CAYLEY AND LANGLEYDS TYPE CORRESPONDENCES
FOR ORTHOGONAL HIGGS BUNDLES

DAVID BARAGLIA AND LAURA P. SCHAPOSNIK

Abstract. Through Cayley and Langlands type correspondences, we give a geometric description of the moduli spaces of real orthogonal and symplectic Higgs bundles of any signature in the regular fibers of the Hitchin fibration. As applications of our methods, we complete the concrete abelianization of real slices corresponding to all quasi-split real forms, and we describe how extra components emerge naturally from the spectral data point of view.

1. Introduction

The moduli space of surface group representations in a reductive Lie group has long been studied, and through nonabelian Hodge theory, Higgs bundles become a natural holomorphic tool through which to understand them. This paper is dedicated to the study of real orthogonal and symplectic Higgs bundles of any signature on a compact Riemann surface \( \Sigma \) of genus \( g \geq 2 \), and through them, of surface group representations into \( \text{SO}(p + q, p) \) and \( \text{Sp}(2p + 2q, 2p) \). Since most of our results have similar proofs in the symplectic and orthogonal setting, we will mainly focus on the moduli space \( \mathcal{M}_{\text{SO}(p+q,p)} \) of \( \text{SO}(p + q, p) \)-Higgs bundles. The corresponding results for the symplectic counterparts \( \text{Sp}(2p + 2q, 2p) \) follow with only minor modifications (see subsection 8.2). A short review of Higgs bundles and the Hitchin fibration is given in section 2.

Cayley and Langlands type correspondences (sections 3–5). In the paper we consider the restriction of the complex orthogonal Hitchin map to \( h : \mathcal{M}_{\text{SO}(p+q,p)} \to \mathcal{A}_{\text{SO}(p+q,p)} \) on the moduli space \( \mathcal{M}_{\text{SO}(p+q,p)} \) of \( \text{SO}(p + q, p) \)-Higgs bundles\(^1\) (see subsection 2.2 for the construction of these maps). Using our Cayley and Langlands type correspondences, we give a geometric description of the regular fibers \( F(a) = h^{-1}(a) \) of the Hitchin map over generic points \( a \in \mathcal{A}_{\text{SO}(p+q,p)} \) (see section 5 and in particular Theorem 2 for details). More precisely, we identify the...
fiber \( F(a) \) with a covering of the product of two moduli spaces:

\[
\mathcal{M}_{\text{Cay}}(a) \times \mathcal{M}_{\text{Lan}}(a).
\]

The covering in question corresponds to certain extension data \( \tau \) as explained below. The Cayley and Langlands moduli spaces \( \mathcal{M}_{\text{Cay}}(a) \) and \( \mathcal{M}_{\text{Lan}}(a) \) and the extension \( \tau \) are given as follows:

- \( \mathcal{M}_{\text{Cay}}(a) \) is a fiber of the Hitchin map for the moduli space of \( K^2 \)-twisted \( \text{GL}(p, \mathbb{R}) \)-Higgs bundles on \( \Sigma \), which can be identified with the moduli space of line bundles \( L \) of order 2 in the Jacobian of an associated spectral curve.
- \( \mathcal{M}_{\text{Lan}}(a) \) is a moduli space of equivariant \( \text{SO}(p+q) \)-bundles \( M \) on an auxiliary double cover \( \pi_C : C \to \Sigma \) satisfying a condition over the fixed points (section 4).
- The reconstruction of the \( \text{SO}(p+q,p) \)-Higgs bundle \( (E, \Phi) \) in from the data in \( \mathcal{M}_{\text{Cay}}(a) \times \mathcal{M}_{\text{Lan}}(a) \) involves taking an extension of the form
  \[
  0 \to V_0 \to E \to F \otimes K^{1/2} \to 0,
  \]
  where \( F \) is the \( \text{Sp}(2p, \mathbb{R}) \)-Higgs bundle associated to the Cayley moduli space, and \( V_0 \) is the invariant direct image under \( \pi_C : C \to \Sigma \) of the equivariant orthogonal bundle on \( C \). The extension data \( \tau \) is given by the extension class defining the above extension. The requirement that \( E \) is an \( \text{SO}(p+q,q) \)-Higgs bundle limits the possible choices of extension to take values in a torsor over the group \( \mathbb{Z}^{4(p-g-1)} \).

**Remark 1.** The \( K^2 \)-twisted \( \text{GL}(p, \mathbb{R}) \)-Higgs bundle in \( \mathcal{M}_{\text{Cay}}(a) \) is related to a maximal \( \text{Sp}(2p, \mathbb{R}) \)-Higgs bundle through the Cayley correspondence—hence the name of the correspondence and of the space. The construction of such \( K^2 \)-twisted \( \text{GL}(p, \mathbb{R}) \)-Higgs bundles from \( \text{SO}(p+q,p) \)-Higgs bundles is carried out in section 3.

**Remark 2.** In the case \( q = 1 \) the procedure explained above takes us from an \( \text{Sp}(2p, \mathbb{R}) \)-Higgs bundle to an \( \text{SO}(p+1,p) \)-Higgs bundle in a way that is related to Langlands duality of the corresponding complex groups \( \text{Sp}(2p, \mathbb{C}) \) and \( \text{SO}(2p+1, \mathbb{C}) \). For this reason we regard the relation between the original \( \text{SO}(p+q,p) \)-Higgs bundle \( (E, \Phi) \) and the equivariant orthogonal bundle \( M \in \mathcal{M}_{\text{Lan}}(a) \) as a Langlands type correspondence.

As will be explained in subsection 4.1, the space \( \mathcal{M}_{\text{Lan}}(a) \) of equivariant bundles on \( C \) is closely related to moduli spaces of quadratic bundles. Quadratic bundles have played an important role when studying Higgs bundles for groups of low rank \([17]\) in the past, and we shall now show that they are fundamental for the analysis of all \( \text{SO}(p+q,p) \)-Higgs bundles, in general.

It has been predicted by Guichard and Wienhard \([18]\) Conjecture 5.6] that additional connected components appear in the moduli space of surface group representations into \( \text{SO}(p+q,p) \) for \( q \geq 1 \). Such components arise from a notion of \( \Theta \)-positive representations and would give further families of higher Teichmüller components. From the perspective of Theorem 2, natural candidates for such components are those containing Higgs bundles whose spectral data \( (L,M,\tau) \) has the form \( (\mathcal{O}, \mathcal{O}^{\otimes q}, \tau) \), as explained in Conjecture 4. To prove that this actually gives extra components, the monodromy action à la \([3,4]\) should be taken into consideration, as well as the behavior over singular fibers. On the symplectic side
Remark 17 addresses the absence of any extra components in the moduli space of Sp(2p + 2q, 2p)-Higgs bundles.

It should also be noted that Guichard and Wienhard explain in [19] that there are additional invariants for Anosov representations, which come from a reduction of the structure group, when the flat bundle is pulled back to the unit tangent bundle. While the main focus of [19] is on maximal symplectic representations, their observation holds, in general. Since Θ-positive representations are Θ-Anosov, additional characteristic classes arise, which should distinguish at least some of the additional components. From the perspective of spectral data we find that in some cases additional invariants can be obtained. More precisely, in subsection 7.2 we obtain an integer invariant associated to certain spectral data in the case q = 1. We expect that this invariant obtained through spectral data can be related to Θ-positivity, although establishing such a relation will require understanding how Θ-positivity can be seen at the level of Higgs bundles.

Characteristic classes (section 6). After introducing the main concepts in section 2 and describing the spectral data associated to Higgs bundles in \( M_{\text{SO}(p+q,p)} \) leading to Theorem 2 in sections 3–5, we study in section 6 the topological invariants, namely, Stiefel–Whitney classes, that can be used to distinguish components of the moduli space of SO(p + q, p)-Higgs bundles. Theorem 3 shows how the Stiefel–Whitney classes of an SO(p + q, p)-Higgs bundle (see subsection 2.3 for the definition of these) can be computed from the Cayley and Langlands type correspondences, and the extension data \((L, M, \tau)\). In particular, while some classes are determined purely by the Cayley data \(L\), there is a Stiefel–Whitney class which depends on all of the components of the triple \((L, M, \tau)\).

Parameterizations of components (sections 7 and 8). From Theorems 2 and 3 one can see that, in general, the moduli spaces \( M_{\text{SO}(p+q,p)} \) are parameterized by both abelian (Cayley) and nonabelian (Langlands) data, providing the first examples of real slices of the Hitchin fibration which have such a property. This should be compared with the moduli spaces of \( G \)-Higgs bundles, where \( G \) is a split real form which needs only abelian data [32], the moduli spaces for \( G = \text{SU}(p, p) \), and SU(p+1, p), which also need only abelian data [31, 34], and the moduli spaces for \( G = \text{SL}(p, \mathbb{H}), \text{SO}(p, \mathbb{H}), \) and \( \text{Sp}(2p, 2p) \), which involve only nonabelian data [25, 33].

For particular values of \( p \) and \( q \) the geometric properties of SO(p + q, p)-Higgs bundles and corresponding representations become concrete through the application of Theorem 2. In this paper we consider some geometric and topological consequences that follow from Theorems 2 and 4 in the following cases.

• **Quasi-split real forms (section 7):** we show in Theorem 4 that the regular fibers of the moduli space of Higgs bundles for the quasi-split real forms SO(p + 2, p) admit the structure of an abelian group of the form

\[
\text{Prym}(C, \Sigma) \times (\mathbb{Z}_2)^{(4p^2+2p)(g-1)+1}.
\]

• **Split real forms (section 7):** evidence of extra components for \( M_{\text{SO}(p+q,p)} \) was given in the Ph.D. thesis [11], and in the case of the moduli space \( M_{\text{SO}(p+1,p)} \), the existence of extra components as per [15, Conjecture 5.6] was shown in [10]. We show here that these extra components emerge naturally from the the extension class \( \tau \), as suggested in [35] and shown in...
Therefore, our methods provide a simple conceptual explanation for the existence of these components. Furthermore, using our spectral data constructions, we are able to write down explicit parameterizations of the Higgs bundles in these components. The spectral data for the extra components that occur when \( q = 1 \) admit a natural generalization to \( q \geq 1 \). We comment in Conjecture 1 that this gives a characterization of extra components that may appear for \( q > 1 \).

- **Groups of Hermitian type (section 8):** the study of many properties of maximal surface group representations into Hermitian groups of rank 2 reduces often to considering the group \( \text{SO}(2 + q, 2) \) \([9]\). Using Theorems 2 and 3 we show that the Cayley data are parameterized by fibers of the moduli space \( \mathcal{M}^\text{max}_{\text{Sp}(4,\mathbb{R})} \) of maximal \( \text{Sp}(4,\mathbb{R}) \)-Higgs bundles. In section 8 we consider the implications for \( \text{SO}(2 + q, 2) \)-Higgs bundles of the extra components in \( \mathcal{M}^\text{max}_{\text{Sp}(4,\mathbb{R})} \) obtained by Gothen \([16]\).

### Some remarks on Langlands duality (section 8)

The moduli space \( \mathcal{M}_{\text{SO}(p+q,p)} \) of \( \text{SO}(p+q,p) \)-Higgs bundles is a so-called \((B,A,A)\)-brane in the moduli space \( \mathcal{M}_{\text{SO}(2p+q,\mathbb{C})} \) of \( \text{SO}(2p+q,\mathbb{C}) \)-Higgs bundles. According to Langlands duality, interpreted as mirror symmetry between the moduli spaces of Higgs bundles for Langlands dual groups, the mirror of \( \mathcal{M}_{\text{SO}(p+q,p)} \) should be a \((B,B,B)\)-brane in the moduli space for the Langlands dual group of \( \text{SO}(2p+q,\mathbb{C}) \). In section 8 building on our previous work, we give a conjectural description of the dual \((B,B,B)\)-brane. In particular, we conjecture that the underlying support of the brane depends only on \( p \), while the moduli space in which the brane is embedded depends on both \( p \) and \( q \).

## 2. Higgs bundles and the Hitchin fibration

Throughout the paper we will consider a compact Riemann surface \( \Sigma \) of genus \( g \geq 2 \) with canonical bundle \( K = T^*\Sigma \). In what follows we recall some of the main properties of complex and real Higgs bundles, as well as the associated Hitchin fibration.

### 2.1. Higgs bundles for complex and real groups

We begin by briefly reviewing the notions of Higgs bundles for real and complex groups which are relevant to this paper. Further details can be found in standard references such as Hitchin \([21,22]\) and Simpson \([36–38]\). Higgs bundles on \( \Sigma \) are pairs \((E, \Phi)\) where

- \( E \) is a holomorphic vector bundle on \( \Sigma \),
- the Higgs field \( \Phi : E \to E \otimes K \) is a holomorphic \( K \)-valued endomorphism.

More generally, for a complex reductive Lie group \( G_C \) we have the following \([22]\).

**Definition 1.** A \( G_C \)-Higgs bundle is a pair \((P, \Phi)\), where \( P \) is a holomorphic principal \( G_C \) bundle and \( \Phi \) is a holomorphic section of \( \text{Ad}(P) \otimes K \), where \( \text{Ad}(P) \) is the adjoint bundle of \( P \).

Higgs bundles were introduced by Hitchin in \([21]\) as solutions of the *Hitchin equations*

\[
F_A + [\Phi, \Phi^*] = 0, \quad \partial_A \Phi = 0,
\]

where \( F_A \) is the curvature of the unitary connection \( \nabla_A = \partial_A + \overline{\partial}_A \) associated to a reduction of structure of \( P \) to the maximal compact subgroup of \( G_C \), and \( \Phi^* \) is
the adjoint of the Higgs field \( \Phi \) induced by this reduction to the maximal compact subgroup. One can construct the moduli space \( \mathcal{M}_{G_C} \) of solutions to the \( G_C \)-Hitchin equations, which admits a natural hyperkähler metric over its smooth points. By the work of Hitchin and Simpson, when \( G_C \) is semisimple, the existence of a unitary connection satisfying the Hitchin equations is equivalent to polystability of the pair \((P, \Phi)\). From this work it also follows that \( \mathcal{M}_{G_C} \) can be identified with the moduli space of polystable \( G_C \)-Higgs bundles. When \( G_C \) is reductive but not semisimple (e.g., \( \text{GL}(n, \mathbb{C}) \)), we will simply take \( \mathcal{M}_{G_C} \) to be the moduli space of polystable \( G_C \)-Higgs bundles.

Given a real form \( G \) of the complex reductive Lie group \( G_C \), we may define \( G \)-Higgs bundles as follows. Let \( H \) be the maximal compact subgroup of \( G \), and consider the Cartan decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) of \( \mathfrak{g} \), where \( \mathfrak{h} \) is the Lie algebra of \( H \) and \( \mathfrak{m} \) its orthogonal complement. This induces a decomposition of the Lie algebra \( \mathfrak{g}_C = \mathfrak{h}_C \oplus \mathfrak{m}_C \) of \( \mathfrak{g}_C \). Note that the Lie algebras satisfy \( [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \), \( [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \), \( [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \), and there is an induced isotropy representation \( \text{Ad}^}_H : H_C \to \text{GL}(\mathfrak{m}_C) \).

**Definition 2.** A principal \( G \)-Higgs bundle is a pair \((P, \Phi)\) where

- \( P \) is a holomorphic principal \( H_C \)-bundle on \( \Sigma \),
- \( \Phi \) is a holomorphic section of \( P \times_{\text{Ad}} \mathfrak{m}_C \otimes K \).

Two \( G \)-Higgs bundles are isomorphic if there is an isomorphism of the underlying principal \( H_C \)-bundles under which the Higgs fields are mapped to one another. Similar to the case of Higgs bundles for complex groups, there are notions of stability, semistability, and polystability for \( G \)-Higgs bundles, and one can see that the polystability of a \( G \)-Higgs bundle for \( G \subset \text{GL}(n, \mathbb{C}) \) is equivalent to the polystability of the corresponding \( \text{GL}(n, \mathbb{C}) \)-Higgs bundle seen through the standard representation (see [33] and references therein for each classical Lie group). However, a \( G \)-Higgs bundle can be stable as a \( G \)-Higgs bundle but not as stable as a \( \text{GL}(n, \mathbb{C}) \)-Higgs bundle. We denote by \( \mathcal{M}_G \) the moduli space of polystable \( G \)-Higgs bundles on \( \Sigma \).

### 2.2. Spectral data and the Hitchin fibration.
A natural way of studying the moduli spaces \( \mathcal{M}_{G_C} \) of \( G_C \)-Higgs bundles is to use the Hitchin fibration [22]. Let \( \{p_1, \ldots, p_k\} \) be a homogeneous basis for the algebra of invariant polynomials on the Lie algebra \( \mathfrak{g}_C \) of \( G_C \), and let \( d_i \) denote the degree of \( p_i \). The Hitchin fibration is then given by

\[
(2.2) \quad h : \mathcal{M}_{G_C} \to \mathcal{A}_{G_C} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}),
\]

where \( h : (E, \Phi) \mapsto (p_1(\Phi), \ldots, p_k(\Phi)) \) is referred to as the Hitchin map: it is a proper map for any choice of basis and makes the moduli space into an integrable system [22].

Each connected component of a generic fiber of the Hitchin map is an abelian variety. In the case of \( \text{GL}(n, \mathbb{C}) \)-Higgs bundles this can be seen using spectral data [6][22]. A \( \text{GL}(n, \mathbb{C}) \)-Higgs bundle \((E, \Phi)\) defines an algebraic curve, called the spectral curve of \((E, \Phi)\):

\[
(2.3) \quad S = \{\det(\eta I - \Phi) = 0\} \subset \text{Tot}(K),
\]

where \( \text{Tot}(K) \) is the total space of \( K \) and \( \eta \) is the tautological section of the pullback of \( K \) on \( \text{Tot}(K) \). We say that \((E, \Phi)\) lies in the regular locus of \( \mathcal{M}_{\text{GL}(n, \mathbb{C})} \) if the
curve \( S \) is nonsingular. Let \( \pi : S \to \Sigma \) denote the natural projection to \( \Sigma \), and let \( \eta \in H^0(S, \pi^*(K)) \) denote the restriction of the tautological section of \( K \) to \( S \). If \((E, \Phi)\) is in the regular locus, then there exists a line bundle \( L \to S \) for which \( E = \pi_*L \), and \( \Phi \) is obtained by pushing down the map \( L \to L \otimes \pi^*(K) \) given by multiplication by \( \eta \). In this way we recover the pair \((E, \Phi)\) from the pair \((S, L)\). We call \((S, L)\) the spectral data associated to the pair \((E, \Phi)\).

Note that the spectral curve \( S \) of the pair \((E, \Phi)\) depends only on the characteristic polynomial of \( \Phi \) and hence depends only on the image of \((E, \Phi)\) under the Hitchin map. In this way we can associate a spectral curve \( S \) to any point \( a \in \mathcal{A}_{GL(n, \mathbb{C})} \) in the base of the Hitchin system. If \( a \) is in the regular locus of \( \mathcal{A}_{GL(n, \mathbb{C})} \)—in other words, if the associated spectral curve \( S \) is smooth—then the spectral data construction identifies the fiber \( h^{-1}(a) \) of the Hitchin system with \( \text{Pic}(S) \), the Picard variety of the spectral curve \( S \). The connected components of \( \text{Pic}(S) \) are, of course, isomorphic to copies of \( \text{Jac}(S) \), the Jacobian of \( S \). In particular, this confirms that the components of the regular fibers are abelian varieties.

**Remark 3.** Throughout the paper our interest will be in understanding the geometry of generic fibers, and thus when considering spectral curves, we shall always assume they are sufficiently generic, in particular that they are smooth. The precise conditions we require are stated in Assumption \( I \).

### 2.3. Complex orthogonal Higgs bundles

Since the core of this paper is on the geometry of the moduli space of orthogonal Higgs bundles, we shall give here a thorough description of these objects and their spectral data. From Definition \( I \) (see, for example, \( \cite{33} \) for the construction which is obtained through the standard representation), an \( \text{SO}(2p + q, \mathbb{C}) \)-Higgs bundle consists of a pair \((E, \Phi)\) where the following is true:

1. \( E \) is a holomorphic vector bundle of rank \( 2p + q \) with a nondegenerate symmetric bilinear form \((v, w)\), together with a trivialization of the determinant bundle \( \Lambda^{2p+q}E \) as a \( \mathbb{Z}_2 \)-line bundle \( (\text{i.e., a trivialization of the principal} \mathbb{Z}_2 \text{-bundle to which} \Lambda^{2p+q}E \text{is associated})\).
2. \( \Phi \in H^0(\Sigma, \text{End}(E) \otimes K) \) is a Higgs field which satisfies \((\Phi v, w) = -(v, \Phi w)\).

We denote by \( \mathcal{M}_{\text{SO}(2p+q, \mathbb{C})} \) the moduli space of \( S \)-equivalence classes of semistable \( \text{SO}(2p + q, \mathbb{C}) \)-Higgs bundles. This moduli space has two connected components, labeled by the second Stiefel–Whitney class \( w_2(E) \in H^2(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2 \), depending on whether or not \( E \) has a lift to a spin bundle \( \cite{28} \).

### 2.4. \( \text{SO}(p + q, p) \)-Higgs bundles

From Definition \( \cite{2} \) an \( \text{SO}(p + q, p) \)-Higgs bundle consists of

1. a rank \( p + q \) orthogonal bundle \((V, Q_V)\),
2. a rank \( p \) orthogonal bundle \((W, Q_W)\),
3. a holomorphic bundle map \( \beta : W \to V \otimes K \),
4. an isomorphism \( \det(V) \cong \det(W) \) as \( \mathbb{Z}_2 \)-line bundles.

In the above, \( V \) and \( W \) are rank \( p + q \) and \( q \) vector bundles, respectively, and \( Q_V \) and \( Q_W \) are their orthogonal structures. When considering isomorphisms of orthogonal bundles, we shall be considering isomorphisms of vector bundles which preserve the orthogonal structures.
Given an $\text{SO}(p + q, p)$-Higgs bundle $(V, W, \beta)$, the associated $\text{SO}(2p + q, \mathbb{C})$-Higgs bundle $(E, \Phi)$ is obtained by setting $E = V \oplus W$ with bilinear form

$$(x, y) \mapsto Q_V(x, x') - Q_W(y, y')$$

and Higgs field $\Phi : E \rightarrow E \otimes K$ given by

$$(2.4) \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where $\gamma = \beta^t$ is the orthogonal transpose of $\beta$, obtained using the orthogonal structures on $V$ and $W$. In the moduli space $M_{\text{SO}(2p + q, \mathbb{C})}$ of $\text{SO}(2p + q, \mathbb{C})$-Higgs bundles, the real $\text{SO}(p, p + q)$-Higgs bundles are fixed points of the involution $\Theta : (E, \Phi) \mapsto (E, -\Phi)$, corresponding to Higgs bundles $(E, \Phi)$ such that there is an isomorphism $f : (E, \Phi) \rightarrow (E, -\Phi)$ induced by an involution $f : E \rightarrow E$ whose $+1$ and $-1$-eigenspaces have dimensions $p + q$ and $p$, respectively.

The curve defined by the characteristic equation of the Higgs field $\Phi$ is a reducible curve: an $\text{SO}(p, p + q)$-Higgs field $\Phi$ always has a zero eigenspace of dimension $\geq q$ since its characteristic polynomial is of the form

$$(2.5) \quad \det(\eta - \Phi) = \eta^q(\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p),$$

where $a_i \in H^0(\Sigma, K^i)$. In the case $q = 1$ it is shown in [24, section 4.1] that for Higgs bundles in generic fibers, the zero eigenspace $E_0$ is isomorphic to $K^{-p}$. We will see in section 4 how a similar characterization of the zero eigenspace can be made for any $q$ in terms of quadratic bundles.

The generically irreducible component of the characteristic polynomial equation (2.5) defines an associated 2$p$-fold spectral curve $\pi : S \rightarrow \Sigma$ whose equation is

$$(2.6) \quad \eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p = 0.$$ 

By Bertini’s theorem the spectral curve $S$ is smooth for generic points in the Hitchin base. Recall from Remark 3 that we are mainly interested in smooth spectral curves. More precisely, in this paper we will mostly be concerned with spectral curves satisfying the following conditions.

**Assumption 1.** Assume that $S$ is smooth and that $a_p$ and $a_{p-1}$ do not simultaneously vanish.

Note that Assumption 1 implies that $a_p$ has only simple zeros. This follows from the Jacobian criterion for smoothness of $S$. We emphasize this point because we will use the fact that $a_p$ has only simple zeros many times throughout the paper.

The curve $S$ has an involution $\sigma$ which acts as $\sigma(\eta) = -\eta$. Thus, we may consider the quotient curve $\overline{S} = S/\sigma$ in the total space of $K^2$, for which $S$ is a double cover $\rho : S \rightarrow \overline{S}$, leading to the following diagram:

$$(2.7) \quad \begin{array}{ccc}
S & \xrightarrow{2:1} & \overline{S} \\
\pi \downarrow & & \rho \downarrow \\
2p:1 & \xrightarrow{p:1} & \Sigma
\end{array}$$
The covers $S$ and $\tilde{S}$ have, respectively, genera
\[
g_S = 1 + 4p^2(g - 1),
g_{\tilde{S}} = (2p^2 - p)(g - 1) + 1.
\]
By the adjunction formula the canonical bundles of these covers are, respectively,
\[
K_S = \pi^*K^{2p} \quad \text{and} \quad K_{\tilde{S}} = \tilde{\pi}^*K^{2p-1}.
\]
Letting $\xi = \eta^2$ denote the tautological section of $K^{2p}$, the quotient curve is given by
\[
\tilde{S} = \{\xi^p + a_1\xi^{p-1} + \cdots + a_{p-1}\xi + a_p = 0\} \subset \text{Tot}(K^{2p})
\]
and will become a key ingredient in the construction of $\text{SO}(p+q,p)$-Higgs bundles.

Note that the smoothness of $S$ is equivalent to the smoothness of $\tilde{S}$, and that $a_p$ has only simple zeros. This follows by comparing the Jacobian criterion for smoothness for $S$ and for $\tilde{S}$. As mentioned in Remark 3, all of the spectral curves considered in the paper are assumed to satisfy this condition. Let us also note that for $S$ smooth, the condition that $a_p$ and $a_{p-1}$ do not simultaneously vanish is equivalent to requiring that the fixed points of $\sigma : S \to \Sigma$.

Let $\mathcal{M}_{\text{SO}(p+q,p)}$ denote the moduli space of semistable $\text{SO}(p+q,p)$-Higgs bundles (see [1] for the construction of these spaces). The restriction of the Hitchin map to the real moduli space can be identified with the map
\[
h : \mathcal{M}_{\text{SO}(p+q,p)} \to A_{\text{SO}(p+q,p)} = \bigoplus_{i=1}^p H^0(\Sigma, K^{2i}), \quad h(E, \Phi) = (a_1, a_2, \ldots, a_p),
\]
where $a_1, \ldots, a_p$ are as in (2.5).

3. A Cayley type correspondence: Associated $K^2$-twisted GL$(p, \mathbb{R})$-Higgs bundles

Cayley correspondences for Higgs bundles have long been studied, and they provide a procedure in which one can obtain a correspondence between the moduli space of Higgs bundles for a Hermitian group of tube type and the moduli space of $K^2$-twisted Higgs bundles for a certain associated group. The interested reader should refer to [7] and references therein for further details of these correspondences.

In what follows we will construct a naturally defined $K^2$-twisted GL$(p, \mathbb{R})$-Higgs bundle associated to $\text{SO}(p+q,p)$-Higgs bundles ($E = V \oplus W$, $\Phi$) satisfying Assumption 1 (for the case $q = 1$; see [35]), providing a Cayley type correspondence. Note that, as mentioned in [11], our aim is to understand the regular fibers of the moduli space $\mathcal{M}_{\text{SO}(p+q,p)}$ in terms of two associated moduli spaces $\mathcal{M}_{\text{Cay}}$, $\mathcal{M}_{\text{Lan}}$. Only when the group $\text{SO}(p+q,p)$ is of Hermitian type (i.e., only for $p = 2$) does one recover the standard Cayley correspondence between $\text{SO}_0(2+q,2)$-Higgs bundles and $K^2$-twisted $\text{SO}_0(1,1) \times \text{SO}(q+1,1)$-Higgs bundles (for the latter see [7, Table C.4.1]). This case will be addressed in section 8 where the moduli spaces $\mathcal{M}_{\text{SO}(2+q,2)}$ are studied in further detail.

3.1. Unitary structure. We remind the reader that Assumption 1 is being assumed throughout the paper. Under Assumption 1 we have that $a_p$ is not identically 0. This means that $\Phi$ generically has rank $2p$, and thus $\beta$ and $\gamma$ both generically have rank $p$. So $\beta$ is generically injective, and $\gamma$ generically has a $q$-dimensional
kernel. We will see there is a canonically induced $U(p, p)$-Higgs bundle which can be obtained by considering the induced bundle

\[(3.1) \quad V_0 := \text{Ker}(\gamma), \quad \text{where} \quad \gamma : V \to W \otimes K.\]

By this we mean that $\mathcal{O}(V_0)$ is the kernel of $\gamma : \mathcal{O}(V) \to \mathcal{O}(W \otimes K)$, defining a subbundle $V_0 \subset V$.

**Proposition 1.** Let $V_1 := V/V_0$. From Definition 2 we obtain an induced $U(p, p)$-Higgs bundle given by

\[(3.2) \quad (E_+ := V_1 \oplus W, \Phi_+),\]

where $\Phi_+$ is determined by the following commutative diagram:

\[
\begin{array}{ccc}
V \oplus W & \xrightarrow{\Phi} & V \oplus W \\
V_1 \oplus W & \xrightarrow{\Phi_+} & V_1 \oplus W.
\end{array}
\]

**Proof.** By the definition of $V_0$, the map $\gamma$ factors as (see, for example, [30, section 4])

\[(3.3) \quad 0 \longrightarrow V_0 \longrightarrow V \longrightarrow V_1 \longrightarrow 0 \quad \text{with} \quad \gamma: V \to W \otimes K, \gamma_+: V_1 \to W \otimes K,
\]

where the top row is a short exact sequence of vector bundles and $\gamma_+ : V_1 \to W \otimes K$ is generically an isomorphism. Dually, we obtain

\[(3.4) \quad 0 \leftarrow V_0^* \otimes K \leftarrow V \otimes K \leftarrow V_1^* \otimes K \leftarrow 0 \quad \text{with} \quad \beta : W \to V_1 \otimes K, \beta_+: W \to V_1 \otimes K.
\]

Define $\beta_+ : W \to V_1 \otimes K$ by the following diagram,

\[(3.5) \quad 0 \longrightarrow V_0 \otimes K \longrightarrow V \otimes K \longrightarrow V_1 \otimes K \longrightarrow 0, \quad \beta : W \to V_1 \otimes K, \beta_+: W \to V_1 \otimes K,
\]

and note that there is a dual diagram:

\[(3.6) \quad 0 \leftarrow V_0^* \otimes K \leftarrow V \leftarrow V_1^* \otimes K \leftarrow 0 \quad \text{with} \quad \beta_+ : W \to V_1^* \otimes K, \beta_+^*: W \to V_1^* \otimes K.
\]

From (3.3) and (3.4) there are two bundle maps given by $\gamma_+ : V_1 \to W \otimes K$ and $\beta_+ : W \to V_1 \otimes K$. The data $(V, W, \beta_+, \gamma_+)$ define a $U(p, p)$-Higgs bundle [34]. The underlying $\text{GL}(2p, \mathbb{C})$-Higgs bundle of this $U(p, p)$-Higgs bundle from equation (3.2) is given by

\[(3.7) \quad E_+ = V_1 \oplus W, \quad \Phi_+ = \begin{pmatrix} 0 & \beta_+ \\ \gamma_+ & 0 \end{pmatrix}.
\]
To finish the proof, we just have to note that $\Phi_+$, as defined by (3.7), agrees with $\Phi_+$ as given in the statement of the proposition. This follows easily from the above commutative diagrams.

Remark 4. From the construction of the $U(p, p)$-Higgs bundle $(E_+ := V_1 \oplus W, \Phi_+)$, we have that

$$\det(\eta - \Phi_+) = \eta^{2p} + a_1 \eta^{p-1} + \cdots + a_p,$$

and thus the $2p$-fold cover $\pi : S \to \Sigma$ is in fact the spectral curve of $(E_+, \Phi_+)$. 

3.2. Symplectic structure. In what follows we show that the $U(p, p)$-Higgs bundle defined in subsection 3.1 gives rise to a real symplectic Higgs bundle.

Lemma 1. Let $(E, \Phi)$ be a $GL(n, \mathbb{C})$-Higgs bundle, and suppose that $\det(\Phi)$ vanishes to first order at $x \in \Sigma$. Then $\ker(\Phi_x)$ is 1-dimensional.

Proof. Suppose on the contrary that $\dim(\ker(\Phi_x)) \geq 2$. In such a case there exist linearly independent $e_1, e_2 \in E_x$ with $\Phi_x(e_1) = \Phi_x(e_2) = 0$. Extending $e_1$ and $e_2$ to a basis $e_1, e_2, \ldots, e_n$ of $E_x$, one can choose a local frame $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ for $E$ with $\tilde{e}_j(x) = e_j$ and a local trivialization of $K$. Then $\Phi(\tilde{e}_j) = \sum_{i=1}^{n} b_{ij} \tilde{e}_i$ for some holomorphic functions $b_{ij}$ on the Riemann surface $\Sigma$. Since $\Phi_x(e_j) = 0$ for $j = 1, 2$, we have that $b_{ij}(x) = 0$ for $j = 1, 2$. Therefore, two columns of the matrix $[b_{ij}]$ vanish at $x$, so $\det[b_{ij}] = \det(\Phi)$ vanishes to at least second order at $x$, a contradiction.

Lemma 2. The map $\gamma_+ : V_1 \to W \otimes K$ is an isomorphism.

Proof. By Lemma 1 and noting that $a_p = \det(\Phi_+)$, over each $x \in \Sigma$ the Higgs field $\Phi_+$ from equation (3.2) satisfies the following:

- If $a_p(x) \neq 0$, then $\ker(\Phi_+) = \{0\}$.
- If $a_p(x) = 0$, then $\ker(\Phi_+)$ is 1-dimensional.

It follows that at each $0$ of $a_p$ either $\beta_+$ or $\gamma_+$ has a 1-dimensional kernel, and the other is injective. But from its definition we see that $\beta_+$ factors as $W \xrightarrow{\gamma_+} V_1^* \otimes K \xrightarrow{\beta_+} V \otimes K \xrightarrow{\gamma_+} V_1 \otimes K$. Therefore, if $\gamma_+$ has a kernel, so does the map $\beta_+$. It follows that $\gamma_+$ is necessarily everywhere injective, and hence $\gamma_+ : V_1 \to W \otimes K$ is an isomorphism.

Proposition 2. For each choice of $K^{1/2}$ the Higgs bundle $(E_+, \Phi_+)$ from (3.1) defines a canonical $Sp(2p, \mathbb{R})$-Higgs bundle $(F, \Phi_F)$ with maximal Toledo invariant, where

$$F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2}), \quad \Phi_F = \begin{pmatrix} 0 & \gamma_+ \circ \beta_+ \\ 1 & 0 \end{pmatrix}.$$

Proof. Recall that in equation (3.7) we constructed the $GL(2p, \mathbb{C})$-Higgs bundle $(E_+, \Phi_+)$. Let $K^{1/2}$ be a fixed choice of square root of $K$. We will now tensor $(E_+, \Phi_+)$ by $K^{-1/2}$ to obtain a Higgs bundle $(F = E_+ \otimes K^{-1/2}, \Phi_F = \Phi_+ \otimes \Id)$. Using $\gamma_+$ to identify $V_1$ with $W \otimes K$, we have

$$F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2}), \quad \Phi_F = \begin{pmatrix} 0 & \beta_F \\ 1 & 0 \end{pmatrix},$$

where $\beta_F := \gamma_+ \circ \beta_+ : W \to V_1 \otimes K \cong W \otimes K^2$. Note that the orthogonal structure $Q_W$ of $W$ satisfies $Q_W(\beta_F a, b) = Q_W(a, \beta_F b)$; indeed, by the diagrams
in (3.3)–(3.6) the map $\beta_F$ can be written as $W \xrightarrow{\beta} V \otimes K \xrightarrow{\gamma} W \otimes K^2$. Then since $\gamma = \beta^t$, we have that $(\gamma \beta)^t = \beta^t \gamma^t = \gamma \beta$. Using the orthogonal structure $Q_W$, one can make $(F, \Phi_F)$ into an $\text{Sp}(2p, \mathbb{R})$-Higgs bundle as follows. The symplectic form $\omega_F$ is defined to be

$$\omega_F((a, b), (c, d)) = Q_W(a, d) - Q_W(b, c).$$

It is straightforward to check that $\omega_F(\Phi_F u, v) = -\omega_F(u, \Phi_F v)$, and so from Definition 1 we have that $(F, \Phi_F)$ is an $\text{Sp}(2p, \mathbb{C})$-Higgs bundle. In fact, this is an $\text{Sp}(2p, \mathbb{R})$-Higgs bundle since $F$ splits into a sum $F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2})$ of Lagrangian subbundles, $\Phi_F$ is off-diagonal with respect to this splitting, and each component of $\Phi_F$ is symmetric. Moreover, since $\Phi_F$ is as in (3.3), from Definition 2 and since $\deg(W \otimes K^{1/2}) = p(g - 1)$, this means that the pair $(F, \Phi_F)$ is an $\text{Sp}(2p, \mathbb{R})$-Higgs bundle with maximal Toledo invariant [14].

### 3.3. Symplectic Cayley correspondence

Under the Cayley correspondence, maximal $\text{Sp}(2p, \mathbb{R})$-Higgs bundles correspond to $K^2$-twisted GL($p, \mathbb{R}$)-Higgs bundles [7]. Recall that a $K^2$-twisted GL($p, \mathbb{R}$)-Higgs bundle is a triple $(W, Q_W, \beta)$ consisting of a rank $p$ orthogonal vector bundle $(W, Q_W)$ and a holomorphic map $\beta : W \to W \otimes K^2$ such that $Q_W(\beta a, b) = Q_W(a, \beta b)$. Therefore, in our case the triple $(W, Q_W, \beta_F)$ determines exactly the $K^2$-twisted GL($p, \mathbb{R}$)-Higgs bundle determined by $(F, \Phi_F)$.

The construction of the $\text{Sp}(2p, \mathbb{R})$-Higgs bundle $(F, \Phi_F)$ from $(V, W, \beta)$ involved the choice of a square root of $K$. Similarly, the Cayley correspondence relating $(F, \Phi_F)$ to $(W, \beta_F)$ requires a choice of square root of $K$. In the discussion above we have chosen the same square root in both instances. With this convention in place one sees that the construction of $(W, \beta_F)$ from $(V, W, \beta)$ does not depend on the choice of this square root, so $(W, \beta_F)$ is canonically associated to $(V, W, \beta)$.

**Remark 5.** One should note that the Cayley partner can be viewed directly from the $U(p, p)$-Higgs bundle $(E_+, \Phi_+)$ for a choice of square root of $K$, as described in [34] Remark 3.7. In particular, the spectral curve of the $K^2$-twisted GL($p, \mathbb{R}$)-Higgs bundle determined by $(F, \Phi_F)$ is given by $\bar{S}$ as in equation (2.7).

The $K^2$-twisted GL($p, \mathbb{R}$)-Higgs bundles associated to the maximal $\text{Sp}(2p, \mathbb{R})$-Higgs bundles obtained from the SO($p + q, p$)-Higgs pairs above can be described also in terms of spectral data. Indeed, since GL($p, \mathbb{R}$) is the split real form of GL($p, \mathbb{C}$), from [33, Theorem 4.12] we have that these Higgs bundles over a smooth spectral curve $\bar{S}$ in the regular locus of the Hitchin fibration are given by the set of 2-torsion points in the Jacobian of $\bar{S}$:

$$\text{Jac}(\bar{S})[2] := \{ L \in \text{Jac}(\bar{S}) \mid L^2 \cong \mathcal{O} \}.$$  

As will be recalled in section 4, the relation between the vector bundle $W$ and the line bundle $L$ is that $W = \pi_\ast(L \otimes \pi^* K^{(p-1)})$. Since $L$ has order 2, it can be viewed as a line bundle with orthogonal structure. Then $W$ inherits an orthogonal structure by relative duality. For an explanation of how relative duality gives an orthogonal structure on $W$, see, for example, [26, page 6].
4. A Langlands type correspondence: Quadratic bundles and the auxiliary spectral curve

Recall that in section 3 the moduli space of $\text{SO}(p+q,p)$-Higgs bundles were shown to have an associated symplectic $\text{Sp}(2p,\mathbb{C})$-Higgs bundle. In this and the following section we will study in the spirit of Langlands duality the problem of reconstructing the $\text{SO}(2p+q,\mathbb{C})$-Higgs bundle starting from $\text{Sp}(2p,\mathbb{C})$. In the case $q = 1$ the group $\text{SO}(2p + 1, \mathbb{C})$ is the Langlands dual of $\text{Sp}(2p, \mathbb{C})$ and, as shown by Hitchin [24], the process of reconstructing the $\text{SO}(2p + 1, \mathbb{C})$-Higgs bundle exhibits the duality of abelian varieties between the regular fibers of the $\text{Sp}(2p, \mathbb{C})$ and $\text{SO}(2p + 1, \mathbb{C})$-Higgs bundle moduli spaces. For $q > 1$ we shall refer to the relation between $\text{Sp}(2p, \mathbb{C})$ and $\text{SO}(2p + q, \mathbb{C})$-Higgs bundles as a Langlands type correspondence.

We will first show in subsection 4.1 that part of the data needed to recover the orthogonal Higgs bundle is parameterized by the moduli space of quadratic bundles $(V_0, Q_0)$ with fixed determinant (the relevant definitions are given in subsection 4.1). We will then show that this space can actually be identified with a certain moduli space $\mathcal{M}_{\text{Lan}}$ of equivariant orthogonal bundles on an auxiliary spectral curve. This is shown in subsection 4.2 by considering the auxiliary double cover

\[(4.1) \quad C = \{\zeta^2 - a_p = 0\} \subset \text{Tot}(K^p),\]

where $\zeta$ is the tautological section of $K^p$ on $\text{Tot}(K^p)$. The space $\mathcal{M}_{\text{Lan}}$ will become fundamental when describing the nonabelianization of orthogonal Higgs bundles in section 5. Thus, in what follows we consider some general properties of quadratic bundles in order to later understand the ones arising from Higgs bundles. Throughout this section we continue to assume that Assumption 1 holds.

### 4.1. Quadratic bundles and double covers

Understanding how vector bundles on Riemann surfaces can be obtained as direct images of bundles on coverings has been of interest for many decades, and the question is closely related to Higgs bundles (e.g., see [6]). In what follows we describe how quadratic bundles on $\Sigma$ can be obtained naturally using double covers of $\Sigma$.

**Definition 3.** A quadratic bundle is a pair $(V_0, Q_0)$, where $V_0$ is a holomorphic vector bundle and $Q_0$ is a holomorphic section of $\text{Sym}^2(V_0^*)$.

**Remark 6.** Given $(V_0, Q_0)$, we may view $Q_0$ as a map $Q_0 : V_0 \to V_0^*$ and take its induced determinant map $\text{det}(Q_0) : \text{det}(V_0) \to \text{det}(V_0^*)$. Thus, $\text{det}(Q_0)$ may be regarded as a section of $\text{det}(V_0)^{-2}$.

**Definition 4.** We say that $(V_0, Q_0)$ is regular if $\text{det}(Q_0)$ has only simple zeros.

**Definition 5.** Let $L$ be a line bundle on $\Sigma$ of nonnegative degree, and let $\delta$ be a holomorphic section of $L^2$ with only simple zeros. We say that $(V_0, Q_0)$ has determinant $(L^*, \delta)$ if there is an isomorphism $\text{det}(V_0) \cong L^*$ under which $\text{det}(Q_0) = \delta$.

**Remark 7.** One should note that $U$-quadratic bundles $(V_0, Q_0)$, given by a holomorphic vector bundle $V_0$ and $Q_0$ a global section of $\text{Sym}^2(V_0^*) \otimes U$ with $U$ being a fixed line bundle on $\Sigma$, were considered in [17] to show that certain spaces of representations into $\text{SO}_0(3, 2)$ with fixed invariants are connected. By contrast, in the present paper we are considering only the case of $U \cong \mathcal{O}$ and showing that this suffices to study the regular locus of the space of $\text{SO}(p + q, p)$ representations for any $p$ and $q \in \mathbb{N}$.  


Given $L$ and $\delta$ as above, we define an associated spectral curve $\pi_C : C \to \Sigma$ in the total space of $L^2$ by
\begin{equation}
C = \{ \zeta \mid \zeta^2 = \delta \} \subset \text{Tot}(L^2).
\end{equation}
Since the curve $\pi_C : C \to \Sigma$ is a double cover, we have a sheet swapping involution $\sigma_C : C \to C$.

**Definition 6.** A $\sigma_C$-equivariant (or simply equivariant) rank $q$ orthogonal bundle on the double cover $C$ is a triple $(M, Q_M, \tilde{\sigma}_C)$, where $(M, Q_M)$ is a rank $q$ orthogonal bundle on the double cover $C$, together with a lift $\tilde{\sigma}_C : M \to M$ of $\sigma_C$ to an involution on $M$ which preserves $Q_M$.

Over a fixed point $r$ of $\sigma_C$ on $C$, i.e., a ramification point of $\pi_C$, the involution $\tilde{\sigma}_C$ acts on the fiber $M_r$ of an equivariant rank $q$ orthogonal bundle $(M, Q_M, \tilde{\sigma}_C)$ as an orthogonal isomorphism. So there is an orthogonal decomposition $M_r = M_r^+ \oplus M_r^-$ into the ±1-eigenspaces of $\tilde{\sigma}_C$. Let $q_+ = \dim(M_r^+)$ and $q_- = \dim(M_r^-)$ so that $q = q_+ + q_-.$

**Definition 7.** We say that an equivariant rank $q$ orthogonal bundle $(M, Q_M, \tilde{\sigma}_C)$ has type $(q_+, q_-)$ at $r$, where $q_+$ and $q_-$ are obtained as above. We denote by $\MC(q_+, q_-)$ the moduli stack of rank $q$ orthogonal bundles $(M, Q_M, \tilde{\sigma}_C)$ on $C$ which have type $(q_+, q_-)$ over each fixed point of $\tilde{\sigma}_C$.

Stability conditions for such bundles will be considered in subsection 5.3. Note that $\MC(q_+, q_-)$ is a smooth Artin stack; in fact, it is an example of a moduli stack $\text{Bun}_G(\Sigma)$ of $G$-torsors, where $G$ is a parahoric Bruhat–Tits group scheme on $\Sigma$ (see [20] Example (3)).

Given an equivariant rank $q$ orthogonal bundle $(M, Q_M, \tilde{\sigma}_C)$ in $\MC(q_+, q_-)$, we define a rank $q$ quadratic bundle $(V_0, Q_0)$ as follows:

- $V_0$ is defined to be the invariant direct image of $M$, i.e., for each open set $U \subseteq \Sigma$, set
  \[ H^0(U, V_0) = H^0(\pi_C^{-1}(U), M)^{\tilde{\sigma}_C}. \]
- $Q_0$ is the restriction of $Q_M$ to $\tilde{\sigma}_C$-invariant sections.

**Proposition 3.** For each line bundle $L$ and holomorphic section $\delta$ of $L^2$ with only simple zeros, there is a bijection between isomorphism classes of rank $q$ quadratic bundles on $\Sigma$ with fixed determinant $(L^*, \delta)$ and equivariant rank $q$ orthogonal bundles on the spectral curve $C$ associated to $(L, \delta)$, with type $(q - 1, 1)$ over each ramification point and with trivial determinant (trivial as an equivariant $\mathbb{Z}_2$-line bundle on $C$). The correspondence is given by taking the invariant direct image as described above.

**Proof.** Let $(M, Q_M, \tilde{\sigma}_C)$ be an equivariant rank $q$ orthogonal bundle on a spectral curve $C$ given as in equation (4.2) with type $(q - 1, 1)$ over each ramification point. Let $(V_0, Q_0)$ be the quadratic bundle on $\Sigma$ given by taking the invariant direct image. We must show that $(V_0, Q_0)$ has determinant $(L^*, \delta)$. Since $V_0$ is the invariant direct image of $M$, it follows that there is a natural map $\pi_C^*(\det(V_0)) \to \det(M)$. This map is an isomorphism away from the ramification points of $\pi_C$ and has first order zeros at the ramification points (because $M$ has type $(q - 1, 1)$). Since the ramification divisor is $\pi_C^*(L)$, it follows that $\det(M) \cong \pi_C^*(\det(V_0)) \otimes \pi_C^*(L)$. Thus, $\det(M)$ is equivariantly trivial if and only if $\det(V_0) = L^*$. We must also show that $\det(Q_0)$ and $\delta$ have the same divisor. Note that the divisor of $\delta$ is exactly...
the branch locus of $C \to \Sigma$. Clearly $Q_0$ is nondegenerate away from the branch points. Consider then a branch point $x \in \Sigma$ and the corresponding ramification point $r \in C$. Since $\delta$ vanishes to first order at $x$, we can choose a local trivialization of $L$ and a local coordinate $z$ centered at $x$ such that $\delta(z) = z$. The tautological section $\zeta$ on $C$ can then be viewed as a local coordinate on $C$ centered at $r$ and satisfying $\zeta^2 = z$. The map $\pi_C$ is given locally by $\pi_C(\zeta) = \zeta^2 = z$, and $\sigma_C$ is given by $\sigma_C(\zeta) = -\zeta$. Let $e_1, \ldots, e_q$ be a local orthonormal frame for $M$. Since $M$ has type $(q-1,1)$ at $r$, we can choose $e_1, \ldots, e_q$ so that $\sigma_C(e_1) = -e_1$ and $\sigma_C(e_j) = e_j$ for $j > 1$. A local frame for the invariant direct image is therefore given by $e'_1 = \zeta e_1$, and $e'_2 = e_2, \ldots, e'_q = e_q$. Note that $Q_0(e'_1, e'_1) = Q_M(\zeta e_1, \zeta e_1) = \zeta^2 = z$ and $Q_0(e'_i, e'_j) = \delta_{ij}$ for $(i,j) \neq (1,1)$. In particular, this shows that $\det(Q_0)$ vanishes to first order at $x$, as required.

Conversely, let $(V_0, Q_0)$ be a quadratic bundle where the divisor of $\det(Q_0)$ is exactly the branch locus of $C \to \Sigma$, and let $x \in \Sigma$ be a branch point. As $\det(Q_0)$ vanishes only to first order at $x$, it follows that $Q_0|_{(V_0)_x}$ has a 1-dimensional null space $N_x \subseteq (V_0)_x$. Let $N$ be the sheaf on $C$ consisting of a direct sum of skyscraper sheaves where, for each ramification point $r \in C$, we take the skyscraper sheaf with fiber $N_x^*$ for $x = \pi_C(r)$, located at $r$.

Then one may define a vector bundle $M$ on $C$ by the following exact sequence of sheaves:

$$0 \to \mathcal{O}(M) \to \mathcal{O}(\pi^*_C(V_0^*)) \to N \to 0,$$

where the map $\mathcal{O}(\pi^*_C(V_0^*)) \to N$ is the direct sum of the maps $(V_0)_x^* \to N_x^*$ dual to the inclusions $N_x \to (V_0)_x$. Around a branch point $x$ we can choose an orthonormal frame $e'_1, e'_2, \ldots, e'_q$ for the bundle $V_0$ where $Q_0(e'_1, e'_1) = z$ and $Q_0(e'_i, e'_j) = \delta_{ij}$ for $(i,j) \neq (1,1)$. Let $f'_1, \ldots, f'_q$ be the dual frame. Note that $Q_0^{-1}$ defines a singular bilinear form on $V_0^*$ with $Q_0^{-1}(f'_1, f'_1) = 1/z$ and $Q_0^{-1}(f'_i, f'_j) = \delta_{ij}$ for $(i,j) \neq (1,1)$. Then a local frame for $M$ is given by $f_1 = \zeta f'_1$ and $f_2 = f'_2, \ldots, f_q = f'_q$.

The restriction $Q_M$ of $Q_0^{-1}$ to $M$ defines an orthogonal structure on $M$. Then, for $\tilde{\sigma}_C$ the restriction to $M$ of the canonical involution on $\pi^*_C(V_0^*)$, we have that

$$\tilde{\sigma}_C(f_1) = \tilde{\sigma}_C(\zeta f'_1) = \sigma_C(\zeta) f'_1 = -\zeta f'_1 = -f_1$$

and $\tilde{\sigma}_C(f_j) = f_j$ for $j > 1$. So $(M, Q_M, \tilde{\sigma}_C)$ has type $(q-1,1)$ at each ramification point. Lastly, note that these two constructions are inverse to one another, so we obtain the desired bijection. \hfill $\Box$

**Proposition 4.** The moduli stack $\mathcal{M}_C(q-1,1)$ has dimension

$$\dim(\mathfrak{so}(q))(g - 1) + (q - 1) \deg(L).$$

**Proof.** Let $\text{Ad}(M) \cong \Lambda^2 M$ denote the adjoint bundle of $M$. By a standard argument in deformation theory [20], the dimension of the stack $\mathcal{M}_C(q-1,1)$ equals $-\chi^{\tilde{\sigma}_C}$, where $\chi^{\tilde{\sigma}_C}$ is the $\tilde{\sigma}_C$-invariant part of the index:

$$\chi^{\tilde{\sigma}_C} := \dim(H^0(C, \text{Ad}(M))^{\tilde{\sigma}_C}) - \dim(H^1(C, \text{Ad}(M))^{\tilde{\sigma}_C}).$$

By the Lefschetz index formula (see [2, Theorem 4.12]) we have

$$-\chi^{\tilde{\sigma}_C} = \frac{1}{2} \dim(\text{Ad}(M))(g_C - 1) - \frac{1}{4} \sum_p \text{Tr}(\tilde{\sigma}_C : \text{Ad}(M)_p \to \text{Ad}(M)_p),$$
where \( g_C \) is the genus of \( C \) and the sum is over the ramification points of \( C \to \Sigma \). Now since \( M \) has type \((q - 1, 1)\) at each ramification point, we see that
\[
\text{Tr}(\delta_C : \text{Ad}(M)_p \to \text{Ad}(M)_p) = \dim(\text{Ad}(M))_p - 2(q - 1) = \dim(\mathfrak{so}(q)) - 2(q - 1).
\]
Since \( \delta \) is a section of \( L^2 \), letting \( d = \deg(L) \), the cover \( C \to \Sigma \) has \( 2d \) ramification points and hence
\[
-\chi^{\delta_C} = \frac{1}{2} \dim(\mathfrak{so}(q))(g_C - 1) - \frac{d}{2} (\dim(\mathfrak{so}(q)) - 2(q - 1))
= \frac{1}{2} \dim(\mathfrak{so}(q))((g_C - 1) - d) + (q - 1)d.
\]
By the Riemann–Hurwitz theorem, we have \((g_C - 1) - d = 2(g - 1)\), so
\[
\chi^{\delta_C} = \dim(\mathfrak{so}(q))(g - 1) + (q - 1)d,
\]
which concludes the proof. \( \square \)

4.2. Quadratic bundles, Higgs bundles, and the auxiliary spectral curve.

From section 3 an \( \text{SO}(p + q, p) \)-Higgs bundle \((V, W, \beta)\) satisfying Assumption 1 has a naturally associated rank \( q \) quadratic bundle \((V_0, Q_0)\) defined as follows:

- \( V_0 = \text{Ker}(\beta^t) \subset V \), defined as in equation (3.1);
- \( Q_0 \) is the restriction of \( Q_V \) to \( V_0 \).

Lemma 3. The symmetric bilinear form \( Q_0 \) is nondegenerate at points where \( a_p \neq 0 \) and has a 1-dimensional null space at zeros of \( a_p \).

Proof. To see this, recall that we have the short exact sequence \( 0 \to V_0 \to V \to V_1 \to 0 \) and the corresponding dual sequence: \( 0 \to V_1^* \to V \to V_0^* \to 0 \). Note also that \( V_1^* \) as a subbundle of \( V \) can be identified with \( V_0^\perp \), the perpendicular of \( V_0 \). Recall that \( \beta_+ \) viewed as a map \( \beta_+ : W \otimes K^{-1} \to V_1 \) is given by the composition
\[
W \otimes K^{-1} \xrightarrow{\gamma^t_1} V_1^* = V_0^\perp \xrightarrow{\gamma_1} V \xrightarrow{\beta_+} V_1 = V/V_0.
\]
From Lemma 2 the map \( \gamma^t_1 \) is everywhere an isomorphism. Moreover, since \( \gamma = \beta^t \), away from the zeros of \( a_p \) we have that \( \beta_+ \) is an isomorphism, and hence \( V_0^\perp \cap V_0 = \{0\} \). Therefore, \( V_0 \) is a nondegenerate subspace of \( V \), and hence \( Q_0 = Q_V|_{V_0} \) is nondegenerate. If \( x \) is a 0 of \( a_p \), then \( \beta_+ \) has a 1-dimensional kernel, and hence \( (V_0)_{x}^\perp \cap (V_0)_x \) is 1-dimensional. Then \( Q_0 \) at \( x \) has a 1-dimensional null space \( N_x = (V_0)_{x}^\perp \cap (V_0)_x \subset (V_0)_x \), as required. \( \square \)

Lemma 4. For an \( \text{O}(p + q, p) \)-Higgs bundle \((V, W, \beta)\) the condition \( \det(V) \cong \det(W) \) is equivalent to requiring that \( \det(V_0) \cong K^{-p} \).

Proof. From the proof of Lemma 3 one has that \( \det(V_0^*)^2 \cong K^{2p} \). Using this and the isomorphism \( V_1 \cong W \otimes K \), it follows that
\[
\det(V) \cong \det(V_0) \otimes \det(V_1) \cong \det(V_0) \otimes \det(W \otimes K) \cong \det(V_0) \otimes \det(W) \otimes K^p.
\]
Hence, \( \det(V) \cong \det(W) \) is equivalent to requiring that \( \det(V_0) \cong K^{-p} \). \( \square \)

From the above analysis we have the following correspondence between Higgs bundles, quadratic bundles, and orthogonal bundles.
Theorem 1. For each $\text{SO}(p+q,p)$-Higgs bundle $(V, W, \beta)$ there is a rank $q$ quadratic bundle $(V_0, Q_0) = (\text{Ker}(\beta^t), Q_V|_{\text{Ker}(\beta^t)})$ which has determinant $(K^{-p}, a_p)$. Letting $\sigma_C : C \to \Sigma$ denote the corresponding curve $\zeta^2 = a_p$, the quadratic bundle $(V_0, Q_0)$ is given as the invariant direct image of an equivariant orthogonal bundle $(M, Q_M, \sigma_C) \in \mathcal{M}_C(q-1,1)$. Moreover, we have that $\det(M) = \mathcal{O}$.

Proof. This result is obtained by putting together Proposition 9 and Lemmas 3 and 4. Note that $\det(M) = \mathcal{O}$ also follows from Proposition 3.

Remark 8. Recall that to define an $\text{SO}(p+q,p)$-Higgs bundle, it is not enough just to have an isomorphism $\det(V) \cong \det(W)$. We must actually fix a choice of isomorphism of $\mathbb{Z}_2$-line bundles. So far we have not identified what this choice corresponds to in terms of the equivariant bundle $(M, Q_M, \sigma_C)$, but we will see in subsection 5.2 that such an isomorphism corresponds precisely to a choice of trivialization $\det(M) \cong \mathcal{O}$ of the $\mathbb{Z}_2$-line bundle $M$ or, in other words, a choice of orientation for $M$.

5. ABELIAN AND NONABELIAN DATA FOR ORTHOGONAL HIGGS BUNDLES

We have described in previous sections how to associate to an $\text{SO}(p+q,p)$-Higgs bundle $(V, W, Q_V, Q_W, \beta)$ a $K^2$-twisted $\text{GL}(p,\mathbb{R})$-Higgs bundle $(W, Q_W, \beta_F)$ via a Cayley type correspondence, as well as a quadratic bundle $(V_0, Q_0)$, as part of the Langlands type correspondence. To complete the Langlands correspondence, we need to understand the extension data required to reconstruct $(V, Q_V)$ from $(W, Q_W, \beta_F)$ and $(V_0, Q_0)$. Finally, we also consider the relation between stability conditions for the $\text{SO}(p+q,p)$-Higgs bundle and the quadratic bundle. We assume throughout this section that $(a_1, \ldots, a_p)$ satisfy Assumption 1.

5.1. The extension data. We first look into how a rank $p+q$ vector bundle $V$ and its orthogonal structure $Q_V$ can be recovered from a $K^2$-twisted $\text{GL}(p,\mathbb{R})$-Higgs bundle $(W, Q_W, \beta_F)$ and a quadratic bundle $(V_0, Q_0)$ as in Theorem 1. In particular, recall from subsection 3.1 that $V_0 = \text{Ker}(\beta^t)$ and $V_1 = V/V_0$, so we have the short exact sequence $0 \to V_0 \to V \to V_1 \to 0$. Recall also that we obtained an isomorphism $V_1 \cong W \otimes K$. Let $D \subset \Sigma$ denote the zero divisor of $a_p$. Then since $Q_0$ is nondegenerate on $\Sigma \setminus D$, we obtain an orthogonal splitting:

\begin{equation}
V \cong V_0^* \oplus V_0 \cong V_0^* \oplus V_0 \cong (W \otimes K^{-1}) \oplus V_0.
\end{equation}

Lemma 5. Let $(V, W, \beta)$ be an $\text{SO}(p+q,p)$-bundle. Then with respect to the splitting (5.1) on $\Sigma \setminus D$, the orthogonal structure on $V$ is given by

\begin{equation}
Q_V((a, b), (c, d)) = Q_W(\beta_F a, c) + Q_0(b, d),
\end{equation}

where we recall that $\beta_F = \gamma \circ \beta$. Conversely, given $(V_0, Q_0)$ and a $K^2$-twisted $\text{GL}(p,\mathbb{R})$ Higgs bundle $(W, Q_W, \beta_F)$, equation (5.2) defines an orthogonal structure on $(W \otimes K^{-1}) \oplus V_0$ over $\Sigma \setminus D$.

Proof. Suppose that we are given an $\text{SO}(p+q,p)$-Higgs bundle $(V, W, \beta)$. The orthogonal structure $Q_V$ on $V$ is equivalent to giving a symmetric map $Q_V : V \to V^*$, i.e., a map $Q_V : (V_1^* \oplus V_0) \to (V_1^* \oplus V_0)$. Since the direct sums are orthogonal, this is equivalent to giving maps $V_0 \to V_0^*$ and $V_1^* \to V_1$. Clearly the map $V_0 \to V_0^*$ is given by $Q_0$. From (3.3) one finds that the map $V_1^* \cong W \otimes K^{-1} \to V_1 \cong W \otimes K$ in (3.3) is given by $\beta_F : W \otimes K^{-1} \to W \otimes K$. In other words, with respect to the splitting $V \cong (W \otimes K^{-1}) \oplus V_0$ we have $Q_V((a, b), (c, d)) = Q_W(\beta_F a, c) + Q_0(b, d)$. 


Conversely, suppose that we are given the pair \((V_0, Q_0)\) and the triple \((W, Q_W, \beta_F)\). Since \(\det(\beta_F) = (-1)^p a_p\), we have that \(\beta_F\) is an isomorphism on \(\Sigma \setminus D\). Moreover, \(Q_0\) is nondegenerate on \(\Sigma \setminus D\), and thus \(Q_V\) is nondegenerate. Furthermore, since \(Q_W(\beta_F a, c) = Q_W(a, \beta_F c)\), the form \(Q_V\) is symmetric. \(\square\)

**Lemma 6.** Let \((V, W, \beta)\) be an \(SO(p + q, p)\)-bundle. Then with respect to the orthogonal splitting \(V \cong (W \otimes K^{-1}) \oplus V_0\) on \(\Sigma \setminus D\) as in (5.1), we have that \(\beta\) and \(\gamma\) are given by

\[
(5.3) \quad \beta(u) = (u, 0), \quad \gamma(v, w) = \beta_F(v).
\]

Conversely, suppose that we are given a rank \(q\) quadratic bundle \((V_0, Q_0)\), an isomorphism \(\det(V_0) \cong K^{-p}\), and a \(K^2\)-twisted \(GL(p, \mathbb{R})\)-Higgs bundle \((W, Q_W, \beta_F)\) with \(\det(Q_0) = \det(\beta_F)\). We obtain an induced \(SO(p + q, p)\)-Higgs field on \(V \oplus W\) on \(\Sigma \setminus D\), where \((V, Q_V)\) is as in Lemma 5 and \(\beta\) and \(\gamma\) are as in (5.3).

**Proof.** From an \(SO(p + q, p)\)-Higgs bundle \((V, W, \beta)\), we have that \(\beta\) is given by the composition

\[
W \cong V_0^* \otimes K \to V \otimes K.
\]

Under the splitting \(V = V_0^* \oplus V_0\) on \(\Sigma \setminus D\) we have that \(V \otimes K = (V_0^* \otimes K) \oplus (V_0 \otimes K) \cong W \oplus (V_0 \otimes K)\), and \(\beta: W \to V \otimes K\) is the inclusion of the first factor. Similarly, \(\gamma: V \to W \otimes K\) is the map on \(V = V_0^* \oplus V_0 \cong (W \otimes K^{-1}) \oplus V_0\) given by

\[
V \cong (W \otimes K^{-1}) \oplus V_0 \to W \otimes K^{-1} \oplus V_0 \cong W \otimes K,
\]

where the first arrow is the projection to the first factor, and hence \(\beta\) and \(\gamma\) are given as in (5.3).

Conversely, from \((V_0, Q_0)\) and \((W, Q_W, \beta_F)\) we have that

\[
Q_V(\beta(u), (v, w)) = Q_V((u, 0), (v, w)) = Q_W(\beta_F u, v) = Q_W(u, \beta_F v) = Q_W(u, \gamma(v, w)).
\]

From the above we have shown that \(\beta^t = \gamma\), as required for an \(SO(p + q, p)\)-Higgs bundle. The condition \(\det(V) = \det(W)\) follows from \(\det(V_0) = K^{-p}\), as in the proof of Lemma 4. \(\square\)

In what follows we extend the constructions of Lemmas 5 and 6 over the divisor \(D\). To do this, we will define \(V\) as an extension of \(V_0^*\) by \(V_0^*\), and hence given by an extension class in \(H^1(\Sigma, \text{Hom}(V_0^*, V_0^*))\). As this extension class needs to be trivial on \(\Sigma \setminus D\), it allows us to define \(Q_V, \beta, \gamma\) as in Lemmas 5 and 6 on the complement of \(D\). We then need to check that \(Q_V, \beta, \gamma\) extend over \(D\). By continuity if such an extension exists, it is uniquely determined. Moreover, we have that \(\text{deg}(V) = 0\), so if \(Q_V\) extends, the extension will automatically be nondegenerate.

We consider \(Q_0\) as a map \(Q_0: V_0 \to V_0^*\), which induces a map between homomorphisms \(Q_0: \text{Hom}(V_0^*, V_0^*) \to \text{Hom}(V_0, V_0^*)\). Then, for \(N\) the vector bundle over the finite set \(D\) whose fiber over \(x \in D\) is given by \(N_x = (V_0)_x \cap (V_0^*)_x \cong (V_0)_x \cap (V_0^*)_x\), there is an exact sequence at each \(x \in D\) given by

\[
0 \to N_x \to (V_0)_x \xrightarrow{Q_0} (V_0^*)_x \to N^*_x \to 0.
\]

From the above sequence we obtain an associated short exact sequence of sheaves:

\[
0 \to \mathcal{O}_N(\text{Hom}(V_0^*, V_0^*)) \xrightarrow{Q_0} \mathcal{O}_N(\text{Hom}(V_0, V_0^*)) \to \mathcal{O}_D(\text{Hom}(N, V_0^*)) \to 0.
\]
We consider the following portion of the associated long exact sequence:
\[
H^0(D, \text{Hom}(N, V_0^*)) \overset{\delta}{\longrightarrow} H^1(\Sigma, \text{Hom}(V_0^*, V_1^*)) \overset{Q_0}{\longrightarrow} H^1(\Sigma, \text{Hom}(N, V_1^*)),
\]
where we use the orthogonal structure on $V$ to make the identification $V_1^* \cong V_0^*$. Let $i : N \to (V_0^*)^1 | D$ be given by the inclusion $N_x = (V_0)_{x} \cap (V_0^*)_{x} \to (V_0^*)_{x}$. Then $i \in H^0(D, \text{Hom}(N, V_0^*))$, and $\delta(i) \in H^1(\Sigma, \text{Hom}(V_0^*, V_1^*))$ is an extension class.

**Proposition 5.** For any $(V, W, \beta)$ the bundle $V$ is given by the extension class $\delta(i) \in H^1(\Sigma, \text{Hom}(V_0^*, V_1^*))$.

**Proof.** Consider an open cover of $\Sigma$ by two open sets $U$ and $\Sigma \setminus D$, where $U$ is a disjoint union of small discs around each point of $D$, and consider Čech cocycles with respect to this cover. On $\Sigma \setminus D$ we have the orthogonal splitting
\[
V \cong V_1^* \oplus V_0 \cong V_1^* \oplus V_0^*,
\]
where $Q_0$ is used to identify $V_0$ and $V_0^*$ on the complement of $D$. On $U$ we choose a local splitting $V \cong V_1^* \oplus V_0^*$ of the short exact sequence
\[
0 \to V_1^* \to V \to V_0^* \to 0.
\]
With respect to this splitting the inclusion $V_0 \to V$ in (5.4) has the form
\[
V_0 \to V_1^* \oplus V_0^*,
\]
where $u$ is some locally defined holomorphic map $u : U \to \text{Hom}(V_0^*, V_1^*)$ such that $u|_N = i$ on $D$. Therefore, the local trivializations on $\Sigma \setminus D$ and $U$ are related on $(\Sigma \setminus D) \cap U$ as follows:
\[
(v, s) \mapsto (v + u(s), Q_0(s)),
\]
where $(v, s) \in V_1^* \oplus V_0$ is a local section defined with respect to the orthogonal splitting and $(v + u(s), Q_0(s)) \in V_1^* \oplus V_0^*$ is the corresponding section in the local splitting given on $U$. So the extension class in $H^1(\Sigma, \text{Hom}(V_0^*, V_1^*))$ defining $V$ is represented by
\[
(w \mapsto u(Q_0^{-1}(w))) \in H^0((\Sigma \setminus D) \cap U, \text{Hom}(V_0^*, V_1^*)).
\]
Since $u|_N = i$ on $D$, it is straightforward to see that this extension class is exactly the element $\delta(i)$. \hfill \square

From Proposition 5 the data required to define the extension $V$ is a map
\[
i : N \to V_1^* \cong W \otimes K^{-1},
\]
defined over $D$. Given $(W, Q_W, \beta_F)$ and $(V_0, Q_0)$, we would like to determine for which such $i$ that the triple $(Q_V, \beta, \gamma)$ defined in Lemma 4 extends over $D$. In order to simplify the subsequent computations, we give an alternative interpretation of the vector bundle $V$.

**Proposition 6.** Let $i \in H^0(D, \text{Hom}(N, V_1^*))$, and let $V$ be the extension of $V_0^*$ by $V_1^*$ determined by $\delta(i)$. Then $\mathcal{O}(V)$ is isomorphic to the sheaf of meromorphic sections of $V_1^* \oplus V_0$ admitting first order poles over $D$ whose residue over $x \in D$ lies in the subspace
\[
\Gamma_x = \{-i(w), w \in N_x \} \subset (V_1^*)_x \oplus (V_0)_x.
\]
The map $\mathcal{O}(V_1^*) \to \mathcal{O}(V)$ is the inclusion, and the map $\mathcal{O}(V) \to \mathcal{O}(V_0^*)$ sends a meromorphic section $(v, s)$ of $V_1^* \oplus V_0$ to $Q_0(s) \in \mathcal{O}(V_0^*)$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. We have just seen that \( V \cong V_1^* \oplus V_0 \) on \( \Sigma \setminus D \), that \( V \cong V_1^* \oplus V_0^* \) on \( U \), with \( U \) being an open neighborhood of the \( D \), and the transition maps \( V_1^* \oplus V_0 \to V_1^* \oplus V_0^* \) have the form
\[
(v, s) \mapsto (v + u(s), Q_0(s)),
\]
where \( u \) is some locally defined holomorphic map \( u : U \to \text{Hom}(V_0, V_1^*) \) such that \( u|_N = i \) on \( D \). The inverse map \( V_1^* \oplus V_0^* \to V_1^* \oplus V_0 \) is thus given by \( (v, s) \mapsto (v - u(Q_0^{-1}(s)), Q_0^{-1}(s)) \). This shows that holomorphic sections of \( V \) can be identified with meromorphic sections of \( V_1^* \oplus V_0 \) with first order poles over \( D \), whose residue over \( x \in D \) lies in the subspace \( \Gamma_x \) given by \( (5.8) \). The rest of the proposition follows immediately. \( \square \)

**Proposition 7.** Given a \( K^2 \)-twisted \( GL(p, \mathbb{R}) \)-bundle \( (W, Q_W, \beta_F) \) and a quadratic bundle \( (V_0, Q_0) \), the induced maps \( \beta \) and \( \gamma \) as in \( \text{(5.3)} \) extend over \( D \) if and only if
\[
\text{(5.9)} \quad i(Ker(Q_0)) \subseteq Ker(\beta_F).
\]

Proof. Let \( \beta \) and \( \gamma \) be defined as in \( (5.3) \). The fact that \( \beta \) extends is automatic from Proposition 6 and \( (5.3) \). In light of Proposition 6 to show that \( \gamma \) extends, we just need to check that if \( (v, w) \) is a meromorphic section of \( V_1^* \oplus V_0 \) whose poles over each \( x \in D \) lie in \( \Gamma_x \), then \( \gamma(v, w) = \beta_F(v) \) is holomorphic (has no poles). Clearly this holds for all \( (v, w) \in \mathcal{O}(V) \) if and only if \( (\beta_F)_x(i(N_x)) = \{0\} \) for each \( x \in D \). \( \square \)

From the above propositions the map \( i : N \to V_1^* \) is required to send \( N \) to the kernel of \( \beta_F : V_1^* \cong W \otimes K^{-1} \to W \otimes K \cong V_1 \). Over \( D \), let \( J \) denote the kernel of \( \beta_F \). Then for each \( x \in D \) the space \( J_x \) is a 1-dimensional space. We require that \( i \) has the form
\[
\text{(5.10)} \quad i : N = Ker(Q_0) \to J = Ker(\beta_F) \subseteq V_1^*.
\]

Since \( N \) and \( J \) are 1-dimensional, there is for each \( x \in D \) a 1-dimensional space of such maps.

**Proposition 8.** Given \( i : N \to J \), the bilinear form \( Q_V \) extends holomorphically if and only if
\[
\text{(5.11)} \quad i^* \left( \frac{Q_W}{a_{p-1}} \right) = \left( \frac{Q_0}{a_p} \right).
\]

**Remark 9.** Condition \( (5.11) \) is to be understood as follows: over a point \( x \), the map \( i_x : N_x \to J_x \) is equivalent to a map \( i_x : N_x \otimes K_x^p \to J_x \otimes K_x^p \). We have that \( Q_W \) is a nondegenerate bilinear form on \( W \cong V_1^* \otimes K \) and \( J_x \subseteq (V_1^*)_x \cong W_x \otimes K_x^{-1} \). Then since \( a_{p-1}(x) \neq 0 \), we have that \( \frac{Q_W}{a_{p-1}} \) is a nondegenerate bilinear form on \( W_x \otimes K_x^{p-1} \cong (V_1^*)_x \otimes K_x^p \) and restricts to a nondegenerate bilinear form on \( J_x \otimes K_x^p \). Similarly, since \( \det(Q_0) \) and \( a_p \) vanish to first order at \( x \), one sees that \( \frac{Q_0}{a_p} \) is a well-defined nondegenerate bilinear form on \( N_x \otimes K_x^p \).

**Remark 10.** Condition \( (5.11) \) can be understood as saying that over each point \( x \), the map \( i_x : N_x \otimes K_x^p \to J_x \otimes K_x^p \) is an orthogonal isomorphism of 1-dimensional orthogonal spaces. Alternatively, \( \frac{Q_W}{a_{p-1}} \oplus \frac{Q_0}{a_p} \) defines an orthogonal structure on the 2-dimensional space \( (J_x \otimes N_x) \otimes K_x^p \), and thus a conformal structure on \( (J_x \otimes N_x) \). Condition \( (5.11) \) is equivalent to saying that \( \Gamma_x \subset J_x \oplus N_x \) given by \( (5.8) \) is an isotropic subspace.
Proof of Proposition 8. By Proposition 6 we just need to show that if \((a, b)\) is a meromorphic section of \(V_1^* \oplus V_0\) whose poles over each \(x \in D\) lie in \(\Gamma_x\), then
\[
Q_V((a, b), (a, b)) = Q_W(\beta_F a, a) + Q_0(b, b)
\]
is holomorphic. For this we consider the local behavior around a given \(x \in D\).

Let \(z\) be a local holomorphic coordinate centered at \(x\), and use \(dz\) to trivialize the canonical bundle. Then we can write the differentials as \(a_p = a_p(z)(dz)^{2p}\) and \(a_{p-1} = a_{p-1}(z)(dz)^{2p-2}\), where by Assumption 1 we have \(a_p(0) = 0\) and \(a'_{p}(0), a_{p-1}(0) \neq 0\). This means that \(\lambda = 0\) is an eigenvalue of \(\beta_F\) with multiplicity 1 at \(z = 0\). Thus, we can find a local orthonormal frame \(e_1, \ldots, e_p\) for \(W\) such that \(\beta_F(z)e_1 = \lambda(z)e_1(dz)^2\), where \(\lambda(z)\) is a locally defined holomorphic function with \(\lambda(0) = 0\). In particular, \(\text{Ker}(\beta_F)|_{z=0}\) is spanned by \(e_1|_{z=0}\). With respect to the given frame, \(\beta_F(z)\) is a matrix of functions times \((dz)^2\). Differentiating \(\beta_F(z)e_1 = \lambda(z)e_1(dz)^2\), we get \(\beta_F'(0)e_1 = \lambda'(0)e_1(dz)^2\).

Let \(\hat{b}\) be a local nonvanishing holomorphic section of \(V_0\) such that \(\hat{b}|_{z=0}\) spans \(N_z\). Choosing \(\hat{b}\) so that \(Q_0(\hat{b}, \hat{b}) = a_p(z)\), we have that the map \(i_x\) is given by \(i_x(\hat{b}) = uc_1\) for some \(u \in \mathbb{C}\). Then \((a, b)\) can be written as
\[
a(z) = a'(z) - cu\frac{z}{e_1}, \quad b(z) = b'(z) + \frac{c}{z}\hat{b},
\]
where \(a'\) and \(b'\) are holomorphic and \(c\) is a constant. Putting these into (5.2), since \((\beta_F e_1)|_{z=0} = 0\), we get that \(Q_V((a, b), (a, b))\)
is holomorphic for all \((a, b)\) if and only if
\[
\frac{1}{z}
\left(c^2 a'_p(0) + c^2 u^2 Q_W(\beta_F'(0)e_1(0), e_1(0)) - cuQ_W(\beta_F(0)v'(0), e_1(0))\right) + \cdots,
\]
where “\(\cdots\)” denotes terms which are holomorphic. But since
\[
Q_W(\beta_F(0)v'(0), e_1(0)) = Q_W(v'(0), \beta_F(0)e_1(0)) = 0,
\]
we have that (5.13) is holomorphic for all \((a, b)\) if and only if
\[
Q_W(\beta_F'(0)i(\hat{b}))(0), i(\hat{b}(0)) = -a'_{p}(0),
\]
where we have used the relation \(ue_1(0) = i(\hat{b}(0))\). But \(\beta_F'(0)i(\hat{b}(0)) = u\beta_F'(0)e_1(0) = u\lambda'(0)e_1(0)(dz)^2 = \lambda'(0)i(\hat{b}(0))(dz)^2\), so this simplifies to
\[
\lambda'(0)Q_W(\lambda'(0)i(\hat{b}(0))(dz)^2 = -a'_{p}(0).
\]
Considering that the characteristic polynomial \(p(y, z) = y^p + \cdots + a_{p-1}(z)y + a_p(z)\) of \(\beta_F(z)\) factors (locally) as \(p(y, z) = (y - \lambda(z))(\bar{p}(y, z))\) for some \(\bar{p}(y, z)\), we find that \(\lambda'(0) = -a'_{p-1}(0)\). Therefore,
\[
\frac{Q_W}{a_{p-1}(0)(dz)^{2p-2}}(i(\hat{b}(0)), i(\hat{b}(0))) = \frac{1}{(dz)^{2p}} = \frac{Q_0}{a_{p}(z)(dz)^{2p}}(\hat{b}(z), \hat{b}(z))|_{z=0},
\]
which is exactly (5.11). \(\square\)

Remark 11. This proposition is the first time that we have needed to use the assumption that \(a_{p-1}(x) \neq 0\) when \(a_p(x) = 0\).

From Proposition 8 we see that given a \(K^2\)-twisted \(\text{GL}(p, \mathbb{R})\)-bundle \((W, Q_W, \beta_F)\) and a quadratic bundle \((V_0, Q_0)\), for each point \(x \in D\) there are exactly two choices of the map \(i_x\) for which the induced triple \((Q_V, \beta, \gamma)\) extends over \(x\).
Recall from subsection 3.3 that the spectral data correspondence between $K^2$-twisted $GL(p,\mathbb{R})$-Higgs bundles in the regular locus of the Hitchin fibration and line bundles of order 2 on the associated quotient spectral curve $\overline{S}$.

**Proposition 9.** Let $(W, Q_W, \beta_F)$ be the $K^2$-twisted $GL(p,\mathbb{R})$-Higgs bundle corresponding to an orthogonal line bundle $L \in \text{Jac}(\overline{S})[2]$, and let $(V_0, Q_0)$ be the quadratic bundle given by the invariant direct image of an equivariant orthogonal bundle $(M, Q_M, \sigma_C) \in \mathcal{M}_C(q-1,1)$. The extension data needed to obtain an $\text{SO}(p+q,p)$-Higgs bundle from $L$ and $(M, Q_M, \sigma_C)$ are an orthogonal isomorphism $\tau_x : M_{\tau} \to L_x$ over each $x \in D$, where $r$ is the ramification point of $\pi_C$ lying over $x$, the point $r'$ is the ramification point of $p : S \to \overline{S}$ over $x$, and $M_{\tau}$ is the $-1$-eigenspace of $\sigma_C$ on $M_r$.

**Proof.** Given $e \in M_{\tau}$, choose a local section $\tilde{e}$ of $M$ such that $\tilde{e}|_{r'} = e$ and such that $\sigma_C(\tilde{e}) = -\tilde{e}$. Recall that $\zeta$ is the tautological section of $\pi_C^*K^p$ on $C$. It follows that $\zeta \tilde{e}$ is a $\pi_C^*K^p$-valued $\sigma_C$-invariant section of $M$ and thus defines a local $K^p$-valued section of $V_0$. The restriction $\zeta \tilde{e}|_x \in (V_0)_x \otimes K^p_x$ is easily seen to lie in $N_x \otimes K^p_x$ and is independent of the choice of extension $\tilde{e}$. In this way we have defined a canonical map $j_x : M_{\tau} \to N_x \otimes K^p_x$. Since $\zeta^2 = a_p$, we find that

$$j_x^* \left( \frac{Q_0}{a_p} \right) = Q_M|_{M_{\tau}}.$$ 

Hence, given $i_x : N_x \to J_x$ as in equation (6.10) and letting $\kappa_x = i_x \circ j_x : M_{\tau} \to J_x \otimes K^p_x$, the condition on $i_x$ from Proposition 8 now becomes

$$(5.14)$$

$$\kappa_x^* \left( \frac{Q_W}{a_{p-1}} \right) = Q_M|_{M_{\tau}}.$$ 

Let $L \in \text{Jac}(S)[2]$ be the line bundle on $\overline{S}$ such that $W = \pi_* (L \otimes \pi^* K^{(p-1)})$ (see [6] for details). Note that by construction $K^p_{\overline{S}} \cong \pi^* K^{p-1}$, and hence one has that $K^p_{\overline{S}} \otimes \pi^* K^{-1} \cong \pi^* K^{2p-2}$. The orthogonal structure on $L$ induces the orthogonal structure on $W$ by relative duality.

Recall from equation (2.8) in subsection 2.4 that the $p$-fold cover $\pi : \overline{S} \to \Sigma$ is given by the equation

$$\xi^p + a_1 \xi^{p-1} + \cdots + a_{p-1} \xi + a_p = 0.$$ 

We have that $J_x \subseteq (V_1)_x \cong W_x \otimes K^{-1}_x$. Thus, $J_x \otimes K_x$ can be identified with the kernel of $(\beta_F)_x : W_x \to W_x \otimes K^2_x$. Since $\beta_F$ is obtained by pushing down the tautological section $\xi$, it follows that $J_x \otimes K_x$ is canonically isomorphic to $L_{r'} \otimes K^{p-1}_{r'}$. Therefore, $J_x \otimes K^p_x$ is canonically isomorphic to $L_{r'} \otimes K^{2p-2}_{r'}$ and $i_x$ corresponds to a map $\kappa_x : M_{\tau} \to L_{r'} \otimes K^{2p-2}_{r'}$.

The derivative $d\pi : T_{\overline{S}} \to \pi^* T_{\Sigma}$ defines a section of $K^p_{\overline{S}} \otimes \pi^* K^{-1}$. Under the isomorphism $K^p_{\overline{S}} \otimes \pi^* K^{-1} \cong \pi^* K^{2p-2}$ one can see that the map $d\pi$ is given by $p\xi^{p-1} + (p-1)a_1 \xi^{p-2} + \cdots + a_{p-1}$. So at the ramification point $r' \in \overline{S}$ we have $d\pi = a_{p-1}(x) \neq 0$, by Assumption [1]. Therefore, relative duality gives

$$Q_W|_{J_x \otimes K_x} = \frac{Q_L}{d\pi} = \frac{Q_L}{a_{p-1}(x)}.$$
where $Q_L$ is the orthogonal structure on $L$. Putting it all together, the data we need consist of maps $\iota_x: M^-_r \to L_{r'} \otimes K^{2p-2}_x$ such that

$$\iota_x^*(\frac{Q_L}{a_{p-1}(x)}) = Q_M|_{M^-}.$$

Consider now the map

$$\tau_x: M^-_r \to L_{r'},$$

defined by setting $\tau_x = \iota_x \otimes a_{p-1}(x)$. Then the above condition in equation (5.15) is simply that $\tau_x$ is an orthogonal isomorphism. So finally, we have identified the extension data: for each point $x \in D$ let $r \in C$ and $r' \in \mathcal{S}$ be the corresponding ramification points. Then we need an orthogonal isomorphism $\tau_x: M^-_r \to L_{r'}$. \qed

**Remark 12.** Replacing $i$ by $-i$ (that is, changing the sign of $i$ at every point of $D$), we get an isomorphic extension and an isomorphic Higgs bundle. To see this, note that the isomorphism sending $i$ to $-i$ is given by acting as $-1$ on $W$ and $V_1$, and as $1$ on $V_0$. This has determinant $1$ since $\text{rank}(V_1) = \text{rank}(W)$.

**5.2. Determinant bundles.** To complete the correspondence between $\text{SO}(p+q,p)$-Higgs bundles and triples $(W, Q_W, \beta_F)$, $(V_0, Q_0)$, $\{\iota_x\}_{x \in D}$, we still need to identify the data giving the isomorphism $\det(V) \cong \det(W)$ of $\mathbb{Z}_2$-line bundles for the induced $\text{SO}(p+q,p)$-Higgs bundle. Since $\det(V)$ and $\det(W)$ are local systems, it is enough to give such an isomorphism on the complement of $D$ and check that this isomorphism extends. Using the fact that on a compact Riemann surface, unitary flat line bundles correspond to degree $0$ holomorphic line bundles, we see that such an isomorphism will extend if and only if $\text{Hom}(\det(V), \det(W)) \cong \mathcal{O}$, i.e., $\det(V) \cong \det(W)$ as holomorphic line bundles. We have already seen in Lemma 3 that this holds if and only if $\det(V_0) \cong K^{-p}$. So in what follows we will assume that $\det(V_0) \cong K^{-p}$ and look for an isomorphism $\det(V) \cong \det(W)$ of $\mathbb{Z}_2$-line bundles away from $D$.

**Proposition 10.** The choice of isomorphism $\det(V) \cong \det(W)$ is given by a choice of orientation for $M$, with the equivariant orthogonal bundle corresponding to $(V_0, Q_0)$.

**Proof.** On the complement of $D$ we have $V \cong V_1^* \oplus V_0 \cong (W \otimes K^{-1}) \oplus V_0$ and

$$Q_V((a, b), (c, d)) = Q_W(\beta_F a, c) + Q_0(b, d).$$

Let $\omega_W$ be a local unit volume form for $W$. Away from $D$ we have $a_p \neq 0$, and we can locally choose a square root $\zeta$ of $a_p$ with $\zeta^2 = a_p$. Then since $\det(\beta_F) = (-1)^p a_p$, we have that $\zeta^{-1} \omega_W$ is a local unit volume form for $(W \otimes K^{-1})$ with respect to the bilinear form sending $\omega_1, \omega_2 \mapsto Q_W(\beta_F \omega_1, \omega_2)$. Since there is an identification $\det(V_0) \cong K^{-p}$, away from $D$ we may choose locally an isomorphism $\psi: K^p \to \det(V_0^*)$ such that the induced map $\psi^2: K^{2p} \to \det(V_0^*)^2$ satisfies $\psi^2(a_p) = \det(Q_0)$. This implies that $\psi^{-1}(\zeta^{-1}) \in \mathcal{O}(\det(V_0))$ is a local unit volume form for $V_0$. Consider then $\varphi: \det(W) \to \det(V)$ defined by

$$\varphi(\omega_W) = (\zeta^{-1} \omega_W) \otimes \psi(\zeta^{-1}).$$

Note that $\varphi$ does not depend on the choice of (locally defined) square root $\zeta$ of $a_p$, Thus, $\varphi$ is globally defined on the complement of $D$ and depends only on $\psi$. In this manner the choice of $\varphi$ is equivalent to the choice of $\psi$. Hence, the
determinant data are given by the choice of an isomorphism $\psi : K^p \to \det(V_0^*)$ such that $\psi^2(a_p) = \det(Q_0)$.

Consider the equivariant orthogonal bundle $(M, Q_M, \hat{\sigma}_C)$ corresponding to $(V_0, Q_0)$. Then the condition $\det(V_0^*) \cong K^p$ is equivalent to requiring that the determinant $\det(M) = \mathcal{O}$, as shown in Theorem 1. Let $\text{vol}_M$ be a global unit volume form on $M$ (there are precisely two choices for $\text{vol}_M$). We also have $\hat{\sigma}_C(\text{vol}_M) = -\text{vol}_M$, which can be seen by considering $\text{vol}_M$ around a ramification point. As before, let $\zeta$ be the tautological section of $\pi_C^*(K^p)$, which satisfies $\zeta^2 = a_p$. Then $\text{vol}_M \zeta$ is an invariant section of $\pi^*(\det(V_0) \otimes K^p)$ and descends to an isomorphism $\psi : K^p \to \det(V_0^*)$ with the desired property. Thus, the choice of isomorphism $\det(V) \cong \det(W)$ corresponds naturally to a choice of orientation for the bundle $M$.

5.3. Stability. To complete the Langlands type correspondence, it remains to relate stability of the $SO(p+q,p)$-Higgs bundle to a stability condition on the quadratic bundle $(V_0, Q_0)$, or equivalently to a stability condition on the corresponding equivariant bundle $(M, Q_M, \hat{\sigma}_C)$. An $SO(p+q,p)$-Higgs bundle $(V, W, Q_V, Q_W, \beta)$ is semistable if and only if for all pairs of isotropic subbundles $V' \subseteq V$ and $W' \subseteq W$ with $\beta(W') \subseteq V' \otimes K$ and $\gamma(V') \subseteq W' \otimes K$, we have $\deg(V') + \deg(W') \leq 0$.

Notice that the conditions on $(V', W')$ are equivalent to saying $V' \oplus W'$ is an isotropic in $E = V \oplus W$ which is invariant under $\Phi$.

Lemma 7. For an $SO(p+q,p)$-Higgs bundle $(V, W, Q_V, Q_W, \beta)$ in a regular fiber of the Hitchin fibration, if $(V', W')$ is a pair of isotropic subbundles with $\beta(W') \subseteq V' \otimes K$ and $\gamma(V') \subseteq W' \otimes K$, then $W' = 0$ and $V' \subseteq V_0$. Therefore, semistability of $(V, W, Q_V, Q_W, \beta)$ in a regular fiber of the Hitchin fibration reduces to

for all isotropic subbundles $V' \subseteq V_0$, we have $\deg(V') \leq 0$.

Proof. Suppose that the spectral curve $\pi : S \to \Sigma$ from subsection 2.4 is smooth. Then its defining polynomial,

$$
\eta^{2p} + a_1 \eta^{2p-2} + \cdots + a_p,
$$

is irreducible, and therefore the only invariant subbundles of $V_1 \oplus W$ are $V_1 \oplus W$ and $\{0\}$. If there were isotropic bundles $V'$ and $W'$ with $\beta(W') \subseteq V' \otimes K$ and $\gamma(V') \subseteq W' \otimes K$, then the image of $V' \oplus W'$ under $V \oplus W \to V_1 \oplus W$ would need to be either $V_1 \oplus W$ or $\{0\}$. However, since $V' \oplus W'$ is isotropic, the image has to be $\{0\}$ as it cannot contain $W$. This implies that $W' = \{0\}$ and $V' \subseteq V_0$.

Let $(M, Q_M, \hat{\sigma}_C)$ be the equivariant orthogonal bundle corresponding to $(V, W, Q_V, Q_W, \beta)$ in a regular fiber of the Hitchin fibration. Then from the above analysis one can understand semistability as follows.

Proposition 11. An $SO(p+q,p)$-Higgs bundle in the regular fibers of the orthogonal Hitchin fibration is semistable if and only if for all $\hat{\sigma}_C$-invariant isotropic subbundles $M' \subseteq M$, we have that $\deg(M') \leq 0$.

Proof. Note that a $\hat{\sigma}_C$-invariant subbundle $M' \subseteq M$ induces a subbundle $V' \subseteq V_0$ by taking the invariant direct image. Conversely, every subbundle $V' \subseteq V_0$ is seen to arise in this manner. Examining behavior of the invariant direct image around branch points, one sees that $\deg(V') = \deg(M') + s$, where $s$ is the number of ramification points $r \in C$ for which $M'_r \cap M^-_r \neq \{0\}$. However, if $M'$ (or equivalently $V'$) is isotropic, then we must have $M'_r \cap M^-_r = \{0\}$ because $M'_r$ is a
1-dimensional nondegenerate subspace of $M_r$, so its only isotropic subspace is $\{0\}$. Therefore $\text{deg}(V') = \text{deg}(M')$ for isotropic subbundles. □

Remark 13. It is interesting to compare the above stability conditions for these orthogonal Higgs bundles with signature with the conditions obtained in [39] for certain unitary Higgs bundles with signature, where Hodge bundles were used as extension bundles.

Remark 14. In the case of $q = 1$, as one would expect, the above conditions become obsolete, as can be seen in [35].

We have now obtained all the results that lead to a geometric description of the intersection of the real slice $\mathcal{M}_{\text{SO}(p+q,p)}$ with the fibers of the SO$(2p+q,\mathbb{C})$ Hitchin fibration.

Theorem 2. There is a one-to-one correspondence between semistable SO$(p+q,p)$-Higgs bundles $(V, W, \beta)$ in the regular fibers satisfying Assumption [4] of the SO$(2p+q,\mathbb{C})$-Hitchin fibration and isomorphism classes of triples $(L, M, \tau)$, where the following are true.

(I) $L \in \text{Jac}(\overline{S})[2]$ is an orthogonal line bundle on the $p$-fold cover

$$\overline{S} := \{\xi^p + a_1 \xi^{p-1} + \cdots + a_{p-1} + a_p = 0\} \subset \text{Tot}(K^2),$$

where $a = \{a_i\}$, with $a_i \in H^0(\Sigma, K^{2i})$, and where $\xi$ is the tautological section of the pullback of $K^2$.

(II) $M$ is an equivariant rank $q$-orthogonal bundle on the 2-fold cover $C = \{\zeta^2 = a_p\} \subset \text{Tot}(K^p)$ of type $(q-1,1)$ over each ramification point, with a choice of orientation, that satisfies the following semistability condition: all invariant isotropic subbundles $M' \subset M$ have degree $\leq 0$.

(III) For each zero $x$ of $a_p$, an orthogonal isomorphism $\tau_x : M_x^{-} \rightarrow L_x^r$, where $r$ and $r'$ are the corresponding zeros of $\xi$ and with $\zeta$ lying over $x$.

Two such triples $(L, M, \tau)$ and $(L', M', \tau')$ lying in the same fiber of the orthogonal Hitchin map correspond to isomorphic SO$(p+q,p)$-Higgs bundles if and only if there is an isomorphism $\psi : L \rightarrow L'$ of orthogonal line bundles, and an isomorphism $\varphi : M \rightarrow M'$ of equivariant orthogonal bundles under which $\tau' = \pm \psi \circ \tau \circ \varphi^{-1}$.

We finish this section by summarizing the reconstruction of the SO$(p+q,p)$-Higgs bundle $(V, W; \beta)$ from its spectral data $(L, M, \tau)$.

(1) The orthogonal bundle $(W, Q_W)$ is obtained by pushing forward $L$, namely, $W = \pi_*(L) \otimes K^{p-1}$, and $Q_W$ is obtained from the orthogonal structure on $L$ by relative duality.

(2) $\beta_F : W \rightarrow W \otimes K^2$ is obtained by pushing down $\xi : L \rightarrow L \otimes \pi^*(K^2)$.

(3) We let $V_1 = W \otimes K = \pi_*(L) \otimes K^p$ and note that $V_1^* = W \otimes K^{-1}$.

(4) The quadratic bundle $(V_0, Q_0)$ is obtained as the invariant direct image of the orthogonal bundle $M$.

(5) The bundle $V$ is obtained as an extension of the bundle $V_1$ by $V_0$. We have that $\mathcal{O}(V)$ can be identified with the sheaf of meromorphic sections of $V_1^* \oplus V_0 \cong (W \otimes K^{-1}) \oplus V_0$ with first order poles whose residues are constrained by the extension data $\{\tau_x\}$ as specified in Proposition [6].

(6) In terms of the above description of $V$, the orthogonal structure $Q_V$ and the maps $\beta : W \rightarrow W \oplus (V_0 \otimes K)$, $\gamma : (W \otimes K^{-1}) \oplus V_0 \rightarrow W \otimes K$ are
given by
\[
Q_V((a, b), (c, d)) = Q_W(\beta F a, c) + Q_0(b, d),
\]
\[
\beta(u) = (u, 0),
\]
\[
\gamma(v, w) = \beta F(v).
\]

6. Characteristic classes

We consider the characteristic classes of SO\((p + q, p)\)-Higgs bundles in terms of the Cayley and Langlands type correspondences. The maximal compact subgroup of SO\((p + q, p)\) is \(S(O(p + q) \times O(p))\), so an SO\((p + q, p)\)-Higgs bundle \((V \oplus W, \Phi)\) carries three topological invariants: the Stiefel–Whitney classes \(\omega_1(W), \omega_2(W)\), and \(\omega_2(V)\). By a K-theoretic approach following the methods of \[5,26,35\], we give a description of these classes in terms of spectral data, leading to Theorem 3.

6.1. Stiefel–Whitney classes of \(W\) via the spectral curve. We will show here that the Stiefel–Whitney classes of \(W\) can be described completely in terms of the associated line bundle \(L \in \text{Jac}(\bar{S})[2]\) of order 2. Choosing a theta characteristic \(K_{1/2}\), we may assign to a vector bundle \(W\) with \(O(n, \mathbb{C})\)-structure an analytic mod 2 index

\[
\varphi_{\Sigma}(W) = \dim H^0(\Sigma, W \otimes K_{1/2}) \pmod{2}.
\]

(6.1)

It follows from \[26\,\text{Theorem 1}\] that

\[
\omega_2(W) = \varphi_{\Sigma}(W) + \varphi_{\Sigma}(\text{det}(W)).
\]

(6.2)

Since we would like to understand the characteristic classes of \(W\) in terms of the corresponding line bundle \(L \in \text{Jac}(\bar{S})[2]\) in item (1) of Theorem \[2\], we will make a few comments here on how these are related. Adopting the notation of \[26\,\text{section 5}\], for any line bundle \(L\) on \(S\) of order 2, we define

\[
\bar{n}_*(L) = \bar{n}_*(L \otimes (K_{\bar{S}} \otimes \bar{n}^*K^*)^{1/2}).
\]

(6.3)

By relative duality \(\bar{n}_*(L)\) inherits an orthogonal structure. In particular, as in section \[3\] we have

\[
W = \bar{n}_*(L).
\]

(6.4)

One is thus in the setting of \[35\,\text{Theorem 8}\], which leads to the following.

**Proposition 12.** The characteristic classes associated to the rank \(p\) vector bundle \(W\) of an SO\((p + q, p)\)-Higgs bundle \((V \oplus W, \Phi)\) with spectral data \(L\) on \(\bar{S}\) as in (1) of Theorem \[2\] are given by

\[
\omega_1(W) = \text{Nm}(L) \in H^1(\Sigma, \mathbb{Z}_2),
\]

\[
\omega_2(W) = \varphi_{\bar{S}}(L) + \varphi_{\Sigma}(\text{Nm}(L)) \in \mathbb{Z}_2,
\]

where \(\varphi_{\Sigma}\) and \(\varphi_{\bar{S}}\) are the analytic mod 2 indices, and \(\text{Nm : Jac}(\bar{S})[2] \rightarrow \text{Jac}(\Sigma)[2]\) the Norm map.
6.2. Stiefel–Whitney classes of $V$. Since we are working with $\text{SO}(p+q,p)$-Higgs bundles, we have $\omega_1(V) = \omega_1(W)$. Hence, all that remains is to compute $\omega_2(V)$, which we will do by computing $\omega_2(V \oplus W)$ and using

$$\omega_2(V \oplus W) = \omega_2(V) + \omega_2(W) + \omega_1(V) \cup \omega_1(W) = \omega_2(V) + \omega_2(W) + \omega_1(W) \cup \omega_1(W).$$

On a compact Riemann surface the mod 2 intersection form is alternating, so we have $\omega_1^2(W) = 0$, and thus

$$\omega_2(V) = \omega_2(V \oplus W).$$

Our strategy for computing $\omega_2(V \oplus W)$ will be to reduce the problem to the $q = 1$ case, which is more manageable, essentially due to the abelian structure of the fibers in this case. Recall that the spectral data consists of the orthogonal line bundle $L \in \text{Jac}(\tilde{S})[2]$, the quadratic bundle $(V_0,Q_0)$, and for each zero $x$ of $a_p$, an orthogonal isomorphism $\tau_x : N_x \otimes K_p^r \rightarrow L_r$, where $r$ is the zero of $\xi$ lying over $x$.

Case $q = 1$. Consider first the $\text{SO}(p+q,p)$-Higgs bundles for $q = 1$. Then since $V_0$ has rank 1 and determinant $K^{-p}$, we have $V_0 \cong K^{-p}$. The quadratic form $Q_0$ can be viewed as multiplication by $a_p$ via $a_p : K^{-p} \otimes K^{-p} \rightarrow \mathbb{C}$. Therefore, if $x$ is a zero of $a_p$, we have $N_x \otimes K_p^r = K^{-p}_x \otimes K_p^r = \mathbb{C}$ equipped with $Q_0/a_p = a_p/a_p = 1$, the standard orthogonal structure. An orthogonal isomorphism $\tau_x : \mathbb{C} \rightarrow L_r$ is equivalent to a choice of unit vector $\tau_x \in L_r$. Given such a collection $\{\tau_x\}$, we seek to determine $\omega_2(V)$. First note that $V$ is an extension of the form

$$0 \rightarrow K^{-p} \rightarrow V \rightarrow V_1 \cong W \otimes K \rightarrow 0,$$

and therefore the $\text{SO}(2p+1,\mathbb{C})$-bundle $V \oplus W$ is an extension of the form

$$0 \rightarrow K^{-p} \rightarrow (V \oplus W) \rightarrow F \otimes K^{1/2} \rightarrow 0,$$

where $F$ is the $\text{Sp}(2p,\mathbb{C})$-bundle $F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2})$. This is a special case of the construction of $\text{SO}(2p+1,\mathbb{C})$-Higgs bundles from $\text{Sp}(2p,\mathbb{C})$ considered in [24]. It now follows from [24] that $\omega_2(V \oplus W)$ can be obtained from $\{\tau_x\}$ in the following manner.

Recall that an $\text{SO}(p+1,p)$-Higgs bundle defines a double cover $\rho : S \rightarrow \tilde{S}$ and the involution $\sigma : S \rightarrow S$ from [27]. Let $\text{Prym}(S,\tilde{S})$ be the corresponding Prym variety, i.e., the subvariety of $\text{Jac}(S)$ given by line bundles $A \in \text{Jac}(S)$ satisfying $\sigma^*A \cong A^*$. Viewing the orthogonal structure on $L \in \text{Jac}(\tilde{S})[2]$ as a map $Q_L : L \rightarrow L^*$, one can see that $\rho^*L \in \text{Prym}(S,\tilde{S})$. Here the isomorphism $\alpha : \sigma^*(\rho^*L) \rightarrow (\rho^*L)^*$ is given by

$$\sigma^*\rho^*L \cong \rho^*L \xrightarrow{\rho^*(Q_L)} \rho^*L^* = (\rho^*L)^*,$$

where the first isomorphism is given as $\rho \circ \sigma = \rho$. Suppose that $N \in \text{Prym}(S,\tilde{S})$ satisfies $N^2 \cong \rho^*L$, and choose a specific isomorphism $\mu : N^2 \rightarrow \rho^*L$. Then there is an isomorphism $\nu : \sigma^*N \rightarrow N^*$ which we can assume is chosen so that there is a commutative diagram

$$\begin{array}{ccc}
\sigma^*\rho^*L & \xrightarrow{\alpha} & \rho^*L^* \\
\sigma^*\mu & & \mu \otimes (-1) \\
\sigma^*N^2 & \xrightarrow{\nu \otimes (-1)} & N^{-2}.
\end{array}$$

For a given $\mu$, this uniquely determines $\nu$ up to an overall sign. Moreover, $\nu$ can be viewed as a section of $N^*\sigma^*(N^*)$. 

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let \( r \in \tilde{S} \) be a zero of \( \xi \), and let \( r' \) be the unique point of \( r' \in S \) lying over \( r \). These are the ramification points of \( S \to \tilde{S} \), and they are also the fixed points of \( \sigma : S \to S \). Thus, \( \nu_{r'}^{-1} \in N_r.N_{\sigma(r')} = N_r^2 \). Then if we set \( \tau_x = \mu_{r'}(\nu_{r'}^{-1}) \in \langle p^*L \rangle_{r'} = L_r \), the above commutative diagram implies that \( Q_{\tau_x}(\tau_x, \tau_x) = 1 \), so \( \{ \tau_x \} \) is a choice of extension data. It follows from [24, section 4.3] that \( \omega_2(V \oplus W) = 0 \) if and only if the extension data \( \{ \tau_x \} \) arise in this way for some \( N \in \text{Prym}(S, \tilde{S}) \). This completely determines \( \omega_2(V \oplus W) \) in terms of \( N \in \text{Prym}(S, \tilde{S}) \) and the extension data \( \{ \tau_x \} \).

The \( q > 1 \) case. The following proposition allows us to reduce the study of characteristic classes of \( \text{SO}(p+q,p) \)-Higgs bundles to the \( q = 1 \) situation:

**Proposition 13.** There exists a \( C^\infty \)-isomorphism of vector bundles

\[
V_0 \cong V_0' \oplus K^{-p},
\]

where \( V_0' \) is a rank \( q-1 \) orthogonal vector bundle such that \( Q_0 \) is the orthogonal direct sum of \( V_0' \) with \( K^{-p} \) equipped with the bilinear form \( a_p : K^{-p} \otimes K^{-p} \to \mathbb{C} \).

**Proof.** In the case \( q = 1 \) we have already seen that \( V_0 \cong K^{-p} \), so let us assume that \( q \geq 2 \). Choose disjoint open discs \( D_x \) around each zero \( x \) of \( a_p \) such that on each \( D_x \) there is a local holomorphic coordinate \( z \) centered at \( x \) with \( a_p = z(dz)^{2p} \).

We can further assume that over each \( D_x \) there is a holomorphic frame \( e_1, \ldots, e_q \) of \( V_0 \) with \( Q_0(e_i, e_j) = \delta_{ij} \) for \( (i, j) \neq (1, 1) \) and \( Q_0(e_1, e_1) = z \). Set \( e' = e_1 \otimes (dz)^p \).

Then \( e' \) is a section of \( V_0 \otimes K^p \) defined on each \( D_x \) and satisfying \( Q_0(e', e') = a_p \).

Moreover, the space \( \Sigma^* = \Sigma \setminus \cup_x D_x \) is homotopy equivalent to a wedge of circles, and \((V_0 \otimes K^p, Q_0/a_p)\) is a rank \( q \) orthogonal bundle on \( \Sigma^* \). Choosing a reduction of structure of \( V_0 \otimes K^p \) to the maximal compact \( O(q) \subset O(q, \mathbb{C}) \), since \( q \geq 2 \), it follows that the fibers of the unit sphere bundle of \( V_0 \otimes K^p \) are connected. Hence, by obstruction theory we can find a smooth section \( e \) of \( V_0 \otimes K^p \) on \( \Sigma^* \) with \( Q_0(e, e) = a_p \).

For each \( x \) let \( D'_x \subset D_x \) be a smaller open disc around \( x \) so that \( D_x \setminus D'_x \) is an annulus. Then since \( q \geq 2 \), we can smoothly extend \( e \) over the annulus so that \( e|_{\partial D'_x} = e' \) and can extend \( e \) over \( D'_x \) to equal \( e' \).

Let \( E \subset V_0 \) be given by \( E = eK^{-p} \subset V_0 \), and consider \( V_0' \) to be defined as the orthogonal complement of \( E \) away from the zeros of \( a_p \) and \( V_0'|_{D'_x} = \text{span}(e_2, \ldots, e_q) \).

Then we have an orthogonal direct sum \( V_0 = V_0' \oplus E \). Note that \( e : K^{-p} \to E \) gives an isomorphism \( E \cong K^{-p} \), and since \( Q_0(e, e) = a_p \), we can identify \( E \) with \( K^{-p} \) equipped with the bilinear form \( a_p : K^{-p} \otimes K^{-p} \to \mathbb{C} \). \( \square \)

Recall from section [13] that the vector bundle \( V \) can be reconstructed as an extension of \( V_0 \) by \( V_1 \) and that the extension class has the form \( \delta(i) \), for some \( i \in H^0(D, \text{Hom}(N, V_1^*)) \). In general, a decomposition \( V_0 \cong V_0' \oplus K^{-p} \) as in Proposition [13] can only be done smoothly and not holomorphically. However, examining the proof of Proposition [13] we see that the isomorphism \( V_0 \cong V_0' \oplus K^{-p} \) can be chosen so that it is holomorphic in a neighborhood of each zero of \( a_p \). The choice of such an isomorphism, \( \varphi : V_0 \to V_0' \oplus K^{-p}, \) in particular induces an identification \( \varphi : N_x \to K^{-p} \), where \( N_x = \text{Ker}(Q_0)_x \subset (V_0)_x \), as before. Hence, we can view \( i \) as \( \varphi(i) \in H^0(D, \text{Hom}(K^{-p}, V_1^*)) \). Replacing \( V(0, Q_0) \) with \( (K^{-p}, a_p) \), we may construct from \( i \) an extension of \( V_1 \) by \( K^{-p} \):

\[
0 \to K^{-p} \to V'' \to V_1 \to 0.
\]
Theorem 3. Choose a smooth splitting $V_0 \cong V_0^1 \oplus K^{-p}$ as in Proposition [13] and use this to identify $N_x$ with $K_x^{-p}$. The extension data $\tau_x : \mathcal{N}_x \otimes K_x^p \to L_x$ for $V$ is then identified with a collection $\{\tau_x\}$ of unit vectors $\tau_x \in L_x$. Let $\delta = 0$ if $\{\tau_x\}$ is induced from some $N \in \text{Prym}(S, \bar{S})$ with $N^2 = \rho^* L$, and $\delta = 1$ otherwise. Then

\[
\begin{align*}
\omega_1(W) &= \text{Nm}(L) \in H^1(\Sigma, \mathbb{Z}_2), \\
\omega_2(W) &= \varphi_S(L) + \varphi_\Sigma(\text{Nm}(L)) \in \mathbb{Z}_2, \\
\omega_2(V) &= \varphi_S(L) + \varphi_\Sigma(\text{Nm}(L)) + \omega_2(V_0') + \delta \in \mathbb{Z}_2,
\end{align*}
\]

where $\varphi_\Sigma$ and $\varphi_S$ are the analytic mod 2 indices, and $\text{Nm} : \text{Jac}(\bar{S})[2] \to \text{Jac}(\Sigma)[2]$ the Norm map.

Proof. From the constructions of $V$ and $V''$ via $i$ in section 5 one sees that we have a $C^\infty$-isomorphism of orthogonal vector bundles

\[ V \cong V_0' \oplus V''. \]

Note that $\text{det}(V'') = K^{-p} \otimes \text{det}(V_1) = K^{-p} \otimes K^p \otimes \text{det}(W) = \text{det}(W)$ (at the level of holomorphic bundles), so $\omega_1(V'' \oplus W) = 0$. Therefore,

\[
\begin{align*}
\omega_2(V \oplus W) &= \omega_2(V_0' \oplus (V'' \oplus W)) \\
&= \omega_2(V_0') + \omega_2(V'' \oplus W) + \omega_1(V_0') \cup \omega_1(V'' \oplus W) \\
&= \omega_2(V_0') + \omega_2(V'' \oplus W).
\end{align*}
\]

But note that $V'' \oplus W$ is an extension $0 \to K^{-p} \to (V'' \oplus W) \to (V_1 \oplus W) = F \otimes K^{1/2} \to 0$, where $F$ is the $\text{Sp}(2p, \mathbb{C})$-bundle $F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2})$, and therefore we have reduced the computation to the $q = 1$ case. Thus, $\omega_2(V'' \oplus W)$ is computed from the extension data $\{\tau_x\}$ as described above. Putting it all together, the theorem follows. $\square$

7. Abelianization for split and quasi-split real forms

We have seen that the spectral data describing regular fibers of the moduli space of $\text{SO}(p+q,p)$-Higgs bundles consist of an abelian part given by the Cayley type correspondence and a nonabelian part, given by the Langlands type correspondence. In what follows we will consider the special cases of the split real form $\text{SO}(p+1,p)$ and the quasi-split real form $\text{SO}(p+2,p)$. For $q = 1$, the Langlands data is trivially abelian, but we will see that for $q = 2$ the Langlands data can also be abelianized, providing a novel description of the intersection of the moduli spaces of $\text{SO}(p+2,p)$-Higgs bundles with the Hitchin fibration.

We first consider the case of the quasi-split real forms $\text{SO}(p+2,p)$, which are not split in subsection 7.1 and then consider the split real forms $\text{SO}(p+1,p)$ in subsection 7.2. Finally, in subsection 7.3 we describe from a geometric perspective the extra components appearing in the $\text{SO}(p+1,p)$ moduli space, which were first suggested to exist by Morse theory calculations [11] and later identified by Collier in his Ph.D. thesis [11] by constructing connected subsets of the moduli space which are both open and closed. These components can be seen from the spectral data, and we will comment on extra components that should appear for any $q > 1$. 

Licensed to AMS.
License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
7.1. Quasi-split real forms which are not split. Here we consider the case \( q = 2 \), that is, Higgs bundles for the quasi-split real form \( SO(p + 2, p) \). From the study of cameral covers, it is known that abelian data should exist describing Higgs bundles for quasi-split real forms \([31]\). Here we will use our spectral data constructions to provide a concrete description of the abelian structure of the fibers, completing the explicit description of abelian data for all Higgs bundles coming from quasi-split real forms (the case of split real forms was described in \([33]\), and the cases of \( U(p, p) \) and \( U(p + 1, p) \) in \([32][34]\), respectively).

Under the assumptions of Theorem 2, SO\((p + 2, p)\)-Higgs bundles in the regular fibers of the SO\((2p + 2, \mathbb{C})\)-Hitchin fibration can be parameterized by triples \((L, M, \tau)\), where \( L \in \text{Jac}(\bar{S})[2] \) is an orthogonal line bundle on \( \bar{S} \), the bundle \( M \) is an equivariant SO\((2, \mathbb{C})\)-bundle on the 2-fold cover \( C \) of type \((1, 1)\) over each fixed point of \( \sigma_C \) such that all invariant isotropic subbundles have degree \( \leq 0 \), and \( \tau = \{\tau_x\} \) is the extension data.

**Theorem 4.** The intersection of the moduli space of SO\((p + 2, p)\)-Higgs bundles with a fiber of the SO\((2p + 2, \mathbb{C})\) Hitchin fibration over a point defining the spectral curve

\[
S := \{\eta^{2p} + a_1 \eta^{2p-2} + \cdots + a_p = 0\}
\]

satisfying Assumption \([1]\) is given by triples \((L, N, \tau)\), where \( L \in \text{Jac}(\bar{S})[2] \), the line bundle \( N \in \text{Prym}(C, \Sigma) \), and \( \tau = \{\tau_x\} \) are the extension data, considered modulo \( \tau \sim -\tau \). There is a natural abelian group structure on such triples, given by

\[
(L, N, \tau)(L', N', \tau') = (L \otimes L', N \otimes N', \tau \otimes \tau').
\]

As a group, this fiber is isomorphic to \( \text{Prym}(C, \Sigma) \times (\mathbb{Z}_2)^{(4p^2 + 2p)(g-1)+1} \). In particular, the fiber is \( 2^{(4p^2 + 2p)(g-1)+1} \) copies of \( \text{Prym}(C, \Sigma) \).

**Proof:** Consider the parameterization of SO\((p + 2, p)\)-Higgs bundles in Theorem 2 as triples \((L, M, \tau)\). As the bundle \( M \) is an SO\((2, \mathbb{C})\)-bundle, we have

\[
M = N \oplus N^*
\]

for some line bundle \( N \) on \( C \). Let \( \delta_C : M \to M \) denote the lift of \( \sigma_C \) to \( M \). Since \( N \) and \( N^* \) are the only isotropic subbundles of \( M \), we must have either \( \delta_C(N) = N \) or \( \delta_C(N) = N^* \). However, the fact that \( M \) is assumed to have type \((1, 1)\) over each fixed point implies that \( \sigma \) must exchange the isotropic subspaces, so \( \delta_C(N) = N^* \). In particular, we have \( \sigma_C(N) \cong N^* \) so that \( N \) belongs to the Prym variety \( \text{Prym}(C, \Sigma) \) of the cover \( C \to \Sigma \).

Conversely, for any \( N \in \text{Prym}(C, \Sigma) \), we obtain a rank 2 equivariant orthogonal bundle \( M = N \oplus N^* \). Notice that since \( \delta_C \) exchanges \( N \) and \( N^* \), there are no invariant isotropic subbundles of \( M \), so \( M \) satisfies the stability condition. Note also that \( N \) and \( N^* \) are distinguished from one another by the choice of an orientation on \( M \) (swapping \( N \) and \( N^* \) reverses the orientation on \( M \)). Fix an isomorphism \( \varphi : \sigma^* N \to N^* \), and let \( r \in C \) be a fixed point of \( \sigma_C \). Then over \( r \) the map \( \varphi \) induces an isomorphism \( \varphi_r : N_r \to N^*_r \), i.e., an orthogonal structure on \( N_r \).

The \(-1\)-eigenspace \( M_r^- \subset N_r \oplus N_r^* \) is given by \( \{(v, -\varphi_r(v)) \mid v \in N_r\} \), and in this way we get an identification \( M_r^- \cong N_r \) as orthogonal spaces. The extension data \( \{\tau_x\} \) can now be viewed as a collection of isometries \( \tau_x : N_r \to L_{r'} \) (\( r' \) is the corresponding point in \( \bar{S} \)). Notice also that the only orientation preserving isometries of \( M = N \oplus N^* \) are given by \((a, b) \mapsto (ca, c^{-1}b)\), where \( c \in \mathbb{C}^* \) is
a constant. Thus, two triples \((L, N, \tau)\) and \((L', N', \tau')\) define isomorphic Higgs bundles if and only if \(L' \cong L, N' \cong N\) and \(\tau' = \pm \tau\).

Only the last statement about the group structure of the fibers remains to be shown. First note that \(\text{Jac}(\bar{S})[2] \cong (\mathbb{Z}_2)^{2g_S}\) and \(g_S = (2p^2 - p)(g - 1) + 1\). Next, note that \(a_p\) has degree \((K^{2p}) = 4p(g - 1)\) zeros; hence the possible choices of \(\tau\) for given \(L\) and \(N\) forms the group \((\mathbb{Z}_2)^{4p(g - 1) - 1}\), where the \(-1\) comes from identifying \(\tau\) and \(-\tau\). Thus, the group of components of the fiber is isomorphic to \((\mathbb{Z}_2)^{2g_S + 4p(g - 1) - 1} = (\mathbb{Z}_2)^{(4p^2 + 2p)(g - 1) + 1}\). The group structure of the fiber is then an extension of \((\mathbb{Z}_2)^{(4p^2 + 2p)(g - 1) + 1}\) by \(\text{Prym}(C, \Sigma)\). The extension must be split since \(\text{Prym}(C, \Sigma)\) is a divisible group (if \(G\) is a divisible abelian group, then \(G\) is an injective \(\mathbb{Z}\)-module), and thus \(\text{Ext}^1_{\mathbb{Z}}(G, A) = 0\) for any abelian group \(A\). \(\square\)

7.2. **Split real forms.** The group \(\text{SO}(p + 1, p)\) is a split real form, and as such the spectral data for the corresponding Higgs bundles can be described using the techniques developed in this paper, as well as by considering them as 2-torsion points in the complex Hitchin fibration \([35]\). In either case it follows directly that we have abelian spectral data.

The techniques developed in the previous sections have allowed us to understand \(\text{SO}(p + 1, p)\)-Higgs bundles \((V \oplus W, \Phi)\), whose Higgs field has a characteristic polynomial of the form

\[
\text{det}(\Phi - \text{Id}p) = \eta(\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_p),
\]

satisfying Assumption 1. Let \(D \subset \Sigma\) be the divisor of zeros of \(a_p\), and let \(D_S \subset \bar{S}\) be the divisor in \(\bar{S}\) given by the zeros of the tautological section \(\xi\) of \(K^2\). For each point \(x \in D\) there is a unique point \(x' \in D_S\) lying over \(x\). In particular, there is a naturally defined bijection between the points of these divisors.

Recall that \(W = \pi_x(L) \otimes K^{p-1}\), where \(L\) is an orthogonal line bundle on \(\bar{S}\). The tautological section \(\xi = \eta^2 : L \to L \otimes K^2\) pushes down to give the map \(\beta_p : W \to W \otimes K^{2}\), and the orthogonal structure on \(L\) induces by relative duality an orthogonal structure \(Q_W\) on \(W\). Moreover, in this case the quadratic bundle \((V_0, 0_\beta)\) is simply \((K^{-p}, a_p)\), where \(a_p\) is viewed as a bilinear form \(a_p : K^{-p} \otimes K^{-p} \to \mathcal{O}\). Then, from Theorem 2, the remaining data required to construct an \(\text{SO}(p + 1, p)\)-Higgs bundle are for each \(x \in D\) a choice of unit vector \(e_x \in L_{x'}\).

**Proposition 14.** The spectral data for \(\text{SO}(p + 1, p)\) consists of

- an orthogonal line bundle \(L\) on \(\bar{S}\),
- for each \(x \in D\) a choice of unit vector \(e_x \in L_{x'}\).

Two such pairs \((L, \{e_x\})\) and \((L', \{e'_x\})\) define isomorphic \(\text{SO}(p + 1, p)\)-Higgs bundles if and only if \(L \cong L'\) as orthogonal line bundles by some isomorphism \(\varphi : L \to L'\) such that \(\varphi(e_x) = \pm e'_x\).

There is a natural abelian group structure on the fibers given by the tensor product. Repeating the counting argument given in Theorem 4 the fibers are isomorphic to the group \((\mathbb{Z}_2)^{(4p^2 + 2p)(g - 1) + 1}\).

Given a line bundle \(L \in \text{Jac}(\bar{S})[2]\), in general, there is no preferred choice of unit vector in \(L_{x'}\). Suppose that we vary the coefficients \((a_1, a_2, \ldots, a_p)\), and hence also the spectral curves \(\bar{S}\) and \(\bar{S}'\) in a continuous family. Suppose that we also continuously vary the orthogonal line bundle \(L\). For some fixed member of the family choose for each \(x \in D\) a unit vector \(e_x \in L_{x'}\). Moving around a noncontractible
loop in the family, we may find that the choice of unit vectors \( \{ e_x \}_{x \in D} \) does not extend over the loop. In other words, we may find a nontrivial monodromy action on the set of choices of unit vectors.

The situation, however, is much simpler in the special case that \( L = \mathcal{O} \) is the trivial line bundle. Then for each \( x \in D \) a choice of unit vector in \( L_x \) is simply a choice of either \( +1 \) or \( -1 \). Thus, for each \( x \in D \) we have \( e_x \in \{ +1, -1 \} \). In this case it is easy to understand the monodromy action on \( \{ e_x \}_{x \in D} \). Namely, if we vary \( (a_1, a_2, \ldots, a_p) \) in some continuous loop within the space of smooth spectral curves, then the zeros of \( a_p \) are moved around by some permutation \( \theta : D \to D \).

The monodromy action on the choice of unit vectors \( \{ e_x \} \) is just the natural action induced by \( \theta \). In particular, we see that monodromy preserves the total number of \( +1 \)'s and \( -1 \)'s. On the other hand, it is easy to see that the full permutation group of \( D \) can be realized by monodromy. To see this, choose a path \( \gamma \) in the regular part of the base of the Hitchin fibration from \( a = (a_1, a_2, \ldots, a_p) \) to a point of the form \( \hat{a} = (0, \ldots, 0, a_p) \), for some \( a_p \in H^0(\Sigma, K^{2p}) \) with only simple zeros. Conjugating loops based at \( \hat{a} \) by \( \gamma \) sets up a bijection between the permutation group of zeros of \( a_p \) and the permutation group of zeros of \( \hat{a}_p \). Thus, we are reduced to the case \( (0, \ldots, 0, a_p) \). It follows from [4, Theorem 4.2] that every transposition of zeros of \( \hat{a}_p \) can be realized by some loop in the regular part of the base of the Hitchin fibration, and thus every permutation can be realized. Therefore the number of \( +1 \)'s is the only monodromy invariant.

Note that \(|D| = \deg(K^{2p}) = 4p(g - 1)|, so the number of \( +1 \)'s is an integer between 0 and \( 4p(g - 1) \). Let \( b_+ \) denote the number of \( +1 \)'s and \( b_- \) the number of \( -1 \)'s, so \( b_+ + b_- = 4p(g - 1) \). Note that replacing \( e_x \) with \( -e_x \) for every \( x \in D \) produces an isomorphic Higgs bundle. This operation exchanges the roles of \( b_+ \) and \( b_- \), so without loss of generality we may assume that \( b_+ \geq b_- \). It follows that there exists an integer \( b \) such that

\[
(7.2) \quad b_+ = 2p(g - 1) + b \quad \text{and} \quad b_- = 2p(g - 1) - b,
\]

where \( 0 \leq b \leq 2p(g - 1) \). Denote by \( D_+ \) and \( D_- \) the set of \( x \in D \) with \( e_x = 1 \) and \( e_x = -1 \), respectively, so that \( K^{2p} = \mathcal{O}(D_+) \otimes \mathcal{O}(D_-) \). Then there exist sections \( s_+ \in \mathcal{O}(D_+) \) and \( s_- \in \mathcal{O}(D_-) \), unique up to scale, which vanish on \( D_+ \) and \( D_- \), respectively. Scaling these sections appropriately, we can assume that

\[
(7.3) \quad s_-s_+ = \frac{a_p}{2},
\]

where the factor of \( 2 \) is introduced for later convenience. Setting \( B = K^{-p}(D_+) \), we have that \( B^* = K^{-p}(D_-) \) and \( \deg(B) = b_+ - 2p(g - 1) = b \).

### 7.3. The extra components for \( \text{SO}(p + 1, p) \) from spectral data.

It is known that the moduli space of \( \text{SO}(p + 1, p) \)-Higgs bundles has extra components not detected by characteristic classes [1111]. The extra components have some similarities with the Hitchin component, although they have nontrivial topology. They have been discovered as a byproduct of the Morse theoretic approach to counting connected components by looking for minima of the Hitchin functional. We will show here that these extra components emerge naturally from the spectral data point of view. Therefore, spectral data provide a simple conceptual explanation for the existence of these components.
We have seen in subsection 7.2 that choosing \( L = \mathcal{O} \) and \( \varepsilon_x = \pm 1 \) for all \( x \in D \) produces components distinguished by an integer invariant \( b \). To see that these components are indeed the extra components of the moduli space of \( \text{SO}(p+1,p) \)-Higgs bundles, we will carry out the reconstruction of the Higgs bundles corresponding to this spectral data and see that they coincide with the components given in [11].

In order to state the theorem, we introduce the following holomorphic differentials \( \{ h_u \} \). Let
\[
\xi p(\xi, x) = \xi(\xi^p + a_1(x)\xi^{p-1} + \cdots + a_p(x))
\]
be the characteristic polynomial of an \( \text{SO}(p+1,p) \)-Higgs bundle. For a given \( x \), let \( \xi_1, \ldots, \xi_p \) be the zeros of \( \xi p(\xi, x) \). Define \( \{ h_u \} \) to be the complete homogeneous symmetric polynomials of \( \xi_1, \ldots, \xi_p \). Namely, for any \( u \geq 1 \)
\[
(7.5) \quad h_u = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_u \leq p} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_u},
\]
and define \( h_0 = 1 \). Then \( h_u \) is a well-defined section of \( H^0(\Sigma, K^n) \). We recall the following version of Newton’s identities, valid for all \( j \geq 1 \):
\[
(7.6) \quad \sum_{u=0}^{j} h_u a_{j-u} = 0,
\]
where we set \( a_0 = 1 \).

**Theorem 5.** Let \((W, V, Q_W, Q_V, \beta, \gamma)\) be the \( \text{SO}(p+1,p) \)-Higgs bundle associated to \( L = \mathcal{O} \), and \( \varepsilon_x = \pm 1 \) according to whether \( x \in D_+ \) or \( x \in D_- \). Moreover, let \( B = K^{-r}(D_+) \), and let \( s_+ \) and \( s_- \) be as in equation (7.3). Then up to isomorphism \((W, V, Q_W, Q_V, \beta, \gamma)\) are given by
\[
W = K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-1)},
\]
\[
V = \left( K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)} \right) \oplus B \oplus B^* = W_0 \oplus B \oplus B^*,
\]
\[
Q_W(w_i, w'_j) = \begin{cases} 0 & \text{if } i + j < p + 1, \\ h_{i+j-(p+1)} w_i w'_j & \text{if } i + j \geq p + 1, \end{cases}
\]
\[
Q_V = Q_{w_0} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\beta(w_1, \ldots, w_p) = (w_1 - w_p a_{p-1}, w_2 - w_p a_{p-2}, \ldots, w_{p-1} - w_p a_1, w_p s_+, -w_p s_-),
\]
\[
\gamma(v_1, \ldots, v_{p-1}, g, h) = (s_+ h - s_- g, v_1, \ldots, v_{p-1}).
\]
Here \( w_i, w'_j, v_i, \) and \( v'_j \) are, respectively, sections of \( K^{p+1-2i}, K^{p+1-2j}, K^{p-2i} \), and \( K^{p-2j} \), the map \( g \) is a section of \( B \), the map \( h \) is a section of \( B^* \), and we identify \( w_j \) and \( v_j \) with the corresponding sections \((0, \ldots, 0, w_j, 0, \ldots, 0)\) and \((0, \ldots, 0, v_j, 0, \ldots, 0)\) of \( W \) and \( W_0 \), and similarly for \( w'_j \) and \( v'_j \). Also, the \( h_j \) are defined as in (7.5).

**Proof.** Since \( W = \pi_*(\mathcal{O}) \otimes K^{p-1} \), we have
\[
W = K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-3)} \oplus K^{-(p-1)},
\]
and therefore
\[ V_1^* = W \otimes K^{-1} = K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)} \oplus K^{-p}. \]

In order to identify \( Q_W \) and \( \beta_F \), we need to make this isomorphism explicit. Consider
\[ w = (w_1, w_2, \ldots, w_p) \in C^{\infty}(\Sigma, K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-3)} \oplus K^{-(p-1)}). \]

Then we identify \( w \) with the section of \( W = \pi_* (O) \otimes K^{p-1} \) given by
\[ (7.7) \quad w_1 + \xi w_2 + \xi^2 w_3 + \cdots + \xi^{p-1} w_p. \]

Recall that \( \beta_F : W \to W \otimes K^2 \) is obtained by pushing down multiplication by \( \xi \). Thus, if \( w \) is given as in (7.7), then
\[ \beta_F w = \xi w_1 + \xi^2 w_2 + \cdots + \xi^{p-1} w_{p-1} + \xi^p w_p = -w_p a_p + \xi (w_1 - w_p a_{p-1}) + \xi^2 (w_2 - w_p a_{p-2}) + \cdots + \xi^{p-1} (w_{p-1} - w_p a_1), \]

where the second line comes from the definition of \( S \) and from rearranging terms. In other words, we have
\[ \beta_F (w_1, w_2, \ldots, w_p) = (0, w_1, w_2, \ldots, w_{p-1}) - w_p (a_p, a_{p-1}, \ldots, a_1). \]

In order to compute \( Q_W \), recall that if \( p(\xi) = (\xi - \xi_1)(\xi - \xi_2) \cdots (\xi - \xi_p) \) is a monic degree \( p \) polynomial with distinct roots, we have
\[ \sum_i \frac{\xi_i^r}{p'(\xi_i)} = \begin{cases} 0 & \text{if } r < p - 1, \\ h_{r-(p-1)} & \text{if } r \geq p - 1, \end{cases} \]

where \( \{ h_a \} \) are the complete homogeneous symmetric polynomials of \( \xi_1, \ldots, \xi_p \), as in (7.2). Then since \( Q_W \) is obtained by relative duality,
\[ Q_W (w, w') (x) = \sum \frac{w(y) w'(y)}{d \pi(y)}, \]

Locally, we can view \( S \) as the zero set of \( p(\xi, x) \) as in (7.4), with \( d \pi = \partial_x p(x, \xi) \), so that
\[ Q_W (w, w') (x) = \sum_{\{ \xi_i | p(\xi_i, x) = 0 \}} \frac{w(\xi_i) w' (\xi_i)}{\partial_x p(x; \xi_i)}. \]

Let \( w_i \) be a section of \( K^{(p+1)-2i} \), and let \( w'_j \) be a section of \( K^{(p+1)-2j} \). As in the statement of the theorem, we identify \( w_i \) and \( w'_j \) with the corresponding sections of \( W \). Then
\[ Q_W (w_i, w'_j) = \sum_{\{ \xi_k | p(\xi_k) = 0 \}} \frac{w_i w'_j \xi_k^{i+j+2}}{p'(\xi_k)} = \begin{cases} 0 & \text{if } i + j < p + 1, \\ h_{i+j-(p+1)} w_i w'_j & \text{if } i + j \geq p + 1. \end{cases} \]

We would like to calculate the quadratic form \( Q_W (\beta_F v, v') \) on \( V_1^* \equiv W \otimes K^{-1} \), and for this it is better to make a change of basis, so we consider the following bundle automorphism:
\[ (7.8) \quad \psi : W \otimes K^{-1} \to W \otimes K^{-1}, \quad \psi(v_1, \ldots, v_p) = (v_1, \ldots, v_p) + v_p (a_{p-1}, \ldots, a_1, 0). \]

Considering \( \psi \) as an isomorphism \( \psi : V_1^* \to W \otimes K^{-1} \), we denote by \( Q_1 \) the quadratic form on \( V_1^* \) obtained by pullback of \( Q_W (\beta_F v, v') \) on \( W \otimes K^{-1} \), that is,
\[ Q_1(v, v') = Q_W (\beta_F \psi(v), \psi(v')). \]
Let $v_i$ be a section of $K^{p-2i}$, and let $v'_j$ be a section of $K^{p-2j}$. If $i, j < p$, one finds that
\[
Q_1(v_i, v'_j) = \begin{cases} 
0 & \text{if } i + j < p, \\
h_{i+j-p}v_iv'_j & \text{if } i + j \geq p.
\end{cases}
\]

If $i < p$ and $j = p$, we have from equation (7.6) that
\[
Q_1(v_i, v'_p) = Q_W((0, \ldots, 0, v_i, 0, \ldots, 0), v'_p(a_p, \ldots, a_1, 1)) = v_iv'_p \sum_{k=0}^{i} a_k h_{1-k} = 0.
\]

Lastly, if $i = j = p$, then one finds that $\beta_F(\psi(v_p)) = -v_p a_p(1, 0, \ldots, 0)$. Therefore,
\[
Q_1(v_p, v'_p) = Q_W(-v_p a_p(1, 0, \ldots, 0), v'_p(a_p, \ldots, a_1, 1)) = -v_p v'_p a_p.
\]

We write
\[
V_1^* = K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)} \oplus K^{-p}
\]
\[
= (K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)}) \oplus K^{-p}
\]
\[
= W_0 \oplus K^{-p},
\]

where
\[
W_0 = K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)}.
\]

Having shown that $W_0$ and $K^{-p}$ are orthogonal with respect to $Q_1$, and that $Q_1(v_p, v'_p) = -a_p v_p v'_p$, denote the restriction of $Q_1(\beta_F, \cdot)$ to $W_0$ by $Q_{W_0}$. By the above calculations this agrees with the definition of $Q_{W_0}$ given in the statement of the theorem. Next, recall that to construct $V$ from special data as in Theorem 2, we first take the bundle
\[
V' = (W \otimes K^{-1}) \oplus V_0 = (W \otimes K^{-1}) \oplus K^{-p} = W_0 \oplus (K^{-p} \oplus K^{-p}),
\]

which has a degenerate quadratic form $Q_{V'}$. This form is the direct sum of $Q_W(\beta_F, \cdot)$ with $a_p$ on the second $K^{-p}$ factor, and thus
\[
Q_{V'} = Q_{W_0} \oplus \begin{pmatrix} -a_p & 0 \\ 0 & a_p \end{pmatrix}.
\]

Next, we make a change of basis $K^{-p} \oplus K^{-p} \rightarrow K^{-p} \oplus K^{-p}$, so that the $K^{-p}$ factor which comes from $W \otimes K^{-1}$ is sent to the antidiagonal $\{(w, -w)\}$ and the factor of $K^{-p}$ which comes from $V_0$ is sent to the diagonal $\{(w, w)\}$. In such a basis $Q_{V'}$ becomes
\[
Q_{V'} = Q_{W_0} \oplus \begin{pmatrix} 0 & a_p/2 \\ a_p/2 & 0 \end{pmatrix}.
\]

In this basis the isotropic subspaces of $K^{-p} \oplus K^{-p}$ are the two $K^{-p}$ summands. Recall from Proposition [3] that to get $V$, we define $\mathcal{O}(V)$ to be the sheaf of meromorphic sections of $V'$ which for each $x \in D$ are allowed to admit first order poles on one of the two isotropic subspaces $\Gamma_x = K^{-p}_x \oplus 0$ or $\Gamma_x = 0 \oplus K^{-p}_x$. The choice of one of these two isotropic subspaces corresponds to whether $e_x = 1$ or $-1$ (which is which is unimportant since changing the sign of every $e_x$ gives an isomorphic Higgs bundle). Thus, we can assume that $D_+$ is the subset corresponding to the first isotropic, and $D_-$ to the second. Therefore,
\[
V = W_0 \oplus K^{-p}(D_+) \oplus K^{-p}(D_-) = W_0 \oplus B \oplus B^*.
\]

The induced quadratic form on $V$ is the direct sum of $Q_{W_0}$ with the natural dual pairing between $B$ and $B^*$. 
Lastly, we need to work out the maps $\beta : W \to V \otimes K$ and $\gamma : V \to W \otimes K$. We have

$$V = W_0 \oplus B \oplus B^* = \left(K^{(p-2)} \oplus K^{(p-4)} \oplus \cdots \oplus K^{-(p-2)}\right) \oplus B \oplus B^*$$

and $W = K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-1)}$. Recall that $\beta$ is defined away from the zeros of $a_p$ by the natural inclusion $W \to (W \otimes K^{-1}) \otimes K \cong V_1^* \otimes K \subset V \otimes K$, and that this extends holomorphically to a map $\beta : W \to V \otimes K$. Bearing in mind that we are using the isomorphism $\psi$ of (7.8) to identify $V_1^*$ with $W \otimes K^{-1}$, we find that $\beta(w_1, w_2, \ldots, w_p) = (w_1 - w_pa_{p-1}, w_2 - w_pa_{p-2}, \ldots, w_{p-1} - w_pa_1, w_ps_+, -w_ps_-)$. Recall also that $\gamma$ is defined away from the zeros of $a_p$ by

$$V = V_1^* \oplus V_0 \xrightarrow{\psi, id} \left(W \otimes K^{-1}\right) \oplus V_0 \to W \otimes K^{-1} \xrightarrow{\beta_p} W \otimes K,$$

where the map $\left(W \otimes K^{-1}\right) \oplus V_0 \to W \otimes K^{-1}$ is a projection to the first factor. This map also extends holomorphically over $D$, and thus we find that $\gamma(v_1, v_2, \ldots, v_{p-1}, g, h) = (s_+h - s_-g, v_1, \ldots, v_{p-1})$. \hfill $\square$

We now explain why the Higgs bundles we have constructed correspond to the extra components described by Collier in [10]. For simplicity we will assume that $b \neq 0$. The case $b = 0$ is similar but more complicated, as the corresponding component is singular. In Theorem 5 the Higgs bundles were constructed under the assumption that $(a_1, \ldots, a_p)$ is a point in the regular locus of the Hitchin base. However, the formulas for $(W, V, Q_V, Q, \beta, \gamma)$ in Theorem 5 clearly make sense for any collection of differentials $(a_1, \ldots, a_{p-1}) \in \bigoplus_{j=1}^{p-1} H^0(\Sigma, K^{2j})$, any degree $b$ line bundle $B$, with $0 < b \leq 2p(g-1)$, and any pair of sections $s_+ \in H^0(\Sigma, BK^p)$, $s_- \in H^0(\Sigma, B^pK^p)$.

We claim that the resulting Higgs bundles are stable whenever $s_- \neq 0$. To show this, we use the same argument as Hitchin used for the Hitchin component [23]; see also [10 Lemma 4.5]. First one considers the Higgs bundles corresponding to $(a_1, \ldots, a_{p-1}, s_+, s_-) = (0, \ldots, 0, s_-)$ for some nonzero $s_- \in H^0(\Sigma, B^pK^p)$. This has the form

$$B \xrightarrow{-s_-} K^{p-1} \xrightarrow{1} K^{p-2} \xrightarrow{1} \cdots \xrightarrow{1} K^{-(p-1)} \xrightarrow{-s_-} B^*.$$

This Higgs bundle is shown to be stable in [10 Lemma 4.3]. Since stability is an open condition, it follows that the Higgs bundle corresponding to any $(a_1, \ldots, a_{p-1}, s_+, s_-)$ with $s_- \neq 0$ and $(a_1, \ldots, a_{p-1}, s_+)$ sufficiently small will be stable. Next, for $\lambda \in \mathbb{C}^*$ one considers the gauge transformation which acts on $W$ as diag$(\lambda^{p-1}, \lambda^{p-3}, \ldots, \lambda^{-(p-1)})$, on $W_0$ as diag$(\lambda^{p-2}, \lambda^{p-4}, \ldots, \lambda^{-(p-2)})$, and on $B \oplus B^*$ as diag$(\lambda^p, \lambda^{-p})$. One checks that performing this gauge transform and rescaling the Higgs field by $\lambda$ has the effect of replacing $(a_1, \ldots, a_{p-1}, s_+, s_-)$ with $(\lambda^2a_1, \lambda^4a_2, \ldots, \lambda^{2p-2}a_{p-1}, \lambda^{2p}s_+, s_-)$. Since gauge transformations and $\mathbb{C}^*$-rescalings of the Higgs field do not affect stability, this shows that all such Higgs bundles with $s_- \neq 0$ are stable.

From the above discussion we have constructed a space of stable $SO(p + 1, p)$-Higgs bundles, as in [10]. Moreover, our parameterization is by the same space as in [10], where it is denoted as $F_b \times \bigoplus_{j=1}^{p-1} H^0(\Sigma, K^{2j})$. One can argue exactly as in [10] that this space is connected (for fixed $b$), open and closed as a subset of the moduli space, and hence defines a connected component. To see that the components we have constructed are the same as those constructed by Collier, it is
enough to show they have at least some Higgs bundles in common. This is easy to check in the case in which \((a_2, \ldots, a_{p-1}, s_+, s_-) = (0, \ldots, 0, s_-)\). Note that in this case we have \(h_0 = 1\) and \(h_i = 0\) for \(i = 1, \ldots, p - 1\). Then it is straightforward to verify that the Higgs bundle constructed in [10, section 4] under the following changes of notation: \(B \mapsto b\), \(d \mapsto -\mu\), \(s_+ \mapsto \nu\), \(a_j \mapsto q_j\), \(V \mapsto W\), \(W \mapsto V\), \(W_0 \mapsto W_0\). We also note here that in the case \(b = 2p(g-1)\) our construction (and that of Collier) reduces to the usual Hitchin component for \(\text{SO}(p+1, p)\).

**Remark** 15. In addition to the components constructed above, [10] constructs another \(2(2^{2g} - 1)\) connected components of the moduli space of \(\text{SO}(p+1, p)\)-Higgs bundles. These are of the form \(W = I \otimes (K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-1)})\), \(V = I \otimes (K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)}) \oplus H = W_0 \oplus H\), where \(I\) is a nontrivial \(\mathbb{Z}_2\)-line bundle on \(\Sigma\), and \(H\) is a rank 2 orthogonal bundle with determinant \(I\). We claim that such Higgs bundles can be obtained through spectral data similar to Theorem 5, except that we take \(L = \pi'(I)\) instead of \(L = \mathcal{O}\). Indeed, if we take \(L = \pi'(I)\), then as in the proof of Theorem 5 we get

\[
W = I \otimes (K^{p-1} \oplus K^{p-3} \oplus \cdots \oplus K^{-(p-1)}),
\]

\[
V^* = I \otimes (K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)}) \oplus (I \otimes K^{-p}) = W_0 \oplus (I \otimes K^{-p}),
\]

where

\[
W_0 = I \otimes (K^{p-2} \oplus K^{p-4} \oplus \cdots \oplus K^{-(p-2)}).
\]

The identification of \(Q_W\) and \(\beta_p\) is essentially the same as in the proof of Theorem 5. Next (still following the proof of Theorem 5), recall that to construct \(V\) one first takes the bundle

\[
V' = (W \otimes K^{-1}) \oplus V_0 = W_0 \oplus (I \otimes K^{-p} \oplus K^{-p}),
\]

which has a degenerate quadratic form of the form

\[
Q_{V'} = Q_{W_0} \oplus \begin{pmatrix} -a_p & 0 \\ 0 & a_p \end{pmatrix}.
\]

Note that this makes sense, since \(I^2 = \mathcal{O}\), so we can view \(-a_p\) as a quadratic form on \(I \otimes K^{-p}\). Next, to get \(V\) from \(V'\), recall that \(\mathcal{O}(V)\) identifies with the sheaf of meromorphic sections of \(V'\) with first order poles at zeros of \(a_p\) lying in certain 1-dimensional subspaces of \(V'\). In fact, these 1-dimensional subspaces are seen to be contained in the subbundle \(H' = (I \otimes K^{-p} \oplus K^{-p})\). Thus, \(V\) will be of the form

\[
V = W_0 \oplus H
\]

for some rank 2 orthogonal bundle \(H\). Since we allow a pole along a 1-dimensional subspace of \(H'\) for each zero of \(a_{2p}\), it follows that \(\mathcal{O}(\det(H))\) can be identified with the sheaf of meromorphic sections of \(\det(H')\) with first order poles over the zeros of \(a_p\). Thus, \(\det(H) = K^{2p} \otimes \det(H') = I\), so \(H\) has determinant \(I\). Arguing as we did above in the case \(I = \mathcal{O}\), it can be shown that our construction coincides with the \(I \neq \mathcal{O}\) components constructed by Collier in [10].

The results of this section suggests the following conjecture for all \(q \geq 1\).

**Conjecture 1.** *The extra components in the moduli space of \(\text{SO}(p+q,q)\)-Higgs bundles conjectured to exist by Guichard and Wienhard [13] Conjecture 5.6 are those connected components containing Higgs bundles whose spectral data \((L,M,\tau)\)
in Theorem 2 has the form \((\pi^* I, O^{\oplus q}, \tau)\), where \(I\) is a nontrivial \(\mathbb{Z}_2\)-line bundle on the Riemann surface \(\Sigma\).

8. Concluding remarks

8.1. Groups of Hermitian type: \(\text{SO}(2+q, 2)\)-Higgs bundles. In the case \(p = 2\) we have the group \(\text{SO}(2+q, 2)\), which is of Hermitian type, and therefore \(\text{SO}(2+q, 2)\)-Higgs bundles \((V, W, \beta)\) carry a Toledo invariant satisfying a Milnor–Wood type inequality. In what follows we show that this has a very concrete interpretation through the methods developed in this paper. By Theorem 2 regular \(\text{SO}(2+q, 2)\)-Higgs bundles are in correspondence with \((L, M, \tau)\) consisting of the following:

(I) \(L \in \text{Jac}(\Sigma)[2]\) is an orthogonal line bundle on the 2-fold cover satisfying Assumption 1 given by \(S = \{\xi^2 + a_1 \xi^{p-1} + a_2 = 0\} \subset \text{Tot}(K^2)\), where \(a_1, a_2 \in H^0(\Sigma, K^{2\ell})\).

(II) \(M\) is an equivariant rank \(q\)-orthogonal bundle on the 2-fold of the Riemann surface \(C = \{\zeta^2 = a_2\} \subset \text{Tot}(K^2)\) of type \((q-1, 1)\) over each ramification point, with a choice of orientation, and such that all invariant isotropic subbundles \(M' \subset M\) have degree \(\leq 0\).

(III) For each zero \(x\) of \(a_2\), an orthogonal isomorphism \(\tau_x : M_x^- \to L_{r'}\), where \(r\) and \(r'\) are the zeros of \(\xi\) and \(\zeta\) lying over \(x\).

From section 3 the data in (I) corresponds to a maximal \(\text{Sp}(4, \mathbb{R})\)-Higgs bundle given by

\[ F = (W \otimes K^{1/2}) \oplus (W \otimes K^{-1/2}), \quad \Phi_F = \begin{pmatrix} 0 & \beta_F \\ \text{Id} & 0 \end{pmatrix}, \quad \text{where } \beta_F = \gamma \circ \beta. \]

From Gothen’s work [16] the moduli space of maximal \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles, or equivalently, of maximal \(\text{Sp}(4, \mathbb{R})\) surface group representations has \(3 \cdot 2^{2g} + 2g - 4\) connected components. Note that \(\omega_1(W) = 0\) if and only if \(W = L \oplus L^*\) for some \(L \in \text{Pic}(\Sigma)\), which we may assume satisfies \(c := \deg(L) \geq 0\). Then there are three types of components:

(a) \(2^{2g}\) Hitchin components \(M_L\) (where the degree of \(L\) is maximal, in which case \(L^2 = K\));

(b) \(2g - 2\) components \(M_{0,c}\) (where \(\omega_1(W) = 0\) and \(c = \deg(L) \in [0, 2g - 2]\) is nonmaximal); and

(c) \(2(2^{2g} - 1)\) components \(M_{\omega_1, \omega_2}\) given by the possible values of \(\langle \omega_1(W), \omega_2(W)\rangle\) with \(\omega_1 \neq 0\).

Note that Higgs bundles for \(\text{SO}_0(2+q, 2)\), the identity component of \(\text{SO}(2+q, 2)\), correspond to the cases in which \(\omega_1(W) = 0\), i.e., components of types (a) and (b). In such cases we have \(W = L \oplus L^*\), and when \(\text{Tr}(\beta F) = 0\) (these are referred to as conformal Higgs bundles in [12]), the induced \(K^2\)-twisted Higgs bundle \((W, \beta_F)\) is then a \(K^2\)-twisted \(\text{SL}(2, \mathbb{R})\)-Higgs bundle (as opposed to \(\text{GL}(2, \mathbb{R})\)).

Via the Cayley type correspondence the above classification of maximal \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles into classes (a)–(c) gives a similar categorization of \(\text{SO}(2+q, 2)\)-Higgs bundles into classes (a)–(c). In order to complete this to a description of connected components, one would also need to understand additional invariants involved in the construction, arising from the quadratic bundle and the extension data.

\[ \text{Recall that the Toledo invariant for such Higgs bundles is defined as } \deg(W \otimes K^{1/2}) = 2g - 2. \]
In the case of maximal Higgs bundles, i.e., those where \((W, \beta_F)\) is of type (a), it was shown in [8] that for \(q \geq 2\) the moduli space of representations into \(\text{SO}_0(2+q, 2)\) with maximal Toledo invariant, i.e., \(\omega_1(W) = 0\) and \(\deg(L) = 2q - 2\), has \(2.2^{2q}\) Hitchin type connected components. Using the description of spectral data in Theorem 2 one can see these components in terms of maximal \(\text{Sp}(4, \mathbb{R})\)-Higgs bundles, where they correspond to the \(2^{2q}\) Hitchin components (the additional factor of 2 comes from the second Stiefel–Whitney class of \(V\)). The \(2g - 2\) components in (b) are those referred to as Gothen components in [11], originally discovered by Gothen in [10]. From Higgs bundles which do not reduce to the identity component of \(\text{SO}(2 + 2q, 2)\), there are \(2(2^{2q} - 1)\) values of \((\omega_1(W), \omega_2(W))\), and from Theorem 2 the other invariants introduced in Theorem 3 should label further components of the moduli space of \(\text{SO}(2 + q, 2)\)-Higgs bundles.

**Remark 16.** Recall that the natural \(\text{Sp}(4, \mathbb{R})\)-Higgs bundle \((F, \Phi_F)\) associated to an \(\text{SO}(2 + q, 2)\)-Higgs bundle \((E, \Phi)\) in section 3 has reduced spectral curve given by \(\bar{S} = \{\xi^2 + a_1 \xi + a_2 = 0\}\). Noting that \(a_1 = -\text{Tr}(\beta_F)\), we see that when the Higgs bundle is conformal, \(a_1 = 0\) and \(\bar{S}\) is given by \(\xi^2 + a_2 = 0\). This is the same as the equation defining the auxiliary spectral curve \(C\), i.e., \(\bar{S} = C\) for conformal \(\text{SO}(2 + q, 2)\)-Higgs bundles.

### 8.2. \(\text{Sp}(2p+2q, 2p)\)-Higgs bundles.

As mentioned in the introduction, most of our results for orthogonal Higgs bundles have corresponding counterparts for symplectic Higgs bundles. In this section we shall discuss these results, but we shall do so briefly since their proofs are very similar to the orthogonal case.

From Definition 2 one finds that an \(\text{Sp}(2p + 2q, 2p)\)-Higgs bundle is a triple \((V, W, \beta)\) given by

1. a rank \(2p + 2q\) symplectic bundle \((V, Q_V)\);
2. a rank \(2p\) symplectic bundle \((W, Q_W)\);
3. a holomorphic bundle map \(\beta : W \to V \otimes K\).

Let \(\gamma : V \to W \otimes K\) be the symplectic transpose of \(\beta\). One can recover the associated \(\text{Sp}(4p + 2q, \mathbb{C})\)-Higgs bundle \((E, \Phi)\) by setting \(E = V \oplus W\) with symplectic form

\[
((x, y), (x', y')) = Q_V(x, x') - Q_W(y, y'),
\]

and Higgs field \(\Phi : E \to E \otimes K\) given by

\[
\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.
\]

The characteristic polynomial of \(\Phi\) is of the form

\[
\det(\eta - \Phi) = \eta^{2q}(\eta^{2p} + a_1 \eta^{2p-2} + \cdots + a_{p-1} \eta^2 + a_p)^2.
\]

We define the spectral curves \(S\) and \(\bar{S}\) exactly as in the orthogonal case. We suppose that Assumption [1] holds, in particular that \(S\) and \(\bar{S}\) are smooth. Following section 2 define \(V_0\) and \(V_1\) as in the orthogonal case, as well as maps \(\gamma_+\) and \(\beta_+\) and commutative diagrams as in Proposition [1].

Given an \(\text{Sp}(2p + 2q, 2p)\)-Higgs bundle \((V, Q_V, W, Q_W, \beta)\) satisfying Assumption [1] we let \(\beta_F = \gamma_+ \beta : W \to W \otimes K^2\), as before. One finds that \(\beta_F\) is symmetric with respect to \(Q_W\), and hence in this case \((W, Q_W, \beta_F)\) is a \(K^2\)-twisted \(\text{GL}(p, \mathbb{H})\)-Higgs bundle. In other words, \(W\) is a rank \(2p\) holomorphic vector bundle, \(Q_W\) is a holomorphic symplectic form on \(W\), and \(\beta_F\) is a \(K^2\)-valued endomorphism of
$Q_W$ which is symmetric with respect to $Q_W$. This is the Cayley data in the symplectic case. As in [25], under Assumption 1 it can be shown that $(W, Q_W, \beta_F)$ corresponds to a principal $Sp(2, \mathbb{C})$-bundle on $\bar{S}$, in other words, a rank 2 symplectic vector bundle $L \to \bar{S}$, which is the symplectic analogue of an orthogonal line bundle.

Next, consider $(V_0, Q_0)$, where $Q_0 = Q_V|_{V_0}$. This is a skew-symmetric quadratic bundle, i.e., the skew-symmetric analogue of a quadratic bundle as defined previously. Over each zero $x$ of $a_p$, the null space of $Q_0$ is a 2-dimensional symplectic subspace $N_x \subseteq (V_0)_x$. Considering again the auxiliary double cover $\pi_C : C \to \Sigma$, one finds that $V_0$ corresponds to an equivariant symplectic bundle $(M, Q_M, \sigma_C)$ on $C$ such that the $-1$-eigenspace $M^-_\tau$ of $\sigma_C$ over a ramification point $r \in C$ is a 2-dimensional symplectic space.

The extension data $\tau$ needed to reconstruct $V$ as an extension of $V_1$ by $V_0$ is easily seen to consist of symplectomorphisms

$$\tau_x : M^-_\tau \to L_r$$

of 2-dimensional symplectic spaces. In particular, for each zero $x$ of $a_p$, the space of such isomorphisms is a torsor over $Sp(2, \mathbb{C}) \cong SL(2, \mathbb{C})$. This is the Langlands data of Theorem 2 in the symplectic case.

**Remark 17.** An interesting point of contrast between the orthogonal and symplectic cases is that the extension data in the orthogonal case uses the group $O(1, \mathbb{C}) \cong \{\pm 1\}$, which is disconnected, while in the symplectic case the extension data use $Sp(2, \mathbb{C}) \cong SL(2, \mathbb{C})$, which is connected. In particular, this explains the absence of any “extra” components in the moduli space of $Sp(2p + 2q, 2p)$-Higgs bundles [15].

**Remark 18.** The case of $q = 0$ is not a split real form, and for these $Sp(2p, 2p)$-Higgs bundles the spectral data was described in [25, 26]. Here the intersection of the moduli space with the regular fibers is given by a $\mathbb{Z}_2$-quotient of a moduli space of semistable rank 2 parabolic bundles on $\bar{S}$, and it corresponds to the $K^2$-twisted $GL(p, \mathbb{H})$-Higgs bundle mentioned above.

### 8.3. Langlands duality

The appearance of Higgs bundles (and flat connections) within string theory and the geometric Langlands program have led researchers to study the derived category of coherent sheaves and the Fukaya category of these moduli spaces. Therefore, it has become fundamental to understand Lagrangian submanifolds of the moduli space of Higgs bundles supporting flat bundles (A-branes) and their dual objects (B-branes).

We conclude the paper with some comments on Langlands duality. This section will be conjectural, as it is currently not understood how the duality should work over singular fibers of the Hitchin fibration.

Let $LG_C$ denote the Langlands dual group of $G_C$. There is a natural identification of invariant polynomials for $G_C$ and $LG_C$, giving an identification $A_{G_C} \cong A_{LG_C}$ of the Hitchin bases. The two moduli spaces $\mathcal{M}_{G_C}$ and $\mathcal{M}_{LG_C}$ are then torus fibrations over a common base, and their nonsingular fibers are dual abelian varieties [13]. Kapustin and Witten give a physical interpretation of this in terms of S-duality, using it as the basis for their approach to the geometric Langlands program [27]. In this approach a crucial role is played by the various types of branes and their transformation under mirror symmetry. While it is understood that Langlands duality exchanges brane types, the exact correspondence is not yet known. Recall
that the moduli space of Higgs bundles for a complex reductive group carries a natural hyperkähler structure \[21\]. So on each of these moduli spaces there are three complex structures \((I, J, K)\) and three symplectic forms \((\omega_I, \omega_J, \omega_K)\). We may therefore speak of branes which are of type \(A\) (Lagrangian submanifold) or type \(B\) (complex submanifold) with respect to each of the three structures. This leads to branes of type \((B, A, A), (A, B, A), (A, A, B),\) and \((B, B, B)\). Let \(G\) be a real form of \(G_C\). Then \(M_G \subset M_{G_C}\) is the support of a \((B, A, A)\)-brane. In \[5\] we made a conjecture for the support of the dual brane, which was motivated by a paper of Nadler’s \[29\], and which we shall recall here for completeness.

**Conjecture 2 (Baraglia and Schaposnik \[5\]).** The support of the dual brane to the \((B, A, A)\) brane \(M_G\) is the moduli space \(\tilde{M}_G \subset M_{G_C}\) of \(\tilde{H}\)-Higgs bundles, where \(\tilde{H}\) is the group associated to the Lie algebra \(\mathfrak{h}\) in \[29\] Table 1.

In the case of \(G = SO(p + q, p)\), the conjecture is as follows.

**Conjecture 3 (Baraglia and Schaposnik \[5\]).** The \((B, A, A)\)-brane of \(M_{SO(p+q,p)}\) inside \(M_{SO(2p+q,C)}\) has a dual \((B,B,B)\)-brane in the Langlands dual moduli space whose support consists of similar spaces embedded through different maps:

- For \(q\) odd the dual support of the \((B,B,B)\)-brane is
  \[ M_{Sp(2p,C)} \subset M_{Sp(2p+q-1,C)}. \]

- For \(q\) even the dual support of the \((B,B,B)\)-brane is
  \[ M_{SO(2p+1,C)} \subset M_{SO(2p+q,C)}. \]

It is interesting to note that, aside from parity, the conjectured support does not depend on \(q\); only the space in which the brane is embedded depends on \(q\). It therefore seems natural to further conjecture as follows.

**Conjecture 4.** The hyperholomorphic sheaf supported on the spaces in Conjecture 3 giving the \((B,B,B)\)-brane, is also independent of the integer \(q\).

It is also interesting to see that the support of the branes for \(q\) odd and even are dual to each other as hyperkähler moduli spaces of complex Higgs bundles.

**Remark 19.** One should note that, in contrast with the \((A,B,A)\)- and \((A,A,B)\)-branes considered in \[3\], for any \(q > 1\) the \((B,A,A)\)-branes studied in this paper lie completely over the singular locus of the \(SO(2p+q,C)\)-Hitchin fibration.

**References**


School of Mathematical Sciences, The University of Adelaide, South Australia
University of Illinois at Chicago, Chicago, Illinois 60607; and FU Berlin, 14195 Berlin, Germany

Email address: david.baraglia@adelaide.edu.au

Email address: schapos@uic.edu