partial^{\ell-1} \gamma_t \equiv 0$ and hence, for an appropriate choice of $\varepsilon$, by induction $\gamma_t \equiv \alpha(t)$ is a constant function for all $t$'s.

But now observe that

$$\alpha(t) + \alpha(t') = \gamma_t(0) + \gamma_{t'}(t) = \gamma(0) - \gamma(t) + \gamma(t) - \gamma(t + t') = \alpha(t + t')$$

and trivially $\alpha(0) = 0$, $\alpha(-t) = -\alpha(t)$, so $\alpha: G \to A$ is a group homomorphism. Moreover, by hypothesis $d_A(0, \alpha(t)) \leq \varepsilon$ for all $t$'s; so by Lemma 5.10 $\alpha \equiv 0$. Hence, $\gamma(x) = \gamma(0) - \alpha(x) = \gamma(0)$ is constant.

Once again, it is clear that this is a special case of the general result. The point is that we can use the weak structure theory to decompose our nilspace $X$ of degree $s$ into a tower of extensions

$$X \to \pi_{s-1}(X) \to \cdots \to \pi_0(X) = \{\ast\}$$

whose fibers are compact abelian groups $A_k(X)$ equipped with a nilspace structure $D_k(A_k)$. By applying Lemma 5.11 on the fibers one at a time, we deduce the result for $X$.

**Proof of Theorem 5.2.** We proceed by induction on $s$. The case $s = 0$ is trivial, since an ergodic nilspace of degree 0 is just the one-point space $\{\ast\}$.

Suppose that $s > 0$. Fix $y \in \pi_{s-1}(X)$ and consider the fiber $\pi_{s-1}(y)$. Clearly the restriction of $\gamma$ to $\pi_{s-1}(y)$ still has the same properties. By the weak structure theorem ([GMV16a Theorem 5.4]) $\pi_{s-1}(y)$ is isomorphic as a cubespace to
\( \mathcal{D}_s(A_s(X)) \), and hence \( \gamma|_{\pi_{s-1}^{-1}(y)} \) can be identified with a map \( \gamma_y : A_s(X) \to A \) such that

\[
\partial^\ell \gamma_y : C^\ell(\mathcal{D}_s(A_s(X))) \to A
\]

is identically 0.

Note, however, that \( C^\ell(\mathcal{D}_1(A_s(X))) \) is contained in \( C^\ell(\mathcal{D}_s(A_s(X))) \). Hence, a fortiori \( \gamma \) satisfies the hypotheses of Lemma 5.11 and so (for a suitable choice of \( \varepsilon \)) is constant.

Since \( y \) was arbitrary, we deduce that \( \gamma \) is constant on fibers of \( \pi_{s-1} \). Hence, \( \gamma \) factors as a map \( \gamma' : \pi_{s-1}(X) \to A \) on a nilspace with a degree of at most \( (s-1) \), and it is clear that \( \gamma' \) inherits all of the properties of \( \gamma \). By inductive hypothesis, \( \gamma' \) is constant, and hence \( \gamma \) is constant.

\[\square\]

Appendix A. A Technicality

We record here a fact that would distract from the discussion if recorded in place.

The following states that a free action by a compact group is “robustly free”, in the sense that only small elements come close to stabilizing points.

**Proposition A.1.** Suppose that \( G \) is a compact metric group acting continuously and freely on a compact metric space \( X \). For all \( \varepsilon \)'s > 0 there exists a \( \delta > 0 \) such that the following holds: if \( x \in X \), \( g \in G \), and \( d_X(x, gx) \leq \delta \), then \( d_G(id, g) \leq \varepsilon \).

**Proof.** Suppose for contradiction that there exists an \( \varepsilon > 0 \) and sequences \( x_i \in X \), \( g_i \in G \) such that \( d_G(id, g_i) \geq \varepsilon \) for all \( i \)'s but \( d_X(x_i, g_ix_i) \to 0 \). Passing to convergent subsequence, we may assume that \( x_i \to x \) and \( g_i \to g \) for some \( x \in X \) and \( g \in G \). Now, \( d_G(id, g) \geq \varepsilon \) and in particular \( g \neq id \). By continuity of the action map \( G \times X \to X \), we have \( g_i x_i \to g.x \) and so \( d_X(x, gx) = 0 \), contradicting the freeness of the action. \[\square\]

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References


\[9\] On this occasion \( \gamma_y \) does not denote a derivative.

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